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The Case for Subsidizing Harm: Constrained and Costly Pigouvian Taxation with Multiple Externalities*

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Abstract

Many activities are subsidized despite generating negative externalities. Examples include needle exchanges and energy production subsidies. We explain this phenomenon by developing a model in which the policymaker faces constraints or costs. We highlight three examples. First, it may be optimal to subsidize a harmful activity if the policymaker cannot set the first-best tax on an externally harmful substitute. Second, it may be optimal to subsidize a harmful production process if the activity mix at lower levels of output uses more harmful activities than the activity mix at higher levels of output. Third, it may be optimal to subsidize a harmful activity if there is a large administrative cost associated with taxing a harmful substitute. We also show how the functional form of the cost of administering a Pigouvian tax affects the optimal tax. When administrative cost is a function of only tax rates, the policymaker should tax each activity. However, an increase in the tax presents a tradeoff: lower externality, but higher administrative cost. A subsidy may be optimal for some externally harmful activities. When administrative cost is a function of only activity levels, it may not be optimal to tax every activity. If it is optimal to tax all of the activities, the policymaker should set the tax equal to the externality plus the marginal administrative cost. If it is not optimal to tax every activity, the complementarity between activities comes into play and it may be optimal to subsidize externally harmful activities.

Keywords: Administrative cost, Corrective taxation, Externality, Optimal taxation, Optimal tax systems, Pigouvian taxation, Second best
JEL Codes: H21, H23

Taxes used to correct externality-generating behaviors are named for Arthur Cecil Pigou, who first described many of their features (Pigou, 1920). Pigouvian taxes improve welfare by aligning private incentives with a notion of public wellbeing. A broad class of policies that influence behavior—including carbon taxes, gasoline taxes, and toll roads—fit into a Pigouvian tax framework.

This article focuses on cases in which harmful activities are subsidized. Important examples include subsidies for proper waste disposal (Fullerton and Kinnaman, 1995), subsidies for needle exchanges, and subsidies for most types of energy production despite the fact that all energy production processes generate negative externalities, as shown in the table below.[1]

Table 1: Energy technologies, externalities, and policies

Technology	Externalities	U.S. Policy
Coal	Greenhouse gases, Acid rain, Hazardous waste, Airborne particulates, Risk of mining accidents	Tax & Subsidize
Oil	Greenhouse gases, Hazardous waste, Airborne particulates, Risk of oil spills,	Tax & Subsidize
Natural Gas	Greenhouse gases, Ecosystem destruction, Airborne particulates, Water contamination	Tax & Subsidize
Nuclear	Hazardous waste, Risk of nuclear meltdown	Subsidize
Hydropower	Ecosystem destruction, Risk of dam failure	Subsidize
Bioenergy	Greenhouse gases	Subsidize
Solar	Toxic production process, Ecosystem destruction	Subsidize
Geothermal	Toxic gases released	Subsidize
Wind	Harm to wildlife, Eyesore	Subsidize

To better understand the circumstances under which a policymaker might optimally subsidize externally harmful activities, we develop a model with multiple externally harmful activities and characterize optimal policy when the policymaker faces constraints and costs.[2] Possible constraints

[1]This is a partial list of the external harms discussed in the World Energy Council's 2016 World Energy Resource Report. Energy subsidies exceeded $600 billion dollars globally in 2017 (Taylor, 2020).

[2]There are many different polluting activities, each of which releases different combinations of pollutants. The

include technological limitations and political realities that place upper bounds on some taxes or make some activities untaxable. Possible costs include resource expenditures to measure activity levels, enforce tax laws, and maintain a tax bureaucracy. Crucially, when a policymaker faces constraints and costs, the complementarity between activities matters. We generalize previous work that examines settings with a single externality and find plausible cases in which the policymaker will optimally subsidize a harmful activity or output because of a constraint or cost.

When possible, the policymaker should use each instrument available to her up to the point at which the marginal social harm of that instrument eclipses the marginal social benefit. In the first best, a tax equal to marginal external harm optimally trades off private net benefit and external harm. But the policymaker does not always have complete discretion to set taxes as she pleases. When the policymaker faces constraints, some tax rates will not equal marginal external harm, leaving some externalities uncorrected. When some taxes are constrained, the marginal benefit of unconstrained taxes includes the change in external harm from activities with constrained taxes. The policymaker should use unconstrained taxes to adjust the level of activities with constrained taxes. As we show, at the social optimum, each unconstrained tax targets its own externality and also the uncorrected externality of all constrained taxes adjusted for the relationship between activities. The complementarity between activities matters, and if a substitutionary relationship exists between an activity with a constrained tax and an activity with an unconstrained tax, a subsidy on a harmful activity may be optimal. This generalizes Wijkander (1985), which studies the case with one externally harmful and untaxable activity.[3]

Another second-best scenario emerges when the policymaker cannot directly tax externally harmful activities but can tax output. As we show, an output tax is an activity tax subject to the constraint that the tax on each activity be proportional to that activity's marginal product. The optimal output tax balances marginal external harm and marginal product. Even if all activities are externally harmful, an output subsidy may be optimal depending on the relationship between activities and the correlation between the activities' marginal products and external harms.

When the policymaker has both constrained activity taxes and an output tax at her disposal, we show that she should use the output tax to mitigate the uncorrected externality from the activities with constrained taxes. This generalizes the the two-part instrument (Fullerton, 1997; Fullerton and Mohr, 2003).

EPA requires factories to report on 650 chemicals, which are produced by several different productive activities, in a Toxic Release Inventory report.

[3]Our analysis generalizes Wijkander (1985) in two ways. First, we allow an arbitrary number of externally harmful activities, which makes an optimal subsidy on a harmful activity possible. Second, we allow the policymaker to face arbitrary constraints on the taxes she may impose. In Wijkander (1985)'s model, the binding constraint on the tax is equal to 0. Our model includes the cases in which the policymaker can tax externally harmful activities but is unable to set the efficient tax because of, for example, distributional concerns or a powerful lobby.

When levying taxes carries an administrative cost, the policymaker must trade off between the net private benefit of the activities, their external harm, and the administrative cost of taxation. The functional form of administrative costs plays an important role in determining the optimal tax system. If administrative costs are a function of only tax rates, we show that the policymaker should tax every activity. However, the administrative cost imposes a tradeoff on the policymaker. Higher tax rates reduce the externality but increase the administrative cost. Administrative costs that increase with tax rates lead to taxes which only partially correct external harm.

If administrative costs are a function of only activity levels, it may no longer be optimal to tax every activity because the reduced external harm may be smaller than the administrative cost and lost private benefit. When it is optimal to tax every activity, the policymaker should set the tax equal to the externality added to the marginal administrative cost.[4] At that tax rate, the private market internalizes both the externality and the administrative cost. When it is not optimal to tax every activity, the policymaker must choose the optimal tax base and tax rates. The analysis of fixed costs follows a similar line of reasoning. An optimal subsidy on a harmful activity is possible under each considered functional forms of administrative cost.

Relevant literature. Classical Pigouvian theory directs policymakers to tax activities that generate external harm, setting the tax equal to the marginal harm. Classical Pigouvian theory rejects the idea that complements and substitutes affect the optimal Pigouvian tax. Sandmo (1976), for example, contends that "the fact that a commodity involves a negative externality is not in itself an argument for taxing other commodities which are complementary with it, nor for subsidizing substitutes." Several other papers reaffirm the superiority of direct taxation in the first best, including Kopczuk (2003).[5]

In the second best of Lipsey and Lancaster (1956), however, scholars—including Wijkander (1985) and Parry (1998)—have found that complementarity may be relevant for optimally correcting external harm.[6] Fullerton and West (2002) study the indirect taxation of a single externality,

[4]Polinsky and Shavell (1982) develop a model with one externality in which administrative costs are fixed per firm and the tax causes the number of firms to vary. Polinsky and Shavell (1982) do not, however, explore administrative costs that are a function of taxes alone.

[5]Green and Sheshinski (1976) show that when there are heterogenous agents facing a congestion externality, it is possible that individuals choose higher levels of the externally harmful activity the more external harm there is, in which case indirect correction may be superior to direction correction. Congestion externalities cause no external harm to the individual who generates them. We study atmosphere externalities, which harm everyone regardless of who generates them.

[6]Our focus is on second-best environments that restrict the instruments available to the policymaker. However, there is a large body of literature that studies the second-best question of how a policymaker should use corrective taxation in the presence of other distortionary taxes, including Kaplow (1990), Bovenberg and De Mooij (1994), Bovenberg and van der Ploeg (1994), Bovenberg and Goulder (1996), Parry (1997), Fullerton (1997), Pirttilä and Tuomala (1997), Cremer, Gahvari and Ladoux (1998), Goulder (1998), Parry (1998), Goulder et al. (1999), Pirttilä (2000), Cremer and Gahvari (2001), Kaplow (2012), Gahvari (2014), and Jacobs and de Mooij (2015). Many of these

specifically when emissions are not taxable but the policymaker can tax gasoline consumption and certain attributes of vehicles. Fullerton and Kinnaman (1995), Fullerton and Mohr (2003), and Fullerton and Wolverton (2005) examine optimal policy when a two-part instrument must be used in lieu of a direct tax on a harmful activity. Jacobsen et al. (2020) use a sufficient statistics approach that enables comparison between imperfect corrective tax policies. Bennear and Stavins (2007) describe the many ways that environmental policy is in the second best. And Burtraw et al. (2003), Groosman, Muller and O'Neill-Toy (2011), Ambec and Coria (2018), Fullerton and Karney (2018) have shown empirically that taxes and regulations aimed at one external harm also affect others.

Since our model includes multiple tax instruments, this paper also fits into the optimal tax system literature. The policymaker must choose the optimal tax system (Slemrod, 1990; Slemrod and Yitzhaki, 2002), selecting both the optimal tax base and the optimal tax rates (Yitzhaki, 1979). As in Mayshar (1991), each implemented and unconstrained tax instrument is used up to the point at which its marginal benefit falls below its marginal harm.

Roadmap. The next section introduces a simple and flexible model with multiple activities each of which may have an associated externality. Unsurprisingly, the first-best Pigouvian tax on each activity is equal to the marginal external harm from each activity. The remaining sections explore Pigouvian taxation when the policymaker faces either constraints or costs, emphasizing scenarios in which the policymaker should optimally subsidize a harmful activity.

Section two describes the optimal tax when the policymaker faces constraints. The policymaker first faces an upper bound to the taxes she may impose. At the social optimum, the policymaker will use unconstrained taxes to effect changes in the levels of activities with constrained taxes. If, for example, a very harmful activity cannot be taxed at a high level, it may be optimal to subsidize a substitute. The policymaker is then constrained to tax only output. Even if all production activities are harmful, the policymaker should subsidize output if the activity mix at lower levels of output uses more harmful activities than the activity mix at higher levels of output. Lastly, we explore the case in which the policymaker faces an upper bound on the activity taxes she may impose but also has recourse to an output tax. In this case, the policymaker should use the output tax and the unconstrained activity taxes to effect changes in the levels of activities with constrained taxes.

