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A Convergent Reconstruction Method for an Elliptic Operator in Potential Form

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We investigate the problem of recovering a potential $q(x)$ in the equation $-\Delta u + q(x)u = 0$ from overspecified boundary data on the unit square in R^2. The potential is characterized as a fixed point of a nonlinear operator, which is shown to be a contraction on a ball in C^α. Uniqueness of $q(x)$ follows, as does convergence of the resulting recovery scheme. Numerical examples, demonstrating the performance of the algorithm, are presented. © 1995 Academic Press, Inc.

1. INTRODUCTION

For the unit square $\Omega = (0, 1) \times (0, 1)$, consider the inverse problem of determining the univariate potential $q(x) \in C^\alpha$ ([0, 1]) in the boundary value problem

$$-\Delta u + q(x)u = 0, \qquad (x, y) \in \Omega, \tag{1.1a}$$

$$u(0, y) = f_0(y), \tag{1.1b}$$

$$u(1, y) = f_1(y), \tag{1.1c}$$

$$u_y(x, 0) = g_0(x), \tag{1.1d}$$

$$u_y(x, 1) = g(x), \tag{1.1e}$$

from the *single* overposed data measurement

$$u(x, 1) = h(x). \tag{1.1f}$$

The purpose of this paper is twofold: to show that conditions can be

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given on the boundary data under which the inverse problem (1.1) has at most one solution q, and to produce a convergent numerical scheme for reconstructing q from the overposed data h. Cannon and Rundell [3] proved uniqueness for such a layered potential on the quarter plane $\{(x,y): x,y > 0\}$, but the techniques used are different than those of the present paper.

The approach taken is to characterize the coefficient $q(x)$ as a fixed point of a nonlinear operator T_h, constructed via the fixed point projection (FPP) method of Pilant and Rundell [14, 15]. (For a discussion of the FPP method, the reader is referred to [16].) It is shown that under suitable conditions, T_h is locally a contraction on the Hölder space C^α ([0, 1]). This result has two consequences: identifiability of $q(x)$ from a single data measurement along the top boundary, and convergence of the reconstruction scheme given by

$$q^{(k)} = T_h\,[q^{(k-1)}].$$

We indicate the dependence of the convergence rate on various quantities by explicitly computing a bound on the contraction constant.

We remark that the main results of this paper are achieved by controlling the norm of the overposed data h in the space $C^{2+\alpha}$, a norm only slightly stronger than that on the space C^α, where q is presumed to lie. Thus, this inverse problem is only mildly ill-posed, relative to the case where $q = q(y)$, which has been shown to be very ill-posed [4, 5]. These results are consistent with the "metatheorem," normally attributed to Cannon [1], which states that the overposed data measurement should be taken (in some sense) "parallel" to the undetermined coefficient.

A closely related problem, which has received more attention, is the so-called *layered conductivity problem*, achieved by replacing the differential equation in (1.1) by

$$-\nabla \cdot (a(x)\, \nabla u) = 0,$$

where the univariate conductivity a is to be determined. Uniqueness questions for this problem under various hypotheses have been studied by Cannon [1] and Cannon and DuChateau [2].

The inverse problem analyzed in this paper is a special case of the general problem of determining $q(\vec{x})$ in

$$-\Delta u + q(\vec{x})\, u = 0, \qquad \vec{x} \in D \subseteq R^n,$$

from data measurements taken on the boundary ∂D of D. For $n \geq 3$, it has been shown under various hypotheses that q is uniquely determined by knowledge of all possible Cauchy data pairs on ∂D [11, 12, 18, 21, 22]. A global uniqueness result was proved by Sylvester and Uhlmann [20]. Partial results for the $n = 2$ case have been obtained [4, 17–20]. The general problem in R^2 was recently answered in the affirmative by Nachman [13] for the conductivity problem.

In practice, one has only a finite number of Cauchy data pairs, so a general $q(x, y)$ cannot be uniquely determined. However, it is reasonable to ask whether, by imposing additional structure on q, unicity can be restored (see, e.g., [6–10]). The present paper represents a step in this direction, in which only *one* Cauchy data pair is given. Additionally, the object being studied may be oriented in such a way that only a portion of its boundary may be accessible for measurement. This is reflected in our assumption that the data is given only on the top boundary of the square.

This paper is organized as follows: In Section 2 we describe the notation used and give assumptions on the data. This section also includes some useful estimates concerning the Green's function for $-\Delta$. In Section 3, the operator T_h is defined and shown to be a self-map on a ball in $C^\alpha([0, 1])$. In Section 4, T_h is shown to be a contraction. The paper concludes with numerical examples, demonstrating the effectiveness of this reconstruction method.

2. Notation and Assumptions

Let $\|\cdot\|_\alpha$ denote the Hölder norm, given by

$$\|f\|_\alpha \equiv \|f\|_\infty + |f|_\alpha,$$

where the α-seminorm $|\cdot|_\alpha$ is defined as

$$|f|_\alpha \equiv \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}, \qquad \alpha \in (0, 1).$$

It will be useful to introduce the function ψ as the harmonic function satisfying the boundary conditions (1.1b)–(1.1e).

We make the following assumptions on the data in (1.1):

A.1. $h \in C^{2+\alpha}([0, 1])$ with $m \equiv \min_{x \in [0,1]} h(x) > 0$.

A.2. The overposed data h is compatible with the "primary" data (1.1b)–(1.1e), in the sense that there exists a $C^{2+\alpha}$ function which attains the boundary values (1.1b)–(1.1f).

A.3. $g(x) = 0$. (This simplifies the presentation.)

A.4. The boundary data is chosen to make $\|\psi_{yy}\|_\infty$ small.

Denote by G the Green's function for $-\Delta$ with homogeneous boundary conditions of the type (1.1b)–(1.1e). G is given by

$$G = G(x, y; \xi, \eta) = \frac{1}{2\pi} \log |\bar{x} - \bar{\xi}| + w(\bar{x} - \bar{\xi}), \tag{2.1}$$

for $\bar{x} = (x, y)$, $\bar{\xi} = (\xi, \eta) \in \Omega$, where w is harmonic in $\bar{\xi}$ for each $\bar{x} \in \Omega$. It follows from Green's theorem that the solution u of (1.1a)–(1.1e) is representable as

$$u(x, y) = \psi(x, y) - \int_0^1 \int_0^1 G(x, y; \xi, \eta) q(\xi) u(\xi, \eta)\, d\eta\, d\xi. \tag{2.2}$$

We will make frequent use of the rather large null space of the operator

$$\mathscr{G}_{yy}[f] \equiv \int_0^1 \int_0^1 G_{yy}(x, y; \xi, \eta) f(\xi, \eta)\, d\eta\, d\xi, \qquad f \in C^\alpha.$$

It follows from the boundary conditions obeyed by G that

$$\int_0^1 G_{yy}(x, y; \xi, \eta)\, d\eta = 0,$$

for all $x, y, \xi \in (0, 1)$, with $x \neq \xi$. As a result,

$$\int_0^1 \int_0^1 G_{yy}(x, y; \xi, \eta) f(\xi)\, d\eta\, d\xi = \int_0^1 f(\xi) \left[\int_0^1 G_{yy}(x, y; \xi, \eta)\, d\eta \right] d\xi = 0$$

for any $f \in C^\alpha$ ([0, 1]). This allows us to form identities such as

$$\begin{aligned} &\int_0^1 \int_0^1 G_{yy}(x, y; \xi, \eta) u(\xi, \eta)\, d\eta\, d\xi \\ &\quad = \int_0^1 \int_0^1 G_{yy}(x, y; \xi, \eta)[u(\xi, \eta) - u(\xi, y)]\, d\eta\, d\xi, \end{aligned}$$

a tactic which will be used repeatedly in the sequel. Similarly, we have the null space properties

$$\mathscr{G}_x[f] \equiv \int_0^1 \int_0^1 G_x(x, y; \xi, \eta) f(\eta)\, d\xi\, d\eta = 0$$

$$\mathscr{G}_{yx}[f] \equiv \int_0^1 \int_0^1 G_{yx}(x, y; \xi, \eta) f(\xi)\, d\eta\, d\xi = 0,$$

for any $f \in C^\alpha$ ([0, 1]).

