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The Commutant of the Fourier–Plancherel Transform

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The signatures below, by the thesis advisor, a departmental reader, and the Honors coordinator for mathematics, certify that this thesis, prepared by Brianna Cantrall, has been approved, as to style and content.

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Preface

Let us take a moment to examine the matrix representation of the following linear transformation from $\ell^2 \to \ell^2$ (defined later),

$$\begin{bmatrix} 1 & & & & \\ & -i & & & \\ & & -1 & & \\ & & & i & \\ & & & & \ddots \end{bmatrix}.$$

One can see that this matrix is unitary and has eigenvalues $\{1, -i, -1, i\}$, each of infinite multiplicity.

Throughout the remainder of this thesis, we will convince the reader that the above linear transformation is actually the Fourier transform. We will compute the commutant, as well as its invariant subspaces. The key to do this relies on the Hermite polynomials.

Why do we recast the Fourier transform from its well-known and well-studied integral form to the matrix form shown above? As we will see, the matrix form allows us to efficiently discover the operator theory of the Fourier transform obfuscated behind an integral that is difficult to compute.

In the Chapter 1, we establish some basic notation about Hilbert spaces and introduce the two Hilbert spaces central to the ideas developped in this thesis, $L^2(\mathbb{R})$ and ℓ^2. We then define the Hermite polynomials and the

Hermite functions in Chapter 2, which we will show form a convenient orthonormal basis for $L^2(\mathbb{R})$. The Hermite polynomials are further employed in Chapter 3, where we establish the Fourier-Plancherel transform $\mathcal{F}$ on $L^2(\mathbb{R})$. A central step in this is to compute the eigenbasis of the Fourier transform, which we show is the set of Hermite functions. After establishing the Fourier transform, we further characterize it in Chapter 4 by defining the set $\{\mathcal{F}\}'$, which contains all bounded linear operators on $L^2(\mathbb{R})$ that commute with $\mathcal{F}$. We continue this characterization in Chapter 5 by defining $\sqrt{\mathcal{F}}$, the set of bounded linear operators on $L^2(\mathbb{R})$ that are square roots of $\mathcal{F}$. Finally, we conclude our analysis of $\mathcal{F}$ in Chapter 6 by describing all of the invariant subspaces of the Fourier transform.

Chapter 1

Preliminaries

1.1. Notation

First, we establish our basic notation.

1.2. Basic Definitions

In this chapter we will introduce important concepts used throughout this thesis. We will not go into technical details and refer the reader to the texts [**13, 14**]. We begin with the very important definition of a *Hilbert space*.

Definition 1.2.1. A *Hilbert space* $\mathcal{H}$ is a vector space over the complex numbers that is endowed with an inner product $\langle \cdot, \cdot \rangle$ such that $\mathcal{H}$ is complete with respect to the norm $\| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle}$ induced by this inner product.

Here *complete* means Cauchy complete in that if $(\mathbf{x}_n)_{n \geqslant 1}$ is a Cauchy sequence in $\mathcal{H}$, then there is a vector $\mathbf{x} \in \mathcal{H}$ such that

$$\|\mathbf{x}_n - \mathbf{x}\| \to 0.$$

All of the Hilbert spaces in this thesis will be *separable*, meaning they have a countable dense set.

Three results used many times in this thesis without much fanfare are the following.

Theorem 1.2.2 (Cauchy–Schwarz Inequality). *If* $\mathbf{x}, \mathbf{y}$ *are vectors in a Hilbert space, then*

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leqslant \|\mathbf{x}\| \|\mathbf{y}\|.$$

Theorem 1.2.3 (Triangle Inequality). *If* $\mathbf{x}, \mathbf{y}$ *are vectors in a Hilbert space, then*

$$\|\mathbf{x} + \mathbf{y}\| \leqslant \|\mathbf{x}\| + \|\mathbf{y}\|.$$

Theorem 1.2.4 (Polarization Identity). *If* $\mathbf{x}$, $\mathbf{y}$ *are vectors in a Hilbert space, then*

$$\langle \mathbf{x}, \mathbf{y} \rangle = \frac{1}{4}(\|\mathbf{x} + \mathbf{y}\|^2 - \|\mathbf{x} - \mathbf{y}\|^2 + i\|\mathbf{x} + \mathbf{y}\|^2 - i\|\mathbf{x} - \mathbf{y}\|^2).$$

The two Hilbert spaces discussed in this thesis are $L^2(\mathbb{R})$ and ℓ^2 . The first, $L^2(\mathbb{R})$, is the space of complex valued Lebesgue measurable functions on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} |f(x)|^2 dx < \infty.$$

The inner product in this space is

$$\langle f, g \rangle = \int_{\mathbb{R}} f(x)\overline{g(x)} dx.$$

An alert reader might have some reservation about the convergence of the integral on the right. However, an application of the Cauchy-Schwarz inequality says that this inner product is well-defined. The corresponding norm on $L^2(\mathbb{R})$ is

$$\|f\| = \sqrt{\langle f, f \rangle} = \Big(\int_{\mathbb{R}} |f(x)|^2 dx \Big)^{\frac{1}{2}}.$$

One can argue that $L^2(\mathbb{R})$ is a vector space and a technical detail called the Riesz-Fischer theorem will show that it is complete, and hence a Hilbert space. It is also separable.

The second Hilbert space discussed in this thesis, ℓ^2, is the set of all complex sequences $\mathbf{a} = (a_n)_{n\geqslant 0}$ such that

$$\sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

Here the inner product is

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{n=0}^{\infty} a_n \overline{b_n}, \quad \mathbf{a} = (a_n)_{n\geqslant 0}, \mathbf{b} = (b_n)_{n\geqslant 0}.$$

Again, there is some reservation about the convergence of the infinite sum on the right, which is resolved by the Cauchy-Schwarz inequality. The corresponding norm is

$$\|\mathbf{a}\| = \sqrt{\langle \mathbf{a}, \mathbf{a} \rangle} = \Big(\sum_{n=0}^{\infty} |a_n|^2 \Big)^{\frac{1}{2}}.$$

This space is also separable.