Section three describes how the optimal tax changes when taxes are administratively costly. Administrative costs may (1) be fixed, (2) be a function of tax rates, (3) be a function of activity levels, or (4) some combination of (1) - (3). This generalizes Polinsky and Shavell (1982), which considers only one activity and, therefore, cannot explore tradeoffs between externally harmful

papers also address distributional issues that we do not explore here. Notably, Bovenberg and Goulder (1996) show that if the revenue from corrective taxes is lump-sum distributed and not used to reduce distortionary taxes, then—when external harm is sufficiently small—it may be optimal for the policymaker to subsidize the externally harmful activity.

activities.[7] Optimal subsidies arise in this context. For example, it may be optimal to subsidize the substitute of an activity that is both very externally harmful and administratively costly to tax.

1 First-best Pigouvian taxation

This section introduces a model of Pigouvian taxation with multiple externalities and explores optimal policy when the policymaker faces neither constraint nor cost. The result is as expected: the optimal tax vector is equal to the optimal vector of externalities.

Let x be an n-dimensional vector of activity levels.[8] Activities have both private costs and private benefits and can only be performed in nonnegative quantities.[9] Let $X \subseteq \mathbb{R}^n_+$ be the possible set of activity levels. We assume X is convex and has finite measure. We follow Polinsky and Shavell (1982) and analyze activity levels, but x could alternatively be interpreted as a vector of consumption goods, production inputs, or emission levels. However, the externality associated with a good depends on how that good is consumed (Sandmo, 1976; Fullerton and West, 2002) and produced (Plott, 1966; Cremer and Gahvari, 2001); similarly, the externality associated with a production input depends on how that input is used; and mapping emissions to private net benefit or utility requires additional assumptions. Thus we believe activity levels to be the most natural and general interpretation.

Following Ramsey (1927), a net benefit function $b : X \to \mathbb{R}$ maps activity levels to private net benefit. We assume b is twice continuously differentiable and strictly concave and achieves its maximum somewhere in the interior of X.[10]

Both external harm and tax burden are linear functions of activity levels. Let e be the n-dimensional vector of external harm per unit of activity, and t be the n-dimensional vector of tax per unit of activity. Both external harm and tax burden are measured in the same units as net private benefit.

[7] As Table 2 describes, Polinsky and Shavell (1982) study a subset of the administrative cost functional forms that we study here.

[8] An activity is defined as a specific action undertaken at a specific time and in a specific place—for example, burning coal in San Fransisco on July 1st, 2020. With this level of specificity, each activity will have a well-defined externality.

[9] Activities have private benefit either directly or—because they result in the production of valuable goods that have a private benefit—indirectly. Thus activities would include both dancing and burning coal. Dancing would have a direct private benefit and burning coal would be used in the production of a good. There are, of course, many goods for which burning coal might be a part of the production process. The private market ensures that the goods with the highest net private benefit are produced. As in Becker (1965), the private benefit of activities may vary with the available goods.

[10] One way to ensure this outcome is to assume that $\lim_{x_i \searrow \partial X} \frac{\partial b}{\partial x_i} = \infty$ and $\lim_{x_i \nearrow \partial X} \frac{\partial b}{\partial x_i} = -\infty$.

This is a flexible model that generalizes many other settings.[11] Notably, the widely used model in which a representative agent optimizes strictly concave utility subject to a weakly convex production possibility frontier—assuming one activity (say leisure) generates no external harm and cannot be taxed—is a special case of the net benefit approach.

Proposition 1. *Let $u(x, x_{n+1}) : \mathbb{R}^{n+1} \to \mathbb{R}$ be a strictly increasing, strictly concave utility function of a representative agent over $n+1$ activities. Let $p(x, x_{n+1}) : \mathbb{R}^{n+1} \to \mathbb{R}$ be a strictly increasing, weakly convex function, such that $p(x, x_{n+1}) = 0$ defines a production possibility frontier. Then*

(i) there exists a function $f(x) : \mathbb{R}^n \to \mathbb{R}$ such that $p(x, x_{n+1}) = 0$ if and only if $x_{n+1} = f(x)$;

(ii) $u(x, f(x)) = b(x)$ defines a strictly concave net benefit function; and

(iii) assuming activity $n+1$ generates no external harm and is untaxed, the agent's best response function to a vector of activity taxes is the same under either the constrained utility maximization problem or the unconstrained net benefit maximization problem.

Proof. See appendix. □

In our general framework, consumers and firms take t as given and make decisions to maximize utility and profit, resulting in an equilibrium level of activities. For brevity, we say the private market solves

$$\max_x b(x) - t^\top x \tag{1}$$

which leads to the first order condition

$$b'(x) - t^\top = 0 \tag{2}$$

Because b is strictly concave it has an invertible Hessian.[12] Therefore, by the implicit mapping theorem, there exists a continuously differentiable function, $x(t) : \mathbb{R}^n \to \mathbb{R}^n$, such that $b'(x(t)) - t^\top = 0$. Note that $b''(x(t))x'(t) = I$, the identity matrix, so $x'(t)$ is invertible, $x'(t)^{-1} = t'(x)$, and $x(t)$ is a bijection. These are useful properties of $x(t)$ that we make use of.

The market equilibrium of activities changes with t; $x(t)$ is the private market's best response function to t. $x'(t)$ shows how the private market's best response to t changes with t and reveals the

[11] The net benefit approach cannot generally model non-competitive settings because, as Buchanan (1969) shows, market power changes the equilibrium level of activities, which may incidentally correct or exacerbate external harm. If so, the relationship between taxes and uncorrected external harm would require additional assumptions to model.

[12] We use matrix calculus and the associated notation, including $^\top$ to denote a matrix's transpose

complementarity between activities. Complementarity, in this context, means how the equilibrium quantity of one activity changes in response to a change in the tax on another. An advantage of the net benefit approach is that it encompasses complementarities that arise from both consumer preferences and production technology, and as Ramsey (1927) notes, the net benefit approach also allows the utility from consumption to vary with the method of production.

We now show that when there are no constraints or costs the policymaker should set tax rates equal to marginal external harm.

Proposition 2. *In the first best, with costless tax administration, $t = e$ uniquely satisfies the first order condition of the policymaker's problem.*

Proof. The policymaker solves

$$\max_t b(x(t)) - e^\top x(t) \tag{3}$$

which leads to the first order condition $b'(x(t^*))x'(t^*) - e^\top x'(t^*) = 0$. Substituting $b'(x(t)) = t^\top$, the private market's first order condition, yields $(t^* - e)^\top x'(t^*) = 0$. $t^* = e$ is clearly a solution, and the invertibility of $x'(t^*)$ ensures that it is the unique solution. □

This is Pigou (1920)'s remarkable result generalized to arbitrary dimensions. When $t = e$, the private market fully internalizes every externality, and the policymaker need only know e to set optimal policy. Using the tax on activity i to induce changes in activity j is not welfare improving because there is no welfare gain from changing the activity level in a market that already internalizes the externality.

Proposition 3. *In the first best, with costless tax administration, $t = e$ is a global maximizer.*[13]

Proof. Consider two arbitrary activity tax vectors, v and w. The change in welfare caused by moving from tax vector v to tax vector w is

$$\Delta b - e^\top \Delta x = b(x(w)) - b(x(v)) - e^\top [x(w) - x(v)] \tag{4}$$

$$= b(x(\gamma(1))) - b(x(\gamma(0))) - e^\top [x(\gamma(1)) - x(\gamma(0))] \tag{5}$$

[13] This is true because the net benefit function is strictly concave, which rules out both perfect substitutes and perfect complements relationships between activities.

where $\gamma : \mathbb{R} \to \mathbb{R}^n, \gamma(r) = v + r(w - v)$ and $r \in [0, 1]$ is a scalar. By the fundamental theorem of calculus

$$= \int_0^1 [b'(x(\gamma(r))) - e^\top] x'(\gamma(r))\gamma'(r)dr \tag{6}$$

Recalling that $b'(x(t)) = t^\top$

$$= \int_0^1 (\gamma(r) - e)^\top x'(\gamma(r))\gamma'(r)dr \tag{7}$$

Noting that $\gamma'(r) = (w - v)$

$$= \int_0^1 (v + r(w - v) - e)^\top x'(\gamma(r))(w - v)dr \tag{8}$$

If $v = e$ (i.e. the policymaker shifts away from the optimal activity tax), then

$$\Delta b - e^\top \Delta x = (w - e)^\top \left[\int_0^1 r x'(\gamma(r))dr \right] (w - e) \tag{9}$$

$$= \sum_{i=1}^n \sum_{j=1}^n (w_i - e_i)(w_j - e_j) \int_0^1 r \tfrac{\partial x_i}{\partial t_j}(\gamma(r))dr \tag{10}$$

The integrand in (9) is a negative definite matrix and the expression is a quadratic form. Thus the optimal activity tax is strictly superior to any other activity tax. □

We may approximate the change in welfare caused by moving from tax vector v to tax vector w by assuming that the average activity level over the affine path between v and w is equal to the average of activity levels at v and w.[14] Starting from (7)

$$\Delta b - e^\top \Delta x = \int_0^1 (\gamma(r) - e)^\top x'(\gamma(r))\gamma'(r)dr \tag{11}$$

Integrating by parts

$$= (w - e)^\top x(w) - (v - e)^\top x(v) - (w - v)^\top \int_0^1 x(\gamma(r))dr \tag{12}$$

[14] Some papers, for example Jacobsen et al. (2020), use the stronger assumption that $x'(\gamma(r))$ is constant for $r \in [0, 1]$. Under this stronger assumption $\frac{1}{2}(w - e)^\top \Delta x = \frac{1}{2}(w - e)^\top x'(e)(w - e)$.

Assuming that $\int_0^1 x(\gamma(r))dr = \frac{1}{2}(x(w) + x(v))$ and rearranging

$$= \frac{1}{2}(w + v - 2e)^\top \Delta x = \frac{1}{2}\sum_{i=1}^{n}(w_i + v_i - 2e_i)\Delta x_i \tag{13}$$

which is the sum of n Harberger trapezoids. To see the welfare lost compared to the first best, set $v = e$

$$= \frac{1}{2}(w - e)^\top \Delta x = \frac{1}{2}\sum_{i=1}^{n}(w_i - e_i)\Delta x_i \tag{14}$$

which is the sum of n Harberger (1964) triangles.

2 Second-best Pigouvian taxation

We now consider a second-best environment, in which the policymaker faces a constraint that prevents the implementation of the first-best tax regime. We use the term *constrained activity taxation* to refer to the case in which the policymaker cannot set activity taxes above a certain level.[15] The policymaker might face constrained activity taxation if a powerful lobby or distributional concerns prevented her from raising activity taxes above a certain level. A special case arises when the policymaker is constrained to leave some activities untaxed because, for example, the measurement of some activities is technologically impossible, prohibitively expensive, or hopelessly susceptible to evasion.[16] Even if no activities can be taxed, the policymaker may be able to tax output, which is easier to measure because of the record keeping associated with market transactions. We use the term *output taxation* to refer to the case in which the policymaker is constrained to tax only output. This section first analyzes constrained activity taxation, then output taxation, and lastly the combined case in which the policymaker has at her disposal both constrained activity taxes and an output tax.