Next, we gather some estimates on integrals which will arise in our analysis. The proofs of the first two lemmas follow from the representation (2.1) via straightforward techniques and will not be presented here.

LEMMA 2.1. *The following integrals are bounded independent of* $(x, y) \in \Omega$:

$$\int_0^1 \int_0^1 |G(x, y; \xi, \eta)| \, d\eta \, d\xi$$

$$\int_0^1 \int_0^1 |G_y(x, y; \xi, \eta)| \, d\eta \, d\xi$$

$$\int_0^1 \int_0^1 |G_{yy}(x, y; \xi, \eta)| \, |x - \xi|^\alpha \, |y - \eta|^m \, d\eta \, d\xi \qquad m, \alpha \geq 0, m + \alpha \geq 2$$

$$\int_0^1 \left| \int_0^1 G_{yy}(x, y; \xi, \eta)(\eta - y) \, d\eta \right| d\xi.$$

LEMMA 2.2. *The following integrals are bounded independent of* $s \in (0, 1)$:

$$\int_0^1 \int_0^1 |G_x(s, 1; \xi, \eta)| \, |s - \xi|^\alpha \, d\eta \, d\xi$$

$$\int_0^1 \int_0^1 |G_{yx}(s, 1; \xi, \eta)| \, |s - \xi|^\alpha \, d\eta \, d\xi$$

COROLLARY 2.3. *For* α, $m > 0$, *each of the constants defined below is finite*:

$$C_1(\alpha, m) \equiv \sup_{x,y} \int_0^1 \int_0^1 |G_{yy}(x,y; \xi, \eta)| \, |x - \xi|^\alpha |y - \eta|^m \, d\eta \, d\xi$$

$$C_2(\alpha) \equiv \sup_{s} \int_0^1 \int_0^1 |G_{yx}(s, 1; \xi, \eta)| \, |s - \xi|^\alpha \, d\eta \, d\xi$$

$$C_3 \equiv \sup_{x,y} \int_0^1 \int_0^1 |G_y(x, y; \xi, \eta)| \, d\eta \, d\xi$$

$$C_4(\alpha) \equiv \sup_{s} \int_0^1 \int_0^1 |G_x(s, 1; \xi, \eta)| \, |s - \xi|^\alpha \, d\eta \, d\xi$$

$$C_5 \equiv \sup_{x,y} \int_0^1 \int_0^1 |G(x, y; \xi, \eta)| \, d\eta \, d\xi.$$

To demonstrate the application of these "null space properties" relevant to our analysis, we prove the estimates in the next two lemmas.

LEMMA 2.4.

$$\|u_{yy}\|_\infty \leq \|\psi_{yy}\|_\infty + C_1(0,1)\|\psi_y\|_\infty\|q\|_\infty + C_3C_1(0,1)\|u\|_\infty\|q\|_\infty^2.$$

Proof. From (2.2), we use the null space property of $\mathcal{G}_{yy}$ to obtain

$$|u_{yy}(x,y)| \leq |\psi_{yy}(x,y)| + \int_0^1 \left| \int_0^1 G_{yy}(x,y;\xi,\eta)q(\xi)u(\xi,\eta)\,d\eta \right| d\xi$$

$$= |\psi_{yy}(x,y)| + \int_0^1 \left| \int_0^1 G_{yy}(x,y;\xi,\eta)q(\xi)[u(\xi,\eta) - u(\xi,y)]\,d\eta \right| d\xi.$$

From Taylor's Theorem we have, for some point $\sigma(\xi)$ between η and y,

$$|u_{yy}(x,y)| \leq |\psi_{yy}(x,y)| + \int_0^1 |q(\xi)|\,|u_\eta(\xi,\sigma(\xi))| \left| \int_0^1 G_{yy}(x,y;\xi,\eta)(\eta - y)\,d\eta \right| d\xi.$$

Consequently,

$$\|u_{yy}\|_\infty \leq \|\psi_{yy}\|_\infty + C_1(0,1)\|u_y\|_\infty\|q\|_\infty. \tag{2.3}$$

Similarly,

$$\|u_y\|_\infty \leq \|\psi_y\|_\infty + C_3\|u\|_\infty\|q\|_\infty. \tag{2.4}$$

Combining estimates (2.3) and (2.4) establishes the lemma. ∎

LEMMA 2.5. *For $p, q \in C^\alpha([0,1])$, denote by $u = u(q)$ the solution to the BVP* (1.1a)–(1.1e) *corresponding to q, and denote by $v = v(p)$ the solution corresponding to p. Then, for $\|q\|_\alpha$, $\|p\|_\alpha$ sufficiently small,*

$$\|u - v\|_\infty \leq \frac{C_5}{1 - C_5\|p\|_\infty}\|u\|_\infty\|q - p\|_\infty, \tag{2.5}$$

$$\|u_y - v_y\|_\infty \leq C_3\|u\|_\infty\|q - p\|_\infty + \frac{C_3C_5}{1 - C_5\|p\|_\infty}\|p\|_\infty\|u\|_\infty\|q - p\|_\infty, \tag{2.6}$$

$$\|u_{yy} - v_{yy}\|_\infty \le \frac{C_1(0,2)}{2 - C_1(0,2)\|p\|_\infty} \|u_{yy}\|_\infty \|q - p\|_\alpha \equiv \overline{M}\|u_{yy}\|_\infty \|q - p\|_\alpha. \quad (2.7)$$

Proof. For $(x, y) \in \Omega$,

$$\begin{aligned} |u(x,y) - v(x,y)| &\le \left| \int_0^1 \int_0^1 G(x, y; \xi, \eta)[q(\xi) - p(\xi)]u(\xi, \eta)\, d\eta\, d\xi \right| \\ &\quad + \left| \int_0^1 \int_0^1 G(x, y; \xi, \eta)p(\xi)[u(\xi, \eta) - v(\xi, \eta)]\, d\eta\, d\xi \right| \\ &\le \|u\|_\infty \|G(x, y; \cdot, \cdot)\|_{L^1} \|q - p\|_\infty \\ &\quad + \|p\|_\infty \|G(x, y; \cdot, \cdot)\|_{L^1} \|u - v\|_\infty. \end{aligned}$$

So,

$$\|u - v\|_\infty \le C_5 \|u\|_\infty \|q - p\|_\infty + C_5 \|p\|_\infty \|u - v\|_\infty,$$

yielding (2.5). Similarly,

$$\|u_y - v_y\|_\infty \le C_3 \|u\|_\infty \|q - p\|_\infty + C_3 \|p\|_\infty \|u - v\|_\infty,$$

which, combined with (2.5), yields (2.6). Finally, the null space property of $\mathcal{G}_{yy}$ allows us to write

$$\begin{aligned} &|u_{yy}(x,y) - v_{yy}(x,y)| \\ &\quad \le \left| \int_0^1 \int_0^1 G_{yy}(x, y; \xi, \eta)[q(\xi) - p(\xi)]u(\xi, \eta)\, d\eta\, d\xi \right| \\ &\quad\quad + \left| \int_0^1 \int_0^1 G_{yy}(x, y; \xi, \eta)p(\xi)[u(\xi, \eta) - v(\xi, \eta)]\, d\eta\, d\xi \right| \\ &\quad = \left| \int_0^1 \int_0^1 G_{yy}(x, y; \xi, \eta)[q(\xi) - p(\xi)][u(\xi, \eta) - u(\xi, 1)]\, d\eta\, d\xi \right| \\ &\quad\quad + \left| \int_0^1 \int_0^1 G_{yy}(x, y; \xi, \eta)p(\xi)[(u(\xi, \eta) - v(\xi, \eta)) \right. \\ &\quad\quad \left. - (u(\xi, 1) - v(\xi, 1))]\, d\eta\, d\xi \right|. \end{aligned}$$

