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CONCRETE EXAMPLES OF $\mathscr{H}(b)$ SPACES

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Abstract. In this paper we give an explicit description of de Branges-Rovnyak spaces $\mathscr{H}(b)$ when b is of the form q^r, where q is a rational outer function in the closed unit ball of H^∞ and r is a positive number.

1. Introduction

The purpose of this paper is to explicitly describe the elements of the de Branges-Rovnyak space $\mathscr{H}(b)$ for certain $b \in \mathbf{b}(H^\infty)$. Here H^∞ denotes the space of bounded analytic functions on the open unit disk $\mathbb{D}$ normed by $\|f\|_\infty := \sup_{z\in\mathbb{D}} |f(z)|$, and $\mathbf{b}(H^\infty) := \{g \in H^\infty : \|g\|_\infty \leqslant 1\}$ is the closed unit ball in H^∞ and, for $b \in \mathbf{b}(H^\infty)$, the *de Branges-Rovnyak space* $\mathscr{H}(b)$ is the reproducing kernel Hilbert space of analytic functions on $\mathbb{D}$ whose kernel is

$$k_\lambda^b(z) := \frac{1 - \overline{b(\lambda)}b(z)}{1 - \overline{\lambda}z}, \qquad \lambda, z \in \mathbb{D}.$$

Besides possessing a fascinating internal structure [9], $\mathscr{H}(b)$ spaces play an important role in several aspects of function theory and operator theory, most importantly, in the model theory for many types of contraction operators [3, 4].

Despite the important role $\mathscr{H}(b)$ spaces play in operator theory, the exact contents of $\mathscr{H}(b)$ often remain mysterious. What functions belong to $\mathscr{H}(b)$? Certainly the kernel functions $k_\lambda^b, \lambda \in \mathbb{D}$, do (and have dense linear span). What else?

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In this paper, we give a precise description of the elements of $\mathscr{H}(b)$ for certain relatively simple b, namely positive powers of rational outer functions. Our description needs the following set up. If $b \in \mathbf{b}(H^\infty)$ is a non-extreme point of $\mathbf{b}(H^\infty)$, equivalently, $\log(1-|b|) \in L^1(\mathbb{T}, m)$ (where $\mathbb{T} := \{\zeta \in \mathbb{C} : |\zeta| = 1\}$ and m Lebesgue measure on $\mathbb{T}$ normalized so that $m(\mathbb{T}) = 1$), then there exists a unique outer function $a \in \mathbf{b}(H^\infty)$, called the *Pythagorean mate* for b, such that $a(0) > 0$ and $|a|^2 + |b|^2 = 1$ almost everywhere on $\mathbb{T}$. The pair (a, b) is said to be a *Pythagorean pair*.

Our first observation says that in certain situations $\mathscr{H}(b^r)$ does not depend on $r > 0$.

Theorem 1.1.

(1) *Suppose $b \in \mathbf{b}(H^\infty)$ is outer. The following are equivalent:*

   (a) *For any $r > 0$ we have $\mathscr{H}(b^r) = \mathscr{H}(b)$ as sets.*

   (b) *$\mathscr{H}(b^2) = \mathscr{H}(b)$ as sets.*

   (c) *$b\mathscr{H}(b) \subset \mathscr{H}((b)$.*

(2) *If b is non-extreme, i.e., $\log(1-|b|) \in L^1(\mathbb{T})$, with Pythagorean mate a, then conditions (a), (b), and (c) are equivalent to the condition*

$$\inf\{|a(z)| + |b(z)| : z \in \mathbb{D}\} > 0. \tag{1.2}$$

(3) *If b extreme, i.e., $\log(1-|b|) \notin L^1(\mathbb{T})$, then conditions (a), (b), and (c) are equivalent to the condition*

$$b \textit{ is invertible in } H^\infty. \tag{1.3}$$

Remark 1.4. (1) Since b is outer, it has no zeros on $\mathbb{D}$ and so we can define b^r by taking any logarithm of b. Note that $b^r \in \mathbf{b}(H^\infty)$.

(2) Statement (a) of Theorem 1.1 says that $\mathscr{H}(b^r) = \mathscr{H}(b)$ as sets. Though the norms on $\mathscr{H}(b^r)$ and $\mathscr{H}(b)$ are different, one sees from the closed graph theorem that they are equivalent.

(3) Statement (c) of the theorem says that b is a *multiplier* of $\mathscr{H}(b)$. We refer the reader to Sarason's book [9] for further information and references about multipliers of $\mathscr{H}(b)$.

(4) By Carleson's corona theorem [6], the condition (1.2) is equivalent to existence of $\phi, \psi \in H^\infty$ so that $a\phi + b\psi = 1$ on $\mathbb{D}$. Such a pair (a, b) satisfying this condition is called a *corona pair*.

When b is a *rational* outer function, or any positive power of a rational function (which is necessarily non-extreme (see Lemma 3.1)), we obtain the following complete description of $\mathscr{H}(b)$ involving the derivatives of the reproducing kernels. Indeed, when $b = q^r$, where q is outer and rational and $r > 0$, we set

$$v_{r,\lambda}^{\ell}(z) := \frac{d^{\ell}}{d\overline{\lambda}^{\ell}} k_{\lambda}^{q^r}(z) = \frac{d^{\ell}}{d\overline{\lambda}^{\ell}} \left(\frac{1 - \overline{q^r(\lambda)} q^r(z)}{1 - \overline{\lambda} z} \right),$$

for any $z \in \mathbb{D}$, $\lambda \in \mathbb{D}^-$, and $\ell \geqslant 0$. We let H^2 denote the classical Hardy space [6]. By means of the Féjer-Riesz theorem (see Section 6), one can prove that if q is a rational function then so is its Pythagorean mate a. In this case, also notice that for $\zeta \in \mathbb{T}$ we have $|q(\zeta)| = 1$ if and only if $a(\zeta) = 0$.