2.1 Constrained activity taxation

There are three possible cases when a policymaker faces constraints on the activity taxes she may set. At the social optimum, (1) the constraints on none of the taxes bind, (2) the constraints on all of the taxes bind, or (3) the constraints on some of the taxes bind. If none of the constraints bind,

[15] We omit analysis of the similar case in which the policymaker must set activity taxes above a certain level.

[16] An common example is undetectable illegal dumping (Fullerton and Kinnaman, 1995).

the policymaker is effectively in the first best. If all the constraints bind, the optimal tax system is defined by the constraints. The non-trivial case, in which some constraints bind, is the one we consider here.

Whenever a tax constraint binds, the external harm of the activity subject to that tax is not corrected. The tax will not set the marginal social benefit of the activity on which it is levied equal to the marginal social cost of that activity. In that case, using taxes with non-binding constraints to change the levels of activities with bound taxes improves welfare. At the optimum, the marginal social benefit of each unbound tax is equal to that tax's marginal social cost, where the marginal social benefit of an unbound tax is the reduction in total external harm and the marginal social cost is the lost private benefit.

To model constrained activity taxation, assume that the policymaker may tax all activities as before but now faces a vector of upper limits, T, on the taxes she may impose. The policymaker's optimization problem is a Lagrangian with a vector of Lagrange multipliers λ:

$$\max_t b(x(t)) - e^\top x(t) + \lambda^\top (T - t) \tag{15}$$

The first order condition is

$$b'(x(t^*))x'(t^*) - e^\top x'(t^*) - \lambda^\top = 0 \tag{16}$$

After substituting the private market's first order condition, $b'(x(t)) = t^\top$, we have

$$(t^* - e)^\top x'(t^*) - \lambda^\top = 0 \tag{17}$$

Before analyzing the case with an arbitrary number of activities, we consider the case with two activities to gain intuition.

Example 1. Let x_s be the quantity of solar power plant activity and x_c be the quantity of coal power plant activity. The policymaker's first order conditions are

$$(t_s^* - e_s)\frac{\partial x_s}{\partial t_s} + (t_c^* - e_c)\frac{\partial x_c}{\partial t_s} - \lambda_s = 0 \tag{18}$$

and

$$(t_s^* - e_s)\frac{\partial x_s}{\partial t_c} + (t_c^* - e_c)\frac{\partial x_c}{\partial t_c} - \lambda_c = 0 \tag{19}$$

Now assume that the constraint on the coal tax binds but the constraint on the solar tax does not.

Then $\lambda_s = 0$ and $t_c^* = T_c$, and (18) may be written

$$(t_s^* - e_s)\tfrac{\partial x_s}{\partial t_s} + (T_c - e_c)\tfrac{\partial x_c}{\partial t_s} = 0 \tag{20}$$

(20) states that, at the optimum, the marginal net benefit from an increase in t_s must be 0. Because t_c is constrained, the policymaker can no longer use corrective taxes to set the marginal benefit of x_c equal to its marginal cost, but the policymaker can increase welfare by using t_s to induce changes in x_c. In other words, the optimal policy should take advantage of the complementarity between activities. The policymaker will opt to under or overcorrect the externality on x_s to reduce the external harm from x_c.

Because t_c is constrained, $\frac{\partial x_c}{\partial t_s} / \frac{\partial x_s}{\partial t_s} = \frac{dx_c}{dx_s}$, which is the tradeoff between x_s and x_c the policymaker faces given the private market's best response function and the binding constraint on t_c. Thus

$$(t_s^* - e_s) + (T_c - e_c)\frac{dx_c}{dx_s} = 0 \tag{21}$$

(21) states that the marginal net private benefit of x_s should equal the marginal external harm from x_s. In the second best, the marginal net private benefit is $t_s^* + T_c\frac{dx_c}{dx_s}$, and the marginal external harm from x_s is $e_s + e_c\frac{dx_c}{dx_s}$. In the first best, $t_c = e_c$ which makes the complementarity between the activities irrelevant, but in (21) $T_c < e_c$ and thus the complementarity matters. In other words, when $T_c < e_c$, x_c has a net marginal social loss, meaning that reductions in x_c result in a social gain. The policymaker should use t_s to attain this gain up to the point at which this gain is offset by the net marginal social loss from under or over correcting x_s.

With some algebraic manipulation, we can isolate t_s^*.

$$t_s^* = e_s + (e_c - T_c)\frac{dx_c}{dx_s} \tag{22}$$

which shows that, at the social optimum, the tax on solar activity is equal to the externality of solar activity added to the uncorrected externality on coal adjusted for the tradeoff between x_s and x_c. (22) also shows the conditions under which the policymaker should optimally subsidize solar plant activity. By assumption both activities are externally harmful, but the tax constraint binds only for coal. Thus, $e_s > 0$ and $(e_c - T_c) > 0$. Since $\frac{\partial x_s}{\partial t_s} < 0$, the sign of $\frac{dx_c}{dx_s}$ depends on $\frac{\partial x_c}{\partial t_s}$, the complementarity between the activities. If the activities are substitutes, then $\frac{\partial x_c}{\partial t_s} > 0$, in which case it is possible that $e_s + (e_c - T_c)\frac{dx_c}{dx_s} < 0$, making a subsidy on solar plant activity optimal. As (22) highlights, the case for a subsidy on solar strengthens when the marginal external harm from solar plant activity is smaller, the uncorrected externality on coal plant activity is larger, the effect of the solar tax on solar plant activity is smaller, and solar and coal are more substitutable.[17]

This example generalizes to n activities, m non-binding tax constraints, and $n-m$ binding tax constraints, but before doing so it will be helpful to define some additional notation and explore the relationship between tax-constrained and tax-unconstrained activities in the second best. Without loss of generality, assume that the taxes with binding constraints are $t_{m+1}\dots t_n$. Separate the Jacobian $x'(t)$ into four partitions as follows.

$$x'(t)=\left(\begin{array}{ccc|ccc} \frac{\partial x_1}{\partial t_1} & \dots & \frac{\partial x_1}{\partial t_m} & \frac{\partial x_1}{\partial t_{m+1}} & \dots & \frac{\partial x_1}{\partial t_n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_m}{\partial t_1} & \dots & \frac{\partial x_m}{\partial t_m} & \frac{\partial x_m}{\partial t_{m+1}} & \dots & \frac{\partial x_m}{\partial t_n} \\ \hline \frac{\partial x_{m+1}}{\partial t_1} & \dots & \frac{\partial x_{m+1}}{\partial t_m} & \frac{\partial x_{m+1}}{\partial t_{m+1}} & \dots & \frac{\partial x_{m+1}}{\partial t_n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial t_1} & \dots & \frac{\partial x_n}{\partial t_m} & \frac{\partial x_n}{\partial t_{m+1}} & \dots & \frac{\partial x_n}{\partial t_n} \end{array}\right)=\left(\begin{array}{c|c} Dx_u & Dx_c \\ \hline {Dx_c}^\top & Dx_b \end{array}\right) \tag{23}$$

Dx_u is the submatrix comprised of all 'unbound' partials $\frac{\partial x_i}{\partial t_j}$, with $i,j\in 1,\dots,m$, Dx_b is the submatrix comprised of all 'bound' partials $\frac{\partial x_i}{\partial t_j}$, with $i,j\in m+1,\dots,n$, and Dx_c is the submatrix comprised of all 'bound crossed with unbound' partials $\frac{\partial x_i}{\partial t_j}$, with $i\in 1,\dots,m, j\in m+1,\dots,n$. Since $\frac{\partial x_i}{\partial t_j}=\frac{\partial x_j}{\partial t_i}$, the fourth submatrix is ${Dx_c}^\top$. Similarly, separate all relevant vectors into two partitions

$$v=\left(\begin{array}{ccc|ccc} v_1 & \dots & v_m & v_{m+1} & \dots & v_n \end{array}\right)=\left(\begin{array}{c|c} v_u & v_b \end{array}\right) \tag{24}$$

such that u again denotes 'unbound' and b denotes 'bound'.

Note that for a vector dt of small changes to t the resulting changes to x would be

$$dx = x'(t)dt \tag{25}$$

[17]The net benefit approach allows a flexible relationship between activities even when there are only two. This can be reconciled to utility maximization by noting that net benefit maximization may be interpreted as utility maximization with an additional activity that is untaxed and causes no external harm (see the appendix). As a consequence, when using a net benefit function, there is no requirement that activities be Hicksian substitutes when there are only two.

which may be rewritten using the partitions defined in (23) and (24) as

$$dx_u = Dx_u dt_u + Dx_c dt_b \tag{26}$$

and

$$dx_b = {Dx_c}^{\top} dt_u + Dx_b dt_b \tag{27}$$

Because Dx_u is a principal submatrix of a negative definite matrix it is invertible. In the second best, where taxes $t_{m+1}, \ldots, t_n$ are constrained, $dt_b = 0$. Thus

$$dt_u = [Dx_u]^{-1} dx_u \tag{28}$$

and therefore

$$dx_b = {Dx_c}^{\top} [Dx_u]^{-1} dx_u \tag{29}$$

In the first best, the policymaker faces no restrictions on the marginal changes to activity levels that she is able to induce with the appropriate marginal changes to tax rates. When some of the activity taxes face binding constraints, the policymaker cannot induce arbitrary marginal changes to activity levels. If the policymaker induces arbitrary changes to the activities with unbound taxes, (29) shows the required changes to the activities with bound taxes that ensure the tax constraints and the private market's best response function remain satisfied. In other words, given the tax constraints, ${Dx_c}^{\top}[Dx_u]^{-1}$ is the matrix of partial derivatives below that maps dx_u to dx_b.

$${Dx_c}^{\top}[Dx_u]^{-1} = \begin{pmatrix} \frac{\partial x_{m+1}}{\partial x_1} & \ldots & \frac{\partial x_{m+1}}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial x_1} & \ldots & \frac{\partial x_n}{\partial x_m} \end{pmatrix} \tag{30}$$

(30) will prove useful for interpreting optimal tax expressions. With this background, we return now to our generalization of example 1.