Using Taylor's Theorem and assumption A.3, we have, for some points $\sigma(\xi)$, $z(\xi) \in (\eta, 1)$,

$$u(\xi, \eta) - u(\xi, 1) = u_{\eta\eta}(\xi, \sigma(\xi)) \frac{(\eta - 1)^2}{2},$$

$$[u(\xi, \eta) - v(\xi, \eta)] - [u(\xi, 1) - v(\xi, 1)]$$
$$= [u_{\eta\eta}(\xi, z(\xi)) - v_{\eta\eta}(\xi, z(\xi))] \frac{(\eta - 1)^2}{2}.$$

Thus,

$$|u_{yy}(x, y) - v_{yy}(x, y)|$$
$$\leq \frac{\|u_{yy}\|_\infty \|q - p\|_\infty}{2} \int_0^1 \int_0^1 |G_{yy}(x, y; \xi, \eta)|(\eta - 1)^2 \, d\eta \, d\xi$$
$$+ \frac{\|p\|_\infty \|u_{yy} - v_{yy}\|_\infty}{2} \int_0^1 \int_0^1 |G_{yy}(x, y; \xi, \eta)|(\eta - 1)^2 \, d\eta \, d\xi,$$

so that

$$\|u_{yy} - v_{yy}\|_\infty \leq \frac{C_1(0, 2)\|u_{yy}\|_\infty}{2} \|q - p\|_\alpha + \frac{C_1(0, 2)\|p\|_\infty}{2} \|u_{yy} - v_{yy}\|_\infty,$$

and (2.7) follows. ∎

3. Iterative Scheme

Let u satisfy (1.1). Following [14, 15], we project the differential equation (1.1a) onto the boundary $y = 1$ and rearrange to obtain

$$q(x) = \frac{u_{xx}(x, 1) + u_{yy}(x, 1)}{u(x, 1)} = \frac{h''(x) + u_{yy}(x, 1)}{h(x)}.$$

Noting that the right-hand side depends on q, define an operator T_h on $C^\alpha([0, 1])$ by

$$T_h[q](x) \equiv \frac{h''(x) + u_{yy}(x, 1)}{h(x)}.$$

If u solves (1.1), then q is a fixed point of T_h. Conversely, denoting by $u(q)$ the solution of (1.1a)–(1.1e) for a given q, we have

THEOREM 3.1. *If $\|q\|_\alpha$ is sufficiently small, then $u(q)$ satisfies* (1.1) *if and only if q is a fixed point of T_h.*

Proof. Let q be a fixed point of T_h. Then,

$$q(x) = \frac{u_{xx}(x, 1; q) + u_{yy}(x, 1; q)}{u(x, 1; q)}.$$

Since $q = T_h[q]$, we conclude

$$q(x)[h(x) - u(x, 1; q)] = h''(x) - u_{xx}(x, 1; q). \tag{3.1}$$

Setting $\beta(x) \equiv h(x) - u(x, 1; q)$, (3.1) and the compatibility conditions on h at $x = 0$ and $x = 1$ imply that β obeys

$$\begin{aligned} -\beta''(x) + q(x)\beta(x) = 0, &\qquad x \in (0, 1) \\ \beta(0) = \beta(1) = 0. & \end{aligned} \tag{3.2}$$

It is known that μ_0, the smallest eigenvalue for (3.2), obeys the bound

$$\pi - \|q\|_\infty \leq \mu_0.$$

Consequently, if $\|q\|_\infty < \pi$, the only solution of (3.2) is trivial. Thus, if $\|q\|_\alpha < \pi$,

$$u(x, 1; q) = h(x);$$

i.e., $u(q)$ solves (1.1). ∎

In light of this result, the inverse problem (1.1) can be restated in terms of fixed points of the operator T_h.

For $R > 0$, let $B_R \equiv \{q \in C^\alpha([0, 1]) : \|q\|_\alpha \leq R\}$, the ball of radius R around zero in $C^\alpha([0, 1])$. To show the operator T_h has a fixed point, we first establish the following result.

THEOREM 3.2. *There exists an $R_0 > 0$ such that $T_h : B_R \to B_R$ for each $R \in (0, R_0)$.*

Proof. We must show that, for some $R > 0$, $\|q\|_\alpha \leq R$ implies $\|T_h[q]\|_\alpha \leq R$. First, $T_h[q]$ can be expressed as

$$T_h[q](x) = \frac{h''(x) - \psi_{xx}(x,1)}{h(x)} - \frac{1}{h(x)} \int_0^1 \int_0^1 G_{yy}(x, 1; \xi, \eta) q(\xi) u(\xi, \eta)\, d\eta\, d\xi. \tag{3.3}$$

Using techniques similar to those employed in the proofs of Lemmas 2.4 and 2.5, we can derive the inequality

$$\|T_h[q]\|_\alpha \le \Lambda_4 \|\psi_{xx}(\cdot, 1) - h''\|_\alpha + \Lambda_5 \|q\|_\alpha + \Lambda_6 \|q\|_\alpha^2 + \Lambda_7 \|q\|_\alpha^3, \tag{3.4}$$

where the Λ_j are independent of q. (The derivation of this estimate can be found in the Appendix.) In light of (3.4), we will have $\|T_h[q]\|_\alpha \le R$ for $\|q\|_\alpha \le R$, provided R obeys

$$\Lambda_4 \|\psi_{xx}(\cdot, 1) - h''\|_\alpha + \Lambda_5 R + \Lambda_6 R^2 + \Lambda_7 R^3 \le R.$$

The factor $\|\psi_{xx}(\cdot, 1) - h''\|_\alpha$ can be controlled by $\|q\|_\alpha$, so we need only consider the (strict) inequality

$$\Lambda_5 R + \Lambda_6 R^2 + \Lambda_7 R^3 < R,$$

or, equivalently,

$$\Lambda_5 + \Lambda_6 R + \Lambda_7 R^2 < 1.$$

This inequality will hold for sufficiently small R, provided $\Lambda_5 < 1$. Λ_5 has the form

$$\Lambda_5 = \frac{A}{m} + B \|\psi_{yy}\|_\infty,$$

where A and B do not increase as $1/m$ and $\|\psi_{yy}\|_\infty$ decrease. The boundary data has been chosen to make $\|\psi_{yy}\|_\infty$ small, so Λ_5 can be made small by increasing m. Thus, T_h is a self-map on B_R, as asserted. ▌

4. Existence and Uniqueness of a Fixed Point

We now state our main result.

THEOREM 4.1. *Under the assumptions outlined, for R sufficiently small, T_h possesses a unique fixed point in B_R.*

COROLLARY 4.2. *Under the assumptions outlined, the overposed boundary value problem* (1.1) *has a unique solution.*

The proof of Theorem 4.1 will show that T_h is a contraction on B_R, from which we conclude

COROLLARY 4.3. *The sequence of iterates defined by*

$$q^{(k)} = T_h[q^{(k-1)}]$$

converges in C^α *to the unique solution* q *of* (1.1).

Proof of Theorem 4.1. For $p, q \in B_R$, denote by $u = u(q)$ the solution of the BVP (1.1a)–(1.1e) corresponding to q, and denote by $v = v(p)$ the solution corresponding to p. In the Appendix, we derive the estimate

$$\|T_h[q] - T_h[p]\|_\alpha \leq \left\{A\|u_{yy}\|_\alpha + \frac{B}{m}\|p\|_\alpha\right\}\|q - p\|_\alpha, \tag{4.1}$$

where A and B do not increase as $\|u_{yy}\|_\alpha$ and $1/m$ decrease. It follows that T_h is a contraction on B_R for some $R > 0$. This proves Theorem 4.1. ∎

Remark. Note the dependence of the contraction constant on the ratio $\|u_{yy}\|_\alpha/m$. This is to be expected, for if the ratio is very small, then

$$T_h[q] = \frac{h''(x) + u_{yy}(x, 1; q)}{h(x)} \approx \frac{h''(x)}{h(x)},$$

and q becomes essentially a readoff.