Theorem 1.5. *Suppose $q \in \mathbf{b}(H^\infty)$ is a rational outer function and r is a positive real number. Then*

(1) *$\mathscr{H}(q^r) = \mathscr{H}(q)$ as sets.*

(2) *If a is the Pythagorean mate for q and a has distinct zeros $\zeta_1, \ldots, \zeta_n$ on $\mathbb{T}$ with corresponding multiplicities $m_1, \ldots, m_n$, then*

   (a) *the functions $v_{r,j}^{\ell} := v_{r,\zeta_j}^{\ell}$ are well-defined and belong to $\mathscr{H}(q^r)$ for $1 \leqslant j \leqslant n$ and $0 \leqslant \ell \leqslant m_j - 1$. Moreover, they are orthogonal to*

$$aH^2 = \left(\prod_{j=1}^{n} (z - \zeta_j)^{m_j} \right) H^2.$$

   (b) *$\mathscr{H}(q^r)$ is equal to*

$$\left(\prod_{j=1}^{n} (z - \zeta_j)^{m_j} \right) H^2 \oplus \bigvee \left\{ v_{r,j}^{\ell} : 0 \leqslant \ell \leqslant m_j - 1, 1 \leqslant j \leqslant n \right\},$$

   where the orthogonal decomposition is in terms of the inner product in $\mathscr{H}(q^r)$.

Writing $v_j^{\ell} = v_{1,j}^{\ell}$, the theorem above implies that $\mathscr{H}(q^r)$ is equal to

$$\left(\prod_{j=1}^{n} (z - \zeta_j)^{m_j} \right) H^2 \dotplus \bigvee \left\{ v_j^{\ell}(z) : 0 \leqslant \ell \leqslant m_j - 1, 1 \leqslant j \leqslant n \right\},$$

where the sum is no longer necessarily orthogonal.

It was shown in [2], and rediscovered in [1], that

$$\mathscr{H}(q) = \left(\prod_{j=1}^{n}(z-\zeta_j)^{m_j}\right) H^2 \dotplus \mathscr{P}_{N-1}, \tag{1.6}$$

where $N = \sum_{j=1}^{n} m_j$, $\mathscr{P}_{N-1}$ is the N-dimensional vector space of polynomials of degree at most $N-1$, and the sum is an algebraic direct sum (not necessarily an orthogonal one). The novelty of our result is that we can precisely identify the orthogonal complement of $aH^2 = \left(\prod_{j=1}^{n}(z-\zeta_j)^{m_j}\right) H^2$ in $\mathscr{H}(q)$ without using (1.6).

In a recent preprint, Lanucha and Nowak [7] examined when an $\mathscr{H}(b)$ space is isomorphic to a Dirichlet type space. Their discussion naturally leads to the situation when a is a polynomial with simple zeros on $\mathbb{T}$ and a similar description of $\mathscr{H}(b)$ for such a.

A key ingredient used to show statement (1) of Theorem 1.5, and an added bonus to our result, is that if a_r is the Pythagorean mate for q^r then the co-analytic Toeplitz operators $T_{\overline{a}}$ and $T_{\overline{a_r}}$ on H^2 have the same range, namely $\mathscr{H}(q)$.

2. Preliminaries

There are several equivalent definitions of the de Branges-Rovnyak space $\mathscr{H}(b)$. We can, for instance, define it in the standard way [8] as the reproducing kernel Hilbert space associated with the (positive definite) reproducing kernel

$$k_{\lambda}^{b}(z) := \frac{1-\overline{b(\lambda)}b(z)}{1-\overline{\lambda}z}, \qquad \lambda, z \in \mathbb{D}.$$

By definition, $f(\lambda) = \langle f, k_{\lambda}^{b}\rangle_b$ for all $f \in \mathscr{H}(b)$ and $\lambda \in \mathbb{D}$, where $\langle \cdot,\cdot\rangle_b$ represents the scalar product in $\mathscr{H}(b)$.

The space $\mathscr{H}(b)$ can also be defined as the range space $(I - T_b T_{\overline{b}})^{1/2} H^2$ equipped with the norm which makes $(I - T_b T_{\overline{b}})^{1/2}$ a partial isometry. Here T_{φ} is the Toeplitz operator on H^2 with symbol $\varphi \in L^{\infty}(\mathbb{T})$ defined by

$$T_{\varphi} f = P_{+}(\varphi f), \qquad f \in H^2,$$

where P_+ is the orthogonal projection of $L^2(\mathbb{T})$ onto H^2. The book [9] is the classic reference for $\mathscr{H}(b)$ spaces.

When $\|b\|_\infty < 1$, $\mathscr{H}(b)$ turns out to be a renormed version of H^2 while if b is an inner function, then $\mathscr{H}(b)$ turns out to be one of the classical and well-studied model spaces $H^2 \ominus bH^2$.

When b is non-extreme and a is its Pythagorean mate, two important (not necessarily closed) vector spaces of functions in $\mathscr{H}(b)$ are

$$\mathscr{M}(a) := T_a H^2 \quad \text{and} \quad \mathscr{M}(\overline{a}) := T_{\overline{a}} H^2.$$

It follows from the Douglas factorization theorem and the operator inequalities

$$T_a T_{\overline{a}} \leqslant T_{\overline{a}} T_a \quad \text{and} \quad T_{\overline{a}} T_a = I - T_{\overline{b}} T_b \leqslant I - T_b T_{\overline{b}} \tag{2.1}$$

that $\mathscr{M}(a) \subset \mathscr{M}(\overline{a}) \subset \mathscr{H}(b)$ (see [9, p. 24]).