Proposition 4. *If tax administration is costless, at the social optimum, each unconstrained tax t_i will equal the marginal externality of activity i added to the uncorrected externality of all activities with bound taxes weighted by the responsiveness of the activity to changes in activity i.*

Partitioning (17) using the partitions in (23) and (24), the policymaker's first order conditions may be written

$$({t_u}^* - e_u)^\top Dx_u + ({t_b}^* - e_b)^\top {Dx_c}^\top - {\lambda_u}^\top = 0 \tag{31}$$

and

$$({t_u}^* - e_u)^\top Dx_c + ({t_b}^* - e_b)^\top Dx_b - {\lambda_b}^\top = 0 \tag{32}$$

Now since the constraint on t_b binds but the constraint on t_u does not, $\lambda_u = 0$ and $t_b^* = T_b$, and thus (31) is equivalent to

$$({t_u}^* - e_u)^\top Dx_u + (T_b - e_b)^\top {Dx_c}^\top = 0 \tag{33}$$

(33) states that, at the optimum, the marginal net benefit from an increase in t_i with $i \in \{1, \ldots, m\}$ must be 0. Again, the optimal policy should take advantage of the complementarity between activities. The policymaker will opt to under or overcorrect the externality on activities with unbound taxes to better trade off external harm and private net benefit. Applying the invertibility of Dx_u, we arrive at

$$({t_u}^* - e_u)^\top + (T_b - e_b)^\top {Dx_c}^\top [Dx_u]^{-1} = 0 \tag{34}$$

Interpreting ${Dx_c}^\top [Dx_u]^{-1}$ as a matrix of partials as in (30), we have for each unconstrained tax, $i \in \{1, \ldots, m\}$,

$$({t_i}^* - e_i) + \sum_{j=m+1}^{n} (T_j - e_j) \tfrac{\partial x_j}{\partial x_i} = 0 \tag{35}$$

(35) shows how the policymaker must trade off the uncorrected externality of each activity with a non-binding tax constraint against the uncorrected externality of all the tax-constrained activities. In the first best, $t_j^* = e_j$, which makes the complementarity between the activities irrelevant, but in

(35) $T_j \neq e_j$ for all $j \in \{m+1, \dots, n\}$ and thus complementarity matters.[18] Isolating t_i^* yields

$$t_i{}^* = e_i + \underbrace{\sum_{j=m+1}^{n} (e_j - T_j)\tfrac{\partial x_j}{\partial x_i}}_{\text{second-best modification}} \tag{36}$$

which shows how the optimal tax compares to the first best. At the social optimum, for taxes without binding constraints, the tax on each activity is equal to the direct marginal external harm of that activity added to the uncorrected externality of all activities with constrained taxes, adjusted for the relationship between activities. (36) also shows the conditions under which the policymaker should optimally subsidize a harmful activity. An optimal subsidy on activity i requires that $e_i + \sum_{j=m+1}^{n}(e_j - T_j)\frac{\partial x_j}{\partial x_i} < 0$, meaning that $(e_j - T_j)$ and $\frac{\partial x_j}{\partial x_i}$ must have opposite signs for at least one j. This requires a substitutionary relationship between at least one of the activities with bound taxes and i. As before, the case for a subsidy on an harmful activity strengthens when the marginal direct external harm from that activity is smaller, the uncorrected externality on activities with binding tax constraints is larger, and small increases of the activity level cause larger increases in the levels of activities with binding tax constraints.[19]

It is instructive to decompose the welfare lost, relative to the first best, when the policymaker faces a binding constraint on some activity taxes. From (9) we know that the lost welfare for an arbitrary activity tax vector w is

$$\Delta b - e^\top \Delta x = (w - e)^\top \left[\int_0^1 r x'(\gamma(r)) dr \right] (w - e) \tag{37}$$

[18] Consider a very harmful activity with a low binding constraint. Optimal policy would require the policymaker to tax any complement above the marginal externality of that complement. But the complement might too face a binding constraint. Thus, with more than two externally harmful activities, it is possible to have a binding constraint $T_j > e_j$, and technically possible to have a knife edge case with a binding constraint $T_j = e_j$.

[19] A special case of this model occurs when some activities are untaxable, in which case those activities have taxes with a binding constraint equal to 0. The uncorrected externality of the untaxable activities is equal to the marginal externality. If all binding constraints are equal to 0, (36) becomes

$$t_i{}^* = e_i + \sum_{j=m+1}^{n} e_j \tfrac{\partial x_j}{\partial x_i}$$

Wijkander (1985) studies this problem when there are three activities, one of which is externally harmful but untaxable.

which may be partitioned using (23) into

$$= \sum_{i=1}^{m}\sum_{j=1}^{m}(t_i^* - e_i)\left[\int_0^1 r\frac{\partial x_i}{\partial t_j}dr\right](t_j^* - e_j) \tag{38}$$

$$+ \sum_{i=1}^{m}\sum_{j=m+1}^{n}(t_i^* - e_i)\left[\int_0^1 r\frac{\partial x_i}{\partial t_j}dr\right](T_j - e_j) \tag{39}$$

$$+ \sum_{i=m+1}^{n}\sum_{j=1}^{m}(T_i - e_i)\left[\int_0^1 r\frac{\partial x_i}{\partial t_j}dr\right](t_j^* - e_j) \tag{40}$$

$$+ \sum_{i=m+1}^{n}\sum_{j=m+1}^{n}(T_i - e_i)\left[\int_0^1 r\frac{\partial x_i}{\partial t_j}dr\right](T_j - e_j) \tag{41}$$

Because $x'(t)$ is negative definite, its principal submatrices must also be negative definite. Thus (38) and (41), which are quadratic forms, must be negative. (41) is the welfare lost because of the uncorrected externality on all activities with bound taxes. (38) is the welfare lost because the policymaker under or overcorrects the externalities on activities with non-binding taxes. The policymaker sacrifices (38) in order to gain (39) and (40)—the welfare increase from optimally using the complementarity between activities. Note that by the symmetry of $x'(t)$, (39) is equal to (40). The larger the uncorrected externality and the larger the magnitude of the cross partial derivatives in (39) and (40), the larger the welfare gain from using the complementarity between activities optimally.

2.2 Output taxation

An output tax cannot generally induce the private market to select the socially optimal combination of activities. Consider, for example, producing electricity from either coal or solar energy. Assume using coal is privately cheaper (by an arbitrarily small amount) but also has a larger externality. If coal and solar energy are perfect substitutes, then coal will be preferred at any output tax rate. Moreover, an output tax may be detrimental if, for example, the externality is generated by an inferior technology, in which case an output tax will increase external harm (Plott, 1966). Nonetheless, given a set of externally harmful activities that generate output, an output tax or subsidy may be a useful tool depending on the marginal products and marginal external harms of the various activities and the relationship between activities. This section formalizes this intuition and analyzes the optimal output tax.

The model used in the previous section is also used here, albeit with some modification. In the previous section, the model made no explicit reference to output. In fact, the model could implicitly include many different outputs.[20] In this section, we restrict the model to one output and introduce the strictly increasing and weakly concave function $q(x) : X \to \mathbb{R}$ which maps the vector of activity levels to the quantity of output produced.[21] We interpret $q(x)$ to map activities to quantity of goods, but there are other potentially useful interpretations.[22] For example, carbon output is an increasing function of burning coal and burning natural gas. A carbon tax will affect both activities but cannot perfectly correct their external harm because both activities cause external harm beyond what a carbon tax captures, including, for example, non-carbon harmful gas emissions. But if the policymaker is constrained to tax carbon output, as opposed to the individual activities, then the intuitions in this section apply.

Let τ be the tax on output. Then the private market's problem is

$$\max_x b(x) - \tau q(x) \tag{42}$$

with first order condition $b'(x(\tau)) - \tau q'(x(\tau)) = 0$, where $x(\tau) : \mathbb{R} \to \mathbb{R}^n$ is the private market's best response function to τ.[23] Taking the derivative of the private market's first order condition with respect to τ yields

$$b''(x(\tau))x'(\tau) - \tau q''(x(\tau))x'(\tau) - q'(x(\tau))^\top = 0 \tag{43}$$

which after some manipulation is equivalent to

$$x'(\tau) = \mathrm{A}q'(x(\tau))^\top \tag{44}$$

where $\mathrm{A} = [b''(x(\tau)) - \tau q''(x(\tau))]^{-1}$ which is a negative definite matrix.[24] $x'(\tau)$ shows how the private market's best response changes with the output tax, which is central to the optimal output tax.

Proposition 5. *An increase in the output tax decreases output.*

[20] One possible interpretation of a tax on a single output is a tax on income or consumption.

[21] As discussed in the appendix, we assume private benefit is a function of output, and private cost is a function of activity levels. There is, thus, a relationship between private net benefit and output.

[22] As noted previously, many activities do not directly increase private benefit but do so indirectly as part of a production process for goods. For example, burning coal does not directly have a private net benefit (in most instances), but the energy derived from burning coal is used for many things that do have a private benefit.

[23] The appendix shows that $x(\tau)$ is well defined.

[24] See the appendix for a proof that A is negative definite. Recall that $x'(t) = [b''(x)]^{-1}$. $x'(\tau)$ is related but, instead of keeping marginal private net benefit equal to the tax vector, it ensures that the marginal private net benefit of each activity is proportional to its marginal product. Moreover, the vector of marginal products play a roll in $x'(\tau)$ because a change in τ effectively changes the tax on each activity in proportion to its marginal product.

Proof. $q'(x(\tau))x'(\tau) = q'(x(\tau))\mathrm{A}q'(x(\tau))^\top < 0$ because A is negative definite and $q'(x(\tau))\mathrm{A}q'(x(\tau))^\top$ is a quadratic form. □

Proposition 6. *An arbitrary output tax τ causes the same private market behavior as an activity tax equal to $\tau q'(x(\tau))$.*

Proof. Recall that the private market's best response to an activity tax is defined by $b'(x(t)) - t^\top = 0$ and the private market's best response to an output tax is defined by $b'(x(\tau)) - \tau q'(x(\tau)) = 0$. Thus an activity tax $t^\top = \tau q'(x(\tau))$ induces the same private market response as the output tax τ.[25] □

Intuitively, for any output tax, the private market will respond as if there were a tax on each activity equal to the output tax times the marginal product of that activity. The effective output tax on each activity is proportional to that activity's marginal product, and thus an output tax cannot target activity externalities unless the external harm of each activity is also proportional to its marginal product.

Proposition 7. *If the only instrument available to the policymaker is an administratively costless output tax, at the social optimum the output tax will be equal to the marginal external harm of the output tax divided by the marginal product of the output tax.*

Proof. The policymaker's problem is

$$\max_{\tau} b(x(\tau)) - e^\top x(\tau) \tag{45}$$

with first order condition $b'(x(\tau^*))x'(\tau^*) - e^\top x'(\tau^*) = 0$, which sets the marginal benefit of the tax equal to the marginal cost of the tax, where the benefit is reduced external harm and the cost is reduced private net benefit. Substituting in the private market optimum results in

$$\tau^* q'(x(\tau^*))x'(\tau^*) = e^\top x'(\tau^*) \tag{46}$$

Dividing both sides by $q'(x(\tau^*))x'(\tau^*)$ yields

$$\tau^* = \frac{e^\top x'(\tau^*)}{q'(x(\tau^*))x'(\tau^*)} = \frac{\sum_{i=1}^n e_i \frac{\partial x_i}{\partial \tau}}{\sum_{i=1}^n \frac{\partial q}{\partial x_i} \frac{\partial x_i}{\partial \tau}} \tag{47}$$

□

[25] This is somewhat abusive notation. $x(t)$ is the mapping from activity taxes to activity levels and $x(\tau)$ is the mapping from output tax to activity levels.