5. NUMERICAL EXAMPLES

The following numerical examples illustrate the effectiveness of the iterative scheme defined in Corollary 4.3. In each case, we discretize the problem by considering the boundary value problem (1.1) on an evenly spaced grid of size $N \times N$. Starting with an initial guess of $q^{(0)} = 0$, we solve the direct problem (1.1a)–(1.1e) for $u(q^{(0)})$. The next update $q^{(1)}$ is then formed via

$$q^{(1)}(x_j) = T_h[q^{(0)}](x_j), \qquad j = 1, ..., N.$$

This procedure is repeated for a prescribed number I of iterations. In each of the following examples, $N = 40$ and $I = 5$.

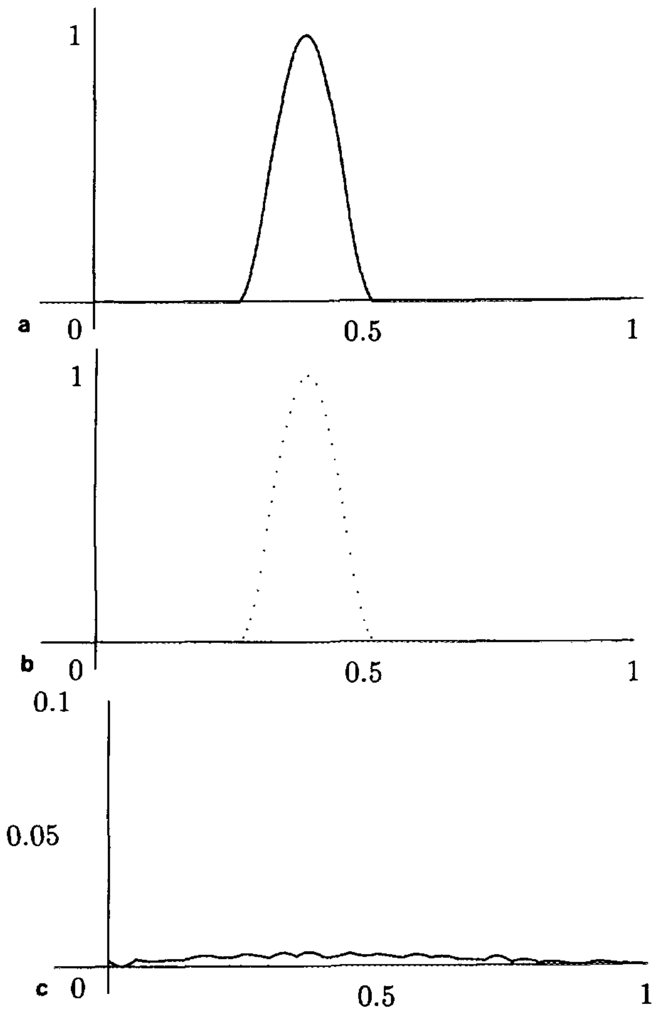

FIG. 1. (a) The reconstructed and (b) the actual q_1. (c) The absolute error in the reconstruction of $q_1(x)$.

Figure 1 shows the reconstruction of the C^1-function,

$$q_1(x) = \begin{cases} 4096(x - \frac{1}{4})^2(x - \frac{1}{2})^2, & \frac{1}{4} \leq x \leq \frac{1}{2}; \\ 0, & \text{otherwise.} \end{cases}$$

Table I shows the relative error in the reconstructions, as well as the value of the residual $\|h - u^l(\cdot, 1)\|_\infty$, where u^l solves the direct problem for the reconstruction q_1^l.

As a second example, we consider a function q_2 which is Hölder continu-

TABLE I

The Relative Supnorm and L^2-Norm Errors in the Reconstruction of q_1, and the Value of the Corresponding Residual

Supnorm	L^2-Norm	Residual
0.0093	0.0052	6.3×10^{-4}

ous, but not continuously differentiable. Figure 2 shows the reconstruction of the function

$$q_2(x) = \begin{cases} x, & 0 \le x < \frac{1}{2}; \\ 1 - x, & \frac{1}{2} \le x \le 1. \end{cases}$$

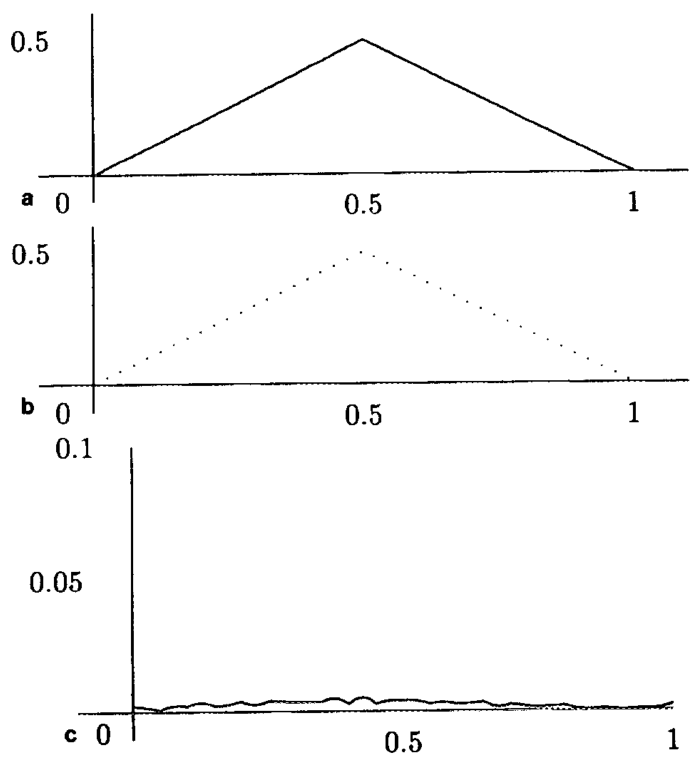

FIG. 2. (a) The reconstructed and (b) the actual q_2. (c) The absolute error in the reconstruction of $q_2(x)$.

TABLE II

The Relative Supnorm and L^2-Norm Errors in the Reconstruction of q_2, and the Value of the Corresponding Residual

Supnorm	L^2-Norm	Residual
0.0102	0.0097	6.6×10^{-4}

Again, Table II reflects both the relative error in the reconstruction and the value of the corresponding residual.

In order to test the effectiveness of our method in reconstructing functions in $L^2(0, 1)$, we consider the discontinuous function

$$q_3(x) = \begin{cases} -1, & 0 \leq x < \frac{1}{2}; \\ 1, & \frac{1}{2} \leq x \leq 1. \end{cases}$$

As before, Fig. 3 and Table III reflect the accuracy of this reconstruction.

6. Conclusions

We have investigated the problem of recovering a univariate potential $q(x)$ in (1.1) on the unit square in R^2 from a single overposed boundary measurement along $y = 1$. We have demonstrated that this inverse problem is only mildly ill-posed, in the sense that the map from the overposed data to the unknown potential is bounded, provided we control the data in a slightly stronger norm. In our case, data from the Hölder space $C^{2+\alpha}$ leads to local existence and uniqueness of a potential in C^α.

We have characterized the solution q of (1.1) as a fixed point of an operator, and have shown that this operator is a contraction near $q = 0$ in C^α. This leads to an iterative scheme which provides very satisfactory

TABLE III

The Relative Supnorm and L^2-Norm Errors in the Reconstruction of q_3, and the Value of the Corresponding Residual