One might wonder why we are indexing our sequences $(a_n)_{n\geqslant 0}$ starting at zero instead of one (which would be more natural). As we will see in the next chapter, we will be writing every function $f \in L^2(\mathbb{R})$ as an infinite linear combination of $(h_n)_{n\geqslant 0}$, the Hermite basis for $L^2(\mathbb{R})$, where the indexing naturally starts at zero (and not one).

With any separable Hilbert space $\mathcal{H}$ comes an *orthonormal basis*.

Definition 1.2.5. An *orthonormal basis* for a separable Hilbert space $\mathcal{H}$ is a sequence of vectors $(\mathbf{x}_n)_{n\geqslant 0}$ in $\mathcal{H}$ such that

$$\langle \mathbf{x}_n, \mathbf{x}_m \rangle = \delta_{m,n} = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

and if $\mathbf{x} \in \mathcal{H}$ and $\langle \mathbf{x}, \mathbf{x}_n \rangle = 0$ for all n, then $\mathbf{x} = \mathbf{0}$.

The first condition in the above definition says that the vectors $\mathbf{x}_n$ are pairwise orthogonal and have norm one, while the second condition says that the linear span of the vectors $\mathbf{x}_n$ is dense in $\mathcal{H}$. A separable Hilbert space always has an orthonormal basis. With an orthonormal basis $(\mathbf{x}_n)_{n\geqslant 0}$, a theorem of Parseval gives that any $\mathbf{x} \in \mathcal{H}$ can be written uniquely as

$$\mathbf{x} = \sum_{n=0}^{\infty} \langle \mathbf{x}, \mathbf{x}_n \rangle \mathbf{x}_n.$$

In the above, the convergence of the infinite sum of vectors is understood in the norm of $\mathcal{H}$. By this we mean

$$\lim_{N\to\infty}\left\|\mathbf{x}-\sum_{n=0}^{N}\langle\mathbf{x},\mathbf{x}_n\rangle\mathbf{x}_n\right\|=0.$$

From here, it follows that

$$\|\mathbf{x}\|^2=\sum_{n=0}^{\infty}|\langle\mathbf{x},\mathbf{x}_n\rangle|^2.$$

For the Hilbert space ℓ^2, an obvious orthonormal basis is $(\mathbf{e}_n)_{n\geqslant 0}$, where

$$\mathbf{e}_n=(0,0,0,\cdots,0,1,0,0,\cdots)$$

such that the 1 appears in the nth slot. For the Hilbert space $L^2(\mathbb{R})$, an orthonormal basis is less clear. This thesis will use the Hermite basis $(h_n)_{n\geqslant 0}$, which we will develop in the next chapter.

A linear transformation T on a Hilbert space $\mathcal{H}$ is said to be *bounded* if there is a $C_T>0$ such that

$$\|T\mathbf{x}\|\leqslant C_T\|\mathbf{x}\|\quad\forall\mathbf{x}\in\mathcal{H}.$$

The optimal constant C_T (the smallest C_T for which $\|T\mathbf{x}\|\leqslant C_T\|\mathbf{x}\|$ for all $\mathbf{x}\in\mathcal{H}$) will be called the *norm* of T and will be denoted by $\|T\|$. If $\mathcal{H}=\mathbb{C}^n$, the norm of a linear transformation T is the largest singular value of the matrix representation of T. The set of all bounded linear transformations (called *bounded operators*) will be denoted by $\mathcal{B}(\mathcal{H})$. With any bounded operator T comes an adjoint $T^*\in\mathcal{B}(\mathcal{H})$ which satisfies

$$\langle T\mathbf{x},\mathbf{y}\rangle=\langle\mathbf{x},T^*\mathbf{y}\rangle\quad\forall\mathbf{x},\mathbf{y}\in\mathcal{H}.$$

When $\mathcal{H}=\mathbb{C}^n$, the adjoint is the usual conjugate transpose of the matrix representation of T. One can show that $\mathcal{B}(\mathcal{H})$ is closed under addition and scalar multiplication, as well as operator composition.

A skeptical reader may wonder why we are focusing our attention on bounded linear operators and not *all* linear operators on a Hilbert space. The reason comes from adjoints, which will be used at various points in this thesis. Without the assumption of a linear transformation being bounded, the adjoint becomes difficult, and sometimes impossible, to define. In addition, we will often prove facts about a linear transformation by first verifying it on a dense set and then extending the result to the entire Hilbert space. This process only works when the linear transformation is bounded.

A $T\in\mathcal{B}(\mathcal{H})$ is *isometric* if

$$\|T\mathbf{x}\|=\|\mathbf{x}\|\quad\forall\mathbf{x}\in\mathcal{H}.$$

Notice that by the Polarization Identity, the above condition is equivalent to

$$\langle T\mathbf{x}, T\mathbf{y}\rangle = \langle \mathbf{x}, \mathbf{y}\rangle \quad \forall \mathbf{x}, \mathbf{y} \in \mathcal{H}.$$

A $T \in \mathcal{B}(\mathcal{H})$ is *unitary* if it is isometric and onto. Notice that T is unitary if and only if $TT^* = T^*T = I$.

Chapter 2

The Hermite Basis

It is a well known fact that $L^2(\mathbb{R})$ is a separable Hilbert space. What is less known is a useful orthonormal basis. In this chapter we will develop the Hermite basis, which will drive the rest of this thesis. Some of the treatment below of the Hermite functions comes from Hsu [**8**].

2.1. The Hermite Functions

Definition 2.1.1. For $n \in \mathbb{N}_0$, the nth *Hermite function* is defined to be

$$\left(\frac{(-1)^n}{n!}\right) e^{\pi x^2} \left(\frac{d}{dx}\right)^n e^{-2\pi x^2}.$$

We will use the notation $\mathbb{N} := \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. This next result helps us relate the Hermite functions with the well-known *Hermite polynomials*.

Theorem 2.1.2. *The nth Hermite function satisfies*

$$H_n(x) e^{-\pi x^2},$$

where $H_n(x)$ is a polynomial of degree n.