At the social optimum, the output tax is equal to a ratio. The numerator of the ratio is the marginal external harm of an increase in the output tax and may be either positive or negative. The denominator is the marginal output attributable to an increase in the output tax, which as shown above, must be negative.

With a single activity, $x'(\tau^*)$ is a negative scalar, which cancels out from the numerator and the denominator, leaving $\tau^* = e/q'(x(\tau^*))$. The optimal output tax achieves the first best because the policymaker can set the output tax such that the marginal harm from the activity is equal to the marginal private net benefit of that activity, $b'(x'(\tau^*)) = \tau^* q'(x(\tau^*)) = e$.

With an arbitrary number of activities, $x'(\tau)$ is vector valued and does not cancel out. The optimal output tax depends on b and q. For intuition, we return to our two activity example.

Example 2. Let x_s be the quantity of solar power plant activity and x_c be the quantity of coal power plant activity. Both activities produce electricity. Assume that the policymaker may tax neither activity but may tax the total quantity of electricity. From (47), the optimal output tax is

$$\tau^* = \frac{e_s \frac{\partial x_s}{\partial \tau} + e_c \frac{\partial x_c}{\partial \tau}}{\frac{\partial q}{\partial x_s} \frac{\partial x_s}{\partial \tau} + \frac{\partial q}{\partial x_c} \frac{\partial x_c}{\partial \tau}} \tag{48}$$

As noted before, the numerator, which is the marginal external harm of the output tax, determines the sign of the optimal output tax. If the numerator is positive, then an output subsidy will be optimal. An increase in the output tax cannot increase both activity levels—at least one of $\frac{\partial x_s}{\partial \tau}$ and $\frac{\partial x_c}{\partial \tau}$ must be negative—and an optimal subsidy requires that one be positive.

Intuitively, an output tax will increase the level of an activity, say coal, if coal is relatively more privately beneficial at lower levels of output. As the appendix shows, $\frac{\partial x_c}{\partial \tau} > 0$ if, for example, (1) solar activity has a large marginal product and scales well, (2) coal has a small marginal product, and (3) either the private marginal cost of coal activity increases with solar activity or the marginal product of coal activity decreases with solar activity or both.[26] Under these conditions, an increase in the output tax would decrease the amount of solar activity, which would either make coal privately cheaper or increase coal's marginal product, making coal more attractive.

If $\frac{\partial x_c}{\partial \tau} > 0$, then a subsidy will be optimal if e_c is sufficiently large so that an increase in the output tax increases total external harm. Stated differently, $e_c \frac{\partial x_c}{\partial \tau} > -e_s \frac{\partial x_s}{\partial \tau}$, the increased external harm from higher coal use outweighs the decreased external harm from lower solar use as the output tax increases.

[26] (3) is a necessary condition. Moreover, since the Hessians of b and q are symmetric, this is the same as saying that either the private marginal cost of solar activity increases with coal activity or the marginal product of solar activity decreases with coal activity or both.

Generally, two intuitive factors influence the optimal output tax. The first is the relationship between the marginal product and the marginal external harm of each activity. If each activity's marginal external harm is exactly proportional to its marginal product, then an output tax can achieve the first best because the optimal output tax implicitly fully corrects the marginal externality of each activity. The weaker the relationship between the marginal external harm and marginal product, the less effective an output tax is in correcting external harm.

The second factor that influences the optimal output tax is $x'(\tau)$. For at least one activity, $\frac{\partial x_i}{\partial \tau} < 0$. It is not possible that a marginal increase in the output tax increases all activity levels. More generally, as (44) shows, the properties of $x'(\tau)$ are pinned down by $q'(x)$, $b''(x)$, and $q''(x)$. The non-diagonal entries in $b''(x)$ and $q''(x)$ are particularly important.

Applying (9), the welfare lost if the policymaker is constrained to tax only output is

$$\Delta b - e^\top \Delta x = (\tau^* q'(x(\tau^*)) - e)^\top \left[\int_0^1 r x' (\gamma(r)) \, dr \right] (\tau^* q'(x(\tau^*)) - e) \tag{49}$$

where $\gamma(r) = e^\top + r(\tau^* q'(x(\tau^*)) - e^\top)$. At the optimal activity tax there are no uncorrected externalities. At the optimal output tax the uncorrected externality vector is $(\tau^* q'(x(\tau^*)) - e)$. Note that the uncorrected externality is small if activities with large external harms also have high marginal products because the output tax discourages activities with high marginal products relatively more.

2.3 Constrained activity taxation with output taxation

When the policymaker faces constrained activity taxation, she may be able to improve welfare by using an output tax to reach the activities with bound taxes. This generalizes the intuition behind the two-part instrument (Fullerton, 1997; Fullerton and Mohr, 2003).

Facing both activity taxes and an output tax, the private market's problem is

$$\max_x b(x) - t^\top x - \tau q(x) \tag{50}$$

with first order condition

$$b'(x(t,\tau)) - t^\top - \tau q'(x(t,\tau)) = 0 \tag{51}$$

where $x(t,\tau) : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the private market's best response function to t and τ. The policymaker's problem is the Lagrangian

$$\mathcal{L} = b(x(t,\tau)) - e^\top x(t,\tau) + \lambda^\top (T - t) \tag{52}$$

with a vector of Lagrange multipliers λ. For ease of exposition, let

$$D_\tau x = \begin{pmatrix} \frac{\partial x_1}{\partial \tau} \\ \vdots \\ \frac{\partial x_n}{\partial \tau} \end{pmatrix} \text{ and } D_t x = \begin{pmatrix} \frac{\partial x_1}{\partial t_1} & \cdots & \frac{\partial x_1}{\partial t_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_1}{\partial t_1} & \cdots & \frac{\partial x_1}{\partial t_1} \end{pmatrix} \tag{53}$$

Proposition 8. *If the policymaker may levy constrained activity taxes and an output tax, at the social optimum the output tax will be equal to the marginal uncorrected external harm of the output tax from the activities with bound taxes, divided by the marginal product of the output tax from the activities with bound taxes.*

The policymaker's first order conditions are

$$b'(x(t^*,\tau^*))D_t x - e^\top D_t x - \lambda^\top = 0 \tag{54}$$

and

$$b'(x(t^*,\tau^*))D_\tau x - e^\top D_\tau x = 0 \tag{55}$$

Assume, as before, that at the optimum the constraints on taxes $\{m+1,\ldots,n\}$ bind. After substituting the private market's first order condition, (51), and applying the partitions described in (23) and (24), the policymaker's first order conditions with respect to t may be written

$$({t_u^*}^\top + \tau^* q_u'(x(t^*,\tau^*)) - e_u^\top)D_t x_u + ({t_b^*}^\top + \tau^* q_b'(x(t^*,\tau^*)) - e_b^\top)D_t {x_c}^\top - \lambda_u^\top = 0 \tag{56}$$

and

$$({t_u^*}^\top + \tau^* q_u'(x(t^*,\tau^*)) - e_u^\top)D_t x_c + ({t_b^*}^\top + \tau^* q_b'(x(t^*,\tau^*)) - e_b^\top)D_t x_b - \lambda_b^\top = 0 \tag{57}$$

The first order condition with respect to τ may be written

$$({t_u^*}^\top + \tau^* q_u'(x(t^*,\tau^*)) - e_u^\top)D_\tau x_u + ({t_b^*}^\top + \tau^* q_b'(x(t^*,\tau^*)) - e_b^\top)D_\tau x_b = 0 \tag{58}$$

Because $\lambda_u = 0$, $t_b^* = T_b$, and $D_t x_u$ is invertible, (56) is equivalent to

$$(t_u^{*\top} + \tau^* q_u'(x(t^*, \tau^*)) - e_u^\top) = -(T_b{}^\top + \tau^* q_b'(x(t^*, \tau^*)) - e_b^\top) D_t x_c{}^\top [D_t x_u]^{-1} \tag{59}$$

which we can substitute into (58) to arrive at

$$(T_b{}^\top + \tau^* q_b'(x(t^*, \tau^*)) - e_b^\top) y = 0 \tag{60}$$

where $y = D_\tau x_b - D_t x_c{}^\top [D_t x_u]^{-1} D_\tau x_u$ maps a small change in τ to a change in x_b given the private market's best response function, the tax constraint the policymaker faces, and the policymaker's first order condition with respect to t. $D_\tau x_b$ is the direct effect of τ on x_b, but changing τ also changes x_u, which given the policymaker's constraint and the private market's best response causes an indirect change to x_b equal to $-D_t x_c{}^\top [D_t x_u]^{-1} D_\tau x_u$. For all $j \in \{m+1, \ldots, n\}$:

$$y_j = \frac{dx_j}{d\tau} = \frac{\partial x_j}{\partial \tau} - \sum_{i=1}^{m} \frac{\partial x_j}{\partial x_i} \frac{\partial x_i}{\partial \tau} \tag{61}$$

Rearranging (60) we arrive at

$$\tau^* = \frac{(e_b - T_b)^\top y}{q_b'(x(t, \tau)) y} = \frac{\sum_{j=m+1}^{n} (e_j - T_j) \frac{dx_j}{d\tau}}{\sum_{j=m+1}^{n} \frac{\partial q}{\partial x_j} \frac{dx_j}{d\tau}} \tag{62}$$

For intuition, we return to our two activity example.

Example 3. Let x_s be the quantity of solar power plant activity and x_c be the quantity of coal power plant activity. Both activities produce electricity. Assume that the policymaker may tax both activities and the total quantity of electricity but faces a binding constraint T_c on the coal tax. From (62), the optimal output tax is

$$\tau^* = \frac{(e_c - T_c) \frac{dx_c}{d\tau}}{\frac{\partial q}{\partial x_c} \frac{dx_c}{d\tau}} = \frac{(e_c - T_c)}{\frac{\partial q}{\partial x_c}} \tag{63}$$

One binding constraint is a special case because it allows y to cancel out. Moreover, the effective activity tax on coal is $T_c + \tau^* \frac{\partial q}{\partial x_c} = e_c$, which means that the combination of the output tax and constrained activity tax achieve the first best. If there is no uncorrected externality on coal, solar will also achieve the first best because the tax on solar is unconstrained. The standard two-part instrument is a special case in which only the activity with the bound tax is externally harmful and $T_b = 0$.

Generally, (62) shows that the optimal output tax trades off the uncorrected externality of all activities with bound taxes against the private net benefit from those activities. The output tax allows the policymaker to increase the tax on all activities with binding tax constraints but only in proportion to those activities' marginal products. The policymaker can undo the effect of the output tax on all the unbound taxes by changing the activity tax. Note that the optimal output tax cannot be a subsidy because, if a subsidy were optimal, then then bound activity taxes would have been set too high, but since they are binding at the constraint they cannot have been set too high.

3 Costly Pigouvian taxation

Even when the policymaker's choice over taxes is unconstrained, optimal tax policy may depend on the relationships between activities if corrective taxes carry an administrative cost, including expenditures on measurement, enforcement, collections, legislation, and litigation. This section highlights how the optimal tax system is influenced by the functional form of administrative cost and the complementarities between activities. We consider both fixed and variable administrative costs.