Supnorm	L^2-Norm	Residual
0.0054	0.0035	5.0×10^{-4}

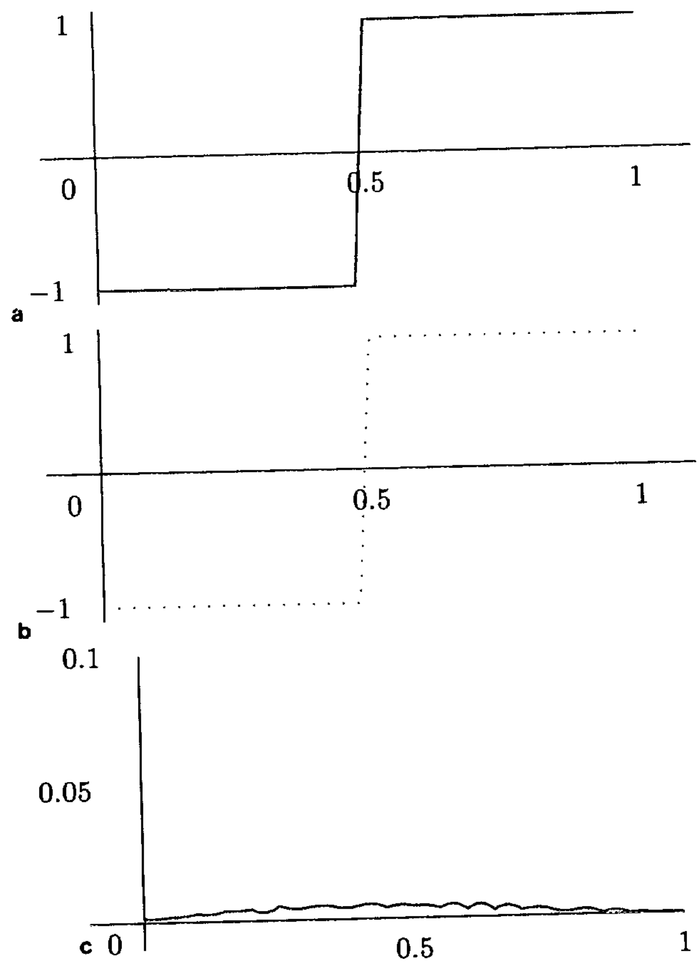

FIG. 3. (a) The reconstructed and (b) the actual q_3. (c) The absolute error in the reconstruction of $q_3(x)$.

reconstructions on example potentials possessing various degrees of smoothness.

APPENDIX

Derivation of Inequality (3.4). From (3.3),

$$|T_h[q](x)| \leq \left|\frac{h''(x) - \psi_{xx}(x, 1)}{h(x)}\right| + \frac{1}{m}\left|\int_0^1 \int_0^1 G_{yy}(x, 1; \xi, \eta)q(\xi)u(\xi, \eta)\, d\eta\, d\xi\right|.$$

From the null space property of the operator $\mathcal{G}_{yy}$, we have

$$\left| \int_0^1 \int_0^1 G_{yy}(x, 1; \xi, \eta) q(\xi) u(\xi, \eta) \, d\eta \, d\xi \right|$$
$$\leq \frac{\|u_{yy}\|_\infty}{2} \|q\|_\infty \int_0^1 \int_0^1 |G_{yy}(x, 1; \xi, \eta)| (\eta - 1)^2 \, d\eta \, d\xi,$$

so that

$$\|T_h[q]\|_\infty \leq \frac{\|h'' - \psi_{xx}(x, 1)\|_\alpha}{m} + \frac{C_1(0, 2) \|u_{yy}\|_\infty \|q\|_\infty}{2}. \tag{A.1}$$

Next, we estimate $|T_h[q]|_\alpha$. For $x_1 \neq x_2$,

$$|T_h[q](x_1) - T_h[q](x_2)|$$
$$\leq \left| \frac{h''(x_1) - \psi_{xx}(x_1, 1)}{h(x_1)} - \frac{h''(x_2) - \psi_{xx}(x_2, 1)}{h(x_2)} \right|$$
$$+ \left| \frac{1}{h(x_1)} \int_0^1 \int_0^1 G_{yy}(x_1, 1; \xi, \eta) q(\xi) u(\xi, \eta) \, d\eta \, d\xi \right. \tag{A.2}$$
$$\left. - \frac{1}{h(x_2)} \int_0^1 \int_0^1 G_{yy}(x_2, 1; \xi, \eta) q(\xi) u(\xi, \eta) \, d\eta \, d\xi \right|.$$

Note that

$$\left| \frac{h''(x_1) - \psi_{xx}(x_1, 1)}{h(x_1)} - \frac{h''(x_2) - \psi_{xx}(x_2, 1)}{h(x_2)} \right| \leq \frac{\|h\|_\alpha \|\psi_{xx}(\cdot, 1) - h''\|_\alpha}{m^2} |x_1 - x_2|^\alpha,$$

and denote by I_1 the integral terms in (A.2). Then,

$$I_1 \leq \left| \frac{1}{h(x_1)} \int_0^1 \int_0^1 [G_{yy}(x_1, 1; \xi, \eta) - G_{yy}(x_2, 1; \xi, \eta)] q(\xi) u(\xi, \eta) \, d\eta \, d\xi \right|$$
$$+ \left| \left(\frac{1}{h(x_1)} - \frac{1}{h(x_2)} \right) \int_0^1 \int_0^1 G_{yy}(x_2, 1; \xi, \eta) q(\xi) u(\xi, \eta) \, d\eta \, d\xi \right|$$
$$\equiv I_{11} + I_{12}.$$

As above,

$$I_{12} = \left| \left(\frac{1}{h(x_1)} - \frac{1}{h(x_2)} \right) \int_0^1 \int_0^1 G_{yy}(x_2, 1; \xi, \eta) q(\xi)[u(\xi, \eta) - u(\xi, 1)]\, d\eta\, d\xi \right|$$

$$\leq \frac{C_1(0, 2)|h|_\alpha \|u_{yy}\|_\infty \|q\|_\infty}{2m^2} |x_1 - x_2|^\alpha.$$

Next, writing I_{11} as

$$I_{11} = \left| \frac{1}{h(x_1)} \int_{x_1}^{x_2} \int_0^1 \int_0^1 G_{yyx}(s, 1; \xi, \eta) q(\xi)[u(\xi, \eta) - u(\xi, 1)]\, d\eta\, d\xi\, ds \right|,$$

Taylor's Theorem yields, for some point $\sigma(\xi)$ in the interval $(\eta, 1)$,

$$I_{11} \leq \frac{1}{m} \int_{x_1}^{x_2} \int_0^1 \left| q(\xi) u_\eta(\xi, \sigma(\xi)) \int_0^1 G_{yyx}(s, 1; \xi, \eta)(\eta - 1)\, d\eta \right| d\xi\, ds. \quad \text{(A.3)}$$

Integration by parts on the innermost integral yields

$$\int_0^1 G_{yyx}(s, 1; \xi, \eta)(\eta - 1)\, d\eta = -G_{yx}(s, 1; \xi, 0) - \phi(s, \xi)$$

$$+ \int_0^1 G_{yx}(s, 1; \xi, \eta)\, d\eta,$$

where $\phi(s, \xi) \equiv G_{yx}(s, 1; \xi, \eta)(\eta - 1)|_{\eta=1}$. Noting that $\int_0^1 |\phi(s, \xi)|\, d\xi = 0$, we see from (A.3) that

$$I_{11} \leq \frac{1}{m} \int_{x_1}^{x_2} \int_0^1 |q(\xi) u_y(\xi, \sigma(\xi)) G_{yx}(s, 1; \xi, 0)|\, d\xi\, ds$$

$$+ \frac{1}{m} \int_{x_1}^{x_2} \int_0^1 \left| \int_0^1 G_{yx}(s, 1; \xi, \eta) q(\xi) u_\eta(\xi, \sigma(\xi))\, d\eta \right| d\xi\, ds$$

$$\leq \frac{\|q\|_\infty \|u_y\|_\infty}{m} \int_{x_1}^{x_2} \int_0^1 |G_{yx}(s, 1; \xi, 0)|\, d\xi\, ds$$

$$+ \frac{1}{m} \int_{x_1}^{x_2} \int_0^1 \int_0^1 |G_{yx}(s, 1; \xi, \eta)[q(\xi) - q(s)] u_\eta(\xi, \sigma(\xi))|\, d\eta\, d\xi\, ds$$

$$+ \frac{1}{m} \int_{x_1}^{x_2} \int_0^1 \int_0^1 |G_{yx}(s, 1; \xi, \eta) q(s)[u_\eta(\xi, \sigma(\xi)) - u_\eta(s, \sigma(s))]|\, d\eta\, d\xi\, ds$$

$$\leq \frac{M_1\|q\|_\infty\|u_y\|_\infty}{m}|x_1 - x_2| + \frac{|q|_\alpha\|u_y\|_\infty}{m}\int_{x_1}^{x_2}\int_0^1\int_0^1 |G_{yx}(s, 1; \xi, \eta)||\xi - s|^\alpha\, d\eta\, d\xi\, ds$$