Proof. Let

$$\left(\frac{(-1)^n}{n!}\right) e^{\pi x^2} \left(\frac{d}{dx}\right)^n e^{-2\pi x^2}$$

be the nth Hermite function. By the Leibnitz formula for the nth derivative of the product of two functions, $(\frac{d}{dx})^n e^{-2\pi x^2}$ will produce some polynomial in x, call it $p(x)$, multiplied by the exponential $e^{-2\pi x^2}$. To obtain the degree condition on $p(x)$, we will use induction. For the base case, observe that when $n = 0$ we have

$$\left(\frac{d}{dx}\right)^0 e^{-2\pi x^2} = e^{-2\pi x^2},$$

so that $p(x) = 1$ with degree 0. Suppose

$$\left(\frac{d}{dx}\right)^k e^{-2\pi x^2} = p(x)e^{-2\pi x^2}$$

for some $k \geqslant 0$ where $p(x)$ is a polynomial of degree k. Observe that

$$\left(\frac{d}{dx}\right)^{k+1} e^{-2\pi x^2} = -4\pi x p(x)e^{-2\pi x^2} + p'(x)e^{-2\pi x^2}.$$

Note that $p'(x)$ will now have degree $k-1$. Then $-4\pi x p(x)$ has degree $k+1$ and thus we can conclude by induction that $p(x)$ is a polynomial of degree n for all $n \in \mathbb{N}_0$.

We now have

$$\left(\frac{(-1)^n}{n!}\right)e^{\pi x^2}\left(\frac{d}{dx}\right)^n e^{-2\pi x^2} = \left(\frac{(-1)^n}{n!}\right)e^{\pi x^2}p(x)e^{-2\pi x^2} = \left(\frac{(-1)^n}{n!}\right)p(x)e^{-\pi x^2}.$$

Define

$$H_n(x) := \frac{(-1)^n}{n!}p(x).$$

Then $H_n(x)$ is a polynomial of degree n. □

The functions $H_n(x)$ are called the *Hermite polynomials*. The first 6 Hermite polynomials are given below.

$$H_0(x) = 1 \qquad H_3(x) = \tfrac{1}{3!}(64\pi^3x^3 - 48\pi^2x)$$
$$H_1(x) = 4\pi x \qquad H_4(x) = \tfrac{1}{4!}(256\pi^4x^4 - 384\pi^3x^2 + 48\pi^2)$$
$$H_2(x) = \tfrac{1}{2!}(16\pi^2x^2 - 4\pi) \qquad H_5(x) = \tfrac{1}{5!}(1024\pi^5x^5 - 2560\pi^4x^3 + 960\pi^3x)$$

This is a well-known class of polynomials that appear as the eigenstates of the quantum harmonic oscillator, in combinatorics as an example of an Appell sequence, and in signal processing as Hermitian wavelets. An important consequence of defining the Hermite polynomials is demonstrated below.

Proposition 2.1.3. *For the Hermite polynomials $(H_k)_{k\geqslant 0}$,*

$$\textit{span}\{H_0, H_1, H_2, \dots, H_n\} = \textit{span}\{1, x, x^2, \dots, x^n\} \quad \textit{for all } n \in \mathbb{N}_0.$$

Proof. Recall that H_n is a polynomial of degree n. Let

$$p(x) = \sum_{k=0}^{n} c_k x^k$$

be a polynomial of degree n. We will proceed by induction. In the case where $n = 0$, $p(x) = c$ for some $c \in \mathbb{C}$. Note that $H_0 = 1$. Then $p(x) = cH_0$ and $\text{span}\{1\} = \text{span}\{H_0\}$. Suppose

$$\text{span}\{1, x, x^2, \ldots, x^k\} = \text{span}\{H_0, H_1, H_2, \ldots, H_k\}$$

for some $k \geqslant 0$. Then

$$H_{k+1} = c_{k+1}x^{k+1} + q(x)$$

for $c_{k+1} \neq 0$ where $q(x)$ is a polynomial of degree less than $k+1$ such that $q(x) \in \text{span}\{H_0, H_1, H_2, \ldots, H_k\}$ by assumption. We then have that

$$x^{k+1} = \frac{1}{c_{k+1}} H_{k+1} - \frac{1}{c_{k+1}} q(x).$$

Thus,

$$x^{k+1} \in \text{span}\{H_0, H_1, H_2, \ldots, H_{k+1}\}$$

and we conclude by induction that the assumption is true for all $n \geqslant 0$. □

We need to know the value of

$$\|H_n e^{-\pi x^2}\| = \Big(\int_{\mathbb{R}} |H_n(x) e^{-\pi x^2}|^2 dx \Big)^{1/2}$$

in order to normalize this function. This is done with the following.

Theorem 2.1.4. $\langle H_n e^{-\pi x^2}, H_n e^{-\pi x^2} \rangle = \frac{(4\pi)^n}{\sqrt{2}n!}$ *for all* $n \in \mathbb{N}_0$.

The proof of this theorem makes use of the orthogonality of the Hermite functions, which we have yet to show. We will delay this proof until this property is established.

Definition 2.1.5. The *normalized Hermite functions* $h_n(x)$ are defined to be

$$h_n(x) = \left(\frac{2^{1/4}\sqrt{n!}}{(4\pi)^{n/2}} \right) H_n e^{-\pi x^2} \quad \text{for } n \in \mathbb{N}_0.$$

We will now show that $\langle h_n, h_m \rangle = 0$ for $m \neq n$; in other words, that the Hermite functions are orthogonal in the inner product of $L^2(\mathbb{R})$. To show this, we will define a useful operator. Let $\mathbb{C}[x]$ denote the set of all polynomials with complex coefficients.

Definition 2.1.6. Let $\mathcal{S}$ be the vector space of functions of the form

$$\mathcal{S} = \{p(x)e^{-\pi x^2} : p(x) \in \mathbb{C}[x]\}.$$

Routine integral estimates will show that $\mathcal{S} \subset L^2(\mathbb{R})$. The following theorem of M. Riesz [**12**] (see also [**1**, Ch. 2]) plays an important role later on.

Theorem 2.1.7 (M. Riesz)**.** *$\mathcal{S}$ is dense in $L^2(\mathbb{R})$.*

Proof. The proof of this result is very technical and beyond the scope of this thesis, so we will only give a brief outline.