3.1 Fixed administrative costs

If there are fixed administrative costs, it may not be optimal to tax every activity, in which case the policymaker faces the constrained activity tax optimization problem discussed in the previous section with binding constraint $t_j = 0$ for all untaxed activities. At the optimum each tax must account not only for the externality on the activity it is levied on but also the externality on every untaxed activity. Optimal subsidies on harmful activities are possible.

If each activity tax has its own fixed cost, the policymaker faces 2^n possible combinations of taxes. Let the fixed cost for the tax on activity i be f_i, the power set of $\{1, ..., n\}$ be Θ, and θ an element in Θ. Then the policymaker's problem is

$$\max_{\theta \in \Theta} \left\{ \max_t b(x(t)) - e^\top x(t) - \sum_{i \in \theta} f_i, \text{ subject to } t_j = 0 \forall j \notin \theta \right\} \tag{64}$$

No closed form solution exists, but the policymaker should choose θ^*, the set of taxes which if set optimally will maximize welfare.

3.2 Variable administrative costs with one activity

In this section, the policymaker must construct an administrative apparatus to collect taxes and enforce tax law. Let c map the activity level and tax rate to administrative cost. Formally c is continuously differentiable and weakly convex and $\arg\min(c) = (0, 0)$. These assumptions are consistent with a policymaker who employs the most effective tax collecting and enforcement resources first. Administrative cost and marginal administrative cost with respect to x are both increasing in activity level. A subsidy should also be costly to administer, so administrative cost is increasing in the tax rate above 0 and decreasing in the tax rate below 0, and marginal administrative cost with respect to t is increasing for all t.

We call the case in which administrative cost increases with activity levels *measurement costs*. Measurement costs arise because it is costly to determine the level of pollution generating activities, regardless of who is making these measurements and even if all parties are behaving honestly. Specific examples of these costs include the monitoring devices and the labor required to design and operate them. We call the case in which administrative cost increases with tax rates *enforcement costs*. Enforcement costs arise if the incentive to evade increases with tax rates and the government prevents this evasion by pouring more resources into tax enforcement, including more auditors, more lawyers, and more evasion detection software. More enforcement resources may be optimal when the policymaker is unable to implement an arbitrarily large penalty for evasion (Cremer and Gahvari, 1993) or large penalties introduce distortions (Kaplow, 1990).[27] We call the case in which administrative cost increases with tax revenue collected *bureaucracy costs*. Bureaucracy costs resemble the Flypaper effect (Hines and Thaler, 1995)—larger revenues induce larger bureaucracies—for whatever reason the money sticks.

The policymaker solves

$$\max_t b(x(t)) - ex(t) - c(x(t), t) \tag{65}$$

which leads to the first order condition $b'(x(t^*))\frac{\partial x}{\partial t} - e\frac{\partial x}{\partial t} - c_1\frac{\partial x}{\partial t} - c_2 = 0$, where c_i denotes the partial derivative of c with respect to its i^{th} argument. At the social optimum, the marginal social benefit of the tax (reduced externality and decreased administrative cost) is equal to the marginal social cost of the tax (reduced private benefit and increased administrative cost). Increasing the tax has an ambiguous effect on administrative cost because increasing the tax increases the enforce-

[27] Although more enforcement requires a resource cost that higher rates do not, there are important cases in which more enforcement is optimal. First, when there are some taxpayers that completely evade and some that do not, then there is an increased distortion because of the higher effective tax rate on those that pay. Second, when evading requires resources (and more resources are spent at higher tax rates), more enforcement may be optimal.

ment cost but decreases the measurement cost. Substituting $b'(x(t^*)) = t^*$, dividing by $\frac{\partial x}{\partial t}$, and rearranging yields

$$t^* = e + c_1 + c_2 \frac{\partial t}{\partial x} \tag{66}$$

The following table presents the necessary condition for the socially optimal tax for several possible functions of administrative cost.

Table 2: Single activity optimal taxes

Case	Cost function	Optimal Pigouvian tax	Previous Reseach
No cost	0	$t^* = e$	Pigou (1920)
Enforcement costs	$c(t)$	$t^* = e + \frac{\partial c}{\partial t}\frac{\partial t}{\partial x}$	
Measurement costs	$c(x)$	$t^* = e + \frac{\partial c}{\partial x}$	Polinsky & Shavell (1982)[28]
Bureaucracy costs	$c(xt) = c(R)$	$t^* = e + \frac{\partial c}{\partial R}\left(t^* + x(t^*)\frac{\partial t}{\partial x}\right)$	Polinsky & Shavell (1982)
Arbitrary costs	$c(x,t)$	$t^* = e + c_1 + c_2 \frac{\partial t}{\partial x}$	

When administrative cost takes the $c(t)$ functional form, the marginal administrative cost at $t = 0$ is 0. Thus the policymaker will always implement a tax. However, the optimal tax is not equal to the externality, and at the optimum the marginal social benefit of the activity is smaller than the marginal social cost of the activity. This is because when administrative cost is a function of tax rates, higher taxes reduce the external harm but increase administrative cost. The marginal benefit of a higher tax is lower external harm, and the marginal cost of a higher tax is lower private net benefit and higher administrative cost. Thus the tax rate is always lower than the case with no administrative cost, and there remains an uncorrected externality.[29] This shows up in the model because $\frac{\partial c}{\partial t}/\frac{\partial x}{\partial t} < 0$.[30]

When administrative cost takes the $c(x)$ functional form, there is a discrete jump in the administrative cost if the policymaker increases the tax from 0—i.e. when the policymaker decides to implement a tax. It is possible that $b(x(0)) - ex(0) > b(x(t^*)) - ex(t^*) - c(x(t^*))$ in which case the

[28] Polinsky and Shavell (1982) consider (1) both intensive and extensive margin responses to taxes, (2) administrative costs that are borne by either firms or the government, and (3) administrative costs that are fixed per firm and administrative costs that are a function of total tax revenue. The intuition behind their result for fixed costs per firm that are borne by the government is similar to our measurement costs case.

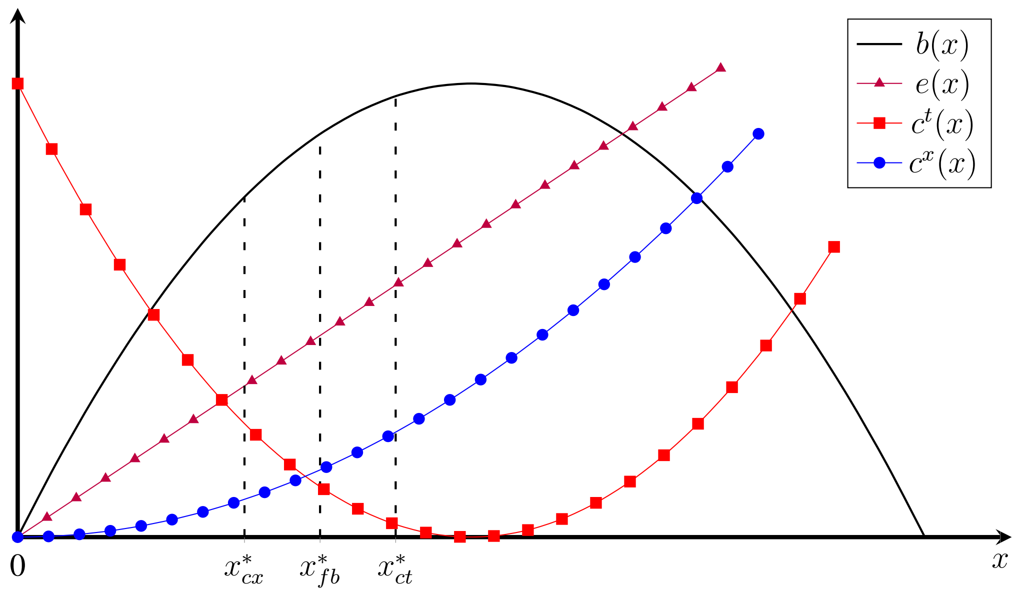
[29] Similarly, if there is a positive externality the subsidy rate is always lower than the case with no administrative cost.

[30] For positive externalities $\frac{\partial c}{\partial t}/\frac{\partial x}{\partial t} > 0$. In either case the optimal tax will never be 0 because the marginal administrative cost is 0 at $t = 0$.

optimal tax is $t = 0$, and the externality is uncorrected. When implementing a tax is optimal, the optimal tax is not equal to the externality. If administrative cost is a function of x, the policymaker should raise the tax until the marginal social benefit of the activity equals the marginal social cost of the activity. Because administrative cost increases only with x, the administrative cost may be interpreted as an additional social cost of the activity that may be corrected just as an external harm. Thus the optimal tax induces the private market to fully internalize the externality and the administrative cost. The tax rate is always higher than the case with no administrative cost.[31] This shows up in the model because $\frac{\partial c}{\partial x} > 0$.

We plot an example of the one-activity case in both x space and t space to provide intuition.

Figure 1: Net benefit, external harm, and administrative cost in x space

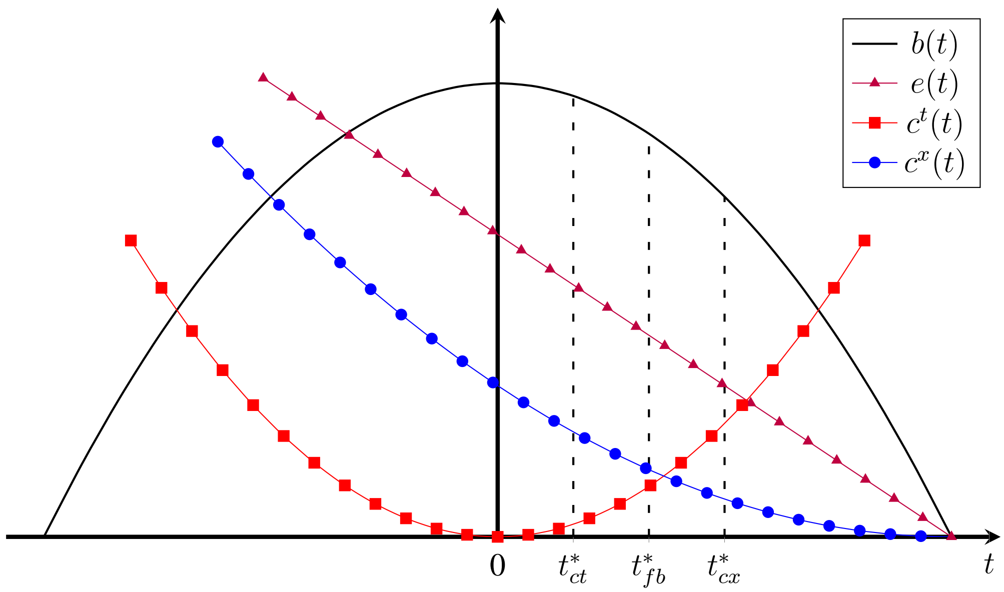


In Figure 1, net benefit, external harm, and measurement costs (costs that are a function of x) are 0 when there is no activity. Net benefit increases with the activity up to the point at which private cost exceeds private benefit, and then it decreases. External harm increases in proportion to the activity. And measurement costs increase convexly with the activity. To plot enforcement costs (costs that are a function of t) in x space, we map each activity level to the tax that would result in the private market choosing that activity level and thus map each activity level to enforcement cost. Enforcement costs are 0 when there is no tax which occurs when private net benefit is at its highest. Increasing or decreasing the activity level would imply, respectively, a subsidy or a tax that would incur an enforcement cost.