$$+ \frac{\|q\|_\infty\|u_{yy}\|_\infty}{m}\int_{x_1}^{x_2}\int_0^1\int_0^1 |G_{yx}(s, 1; \xi, \eta)||\xi - s|\, d\eta\, d\xi\, ds$$

$$\leq \left\{\frac{M_1\|q\|_\infty\|u_y\|_\infty}{m} + \frac{C_2(\alpha)|q|_\alpha\|u_y\|_\infty}{m} + \frac{C_2(1)\|q\|_\infty\|u_{yy}\|_\infty}{m}\right\}|x_1 - x_2|,$$

where

$$M_1 \equiv \sup_s \int_0^1 |G_{yx}(s, 1; \xi, 0)|\, d\xi.$$

Thus, we have the estimate

$$|T_h[q](x_1) - T_h[q](x_2)|$$
$$\leq \left\{\frac{\|h\|_\alpha\|\psi_{xx}(\cdot, 1) - h''\|_\alpha}{m^2} + \frac{C_1(0, 2)|h|_\alpha\|u_{yy}\|_\infty\|q\|_\infty}{2m^2}\right\}|x_1 - x_2|^\alpha$$
$$+ \left\{\frac{M_1\|q\|_\infty\|u_y\|_\infty}{m} + \frac{C_2(\alpha)|q|_\alpha\|u_y\|_\infty}{m} + \frac{C_2(1)\|q\|_\infty\|u_{yy}\|_\infty}{m}\right\}|x_1 - x_2|.$$

Dividing both sides by $|x_1 - x_2||^\alpha > 0$ and taking suprema over $x_1 \neq x_2$ yields

$$|T_h[q]|_\alpha \leq \frac{\|h\|_\alpha\|\psi_{xx}(\cdot, 1) - h''\|_\alpha}{m^2} + \left\{\frac{C_1(0, 2)|h|_\alpha\|u_{yy}\|_\infty\|q\|_\infty}{2m^2}\right.$$
$$\left. + \frac{M_1\|u_y\|_\infty}{m} + \frac{C_2(\alpha)\|u_y\|_\infty}{m} + \frac{C_2(1)\|u_{yy}\|_\infty}{m}\right\}\|q\|_\alpha.$$

Combining this with estimate (A.1) gives

$$\|T_h[q]\|_\alpha \leq \Lambda_1\|\psi_{xx}(\cdot, 1) - h''\|_\alpha + \Lambda_2\|u_y\|_\infty\|q\|_\alpha + \Lambda_3\|u_{yy}\|_\infty\|q\|_\alpha,$$

where the Λ_j are independent of q. Lemma 2.4 and the estimate (2.4) then yield

$$\|T_h[q]\|_\alpha \leq \Lambda_4\|\psi_{xx}(\cdot, 1) - h''\|_\alpha + \Lambda_5\|q\|_\alpha + \Lambda_6\|q\|_\alpha^2 + \Lambda_7\|q\|_\alpha^3,$$

where

$$\Lambda_5 = \left(\frac{M_1 + C_2(\alpha)}{m}\right)\|\psi_y\|_\infty + \left(\frac{C_2(1)}{m} + \frac{C_1(0,2)|h|_\alpha}{2m^2} + \frac{C_1(0,2)}{2}\right)\|\psi_{yy}\|_\infty.$$

This proves inequality (3.4). ∎

Proof of Inequality (4.1). For $x \in (0, 1)$,

$$\begin{aligned}
&|T_h[q](x) - T_h[p](x)| \\
&\quad = \frac{u_{yy}(x,1) - v_{yy}(x,1)}{h(x)} \\
&\quad = \left|\frac{1}{h(x)}\int_0^1\int_0^1 G_{yy}(x,1;\xi,\eta)[q(\xi)u(\xi,\eta) - p(\xi)v(\xi,\eta)]\,d\eta\,d\xi\right| \\
&\quad \le \left|\frac{1}{h(x)}\int_0^1\int_0^1 G_{yy}(x,1;\xi,\eta)u(\xi,\eta)[q(\xi) - p(\xi)]\,d\eta\,d\xi\right| \\
&\qquad + \left|\frac{1}{h(x)}\int_0^1\int_0^1 G_{yy}(x,1;\xi,\eta)p(\xi)[u(\xi,\eta) - v(\xi,\eta)]\,d\eta\,d\xi\right| \\
&\quad \equiv I_2 + I_3.
\end{aligned}$$

First, from Taylor's Theorem and the null space property of $\mathcal{G}_{yy}$, we have

$$\begin{aligned}
I_2 &\le \left|\int_0^1\int_0^1 G_{yy}(x,1;\xi,\eta)[(q(\xi) - p(\xi)) - (q(x) - p(x))]u(\xi,\eta)\,d\eta\,d\xi\right| \\
&\quad + \left|\int_0^1\int_0^1 G_{yy}(x,1;\xi,\eta)[q(x) - p(x)]u(\xi,\eta)\,d\eta\,d\xi\right| \\
&= \left|\int_0^1\int_0^1 G_{yy}(x,1;\xi,\eta)[(q(\xi) - p(\xi))\right. \\
&\quad \left. - (q(x) - p(x))][u(\xi,\eta) - u(\xi,1)]\,d\eta\,d\xi\right| \\
&\quad + \left|\int_0^1\int_0^1 G_{yy}(x,1;\xi,\eta)[q(x) - p(x)][u(\xi,\eta) - u(\xi,1)]\,d\eta\,d\xi\right| \\
&\le \frac{\|u_{yy}\|_\infty|q-p|_\alpha}{2}\int_0^1\int_0^1 |G_{yy}(x,1;\xi,\eta)|\,|\xi - x|^\alpha(\eta - 1)^2\,d\eta\,d\xi \\
&\quad + \frac{\|u_{yy}\|_\infty\|q-p\|_\infty}{2}\int_0^1\int_0^1 |G_{yy}(x,1;\xi,\eta)|(\eta - 1)^2\,d\eta\,d\xi
\end{aligned}$$

$$\leq \frac{(C_1(\alpha,2) + C_1(0,2))}{2} \|u_{yy}\|_\infty \|q - p\|_\alpha. \tag{A.4}$$

Next, write

$$\begin{aligned} I_3 &= \left| \frac{1}{h(x)} \int_0^1 \int_0^1 G_{yy}(x,1;\xi,\eta) p(\xi)[(u(\xi,\eta) - v(\xi,\eta)) \right. \\ &\quad \left. - (u(\xi,1) - v(\xi,1))]\, d\eta\, d\xi \right| \\ &\leq \frac{\|p\|_\infty}{m} \int_0^1 \int_0^1 |G_{yy}(x,1;\xi,\eta)| |(u(\xi,\eta) - v(\xi,\eta)) \\ &\quad - (u(\xi,1) - v(\xi,1))|\, d\eta\, d\xi. \end{aligned} \tag{A.5}$$

Now, by the Mean Value Theorem,

$$|(u(\xi,\eta) - v(\xi,\eta)) - (u(\xi,1) - v(\xi,1))| \leq \|u_y - v_y\|_\infty (1 - \eta).$$

Further, for $(x, y) \in \Omega$,

$$\begin{aligned} |u_y(x,y) - v_y(x,y)| &= \left| [u_y(x,1) - v_y(x,1)] - \int_y^1 [u_{yy}(x,s) - v_{yy}(x,s)]\, ds \right| \\ &\leq \int_y^1 |u_{yy}(x,s) - v_{yy}(x,s)|\, d|s| \leq \|u_{yy} - v_{yy}\|_\infty (1 - y). \end{aligned}$$

By combining these estimates with (A.5) and Lemma 2.5, we obtain

$$\begin{aligned} I_3 &\leq \frac{\|p\|_\infty \|u_{yy} - v_{yy}\|_\infty}{m} \int_0^1 \int_0^1 |G_{yy}(x,1;\xi,\eta)| (\eta - 1)^2\, d\eta\, d\xi \\ &\leq \frac{C_1(0,2)\|p\|_\infty}{m} \|u_{yy} - v_{yy}\|_\infty \\ &\leq C_1(0,2)\overline{M}\|p\|_\infty \|u_{yy}\|_\infty \|q - p\|_\alpha. \end{aligned}$$