For technical reasons, we will make use of the space $\mathscr{H}(\overline{b})$ which, for any $b \in \mathbf{b}(H^\infty)$, is defined similarly as with $\mathscr{H}(b)$ but as the range space $(I - T_{\overline{b}} T_b)^{1/2} H^2$. The operator inequalities from (2.1) show that $\mathscr{H}(\overline{b})$ is contractively contained in $\mathscr{H}(b)$.

3. Corona pairs

This following lemma is well-known but we record it here along with a proof for the sake of completeness and for the discussion of the examples in Section 6.

Lemma 3.1. *Suppose $q \in \mathbf{b}(H^\infty)$ is rational and not inner. Then q is non-extreme and, if a is the Pythagorean mate for q, then a is also rational.*

Proof. Since q is rational then $q = p_1/p_2$ where p_1 and p_2 are analytic polynomials and p_2 has no zeros on $\mathbb{D}^-$. We can, of course, choose p_2 such that $p_2(0) > 0$. Since $q \in \mathbf{b}(H^\infty)$, we see that $1 - |q(e^{i\theta})|^2 \geqslant 0$ for all θ and so $|p_2(e^{i\theta})|^2 - |p_1(e^{i\theta})|^2$ is a non-negative trigonometric polynomial. Furthermore, $|p_2(e^{i\theta})|^2 - |p_1(e^{i\theta})|^2$ is not the zero function since we are assuming that q is not an inner function. By the Féjer-Riesz theorem, $|p_2(e^{i\theta})|^2 - |p_1(e^{i\theta})|^2 = |p(e^{i\theta})|^2$, where p is an analytic polynomial which is zero free in $\mathbb{D}$ and $p(0) > 0$.

Let $a = p/p_2$. Note that a is rational and zero free in $\mathbb{D}$, hence outer. Moreover, $a(0) > 0$.

Furthermore, on $\mathbb{T}$ we have

$$|a|^2 = \left|\frac{p}{p_2}\right|^2 = \frac{|p_2|^2 - |p_1|^2}{|p_2|^2} = 1 - \left|\frac{p_1}{p_2}\right|^2 = 1 - |q|^2.$$

This means that (a, q) is a Pythagorean pair which, in particular, implies that q is non-extreme. $\square$

Lemma 3.2. *Suppose $b \in \mathbf{b}(H^\infty)$ is outer and r is a positive real number. Then b and b^r are simultaneously non-extreme. Moreover, if a_r is the Pythagorean mate for b^r, the pairs (a, b) and (a_r, b^r) are simultaneously corona.*

Proof. Since

$$\frac{1 - x^r}{1 - x} \asymp 1, \quad x \in [0, 1), \tag{3.3}$$

we see that $1 - |b|^r \asymp 1 - |b|$ when $b \in \mathbf{b}(H^\infty)$, from which we deduce the first part of the Lemma.

Now observe that

$$\frac{|a|^2}{|a_r|^2} = \frac{1 - |b|^2}{1 - |b^2|^r} \asymp 1,$$

and since a and a_r are outer, Smirnov's theorem (which says that if the boundary function for the quotient of two outer functions is bounded on $\mathbb{T}$, then $f \in H^\infty$), shows that a/a_r is invertible in H^∞. Thus both expressions

$$\inf_{z \in \mathbb{D}} (|a(z)| + |b(z)|) \quad \text{and} \quad \inf_{z \in \mathbb{D}} (|a_r(z)| + |b^r(z)|)$$

are strictly positive (or not) simultaneously. Indeed, if there is a sequence $\{z_n\}_{n \geqslant 1}$ in $\mathbb{D}$ such that one expression goes to 0 then, since both $a(z_n)$ and $b(z_n)$ go to zero, the other expression will go to zero as well. $\square$

A special situation where b forms a corona pair with its Pythagorean mate is when b is rational.

Lemma 3.4. *Suppose $q \in \mathbf{b}(H^\infty)$ is rational and not inner. If a is the Pythagorean mate for q, then (a, q) is a corona pair.*

Proof. According to the proof of Lemma 3.1, we know that a is rational, $a = p/p_2$, where p and p_2 are polynomials, p_2 has no zeros in $\mathbb{D}^-$ and p is zero free in $\mathbb{D}$. In particular, a is analytic in an open neighborhood of $\mathbb{D}^-$ and thus has a finite number of zeros on $\mathbb{T}$, say $\{\zeta_1, \dots, \zeta_n\}$. Note that, due to the identity $|a|^2 + |q|^2 = 1$ on $\mathbb{T}$, the zeros of a (on $\mathbb{T}$) must lie where q is unimodular on $\mathbb{T}$.

Let D_j be disjoint open disks with center at the zeros ζ_j of a and let

$$F = \mathbb{D}^- \setminus \bigcup_{j=1}^{n} D_j.$$

By making the disks smaller, one can, by using the continuity of $|q|$ on $\mathbb{D}^-$, arrange things so that $|q| \geqslant \frac{1}{2}$ on each $D_j \cap \mathbb{D}^-$.

Notice that F is closed and omits all of the zeros of a in $\mathbb{D}^-$ and so

$$\inf_{z \in F} |a(z)| = \delta > 0.$$

Thus

$$\inf_{z \in \mathbb{D}} (|a(z)| + |q(z)|) \geqslant \min(\tfrac{1}{2}, \delta) > 0$$

concluding the proof. □

The first statement of Theorem 1.1 depends on the following two results. The first is from Sarason's book [9, p. 62].