First, it suffices to show that the polynomials are dense in $L^2(wdx)$, where $w(x) = e^{-2\pi x^2}$. To prove this, we must first show that the set

$$\sum_{j=1}^{N} \frac{c_j}{x - a_j} \tag{2.1.8}$$

for $c_j \in \mathbb{C}$ and $a_j \notin \mathbb{R}$ is dense in $L^2(wdx)$. By basic orthogonality of Hilbert spaces, it suffices to show that if $g \in L^2(wdx)$ and

$$\int_{-\infty}^{\infty} \frac{g(x)}{x - z} w(x) dx = 0$$

for all $z \notin \mathbb{R}$, then $g = 0$. One can verify this fact by the Poisson Integral Formula, a Theorem of Fatou [**5**] (see also [**7**, p. 34]), and the solution of the Dirichlet problem. Next, use the Gram Schmidt method on the set $\{1, x, x^2, \ldots\}$ with respect to $L^2(wdx)$ to produce an orthonormal set of polynomials $\{p_0, p_1, p_2, \ldots\}$. Fix $z \notin \mathbb{R}$ and let

$$a_n = \langle \frac{1}{x - z}, p_n \rangle_{L^2(wdx)}.$$

One can verify that Parseval's theorem holds for this z, i.e.,

$$\sum_{n=0}^{\infty} |a_n|^2 = \int_{-\infty}^{\infty} \frac{1}{|x - z|^2} e^{-2\pi x^2} dx.$$

This is a significant part of Riesz's technique and though it has the name Parseval's theorem, it is not quite Parseval's theorem. Finally, it follows by orthogonality that

$$\int_{-\infty}^{\infty} \Big| \frac{1}{x - z} - \sum_{n=0}^{N} \langle \frac{1}{x - z}, p_n \rangle p_n \Big|^2 e^{-2\pi x^2} dx = \int_{-\infty}^{\infty} \frac{1}{|x - z|^2} e^{-2\pi x^2} dx - \sum_{n=0}^{N} |a_n|^2.$$

Now, let $N \to \infty$ to see that $\frac{1}{x-z}$ can be approximated by a sequence of polynomials. From here, it follows that any function of the form (2.1.8) can be approximated by a sequence of polynomials. □

Definition 2.1.9. Define the linear transformation $K : \mathcal{S} \to \mathcal{S}$ by

$$Kf(x) = -\frac{d^2 f}{dx^2}(x) + 4\pi^2 x^2 f(x).$$

We will now show that the Hermite functions are eigenfunctions of this linear transformation. To establish this, we need a few identities.

Lemma 2.1.10. *Define* $h_{-1}(x) := 0$*. Then for* $n \in \mathbb{N}_0$*, the Hermite functions satisfy the following identities:*

(i) $\left(\frac{d}{dx} - 2\pi x\right)h_n = h_n' - 2\pi x h_n = -(n+1)h_{n+1}$

(ii) $\left(\frac{d}{dx} + 2\pi x\right)h_n = h_n' + 2\pi x h_n = 4\pi h_{n-1}$

Proof. We will first prove (i). Recall

$$h_n(x) = \sigma_n \frac{(-1)^n}{n!} e^{\pi x^2} \frac{d}{dx} e^{-2\pi x^2}$$

where $\sigma_n = \frac{2^{1/4}\sqrt{n!}}{(4\pi)^{n/2}}$ is the normalizing constant. Then

$$\begin{aligned}
h_n'(x) &= 2\pi x \sigma_n \frac{(-1)^n}{n!} e^{\pi x^2} \left(\frac{d}{dx}\right)^n e^{-2\pi x^2} + \sigma_n \frac{(-1)^n}{n!} e^{\pi x^2} \left(\frac{d}{dx}\right)^{n+1} e^{-2\pi x^2} \\
&= 2\pi x h_n(x) - (n+1)\sigma_n \frac{(-1)^{n+1}}{(n+1)!} e^{\pi x^2} \left(\frac{d}{dx}\right)^{n+1} e^{-2\pi x^2} \\
&= 2\pi x h_n(x) - (n+1)h_{n+1}(x).
\end{aligned}$$

Thus,

$$h_n'(x) - 2\pi x h_n(x) = -(n+1)h_{n+1}(x).$$

We will now prove (ii). To begin, define $h_{-1}(x) := 0$. Next we establish the following identity,

$$4\pi x \left(\frac{d}{dx}\right)^n (e^{-2\pi x^2}) = -\left(\frac{d}{dx}\right)^{n+1} (e^{-2\pi x^2}) - 4\pi n \left(\frac{d}{dx}\right)^{n-1} (e^{-2\pi x^2}).$$

We can show that this holds for $n = 1$ as follows,

$$\begin{aligned}
4\pi x \left(\frac{d}{dx}\right)(e^{-2\pi x^2}) &= -\left(\frac{d}{dx}\right)^2 (e^{-2\pi x^2}) - 4\pi(e^{-2\pi x^2}) \\
4\pi x(-4\pi x)e^{-2\pi x^2} &= -\frac{d}{dx}(-4\pi x e^{-2\pi x^2}) - 4\pi e^{-2\pi x^2} \\
-16\pi^2 x^2 e^{-2\pi x^2} &= 4\pi e^{-2\pi x^2} - 16\pi^2 x^2 e^{-2\pi x^2} - 4\pi e^{-2\pi x^2} \\
-16\pi^2 x^2 e^{-2\pi x^2} &= -16\pi^2 x^2 e^{-2\pi x^2}.
\end{aligned}$$

We will prove this identity holds for $n \geqslant 1$ by induction. Suppose

$$4\pi x \left(\frac{d}{dx}\right)^k (e^{-2\pi x^2}) = -\left(\frac{d}{dx}\right)^{k+1} (e^{-2\pi x^2}) - 4\pi k \left(\frac{d}{dx}\right)^{k-1} (e^{-2\pi x^2})$$

for some $k \geqslant 1$. Taking the derivative with respect to x on both sides of the previous line shows that

$$4\pi\left(\frac{d}{dx}\right)^{k}(e^{-2\pi x^2}) + 4\pi x\left(\frac{d}{dx}\right)^{k+1}(e^{-2\pi x^2})$$

is equal to

$$-\left(\frac{d}{dx}\right)^{k+2}(e^{-2\pi x^2}) - 4\pi k\left(\frac{d}{dx}\right)^{k}(e^{-2\pi x^2}).$$

Combining these two equations and grouping like terms gives

$$4\pi x\left(\frac{d}{dx}\right)^{k+1}(e^{-2\pi x^2}) = -\left(\frac{d}{dx}\right)^{k+2}(e^{-2\pi x^2}) - 4\pi(k+1)\left(\frac{d}{dx}\right)^{k}(e^{-2\pi x^2}).$$

We conclude by induction that the identity is true for all $n \geqslant 1$.