Combining this with (A.4) yields

$$\|T_h[q] - T_h[p]\|_\infty \leq \left\{ \frac{C_1(\alpha,2) + C_1(0,2)}{2} + \overline{M}\|p\|_\infty C_1(0,2) \right\} \|u_{yy}\|_\infty \|q - p\|_\alpha. \tag{A.6}$$

Next, we estimate $|T_h[q] - T_h[p]|_\alpha$. For $x_1 \neq x_2$,

$$
\begin{aligned}
&|\{T_h[q](x_1) - T_h[p](x_1)\} - \{T_h[q](x_2) - T_h[p](x_2)\}| \\
&\quad \le \left| \frac{1}{h(x_1)} \int_0^1 \int_0^1 [G_{yy}(x_1, 1; \xi, \eta) - G_{yy}(x_2, 1; \xi, \eta)] \right. \\
&\qquad \left. \times [q(\xi)u(\xi, \eta) - p(\xi)v(\xi, \eta)]\, d\eta\, d\xi \right| \\
&\qquad + \left| \left(\frac{1}{h(x_1)} - \frac{1}{h(x_2)} \right) \int_0^1 \int_0^1 G_{yy}(x_2, 1; \xi, \eta) \right. \\
&\qquad \left. \times [q(\xi)u(\xi, \eta) - p(\xi)v(\xi, \eta)]\, d\eta\, d\xi \right| \\
&\quad \le \left| \frac{1}{h(x_1)} \int_0^1 \int_0^1 [G_{yy}(x_1, 1; \xi, \eta) - G_{yy}(x_2, 1; \xi, \eta)] \right. \\
&\qquad \left. \times [q(\xi) - p(\xi)]u(\xi, \eta)\, d\eta\, d\xi \right| \\
&\qquad + \left| \frac{1}{h(x_1)} \int_0^1 \int_0^1 [G_{yy}(x_1, 1; \xi, \eta) - G_{yy}(x_2, 1; \xi, \eta)]p(\xi) \right. \\
&\qquad \left. \times [u(\xi, \eta) - v(\xi, \eta)]\, d\eta\, d\xi \right| \\
&\qquad + \left| \left(\frac{h(x_2) - h(x_1)}{h(x_1)h(x_2)} \right) \int_0^1 \int_0^1 G_{yy}(x_2, 1; \xi, \eta) \right. \\
&\qquad \left. \times [q(\xi) - p(\xi)]u(\xi, \eta)\, d\eta\, d\xi \right| \\
&\qquad + \left| \left(\frac{h(x_2) - h(x_1)}{h(x_1)h(x_2)} \right) \int_0^1 \int_0^1 G_{yy}(x_2, 1; \xi, \eta)p(\xi) \right. \\
&\qquad \left. \times [u(\xi, \eta) - v(\xi, \eta)]\, d\eta\, d\xi \right| \\
&\quad \equiv I_4 + I_5 + I_6 + I_7.
\end{aligned}
$$

First,

$$I_7 = \left| \left(\frac{h(x_2) - h(x_1)}{h(x_1)h(x_2)} \right) \int_0^1 \int_0^1 G_{yy}(x_2, 1; \xi, \eta) p(\xi)[(u(\xi, \eta) - v(\xi, \eta)) \right.$$

$$\left. - (u(\xi, 1) - v(\xi, 1))] \, d\eta \, d\xi \right|$$

$$\leq \frac{\|p\|_\infty \|u_{yy} - v_{yy}\|_\infty}{2m^2} |h(x_1) - h(x_2)| \int_0^1 \int_0^1 |G_{yy}(x_2, 1; \xi, \eta)| (\eta - 1)^2 \, d\eta \, d\xi$$

$$\leq \frac{C_1(0,2) \|p\|_\infty |h|_\alpha}{2m^2} \|u_{yy} - v_{yy}\|_\infty |x_1 - x_2|^\alpha$$

$$\leq \frac{C_1(0,2) \|p\|_\infty |h|_\alpha \overline{M}}{2m^2} \|u_{yy}\|_\infty \|q - p\|_\alpha |x_1 - x_2|^\alpha.$$

Next, we write I_6 as

$$I_6 \leq \left| \left(\frac{h(x_2) - h(x_1)}{h(x_1)h(x_2)} \right) \int_0^1 \int_0^1 G_{yy}(x_2, 1; \xi, \eta) \right.$$

$$\left. \times [(q(\xi) - p(\xi)) - (q(x) - p(x))] u(\xi, \eta) \, d\eta \, d\xi \right|$$

$$+ \left| \left(\frac{h(x_2) - h(x_1)}{h(x_1)h(x_2)} \right) \int_0^1 \int_0^1 G_{yy}(x_2, 1; \xi, \eta)[q(x) - p(x)] u(\xi, \eta) \, d\eta \, d\xi \right|.$$

As in the estimate of integral I_2, we have

$$I_6 \leq \frac{C_1(\alpha, 2)|h|_\alpha}{2m^2} \|u_{yy}\|_\infty |q - p|_\alpha |x_1 - x_2|^\alpha$$

$$+ \left| \frac{h(x_2) - h(x_1)}{h(x_1)h(x_2)} \right| \left| \int_0^1 \int_0^1 G_{yy}(x_2, 1; \xi, \eta) \right.$$

$$\left. \times [q(x) - p(x)][u(\xi, \eta) - u(\xi, 1)] \, d\eta \, d\xi \right|$$

$$\leq \left(\frac{C_1(\alpha, 2) + C_1(0, 2)}{2m^2} \right) |h|_\alpha \|u_{yy}\|_\infty \|q - p\|_\alpha |x_1 - x_2|^\alpha.$$

Continuing, we have

$$I_5 = \left| \frac{1}{h(x_1)} \int_0^1 \int_0^1 [G_{yy}(x_1, 1; \xi, \eta) - G_{yy}(x_2, 1; \xi, \eta)] p(\xi) \right.$$

$$\times [(u(\xi,\eta) - v(\xi,\eta)) - (u(\xi,1) - v(\xi,1))]\, d\eta\, d\xi \Bigg|$$

$$= \left| \frac{1}{h(x_1)} \int_{x_1}^{x_2} \int_0^1 p(\xi)[u_\eta(\xi,\sigma(\xi)) - v_\eta(\xi,\sigma(\xi))] \right.$$

$$\left. \times \int_0^1 G_{yyx}(s,1;\xi,\eta)(\eta - 1)\, d\eta\, d\xi\, ds \right|.$$

As in the estimate of integral I_{11}, we integrate by parts on the innermost integral to obtain

$$I_5 \leq \frac{1}{m} \left| \int_{x_1}^{x_2} \int_0^1 G_{yx}(s,1;\xi,0)p(\xi)[u_\eta(\xi,\sigma(\xi)) - v_\eta(\xi,\sigma(\xi))]\, d\xi\, ds \right|$$

$$+ \frac{1}{m} \left| \int_{x_1}^{x_2} \int_0^1 \int_0^1 G_{yx}(s,1;\xi,\eta)p(\xi)[u_\eta(\xi,\sigma(\xi)) - v_\eta(\xi,\sigma(\xi))]\, d\xi\, ds \right|$$

$$\equiv I_{51} + I_{52}$$

As before, in light of Lemma 2.5,

$$I_{51} \leq \frac{M_1 \|p\|_\infty \|u_y - v_y\|_\infty}{m} |x_1 - x_2|$$

$$\leq \frac{C_3 M_1 \|p\|_\infty}{m} \left\{ 1 + \left(\frac{C_5 \|p\|_\infty}{1 - C_5 \|p\|_\infty} \right) \right\} \|u\|_\infty \|q - p\|_\alpha |x_1 - x_2|$$