Proposition 3.5. *For $b \in \mathbf{b}(H^\infty)$ and non-extreme, the following are equivalent:*

(1) *(a, b) is a corona pair;*

(2) *$\mathscr{H}(b) = \mathscr{M}(\overline{a})$.*

The second is the following.

Proposition 3.6. *If $a, a_1 \in H^\infty$ are two outer functions such that a/a_1 and a_1/a belong to L^∞, then $\mathscr{M}(\overline{a}) = \mathscr{M}(\overline{a_1})$.*

Proof. Again, by Smirnov's theorem, we know that a/a_1 and a_1/a belong to H^∞. Thus T_{a/a_1}, and hence $T_{\overline{a/a_1}}$, are invertible operators on H^2. From here we get

$$\mathscr{M}(\overline{a}) = T_{\overline{a}} H^2 = T_{\overline{a_1}} T_{\overline{a/a_1}} H^2 = T_{\overline{a_1}} H^2 = \mathscr{M}(\overline{a_1}). \quad \square$$

4. $\mathscr{H}(b^r) = \mathscr{H}(b)$ AS SETS

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. The implication $(a) \implies (b)$ is trivial.

To show $(b) \implies (c)$ note from [9, I-10] we have

$$\mathscr{H}(b^2) = \mathscr{H}(b) + b\mathscr{H}(b).$$

But since we are assuming that $\mathscr{H}(b^2) = \mathscr{H}(b)$ is follows that $b\mathscr{H}(b) \subset \mathscr{H}(b)$.

For the implication $(c) \implies (1.2)$, we use the fact that b is non-extreme and [9, VIII-1, VIII-7] to see that b being a multiplier of $\mathscr{H}(b)$ is equivalent to (a, b) being a corona pair.

To show that $(1.2) \implies (a)$, we proceed as follows. By Lemma 3.2 we know that since (a, b) is a corona pair, then so is (a_r, b^r). Thus from Proposition 3.5 we see that $\mathscr{H}(b) = \mathscr{M}(\overline{a})$ and $\mathscr{H}(b^r) = \mathscr{M}(\overline{a_r})$. As in the proof of Lemma 3.2 a/a_r and a_r/a belong to H^∞ so, by Proposition 3.6, we get $\mathscr{M}(\overline{a}) = \mathscr{M}(\overline{a_r})$. Putting this all together we get the desired set equality $\mathscr{H}(b^r) = \mathscr{H}(b)$.

For the implication $(c) \implies (1.3)$, we use the fact that b is extreme and [9, VIII-1, VIII-5] to see that b being a multiplier of $\mathscr{H}(b)$ is equivalent to b being an invertible element of H^∞.

It remains to show $(1.3) \implies (a)$. Assuming that b is invertible in H^∞, we use, once again, [9, VIII-1] to see that $\mathscr{H}(b) = \mathscr{H}(\bar{b})$. But, since b is invertible in H^∞, then so is b^r and we thus also have $\mathscr{H}(b^r) = \mathscr{H}(\overline{b^r})$. Remember that

$$\frac{1 - |b|^2}{1 - |b^r|^2} \asymp 1$$

and thus there are two constants $c_1, c_2 > 0$ such that

$$c_1(I - T_{\overline{b}} T_b) \leqslant I - T_{\overline{b^r}} T_{b^r} \leqslant c_2(I - T_{\overline{b}} T_b).$$

The Douglas factorization theorem implies that $\mathscr{H}(\bar{b}) = \mathscr{H}(\overline{b^r})$ which concludes the proof. □

Remark 4.1. In Theorem 1.1 we see from the above proofs that one can add the condition $\mathscr{H}(b) = \mathscr{H}(\overline{b})$ to the list of equivalent conditions.

5. The contents of $\mathscr{H}(b)$

We now can give the proof of Theorem 1.5. Indeed, statement (1) of the theorem follows from Lemma 3.4 and Theorem 1.1. Let us consider statement (2).

In [5] it was shown, for an outer function b, that if $\zeta \in \mathbb{T}$ and

$$\int_{\mathbb{T}} \frac{|\log|b(w)||}{|w - \zeta|^{2n+2}} dm(w) < \infty, \tag{5.1}$$

then every function in $\mathscr{H}(b)$, as well as its derivatives up to order n, has a finite non-tangential limit at ζ.

Recalling the notation $v_{r,\lambda}^{\ell}$ for the ℓ-th derivative in the variable $\overline{\lambda}$ of the reproducing kernel in $\mathscr{H}(q^r)$, the results of [5] also show that $v_{r,\zeta}^{\ell} \in \mathscr{H}(q^r)$, $0 \leqslant \ell \leqslant n$, and

$$f^{(\ell)}(\zeta) = \langle f, v_{r,\zeta}^{\ell} \rangle_{q^r}, \quad f \in \mathscr{H}(q^r),\ 0 \leqslant \ell \leqslant n.$$

Let us check condition (5.1) for our situation. Since q is rational its Pythagorean mate a is also rational and can be written as

$$a(z) = s(z) \prod_{j=1}^{n} (z - \zeta_j)^{m_j}, \tag{5.2}$$

where s is a rational function whose poles and zeros lie on the complement of $\mathbb{D}^-$. Pick $w = e^{it}$ near one of the zeros $\zeta_j = e^{i\theta_j}$ of a. Then

$$\begin{aligned} |\log |q^r(e^{it})|| &\asymp |\log |q(e^{it})|^2| = |\log(1 - |a(e^{it})|^2)| \\ &\asymp |a(e^{it})|^2 \asymp |e^{it} - e^{i\theta_j}|^{2m_j} \end{aligned}$$

This means that for t near θ_j we have

$$\frac{|\log |q^r(e^{it})||}{|e^{it} - e^{i\theta_j}|^{2(m_j-1)+2}} \asymp 1$$

and so, by (5.1), every function in $\mathscr{H}(b)$ as well as its derivatives up the order $m_j - 1$ admits non-tangential limits at ζ_j, and $v_{r,\zeta_j}^{\ell} \in \mathscr{H}(q^r)$ for all $0 \leqslant \ell \leqslant m_j - 1$.