Multiplying each side of this identity by $\frac{(-1)^n}{n!}e^{\pi x^2}$ and rearranging terms yields that

$$4\pi x\frac{(-1)^n}{n!}e^{\pi x^2}\left(\frac{d}{dx}\right)^{n}e^{-2\pi x^2} + \frac{(-1)^n}{n!}e^{\pi x^2}\left(\frac{d}{dx}\right)^{n+1}e^{-2\pi x^2} \tag{2.1.11}$$

is equal to

$$4\pi\frac{(-1)^{n-1}}{(n-1)!}e^{\pi x^2}\left(\frac{d}{dx}\right)^{n-1}e^{-2\pi x^2}.$$

Note that the right hand side of the above equation is equal to $4\pi H_{n-1}e^{-\pi x^2}$. Additionally, we have that

$$\begin{aligned} h_n' &= \frac{d}{dx}\left(\sigma_n\frac{(-1)^n}{n!}e^{\pi x^2}\left(\frac{d}{dx}\right)^{n}e^{-2\pi x^2}\right) \\ &= 2\pi x\sigma_n\frac{(-1)^n}{n!}e^{\pi x^2}\left(\frac{d}{dx}\right)^{n}e^{-2\pi x^2} + \sigma_n\frac{(-1)^n}{n!}e^{\pi x^2}\left(\frac{d}{dx}\right)^{n+1}e^{-2\pi x^2}. \end{aligned}$$

Adding $2\pi x h_n$ gives

$$h_n' + 2\pi x h_n = 4\pi x\sigma_n\frac{(-1)^n}{n!}e^{\pi x^2}\left(\frac{d}{dx}\right)^{n}e^{-2\pi x^2} + \sigma_n\frac{(-1)^n}{n!}e^{\pi x^2}\left(\frac{d}{dx}\right)^{n+1}e^{-2\pi x^2}.$$

Then, normalizing the functions in equation (2.1.11), we have

$$h_n' + 2\pi x h_n = 4\pi h_{n-1}$$

as desired. □

We are now ready to present the following theorem which identifies each Hermite function h_n as an eigenfunction of K.

Theorem 2.1.12. *For $n \in \mathbb{N}_0$, $Kh_n = 4\pi(n+\frac{1}{2})h_n$.*

Proof. To begin, observe that

$$\begin{aligned}\left(\frac{d}{dx}+2\pi x\right)\left(\frac{d}{dx}-2\pi x\right)f &= \left(\frac{d}{dx}+2\pi x\right)(f'-2\pi xf)\\ &= \frac{d}{dx}(f'-2\pi xf)+2\pi xf'-4\pi^2x^2f\\ &= f''-2\pi xf'-2\pi f+2\pi xf'-4\pi^2x^2f\\ &= f''-2\pi f-4\pi^2x^2f.\end{aligned}$$

Now,

$$\begin{aligned}-\left(\frac{d}{dx}+2\pi x\right)\left(\frac{d}{dx}-2\pi x\right)f-2\pi f &= -f''+2\pi f+4\pi^2x^2f-2\pi f\\ &= -f''+4\pi x^2 f\\ &= K(f).\end{aligned}$$

Apply this to the Hermite functions h_n to obtain

$$\begin{aligned}K(h_n) &= -\left(\frac{d}{dx}+2\pi x\right)\left(\frac{d}{dx}-2\pi x\right)h_n-2\pi h_n\\ &= -\left(\frac{d}{dx}+2\pi x\right)(-(n+1)h_{n+1})-2\pi h_n\\ &= 4\pi(n+1)h_n-2\pi h_n\\ &= 4\pi nh_n+2\pi h_n\\ &= 4\pi\left(n+\frac{1}{2}\right)h_n.\end{aligned}$$

Thus, the Hermite functions are eigenfunctions of K with corresponding eigenvalues $4\pi(n+\frac{1}{2})$. ☐

Theorem 2.1.12 shows that the Hermite functions are eigenfunctions of a linear transformation, each from a distinct eigenspace. To finally prove that they are orthogonal functions, all that is left to show is that K is Hermitian on $\mathcal{S}$.

Theorem 2.1.13. *For $f, g \in \mathcal{S}$, $\langle Kf, g\rangle = \langle f, Kg\rangle$.*

Proof. Observe that

$$\begin{aligned}\langle Kf, g\rangle &= \int_{-\infty}^{\infty}(f''(x)+4\pi^2x^2f(x))\overline{g(x)}dx\\ &= \int_{-\infty}^{\infty}f''(x)\overline{g(x)}dx+\int_{-\infty}^{\infty}4\pi^2x^2f(x)\overline{g(x)}dx.\end{aligned}$$

Then we must show that we can pass $\frac{d^2}{dx^2}$ to $\overline{g(x)}$ in the first integral. We will use integration by parts. Let $u=\overline{g(x)}$, $du=\overline{g'(x)}dx$, $dv=f''(x)dx$, and $v = f'(x)$. Then

$$\int_{-\infty}^{\infty} f''(x)\overline{g(x)}dx = \overline{g(x)}f'(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x)\overline{g'(x)}dx$$
$$= -\int_{-\infty}^{\infty} f'(x)\overline{g'(x)}dx$$

Note that because $f, g \in \mathcal{S}$, we have $f(-\infty) = f(\infty) = 0$ and the same for g so that $\overline{g(x)}f'(x)|_{-\infty}^{\infty} = 0$.