[31] However, the optimal subsidy is always smaller than the case with no administrative cost.

The first-best activity level, x^*_{fb}, occurs where private net benefit less external harm is maximized. This happens at an activity level below the private maximum because the optimal tax causes the private market to internalize the external harm. If there are measurement costs, the optimal activity level, x^*_{cx}, occurs where private net benefit less external harm and measurement cost is maximized. This happens at an activity level below the first best because measurement costs increase with the activity level, giving the policymaker additional incentive to decrease activity levels. The optimal tax causes the private market to internalize the external harm and the administrative cost. If there are enforcement costs, the optimal activity level, x^*_{ct}, occurs where private net benefit less external harm and enforcement cost is maximized. This happens at an activity level above the first best because, from the first-best activity level, enforcement costs decrease with activity giving the policymaker an incentive to increase activity levels to lower administrative cost. When the policymaker faces both measurement and enforcement costs, the optimal activity level may be above or below the first best.

Figure 2: Net benefit, external harm, and administrative cost in t space



In Figure 2, net benefit, external harm, and measurement costs (costs that are a function of x) are plotted in t space by mapping each tax to the activity level that the private market would choose at that tax rate and then mapping tax to net benefit, external harm, and measurement costs. At a high tax, there is no activity and thus no private net benefit, no external harm, and no measurement cost. As taxes are lowered, private net benefit, external harm, and measurement costs increase. Private net benefit reaches its maximum at a tax of 0. Enforcement costs (costs that are a function of t) are minimized at a tax of 0 and increase with larger taxes or subsidies.[32]

The first-best tax, t^*_{fb}, occurs where private net benefit less external harm is maximized. This happens at a tax above the private maximum because the optimal tax causes the private market to internalize the external harm. If there are measurement costs, the optimal tax, t^*_{cx}, occurs where private net benefit less external harm and measurement cost is maximized. This happens at a tax above the first best because measurement costs decrease with the tax giving the policymaker additional incentive to increase the tax. If there are enforcement costs, the optimal tax, t^*_{ct}, occurs where private net benefit less external harm and enforcement cost is maximized. This happens at a tax below the first best because, from the first-best tax, enforcement costs increase with the tax giving the policymaker an incentive to decrease the tax to lower administrative cost. When the policymaker faces both measurement and enforcement costs, the optimal tax may be above or below the first best.

3.3 Variable administrative costs with multiple activities

When there are multiple externally harmful activities and administrative costs, the policymaker must trade off between private net benefit, external harm, and administrative cost. The policymaker may use the complementarities between activities to minimize particularly externally harmful or administratively costly activities, potentially resulting in the optimal subsidization of a harmful activity.

The cost function $c(x,t)$, $c : X \times \mathbb{R}^n \to \mathbb{R}$, is the multidimensional analogue of the single activity case. We assume that $c(x,t)$ is continuously differentiable, weakly convex, and has $\arg\min(c) = (0,0)$. The private market problem is the same as in the multiple activity, complete taxation, costless administration case. The policymaker solves

$$\max_t b(x(t)) - e^\top x(t) - c(x(t),t) \tag{67}$$

which leads to the first order condition $b'(x(t^*))x'(t^*) - e^\top x'(t^*) - c_1(x(t^*),t^*)x'(t^*) - c_2(x(t^*),t^*) = 0$, where c_1 denotes the partial derivative of c with respect to its first vector argument and c_2 is the partial derivative of c with respect to its second vector argument. Substituting in the private market first order condition and applying $x'(t^*)$'s invertibility yields

$$t^{*\top} = e^\top + c_1 + c_2 x'(t^*)^{-1} \tag{68}$$

[32] Subsidies distort the private market in the same way that taxes do. Interpreting the net benefit model as a utility maximization problem in which one activity does not appear in b (see the appendix), the subsidy inefficiently increases x and decreases the implicit activity.

The table below lists special cases.

Table 3: Multiple activity optimal taxes

Case	Cost function	Optimal Pigouvian tax matrix notation	Optimal Pigouvian tax element notation
No administrative cost	$c = 0$	$t^{*\top} = e^\top$	$t_i^* = e_i$
Enforcement costs	$c = c(t)$	$t^{*\top} = e^\top + c'(t^*)x'(t^*)^{-1}$	$t_i^* = e_i + \sum_j \frac{\partial c}{\partial t_j}\frac{\partial t_j}{\partial x_i}$
Measurement costs[33]	$c = c(x)$	$t^{*\top} = e^\top + c'(x(t^*))$	$t_i^* = e_i + \frac{\partial c}{\partial x_i}$

For intuition, we return to our two activity example considering both the enforcement cost and measurement cost cases.

Example 4. Let x_s be the quantity of solar power plant activity and x_c be the quantity of coal power plant activity. Assume that the policymaker may tax both activities but that doing so carries an administrative cost that is a function of t. From Table 3, at the social optimum

$$t_s^* = e_s + \frac{\partial c}{\partial t_s}\frac{\partial t_s}{\partial x_s} + \frac{\partial c}{\partial t_c}\frac{\partial t_c}{\partial x_s} \tag{69}$$

and

$$t_c^* = e_c + \frac{\partial c}{\partial t_s}\frac{\partial t_s}{\partial x_c} + \frac{\partial c}{\partial t_c}\frac{\partial t_c}{\partial x_c} \tag{70}$$

The tax on solar activity increases with e_s and decreases with $\frac{\partial c}{\partial t_s}$ because $\frac{\partial t_s}{\partial x_s} < 0$. This is the intuition from the single activity case: the higher the marginal administrative cost of taxing solar activity, the lower the optimal tax on solar activity. The effect of $\frac{\partial c}{\partial t_c}$ on the solar tax depends on whether solar and coal are complements or substitutes. If $\frac{\partial t_c}{\partial x_s} < 0$, then solar and coal are substitutes, in which case the optimal tax on solar decreases with $\frac{\partial c}{\partial t_c}$.[34] If coal and solar are substitutes, then the higher the marginal administrative cost of taxing coal activity, the lower the optimal tax on solar activity because a lower tax on solar discourages the use of substitutes. The policymaker should use a lower solar tax to avoid the marginal cost of taxing coal. If coal and solar are complements, then the higher the marginal administrative cost of taxing coal activity, the higher the optimal tax on solar activity because a higher tax on solar discourages the use of

[33] This assumes it is optimal to tax all activities.

[34] Because $t'(x)^{-1} = x'(t)$, the sign of $\frac{\partial t_c}{\partial x_s}$ will be the opposite of the sign of $\frac{\partial x_c}{\partial t_s}$. If the policymaker wishes to marginally increase x_c holding x_s constant and the two are substitutes, she must decrease both taxes. Thus $\frac{\partial t_c}{\partial x_s} < 0$ if solar and coal are substitutes.

complements. The policymaker will use the relationship between the activities to optimally trade off external harm, administrative cost, and private net benefit. Note also that the external harm of coal may have an effect on the optimal solar tax. An exogenous increase in e_c would increase the optimal tax on coal. If administrative cost is strictly convex with respect to the coal tax, then a higher coal tax would increase the marginal administrative cost of the coal tax, which would change the optimal solar tax.[35]

(69) shows under what conditions a subsidy on solar would be optimal. An optimal subsidy requires that $e_s + \frac{\partial c}{\partial t_s}\frac{\partial t_s}{\partial x_s} + \frac{\partial c}{\partial t_c}\frac{\partial t_c}{\partial x_s} < 0$. If there is a subsidy on coal, then $\frac{\partial c}{\partial t_s} < 0$ because decreasing the subsidy (increasing the tax) reduces administrative cost. Thus an optimal subsidy requires that solar and coal be substitutes. The case for a subsidy on solar strengthens when the marginal external harm from solar is smaller, the marginal administrative cost of the solar tax is smaller, the private market solar activity response to the tax on solar is smaller, the marginal administrative cost of the coal tax is larger, and the private market coal activity response to the tax on solar is larger. As noted before, if $c(t)$ is strictly convex, the case for a solar subsidy strengthens when the marginal external harm of coal is larger.

Example 5. Now assume that the policymaker may tax both activities but that doing so carries an administrative cost that is a function of x. From Table 3, assuming it is optimal to tax all activities, at the social optimum

$$t_s^* = e_s + \frac{\partial c}{\partial x_s} \tag{71}$$

and

$$t_c^* = e_c + \frac{\partial c}{\partial x_c} \tag{72}$$

Just as in the single activity case, the administrative cost may be interpreted as an additional social cost of the activity which can be corrected like an external harm. Thus the optimal tax induces the private market to fully internalize the externality and the administrative cost. Both taxes will be greater than in the case with no administrative costs, and a subsidy on a harmful activity will never be optimal.

However, when administrative cost is a function of only activity levels, it may not be optimal to tax every activity. This is most apparent when the administrative cost of taxing an activity is avoidable when that activity is untaxed. If the policymaker decides to tax a particular activity,

[35]Depending on the third derivative of b, there are other dynamics in the model by which changes in the marginal external harm of one activity would affect the optimal tax on another.

there will be a discrete increase in the administrative cost. There are four possible cases: tax both, tax neither, tax solar only, and tax coal only. If the policymaker taxes only one of the activities, then she is optimizing under a binding constraint, and the earlier analysis applies. Just as before, the complementarities between activities are relevant factors in determining the optimal tax, and a subsidy may be optimal.

Both of these examples generalize. When administrative cost is a function of tax rates only, the optimal tax on each activity is equal to the externality generated by that activity plus the sum of all the marginal administrative costs weighted by the responsiveness of the tax rate to changes in activity. The policymaker should tax each activity because the marginal cost of increasing or decreasing a tax of 0 is 0.[36] However, just as as in the single activity case, an increase in the tax presents a tradeoff: lower externality, but higher administrative cost. Thus, optimal taxes do not induce the private market to fully internalize the externality. The policymaker should raise (or lower) taxes until the marginal social benefit of the tax is equal to the tax's marginal social harm. In particular, the policymaker should use taxes with low marginal administrative cost to change other activity levels. A subsidy may be optimal for some externally harmful activities.[37] c' will have negative entries whenever there is a subsidy because the cost of administration will decrease the less negative the tax becomes.