The following interesting observation will be very useful in the proof of our main theorem.

Lemma 5.3. *Suppose $a(z) = \prod_{j=1}^{n} (z - \zeta_j)^{m_j}$, where $\zeta_j \in \mathbb{T}$ and m_j is the corresponding multiplicity. If the non-tangential limits if an $f = T_{\overline{a}} g \in \mathscr{M}(\overline{a})$, along with the non-tangential limits of its derivatives up to order $m_j - 1$, vanish at every point ζ_j, $j = 1, \ldots, n$, then*

$$\widehat{g}(0) = \widehat{g}(1) = \cdots = \widehat{g}(N-1) = 0, \tag{5.4}$$

where $N = \sum_{j=1}^{n} m_j$.

Proof of Lemma. We prove (5.4) as follows. Consider the kernels

$$k_{\lambda,\ell}(z) = c_\ell \frac{z^\ell}{(1 - \overline{\lambda} z)^{\ell+1}},$$

where c_ℓ is adjusted so that these are the reproducing kernels for ℓ-th derivatives at point $\lambda \in \mathbb{D}$ in the Hardy space H^2, that is to say,

$$f^{(\ell)}(\lambda) = \langle f, k_{\lambda,\ell}\rangle_{H^2} = \int_{\mathbb{T}} f(\zeta)\overline{k_{\lambda,\ell}(\zeta)}dm(\zeta), \qquad f \in H^2.$$

Observe, for $1 \leqslant j \leqslant n$ and $0 \leqslant \ell \leqslant m_j - 1$, that

$$\begin{aligned} a(z)k_{t\zeta_j,\ell}(z) &= c_\ell \frac{z^\ell (z-\zeta_j)^{m_j}}{(1-t\overline{\zeta_j}z)^{\ell+1}} \prod_{k\neq j}(z-\zeta_k)^{m_k} \\ &= c_\ell z^\ell (z-\zeta_j)^{m_j-(\ell+1)} \left(\frac{z-\zeta_j}{1-t\overline{\zeta_j}z}\right)^{\ell+1} \prod_{k\neq j}(z-\zeta_k)^{m_k}. \end{aligned}$$

Writing

$$\frac{z-\zeta_j}{1-t\overline{\zeta_j}z} = -\zeta_j\left(1-\overline{\zeta_j}z\frac{1-t}{1-t\overline{\zeta_i}z}\right),$$

we see that $a(z)k_{t\zeta_j,\ell}(z)$ is uniformly bounded in $z \in \mathbb{D}$ and $t \in [0,1)$, and moreover

$$\frac{z-\zeta_j}{1-t\overline{\zeta_j}z} \to -\overline{\zeta_j}, \quad t \to 1,$$

for every z. Thus, by the dominated convergence theorem,

$$ak_{t,\ell} \to cz^\ell (z-\zeta_j)^{m_j-(\ell+1)} \prod_{k\neq j}(z-\zeta_k)^{m_k}$$

in the norm of H^2, where c is some non zero constant depending on ℓ and j.

Choose any function $f = T_{\overline{a}}g \in \mathscr{M}(\overline{a})$ with $(T_{\overline{a}}g)^{(\ell)}(\zeta_j) = 0$ for all $1 \leqslant j \leqslant n$, $0 \leqslant \ell \leqslant m_j - 1$. Recall that $\mathscr{M}(\overline{a}) \subset \mathscr{H}(b)$ and so f, as well as all its derivatives up to order $m_j - 1$, admits non-tangential limits at ζ_j for all $1 \leqslant j \leqslant n$. Then

$$\begin{aligned} 0 = (T_{\overline{a}}g)^{(\ell)}(\zeta_j) &= \lim_{t\to 1^-}(T_{\overline{a}}g)^{(\ell)}(t\zeta_j) = \lim_{t\to 1^-}\langle T_{\overline{a}}g, k_{t\zeta_j,\ell}\rangle_{H^2} \\ &= \lim_{t\to 1^-}\langle g, ak_{t\zeta_j,\ell}\rangle_{H^2} \\ &= \overline{c}\langle g, cz^\ell(z-\zeta_j)^{m_j-(\ell+1)}\prod_{k\neq j}(z-\zeta_k)^{m_k}\rangle_{H^2}. \end{aligned}$$

In order to prove the lemma, it suffices to show that the set

$$\left\{\varphi_{j,\ell}(z) := z^\ell (z-\zeta_j)^{m_j-(\ell+1)}\prod_{k\neq j}(z-\zeta_k)^{m_k}\right\},$$

where $j = 1, \ldots, n$ and $\ell = 0, \ldots, m_j - 1$, is a basis for the space of polynomials of degree at most $N-1$. Clearly each $\varphi_{j,\ell}$ is a polynomial

of degree $N-1$ and there are $N-1$ functions $\varphi_{j,\ell}$. It remains to show that the elements of this family are linearly independent. Obviously, for fixed $1 \leqslant r \leqslant n$ and $0 \leqslant k \leqslant m_r - 1$, we have

$$\varphi_{j,\ell}^{(k)}(\zeta_r) = 0, \quad j \neq r,\ 0 \leqslant k \leqslant m_r - 1,$$

and

$$\varphi_{r,\ell}^{(k)}(\zeta_r) = 0, \quad 0 \leqslant k \leqslant m_r - (\ell + 2). \tag{5.5}$$