Performing integration by parts once more yields

$$\int_{-\infty}^{\infty} f''(x)\overline{g(x)}dx = \int_{-\infty}^{\infty} f(x)\overline{g''(x)}dx.$$

Then we have

$$\langle Kf, g\rangle = \int_{-\infty}^{\infty} f(x)\overline{g''(x)}dx + \int_{-\infty}^{\infty} 4\pi^2x^2f(x)\overline{g(x)}dx$$
$$= \int_{-\infty}^{\infty} f(x)(\overline{g''(x)} + 4\pi^2x^2\overline{g(x)})dx$$
$$= \langle f, Kg\rangle$$

and thus K is Hermitian. □

This yields the following orthogonality relation.

Corollary 2.1.14. *For all $m, n \in \mathbb{N}_0$ with $m \neq n$,*

$$\langle h_n, h_m\rangle = 0.$$

Proof. Let $c_n = 4\pi(n + \frac{1}{2})$. Then for $n \neq m$,

$$\begin{aligned} c_n\langle h_n, h_m\rangle &= \langle c_nh_n, h_m\rangle \\ &= \langle Kh_n, h_m\rangle \\ &= \langle h_n, Kh_m\rangle \\ &= \langle h_n, c_mh_m\rangle \\ &= c_m\langle h_n, h_m\rangle. \end{aligned}$$

Since $c_m \neq c_n$, it must be the case that $\langle h_n, h_m\rangle = 0$. □

The last step in showing that the Hermite functions form an orthonormal basis is to show they are complete. This is done with the following.

Proposition 2.1.15. *Suppose $f \in L^2(\mathbb{R})$ satisfies $\langle f, h_n\rangle = 0$ for all Hermite functions $(h_n)_{n\geqslant 0}$. Then $f = 0$.*

Proof. Since $\mathcal{S}$ is dense in $L^2(\mathbb{R})$, it is enough to prove that when $\langle f, h_n\rangle = 0$ for all $n \in \mathbb{N}_0$, $\langle f, s\rangle = 0$ for all $s \in \mathcal{S}$.

Suppose $f \in L^2(\mathbb{R})$ satisfies $\langle f, h_n\rangle = 0$ for all $n \in \mathbb{N}_0$. By Proposition 2.1.3, we can take linear combinations of Hermite polynomials to produce any x^k for $k \geqslant 0$ so that $c_0H_0+c_1H_1+\cdots+c_kH_k = x^k$. Then $\langle f, x^k e^{-\pi x^2}\rangle = 0$.

Then for any polynomial of degree N where $p(x) = \sum_{n=0}^{N} a_n x^n$,

$$\begin{aligned}\langle f, p(x)e^{-\pi x^2}\rangle &= \langle f, (a_0 + a_1x + \cdots + a_Nx^N)e^{-\pi x^2}\rangle \\ &= \sum_{n=0}^{N} a_n\langle f, x^n e^{-\pi x^2}\rangle \\ &= 0.\end{aligned}$$

□

In the next chapter we will show that, in fact, the Hermite functions form an eigenbasis for the Fourier–Plancherel transform.

We now return to the proof of the normalizing constant for the Hermite functions delayed in this section.

Proof of Theorem 2.1.4. We will begin by showing that for the Hermite polynomial H_n of degree n, the leading coefficient is $\frac{(4\pi)^n}{n!}$. One already saw this for the first few Hermite polynomials shown earlier in this chapter. We will proceed by induction. For the case where $n = 0$, observe

$$H_0e^{-\pi x^2} = \left(\frac{(-1)^0}{0!}\right)e^{\pi x^2}\left(\frac{d}{dx}\right)^0 e^{-2\pi x^2} = e^{-\pi x^2},$$

and thus the leading coefficient of H_0 is $\frac{(4\pi)^0}{0!} = 1$. Suppose

$$H_ke^{-\pi x^2} = \left(\frac{(-1)^k}{k!}\right)e^{\pi x^2}\left(\frac{d}{dx}\right)^k e^{-2\pi x^2} = e^{-\pi x^2}\left(\frac{(4\pi)^k}{k!}x^k + \cdots\right)$$

for all $k \geqslant 0$. Taking a derivative with respect to x of the above equation gives

$$2\pi x\left(\frac{(-1)^k}{k!}\right)e^{\pi x^2}\left(\frac{d}{dx}\right)^k e^{-2\pi x^2} + \left(\frac{(-1)^k}{k!}\right)e^{\pi x^2}\left(\frac{d}{dx}\right)^{k+1} e^{-2\pi x^2}$$

is equal to

$$-2\pi x e^{-\pi x^2}\left(\frac{(4\pi)^k}{k!}x^k + \cdots\right) + e^{-\pi x^2}\left(\frac{(4\pi)^k}{k!}kx^{k-1} + \cdots\right).$$

Note that the second term on the right hand side of the above equation includes only lower order terms of x and will thus be omitted from the following computations since we are only interested in the leading coefficient of H_n. Multiplying the above by $\frac{-1}{k+1}$ and combining terms gives

$$\frac{-2\pi x}{k+1}\left(\frac{(-1)^k}{k!}\right)e^{\pi x^2}\left(\frac{d}{dx}\right)^k e^{-2\pi x^2}+\left(\frac{(-1)^{k+1}}{(k+1)!}\right)e^{\pi x^2}\left(\frac{d}{dx}\right)^{k+1}e^{-2\pi x^2}$$

is equal to

$$2e^{-\pi x^2}\left(\frac{4^k\pi^{k+1}}{(k+1)!}x^{k+1}+\cdots\right)+\cdots.$$

Then by Theorem 2.1.2, this is equivalent to

$$\frac{-2\pi x}{k+1}H_k e^{-\pi x^2}+H_{k+1}e^{-\pi x^2}=2e^{-\pi x^2}\left(\frac{4^k\pi^{k+1}}{(k+1)!}x^{k+1}+\cdots\right)+\cdots.$$