When administrative cost is a function of only activity levels, it may not be optimal to tax every activity. Just as in the fixed cost case, the policymaker must optimize 2^n different problems, each with a different variable cost function, depending on which activities are taxed. If it is optimal to tax all of the activities, the policymaker should set the tax equal to the externality plus the marginal administrative cost. At this tax, the private market internalizes both the externality and the administrative tax. If it is not optimal to tax every activity, the complementarity between activities comes into play, and it may be optimal to subsidize externally harmful activities.

This section could be generalized to study the optimal tax system when the policymaker may implement activity taxes and an output tax, both of which carry administrative costs. Intuitively, the policymaker should choose the bundle of taxes that best trade off private net benefit, external harm, and administrative cost. If the administrative cost of the output tax were low relative to the administrative cost of the activity taxes, the policymaker might optimally tax output to reach taxes with particularly high activity administrative costs and selectively tax or subsidize a subset of activities.

[36]Theoretically, the knife-edge case in which $t_i^* = e_i + \sum_j \frac{\partial c}{\partial t_j}\frac{\partial t_j}{\partial x_i} = 0$ is possible.

[37]The matrix $t'(x(t^*)) = b''(x(t^*))$ describes the effect of a change in the activity vector on tax rates at the optimal tax rate. Because b is concave, this matrix is negative definite, so the diagonals are all negative. An increase in t_i will, thus, reduce x_i although it may increase or have no effect on x_j. This implies that t^* may have negative entries.

4 Conclusion

Pigou's seminal insight demonstrated that in a first-best world taxes can fully correct externalities. This paper extends his insight into a world in which the policymaker faces constraints and costs. In this world, the policymaker must trade off the administrative cost of the tax instruments available to her against external harm and lost private net benefit, and she must take advantage of complementarities to achieve optimal tax outcomes. As we show, there are several cases in which an externality cannot or should not be fully corrected and even cases in which a harmful activity should be subsidized.

Our paper also suggests that optimal policy requires more than determining marginal external harm. When taxes are constrained or costly, the net private benefit function and administrative cost function are necessary for policy decisions. While the examples of subsidizing harm noted above suggest that policymakers may intuitively understand how to set optimal policy, our hope is that this paper helps formalize that intuition. Ideally, policymakers will collect the relevant market and administrative cost data to be able to set optimal policy more precisely. Future work could estimate the relevant parameters to determine whether the many current subsidies on harmful activities are indeed optimal.

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A Net benefit and utility maximization

Let $u(x, x_{n+1}) : \mathbb{R}^{n+1} \to \mathbb{R}$ be a strictly increasing, strictly concave representative agent's utility function over $n+1$ activities. Let $p(x, x_{n+1}) : \mathbb{R}^{n+1} \to \mathbb{R}$ be a strictly increasing, weakly convex function, such that $p(x, x_{n+1}) = 0$ defines a production possibility frontier. Then

(i) There exists a function $f(x) : \mathbb{R}^n \to \mathbb{R}$ such that $p(x, x_{n+1}) = 0$ if and only if $x_{n+1} = f(x)$;

(ii) $u(x, f(x)) = b(x)$ defines a strictly concave net benefit function; and

(iii) Assuming activity $n+1$ generates no external harm and is untaxed, the agent's best response to a vector of activity taxes is the same under either the constrained utility maximization problem or the unconstrained net benefit maximization problem.

Proof. (i) Under the implicit mapping theorem, there exists a function f such that $p(x, x_{n+1}) = 0$ if and only if $x_{n+1} = f(x)$ and $f_i = -p_i/p_{n+1}$.

(ii) First, the weak convexity of p implies the weak concavity of f.

$$p(y, y_{n+1}) - p(x, x_{n+1}) \geq \sum_{i=1}^{n+1} p_i(x, x_{n+1})(y_i - x_i) \tag{73}$$

Thus $\forall (x, x_{n+1}), (y, y_{n+1})$ such that $p(x, x_{n+1}) = p(y, y_{n+1}) = 0$, we have

$$0 \geq \sum_{i=1}^{n+1} p_i(x, x_{n+1})(y_i - x_i) \implies \tag{74}$$

$$-p_{n+1}(x, x_{n+1})(y_{n+1} - x_{n+1}) \geq \sum_{i=1}^{n} p_i(x, x_{n+1})(y_i - x_i) \implies \tag{75}$$

$$(y_{n+1} - x_{n+1}) \leq \sum_{i=1}^{n} (-p_i(x, x_{n+1}))/(p_{n+1}(x, x_{n+1}))(y_i - x_i) \implies \tag{76}$$

$$f(y) - f(x) \leq \sum_{i=1}^{n} f_i(x)(y_i - x_i) \tag{77}$$

Second, the weak concavity of f and strict concavity of u imply the strict concavity of b.

$$\begin{aligned}
\forall\theta \in [0,1] : b\left(\theta x + [1-\theta]y\right) &= u(\theta x + [1-\theta]y, f(\theta x + [1-\theta]y)) && (78)\\
&\geq u(\theta x + [1-\theta]y, \theta f(x) + [1-\theta] f(y)) && (79)\\
&= u(\theta x + [1-\theta]y, \theta x_{n+1} + [1-\theta] y_{n+1}) && (80)\\
&> \theta u(x, x_{n+1}) + [1-\theta] u(y, y_{n+1}) && (81)\\
&= \theta b(x) + [1-\theta] b(y) && (82)
\end{aligned}$$

(iii) With a vector of taxes t, the agent's utility maximization problem is a Lagrangian with Lagrange multiplier μ,

$$\mathcal{L} = u(x, x_{n+1}) - t^{\top} x - \mu p(x, x_{n+1}) \quad (83)$$

with first order conditions

$$u_i - t_i - \mu p_i = 0 \text{ for all } i \in \{1, \ldots, n\} \text{ and } u_{n+1} - \mu p_{n+1} = 0 \quad (84)$$

which may be combined to yield

$$u_i - t_i - u_{n+1}\frac{p_i}{p_{n+1}} = 0 \quad (85)$$

With a vector of taxes t, the agent's net benefit maximization problem is $\max_x u(x, f(x)) - t^{\top} x$ with first order conditions

$$u_i + u_{n+1} f_i - t_i = 0 \text{ for all } i \in \{1, \ldots, n\} \quad (86)$$

From (i) $f_i = -p_i/p_{n+1}$. Thus

$$u_i - t_i - u_{n+1}\frac{p_i}{p_{n+1}} = 0 \quad (87)$$

Since the two problems have the same first order conditions, they imply the same agent best response function. □

B Private net benefit and output

A rigorous investigation of the optimal output tax requires a formal relationship between private net benefit and output. We assume that private net benefit is the benefit of output less the cost of activities. Thus when output is explicitly incorporated into the model, we require two additional functions. Strictly increasing, strictly concave v maps output to private benefit. Strictly increasing, strictly convex g maps the activities to private cost. Thus $b(x) = v(q(x)) - g(x)$ and the private market's problem is

$$\max_x v(q(x)) - g(x) - \tau q(x) \tag{88}$$

with first order condition

$$v'(q(x(\tau))q'(x(\tau)) - g'(x(\tau)) - \tau q'(x(\tau)) = 0 \tag{89}$$

Under the implicit mapping theorem, $x(\tau)$ exists and is continuously differentiable everywhere that the derivative of first order condition with respect to x has a non-zero determinant. The derivative of first order condition with respect to x is

$$v'(q(x))q''(x) + (q'(x))^\top v''(q(x))q'(x) - g''(x) - \tau q''(x) \tag{90}$$

$$=[v'(q(x)) - \tau]q''(x) + v''(q(x))(q'(x))^\top q'(x) - g''(x) \tag{91}$$

$v'(q(x)) - \tau > 0$ under the first order condition. $q''(x)$ is negative semidefinite by assumption. $v''(q(x)) < 0$ by assumption. $(q'(x))^\top q'(x)$ is positive semidefinite because for any vector v, $(v^\top q'(x)^\top)(q'(x)v) \geq 0$. And $g''(x)$ is positive definite by assumption. Thus

$$= \underbrace{[v'(q(x)) - \tau]}_{+} \underbrace{q''(x)}_{\text{NSD}} + \underbrace{v''(q(x))}_{-} \underbrace{(q'(x))^\top q'(x)}_{\text{PSD}} - \underbrace{g''(x)}_{\text{PD}} \tag{92}$$

Because a positive (semi)definite matrix multiplied by a negative scalar is a negative (semi)definite matrix and since the sum of a negative definite matrix and a negative semidefinite matrix is a negative definite matrix, the sum here is negative definite. Since a negative definite matrix has a non-zero determinant, $x(\tau)$ exists and is continuously differentiable. We have shown that $b''(x) - \tau q''(x)$ is negative definite. Thus, its inverse, A, is also negative definite.

The additional structure that v and g provide to the problem make it clearer under what circumstances the level of an activity increases with the output tax. In our solar and coal power example we have

$$x'(\tau) = \mathrm{A} q'(x(\tau))^\top \tag{93}$$

$$= \frac{1}{\det(\mathrm{A}^{-1})} \begin{pmatrix} (v'-\tau)q_{cc} + v'' q_c q_c - g_{cc} & -(v'-\tau)q_{sc} - v'' q_s q_c + g_{sc} \\ -(v'-\tau)q_{cs} - v'' q_c q_s + g_{cs} & (v'-\tau)q_{ss} + v'' q_s q_s - g_{ss} \end{pmatrix} \begin{pmatrix} q_s \\ q_c \end{pmatrix} \tag{94}$$

$$= \frac{1}{\det(\mathrm{A}^{-1})} \begin{pmatrix} q_s(v'-\tau)q_{cc} + q_s v'' q_c q_c - q_s g_{cc} - q_c(v'-\tau)q_{sc} - q_c v'' q_s q_c + q_c g_{sc} \\ -q_s(v'-\tau)q_{cs} - q_s v'' q_c q_s + q_s g_{cs} + q_c(v'-\tau)q_{ss} + q_c v'' q_s q_s - q_c g_{ss} \end{pmatrix} \tag{95}$$

$$\frac{\partial x_c}{\partial \tau} = \frac{1}{\det(\mathrm{A}^{-1})} \left(-q_s(v'-\tau)q_{cs} + q_s g_{cs} + q_c(v'-\tau)q_{ss} - q_c g_{ss}\right) \tag{96}$$

Since $\frac{1}{\det(\mathrm{A}^{-1})} > 0$, $\frac{\partial x_c}{\partial \tau} > 0$ requires

$$\underbrace{-q_s(v'-\tau)}_{\text{negative}} q_{cs} + \underbrace{q_s}_{\text{positive}} g_{cs} + \underbrace{q_c(v'-\tau)q_{ss}}_{\text{negative}} + \underbrace{-q_c g_{ss}}_{\text{negative}} > 0$$

We know that $(v'-\tau) > 0$, $q_i > 0$, $q_{ss} < 0$, and $g_{ss} > 0$. Thus for $\frac{\partial x_c}{\partial \tau} > 0$ we require either $q_{sc} < 0$ or $g_{sc} > 0$. Having q_s be large, q_c be small, q_{ss} be small, and g_{ss} be small will also help push $\frac{\partial x_c}{\partial \tau}$ above zero.