In particular, if $\sum_{j,\ell} \alpha_{j,\ell}\varphi_{j,\ell} = 0$, then, for fixed r and $0 \leqslant k \leqslant m_r - 1$, $\sum_{j,\ell} \alpha_{j,\ell}\varphi_{j,\ell}^{(k)}(\zeta_r) = 0$ which reduces to $\sum_{\ell} \alpha_{r,\ell}\varphi_{r,\ell}^{(k)}(\zeta_r) = 0$. Writing $\varphi_{j,\ell}(z) = (z-\zeta_j)^{m_j-(\ell+1)}p_{j,\ell}(z)$, where $p_{j,\ell}$ does not vanish at ζ_j, Leibniz's formula gives

$$\varphi_{j,\ell}(z)^{m_j-(\ell+1)}(z) = \sum_{k=0}^{m_j-(\ell+1)} \binom{m_j-(\ell+1)}{k} \times \frac{(m_j-(\ell+1))!}{(m_j-(\ell+1)-k)!}(z-\zeta_j)^{m_j-(\ell+1)-k}p_{j,\ell}^{(m_j-(\ell+1)-k)}(z).$$

Evaluating this expression at ζ_j makes all terms of the sum vanish except for $k = m_j - (\ell+1)$, and thus

$$\varphi_{j,\ell}(z)^{m_j-(\ell+1)}(\zeta_j) = (m_j-(\ell+1))!p_{j,\ell}(\zeta_j) \neq 0.$$

This together with (5.5) generates a triangular system of linear equations with non-zero diagonal entries. Thus $\alpha_{r,\ell} = 0$, $0 \leqslant \ell \leqslant m_r - 1$. $\square$

We are now in a position to prove Theorem 1.5.

Our arguments so far yield

$$\mathscr{H}(q^r) = \mathscr{M}(\overline{a_r}) \tag{5.6}$$

and

$$\mathscr{M}(\overline{a_r}) = \mathscr{M}(\overline{a}) \supset \mathscr{M}(a) + \bigvee\{v_{r,\zeta_j}^{\ell} : 1 \leqslant j \leqslant n, 0 \leqslant \ell \leqslant m_j - 1\}. \tag{5.7}$$

First we show that the sum is orthogonal in the $\mathscr{H}(q^r)$ inner product:

$$v_{r,\zeta_j}^{\ell} \perp \mathscr{M}(a), \qquad 1 \leqslant j \leqslant n,\ 0 \leqslant \ell \leqslant m_j - 1.$$

Indeed, for each $f \in \mathscr{H}(q^r)$ the radial limits $f^{(\ell)}(t\zeta_j)$ exist as $t \to 1^-$. Since

$$f^{(\ell)}(t\zeta_j) = \langle f, v_{r,t\zeta_j}^{\ell}\rangle_{q^r},$$

we can apply the principle of uniform boundedness to see that $\|v_{r,t\zeta_j}^{\ell}\|_{q^r}$ is uniformly bounded as $t \to 1^-$. Since $v_{r,t\zeta_j}^{\ell}$ converges pointwise to v_{r,ζ_j}^{ℓ} as $t \to 1^-$ we see that $v_{r,t\zeta_j}^{\ell}$ converges weakly to v_{r,ζ_j}^{ℓ}. Thus, since $v_{r,t\zeta_j}^{\ell}$

reproduces the ℓ-th derivative of $\mathscr{H}(q^r)$-functions at point $t\zeta_j$, for any $g \in H^2$, we have

$$\begin{aligned}\langle ag, v^{\ell}_{r,\zeta_j}\rangle_{q^r} &= \lim_{t\to 1^-}\langle ag, v^{\ell}_{r,t\zeta_j}\rangle_{q^r} = \lim_{t\to 1}(ag)^{(\ell)}(t\zeta_j)\\ &= \lim_{t\to 1}\sum_{p=0}^{\ell}\binom{\ell}{p}a^{(p)}(t\zeta_j)g^{(\ell-p)}(t\zeta_j).\end{aligned}$$

Using the estimate

$$|a^{(p)}(t\zeta_j)| \lesssim (1-t)^{m_j-p}$$

along with the following standard H^2 estimate on the growth of the derivative of an H^2 function

$$|g^{(\ell-p)}(t\zeta_j)| \lesssim \frac{1}{(1-t)^{(\ell-p)+1/2}},$$

we see that

$$|a^{(p)}(t\zeta_j)g^{(\ell-p)}(t\zeta_j)| \lesssim (1-t)^{m_j-p-((\ell-p)+1/2)}.$$

But since $0 \leqslant \ell \leqslant m_j - 1$ we see that

$$m_j - p - ((\ell-p)+1/2) = m_j - \ell - \tfrac{1}{2} \geqslant \tfrac{1}{2}$$

and so

$$\lim_{t\to 1^-}|a^{(p)}(t\zeta_j)g^{(\ell-p)}(t\zeta_j)| = 0.$$

Thus $\langle ag, v^{\ell}_{r,\zeta_j}\rangle_{q^r} = 0$ and $v^{\ell}_{r,\zeta_j} \perp \mathscr{M}(a)$ in $\mathscr{H}(q^r)$, for all $0 \leqslant \ell \leqslant m_j-1$.

This upgrades (5.6) and (5.7) to

$$\mathscr{H}(q^r) = \mathscr{M}(\overline{a}) \supset \mathscr{M}(a) \oplus \bigvee\{v^{\ell}_{r,\zeta_j} : 1 \leqslant j \leqslant n, 0 \leqslant \ell \leqslant m_j-1\}, \tag{5.8}$$

and orthogonality is with respect to the norm in $\mathscr{H}(q^r)$.