By the induction hypothesis,

$$H_k e^{-\pi x^2}=e^{-\pi x^2}\left(\frac{(4\pi)^k}{k!}x^k+\cdots\right)$$

so that

$$\frac{-2\pi x}{k+1}e^{-\pi x^2}\left(\frac{(4\pi)^k}{k!}x^k+\cdots\right)+H_{k+1}e^{-\pi x^2}$$

is equal to

$$2e^{-\pi x^2}\left(\frac{4^k\pi^{k+1}}{(k+1)!}x^{k+1}+\cdots\right)+\cdots.$$

Combining terms, we have

$$-2e^{-\pi x^2}\left(\frac{4^k\pi^{k+1}}{(k+1)!}x^{k+1}+\cdots\right)+H_{k+1}e^{-\pi x^2}$$

is equal to

$$2e^{-\pi x^2}\left(\frac{4^k\pi^{k+1}}{(k+1)!}x^{k+1}+\cdots\right)+\cdots.$$

Thus,

$$H_{k+1}e^{-\pi x^2}=e^{-\pi x^2}\left(\frac{(4\pi)^{k+1}}{(k+1)!}x^{k+1}+\cdots\right).$$

We can conclude by induction that the leading coefficient of H_n is $\frac{(4\pi)^n}{n!}$ for all $n \geqslant 0$. Define

$$a_n := \frac{(4\pi)^n}{n!}.$$

We will now show

$$\langle H_n e^{-\pi x^2}, H_n e^{-\pi x^2}\rangle = \frac{a_n}{\sqrt{2}}.$$

By the orthogonality of the Hermite functions, for $k < n$ we have

$$\left\langle e^{-\pi x^2} \sum_{j=0}^{k} c_j H_j, H_n e^{-\pi x^2} \right\rangle = 0$$

for $c_j \in \mathbb{C}$. Let $H_n = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$ by Proposition 2.1.3. Then

$$\begin{aligned}\langle H_n e^{-\pi x^2}, H_n e^{-\pi x^2}\rangle &= \langle a_n x^n e^{-\pi x^2} + a_{n-1}x^{n-1}e^{-\pi x^2} + \ldots, H_n e^{-\pi x^2}\rangle \\ &= a_n \langle x^n e^{-\pi x^2}, e^{-\pi x^2} H_n\rangle \\ &= a_n \int_{-\infty}^{\infty} x^n e^{-\pi x^2} \frac{(-1)^n}{n!} e^{\pi x^2} \left(\frac{d}{dx}\right)^n e^{-2\pi x^2} dx \\ &= \frac{a_n(-1)^n}{n!} \int_{-\infty}^{\infty} x^n \left(\frac{d}{dx}\right)^n e^{-2\pi x^2} dx.\end{aligned}$$

We will proceed by integration by parts. Let $u = x^n$, $du = nx^{n-1}dx$, $v = (\frac{d}{dx})^{n-1}e^{-2\pi x^2}$, and $dv = (\frac{d}{dx})^n e^{-2\pi x^2} dx$. Then

$$\frac{a_n(-1)^n}{n!} \int_{-\infty}^{\infty} x^n \left(\frac{d}{dx}\right)^n e^{-2\pi x^2} dx$$

is equal to

$$\frac{a_n(-1)^n}{n!} x^n \left(\frac{d}{dx}\right)^{n-1} e^{-2\pi x^2}\Big|_{-\infty}^{\infty} - \frac{a_n(-1)^n}{n!} \int_{-\infty}^{\infty} nx^{n-1} \left(\frac{d}{dx}\right)^{n-1} e^{-2\pi x^2} dx.$$

Note that the first term on the right hand side of this equation is an element of $\mathcal{S}$ and will thus go to 0. Then

$$\frac{a_n(-1)^n}{n!} \int_{-\infty}^{\infty} x^n \left(\frac{d}{dx}\right)^n e^{-2\pi x^2} dx = \frac{-a_n(-1)^n}{(n-1)!} \int_{-\infty}^{\infty} x^{n-1} \left(\frac{d}{dx}\right)^{n-1} e^{-2\pi x^2} dx.$$

Applying integration by parts n more times will give

$$\begin{aligned}\frac{a_n(-1)^n}{n!} \int_{-\infty}^{\infty} x^n \left(\frac{d}{dx}\right)^n e^{-2\pi x^2} dx &= a_n \int_{-\infty}^{\infty} e^{-2\pi x^2} dx \\ &= a_n \int_{-\infty}^{\infty} e^{-\pi(\sqrt{2}x)^2} dx.\end{aligned}$$

Using u-substitution, let $u = \sqrt{2}x$ and $du = \sqrt{2}$. Then

$$\begin{aligned}a_n \int_{-\infty}^{\infty} e^{-\pi(\sqrt{2}x)^2} dx &= a_n \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} e^{-\pi u^2} du \\ &= \frac{a_n}{\sqrt{2}}\end{aligned}$$

by Proposition 3.1.1 (i) (we prove this proposition in the following chapter) and the theorem follows. □

Chapter 3

The Fourier–Plancherel Transform

3.1. An alternate definition of the Fourier–Plancherel transform

Most people define the Fourier–Plancherel transform on $L^2(\mathbb{R})$ via the integral formula

$$(\mathcal{F}f)(t) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ixt}dx.$$

However, there are technical difficulties with convergence of the integral since, for particular $f \in L^2(\mathbb{R})$, the integral may not converge for all values of t. Some people get around this issue by understanding the integral in the Fourier transform as a limit in the mean, i.e.,

$$\lim_{N\to\infty} \left\| \mathcal{F}f - \int_{-N}^{N} f(x)e^{-2\pi ixt}dx \right\| = 0,$$

where the norm above is the $L^2(\mathbb{R})$ norm, and use the notation

$$\text{l. i. m.} \int_{-\infty}^{\infty} f(x)e^{-2\pi ix^2}dx.$$

To avoid these difficulties, and to avoid the integral representation of the Fourier transform altogether, we will define the Fourier transform via the Hermite basis $(h_n)_{n\geqslant 0}$ discussed in the previous chapter. Recall that

$$\mathcal{S} = \{p(x)e^{-\pi x^2} : p(x) \in \mathbb{C}[x]\}$$

is a dense subset of $L^2(\mathbb{R})$ by Theorem 2.1.7.

For $f \in \mathcal{S}$, define

$$(\mathcal{F}f)(t) := \int_{-\infty}^{\infty} f(x)e^{-2\pi ixt}dx.$$

Note that there is no issue of convergence in this integral since $f \in \mathcal{S}$ rapidly decays to zero near infinity. We now wish to show that $\mathcal{F} : \mathcal{S} \to \mathcal{S}$. To do this, we first present a few integral identities.