$$\equiv M_1 K_1(p) \|p\|_\infty \|u\|_\infty \|q - p\|_\alpha |x_1 - x_2|.$$

Making use of the null space property of $\mathcal{G}_{yx}$, we bound I_{52} by

$$I_{52} \leq \frac{1}{m} \left| \int_{x_1}^{x_2} \int_0^1 \int_0^1 G_{yx}(s,1;\xi,\eta)[p(\xi) - p(s)] \right.$$

$$\left. \times [u_\eta(\xi,\sigma(\xi)) - v_\eta(\xi,\sigma(\xi))]\, d\eta\, d\xi\, ds \right|$$

$$+ \frac{1}{m} \left| \int_{x_1}^{x_2} \int_0^1 \int_0^1 G_{yx}(s,1;\xi,\eta)p(\xi)[(u_\eta(\xi,\sigma(\xi)) - v_\eta(\xi,\sigma(\xi))) \right.$$

$$\left. - (u_\eta(s,\sigma)) - v_\eta(s,\sigma(s)))]\, d\eta\, d\xi\, ds \right|$$

$$\le \frac{C_2(\alpha)|p|_\alpha}{m}\|u_y - v_y\|_\infty|x_1 - x_2| + \frac{C_2(1)\|p\|_\infty}{m}\|u_{yy} - v_{yy}\|_\infty|x_1 - x_2|$$

$$\le \left\{\frac{C_3C_2(\alpha)\|u\|_\infty}{m} + C_2(\alpha)K_1(p)\|u\|_\infty + \frac{C_2(1)}{m}\overline{M}\|u_{yy}\|_\infty\right\}$$

$$\times \|p\|_\alpha\|q - p\|_\alpha|x_1 - x_2|$$

$$\equiv K_2(p, q)\|p\|_\alpha\|q - p\|_\alpha|x_1 - x_2|.$$

Combining this with (A.7) yields

$$I_5 \le \{M_1K_1(p)\|u\|_\infty + K_2(p, q)\}\|p\|_\alpha\|q - p\|_\alpha|x_1 - x_2|.$$

Finally, consider I_4:

$$I_4 = \left|\frac{1}{h(x_1)}\int_0^1\int_0^1 [G_{yy}(x_1, 1; \xi, \eta) - G_{yy}(x_2, 1; \xi, \eta)]\right.$$

$$\left.\times [q(\xi) - p(\xi)][u(\xi, \eta) - u(\xi, 1)]\, d\eta\, d\xi\, ds\right|$$

$$\le \frac{1}{2m}\left|\int_{x_1}^{x_2}\int_0^1 [q(\xi) - p(\xi)]u_{\eta\eta}(\xi, z(\xi))\right.$$

$$\left.\times\int_0^1 G_{yyx}(s, 1; \xi, \eta)(\eta - 1)^2\, d\eta\, d\xi\, ds\right|.$$

Integrating by parts twice with respect to η yields

$$I_4 \le \frac{1}{2m}\left|\int_{x_1}^{x_1}\int_0^1 [q(\xi) - p(\xi)]u_{\eta\eta}(\xi, z(\xi))\right.$$

$$\left.\times\left\{\phi_1(s, \xi) + \phi_2(s, \xi) + G_{yx}(s, 1; \xi, 0) - 2G_x(s, 1; \xi, 0)\right\} d\eta\, d\xi\right|$$

$$+ \frac{1}{2m}\left|\int_{x_1}^{x_2}\int_0^1\int_0^1 G_x(s, 1; \xi, \eta)[q(\xi) - p(\xi)]\right.$$

$$\left.\times\, u_{\eta\eta}(\xi, z(\xi))\, d\eta\, d\xi\, ds\right| \equiv I_{41} + I_{42},$$

where $\phi_1(s, \xi) \equiv -G_{yx}(s, 1; \xi, \eta)(\eta - 1)^2|_{\eta=1}$ and $\phi_2(s, \xi) \equiv -2G_x(s, 1; \xi, \eta)(\eta - 1)|_{\eta=1}$. Noting that $\int_0^1 |\phi_1(s, \xi)|\, d\xi = \int_0^1 |\phi_2(s, \xi)|\, d\xi = 0$, we have

$$I_{41} \leq \left(\frac{M_1 + M_2}{2m}\right) \|u_{yy}\|_\infty \|q - p\|_\infty |x_1 - x_2|, \tag{A.8}$$

where $M_2 \equiv 2\|G_x(\cdot, 1; \cdot, 0)\|_\infty$. Next, we can use the null space property of $\mathcal{G}_x$ to estimate I_{42} by

$$\begin{aligned} I_{42} &= \frac{1}{2m} \left| \int_{x_1}^{x_2} \int_0^1 \int_0^1 G_x(s, 1; \xi, \eta)[(q(\xi) - p(\xi))u_{\eta\eta}(\xi, z(\xi)) \right. \\ &\quad \left. - (q(s) - p(s))u_{\eta\eta}(s, z(s))]\, d\eta\, d\xi\, ds \right| \\ &\leq \frac{C_4(\alpha)\|u_{yy}\|_\alpha}{2m} \|q - p\|_\alpha |x_1 - x_2|. \end{aligned}$$

This, combined with (A.8) yields

$$I_4 \leq \left\{\frac{C_4(\alpha) + M_1 + M_2}{2m}\right\} \|u_{yy}\|_\alpha \|q - p\|_\alpha |x_1 - x_2|.$$

Collecting all of these estimates, we have

$$\begin{aligned} &|[T_h[q](x_1) - T_h[p](x_1)] - [T_h[q](x_2) - T_h[p](x_2)]| \\ &\quad \leq \left\{\frac{C_4(\alpha) + M_1 + M_2}{2m}\right\} \|u_{yy}\|_\alpha \|q - p\|_\alpha |x_1 - x_2| \\ &\qquad + \{M_1 K_1(p)\|u\|_\infty + K_2(p, q)\}\|p\|_\alpha \|q - p\|_\alpha |x_1 - x_2| \\ &\qquad + \left\{\frac{C_1(\alpha, 2) + C_1(0, 2)}{2m^2}\right\} |h|_\alpha \|u_{yy}\|_\infty \|q - p\|_\alpha |x_1 - x_2|^\alpha \\ &\qquad + \left\{\frac{C_1(0, 2)\overline{M}|h|_\alpha}{2m^2}\right\} \|u_{yy}\|_\infty \|p\|_\infty \|q - p\|_\alpha |x_1 - x_2|^\alpha, \end{aligned}$$

which leads to

$$\begin{aligned} &|T_h[q] - T_h[p]\|_\alpha \\ &\quad \leq \left\{\frac{C_4(\alpha) + M_1 + M_2}{2m} + \frac{C_1(\alpha, 2) + C_1(0, 2)}{2m^2} |h|_\alpha + \frac{C_1(0, 2)\overline{M}}{2m^2} |h|_\alpha \right. \end{aligned}$$

$$\times \|p\|_\infty \Big\} \|u_{yy}\|_\alpha \|q - p\|_\alpha$$
$$+ \{M_1 K_1(p)\|u\|_\infty + K_2(p, q)\} \|p\|_\alpha \|q - p\|_\alpha.$$

Combined with (A.6), this gives us

$$\begin{aligned}
&\|T_h[q] - T_h[p]\|_\alpha \\
&\quad \le \left\{ \frac{C_1(\alpha, 2) + C_1(0, 2)}{2} + \overline{M} C_1(0, 2)\|p\|_\infty + \frac{C_4(\alpha) + M_1 + M_2}{2m} \right. \\
&\qquad \left. + \frac{C_1(\alpha, 2) + C_1(0, 2)}{2m^2} |h|_\alpha + \frac{C_1(0, 2)\overline{M}}{2m^2} |h|_\alpha \|p\|_\infty \right\} \|u_{yy}\|_\alpha \|q - p\|_\alpha \\
&\qquad + \{M_1 K_1(p)\|u\|_\infty + K_2(p, q)\}\|p\|_\alpha \|q - p\|_\alpha \\
&\quad \equiv \left\{ A\|u_{yy}\|_\alpha + \frac{B}{m}\|p\|_\alpha \right\} \|q - p\|_\alpha,
\end{aligned}$$

which is (4.1). ▮

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