To show equality in (5.8), our second step is to show that if $f \in \mathscr{M}(\overline{a})$ and $f \perp v^{\ell}_{r,\zeta_j}$ for all $1 \leqslant j \leqslant n, 0 \leqslant \ell \leqslant m_j - 1$, then $f \in \mathscr{M}(a)$. Since $\mathscr{M}(\overline{a}) = T_{\overline{a}}H^2$ this is equivalent to prove that if $g \in H^2$ and

$$0 = (T_{\overline{a}}g)^{(\ell)}(\zeta_j) = \lim_{t\to 1^-}(T_{\overline{a}}g)^{(\ell)}(t\zeta_j)$$

for all $1 \leqslant j \leqslant n, 0 \leqslant \ell \leqslant m_j-1$ then $T_{\overline{a}}g \in \mathscr{M}(a)$. To simplify matters a bit, let us recall the formula for a from (5.2). Since s is a rational function with zeros and poles outside $\mathbb{D}^-$ then certainly the Toeplitz operators $T_{1/s}$ and $T_{\overline{1/s}}$ are invertible, and so $\mathscr{M}(\overline{a}) = \mathscr{M}(\overline{a/s})$. We can therefore make the simplifying assumption that

$$a(z) = \prod_{j=1}^{n}(z-\zeta_j)^{m_j}.$$

We will show that

$$(T_{\overline{a}}g)^{(\ell)}(\zeta_j) = 0,\ 1 \leqslant j \leqslant n, 0 \leqslant \ell \leqslant m-1 \implies T_{\overline{a}}g \in aH^2. \tag{5.9}$$

With $N = \sum_{j=1}^{n} m_j$, one can verify the identify

$$\overline{a(\zeta)} = \overline{\zeta}^N a(\zeta) \prod_{j=1}^{n} (-\overline{\zeta_j})^{m_j}, \qquad \zeta \in \mathbb{T}.$$

Thus

$$T_{\overline{a}}g = \prod_{j=1}^{n} (-\overline{\zeta_j})^{m_j} P_+(a\overline{\zeta}^N g).$$

By Lemma 5.3, we have $\widehat{g}(0) = \widehat{g}(1) = \cdots = \widehat{g}(N-1) = 0$, which shows that $\overline{\zeta}^N g \in H^2$ and so

$$T_{\overline{a}}g = \Big(\prod_{j=1}^{n} (-\overline{\zeta_j})^{m_j}\Big) P_+(a\overline{\zeta}^N g) \in aH^2.$$

This completes the proof. □

6. Examples

Example 6.1. Consider the function

$$q(z) = \frac{1}{2}(1+z)$$

and notice that q is outer and $\|q\|_\infty = 1$. One can easily guess the Pythagorean mate for q to be $a(z) = \frac{1}{2}(1-z)$. The function $a(z)$ has one zero of order 1 at $z = 1$ and a computation reveals that

$$v^0_{1,1}(z) = \frac{1 - \overline{q(1)}q(z)}{1-z} = \frac{1}{2}.$$

In this case

$$\mathscr{H}(q) = (z-1)H^2 \oplus \mathbb{C}.$$

Moreover, for any $r > 0$ we get $\mathscr{H}(q^r) = \mathscr{H}(q)$ and

$$\mathscr{H}(q^r) = (z-1)H^2 \dotplus \mathbb{C} = (z-1)H^2 \oplus \mathbb{C}\frac{1 - \Big(\dfrac{1+z}{2}\Big)^r}{1-z}.$$

For more general q we need to review the proof of the Féjer-Riesz theorem which says that if

$$w(e^{i\theta}) = \sum_{j=-n}^{n} c_j e^{ij\theta}$$

is a non-zero trigonometric polynomial which assumes non-negative values for all θ, then there is an analytic polynomial

$$p(z) = \sum_{j=0}^{n} a_j z^j$$

so that $w(e^{i\theta}) = |p(e^{i\theta})|^2$. Since the proof gives us the algorithm for computing p, we give a quick sketch. Indeed, as a function of the complex variable z, we see that if

$$w(z) = \sum_{j=-n}^{n} c_j z^j$$

then $\overline{w(1/\overline{z})} = w(z)$, $z \in \mathbb{T}$. Assuming that $c_{-n} \neq 0$ we see that $s(z) = z^n w(z)$, $z \in \mathbb{C}$, is a polynomial of degree $2n$ and the roots of s occur in the pairs $\alpha, 1/\overline{\alpha}$ of equal multiplicity. It follows that

$$w(z) = c \prod_{j=1}^{n} (z - \alpha_j)(\frac{1}{z} - \overline{\alpha_j})$$

for some positive constant c and where $\alpha_1, \ldots, \alpha_n$ satisfy $|\alpha_j| \geqslant 1$ for $1 \leqslant j \leqslant n$. The desired polynomial p is

$$p(z) = \sqrt{c} \prod_{j=1}^{n} (z - \alpha_j).$$

Note that p is zero free in $\mathbb{D}$ and we can multiply p by a unimodular constant so that $p(0) > 0$.

Recall from the proof of Lemma (3.1) that if $q = p_1/p_2$ is rational then the Pythagorean mate a for q is given by $a = p/p_2$, where p is the analytic polynomial (guaranteed by the Féjer-Riesz theorem) which satisfies $|p(e^{i\theta})|^2 = w(e^{i\theta}) = |p_2(e^{i\theta})|^2 - |p_1(e^{i\theta})|^2 \geqslant 0$, and p is chosen to that $a(0) > 0$.

Example 6.2. Consider the function

$$q(z) = \frac{1}{2}(1 - z)(1 + z)$$

and note that $q \in \mathbf{b}(H^\infty)$ and is outer. A computation shows that

$$1 - |q(e^{it})|^2 = \frac{1}{4}e^{-2it} + \frac{1}{4}e^{2it} + \frac{1}{2}.$$

Define

$$w(z) = \frac{z^{-2}}{4} + \frac{z^2}{4} + \frac{1}{2}$$

and

$$s(z) = z^2 w(z) = \frac{z^4}{4} + \frac{z^2}{2} + \frac{1}{4} = \frac{1}{4}(z-i)^2(z+i)^2.$$

Notice how the zeros occur in pairs $i = 1/\bar{i}$ and $-i = 1/\overline{-i}$ as guaranteed by the above proof of the Féjer-Riesz theorem. Thus the Pythagorean mate a for q is of the form $a(z) = c(z-i)(z+i)$ for some c adjusted so that $a(0) > 0$ and $1 - |q(e^{i\theta})|^2 = |a(e^{i\theta})|^2$. One can check by direct calculation that $c = 1/2$ works and so $a(z) = \frac{1}{2}(z-i)(z+i)$. Of course the exact value of c is not important for our calculations since we only need to identify the zeros of a along with their multiplicities.

The zeros of a are at $z = i$ and $z = -i$ and each has order one. Thus

$$\mathscr{H}(q) = (z-i)(z+i)H^2 \oplus \bigvee\{v^0_{1,i}, v^0_{1,-i}\},$$

where the kernels can be computed directly as

$$v^0_{1,i}(z) = \frac{1}{2i}(z+i), \qquad v^0_{1,-i}(z) = \frac{1}{2i}(z-i).$$

Again, as in the previous example, $\mathscr{H}(q^r) = \mathscr{H}(q)$ and so

$$\mathscr{H}(q^r) = (z-i)(z+i)H^2 \dot{+} \bigvee\{z+i, z-i\}.$$

Example 6.3. Consider the function

$$q(z) = \frac{1}{4}(z+1)^2$$

and note that q is outer and belongs to $\mathbf{b}(H^\infty)$. Following our Fejer-Riesz computations as in the previous example, note that

$$1 - |q(e^{it})|^2 = -\frac{e^{-it}}{4} - \frac{e^{it}}{4} - \frac{1}{16}e^{-2it} - \frac{1}{16}e^{2it} + \frac{5}{8}.$$

Define

$$w(z) = -\frac{z^2}{16} - \frac{1}{16z^2} - \frac{z}{4} - \frac{1}{4z} + \frac{5}{8}$$

and

$$\begin{aligned} s(z) &= z^2 w(z) \\ &= -\frac{z^4}{16} - \frac{z^3}{4} + \frac{5z^2}{8} - \frac{z}{4} - \frac{1}{16} \\ &= -\frac{1}{16}(-1+z)^2(1+6z+z^2). \end{aligned}$$

The zeros of s are at

$z = -1, z = -1, z = -3-2\sqrt{2} \approx -5.82843, z = -3+2\sqrt{2} \approx -0.171573.$

Notice how these roots occur in the pairs $\alpha, 1/\overline{\alpha}$. The function a is then $a(z) = c(z-1)(z+3+2\sqrt{2})$ for some appropriate constant c. There is one zero of a at $z = 1$ with multiplicity one and so

$$\mathscr{H}(q) = (z-1)H^2 \oplus \mathbb{C}v^0_{1,1}(z).$$

The kernel can be computed to be

$$v^0_{1,1}(z) = \frac{z+3}{4}.$$

As in our previous examples, note that

$$\mathscr{H}(q^r) = (z-1)H^2 \dotplus \mathbb{C}(z+3).$$

Observe that the q from this example is the square of the q from Example 6.1 and thus the corresponding spaces should be the same. Indeed, a little algebra will show that

$$(z-1)H^2 \dotplus \mathbb{C} = (z-1)H^2 \dotplus \mathbb{C}(z+3).$$

Example 6.4. Reversing the roles of a and q in the preceding example:

$$a(z) = \frac{1}{4}(z+1)^2, \qquad q(z) = c(z-1)(z+3+2\sqrt{2}),$$

with suitable c so that $\|q\|_\infty = 1$ (the maximum modulus on $\mathbb{D}^-$ being attained at -1, one has $c = (4(1+\sqrt{2}))^{-1}$, and $q(-1) = -1$ corresponding to the normalization $q(0) > 0$), we obtain a function a with double zero, and so

$$\mathscr{H}(q) = (z+1)^2H^2 \oplus \bigvee\{v^0_{1,-1}, v^1_{1,-1}\}$$

where

$$v^0_{1,-1}(z) = \frac{1-\overline{q(-1)}q(z)}{1-\overline{(-1)}z} = \frac{1+q(z)}{1+z}.$$

Using the facts that $q(-1) = -1$, $q'(z) = c(2+2\sqrt{2})$, and $q'(-1) = -1/2$, we obtain

$$v^1_{1,-1}(z) = \frac{\frac{1}{2}q(z)(1+z) + z(1+q(z))}{(1+z)^2} = \frac{1}{2}\frac{q(z)(1+3z)+2z}{(1+z)^2}.$$

Question 6.5. So far we have computed the exact contents of $\mathscr{H}(b)$ when b outer, rational, and non-extreme. Can one compute the contents of $\mathscr{H}(b)$ when b is outer and extreme. For example if b is the outer function corresponding to the outer function which satisfies $|b(e^{i\theta})| = 1$ for $0 \leqslant \theta \leqslant \pi$ and $|b(e^{i\theta})| = \frac{1}{2}$ for $\pi < \theta < 2\pi$, can one describe the functions in $\mathscr{H}(b)$?

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