Proposition 3.1.1. *The following identities are true:*

(i) $\displaystyle\int_{-\infty}^{\infty} e^{-\pi x^2}dx = 1$

(ii) $\displaystyle\int_{-\infty}^{\infty} e^{-\pi x^2}e^{-2\pi ixt}dx = e^{-\pi t^2}$ *for all* $t \in \mathbb{R}$

Proof. We begin by proving (i). Observe that

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-\pi t^2}dt &= \sqrt{\int_{-\infty}^{\infty} e^{-\pi t^2}dt \int_{-\infty}^{\infty} e^{-\pi s^2}ds} \\
&= \sqrt{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\pi(s^2+t^2)}dtds} \\
&= \sqrt{\int_{0}^{\infty}\int_{0}^{2\pi} e^{-\pi r^2}rd\theta dr} \\
&= \sqrt{2\pi}\sqrt{\int_{0}^{\infty} e^{-\pi r^2}rdr}
\end{aligned}$$

after conversion to polar coordinates. Using u-substitution, let $u = -\pi r^2$ and $du = -2\pi rdr$. Then

$$\begin{aligned}
\sqrt{2\pi}\sqrt{\int_{0}^{\infty} e^{-\pi r^2}rdr} &= \sqrt{2\pi}\sqrt{\frac{1}{2\pi}\int_{-\infty}^{0} e^{u}du} \\
&= e^u\Big|_{-\infty}^{0} \\
&= 1.
\end{aligned}$$

We will now prove (ii). Using integration by parts, let $u = e^{-\pi x^2}$ and $dv = e^{-2\pi ixt}dt$ so that

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-\pi x^2}e^{-2\pi ixt}dx &= e^{-\pi x^2}\frac{e^{-2\pi ixt}}{-2\pi it}\Bigg|_{-\infty}^{\infty} - \frac{2\pi}{2\pi it}\int_{-\infty}^{\infty} xe^{-\pi x^2}e^{-2\pi ixt}dx \\
&= \frac{-1}{it}\int_{-\infty}^{\infty} xe^{-\pi x^2-2\pi ixt}dx.
\end{aligned}$$

Let us examine $-\pi x^2 - 2\pi ixt$. Completing the square yields

$$\begin{aligned} -\pi x^2 - 2\pi ixt &= -\pi(x^2 + 2ixt) \\ &= -\pi(x^2 + 2ixt - t^2 + t^2) \\ &= -\pi((x+it)^2 + t^2). \end{aligned}$$

We then have

$$\begin{aligned} \frac{-1}{it}\int_{-\infty}^{\infty} xe^{-\pi x^2 - 2\pi ixt}dx &= \frac{-1}{it}\int_{-\infty}^{\infty} xe^{-\pi(x+it)^2}e^{-\pi t^2}dx \\ &= \frac{-e^{-\pi t^2}}{it}\int_{-\infty}^{\infty} xe^{-\pi(x+it)^2}dx. \end{aligned}$$

Using u-substitution, let $u = x + it$ and $du = dx$. Then

$$\begin{aligned} \frac{-e^{-\pi t^2}}{it}\int_{-\infty}^{\infty} xe^{-\pi(x+it)^2}dx &= \frac{-e^{-\pi t^2}}{it}\int_{-\infty}^{\infty}(u - it)e^{-\pi u^2}du \\ &= \frac{-e^{-\pi t^2}}{it}\left(\int_{-\infty}^{\infty} ue^{-\pi u^2}du - \int_{-\infty}^{\infty} ite^{-\pi u^2}du\right) \\ &= \frac{-e^{-\pi t^2}}{it}(-it) \\ &= e^{-\pi t^2} \end{aligned}$$

invoking property (i). Note that

$$\int_{-\infty}^{\infty} ue^{-\pi u^2}du = 0$$

since the integrand is an odd function. □

Proposition 3.1.2. *If $f \in \mathcal{S}$, then $\mathcal{F}f \in \mathcal{S}$.*

Proof. Let $f \in \mathcal{S}$. Then $f(x) = p(x)e^{-\pi x^2}$, where $p(x)$ is a polynomial of degree N. The Fourier transform of f will then depend on how $\mathcal{F}$ acts on $x^n e^{-\pi x^2}$ for all $n \leqslant N$. We will proceed with a proof by induction. For $n = 0$, we have

$$\int_{-\infty}^{\infty} e^{-\pi x^2}e^{-2\pi ixt}dx = e^{-\pi t^2},$$

invoking Proposition 3.1.1 (ii), which is in $\mathcal{S}$. Suppose

$$\int_{-\infty}^{\infty} x^k e^{-\pi x^2}e^{-2\pi ixt}dx \in \mathcal{S}$$

for some $k \geqslant 0$. Taking a derivative with respect to t gives

$$\frac{d}{dt}\int_{-\infty}^{\infty} x^k e^{-\pi x^2}e^{-2\pi ixt}dx = \int_{-\infty}^{\infty} x^k e^{-\pi x^2}(-2\pi ix)e^{-2\pi ixt}dx$$

$$= -2\pi i \int_{-\infty}^{\infty} x^{k+1} e^{-\pi x^2} e^{-2\pi i x t} dx,$$

which must be in $\mathcal{S}$, as any derivative of an element in $\mathcal{S}$ is also in $\mathcal{S}$ by Definition 2.1.6. Then $\mathcal{F}(x^n e^{-\pi x^2}) \in \mathcal{S}$ for all $n \leqslant N$ and thus $\mathcal{F}f \in \mathcal{S}$ for all $f \in \mathcal{S}$. □

Then if $f = p(x)e^{-\pi x^2}$ where $p(x)$ is a polynomial of degree N, use the orthogonality of $(h_n)_{n\geqslant 0}$ to see that

$$f = \sum_{n=0}^{N} \langle f, h_n \rangle h_n$$

(note that $\langle f, h_m \rangle = 0$ for all $m \geqslant N+1$). In Section 3.2, we will show that $\mathcal{F}h_n = (-i)^n h_n$. Assuming this, we see that

$$\|\mathcal{F}f\| = \sum_{n=0}^{N} |\langle f, h_n \rangle|^2 = \|f\|^2.$$

Thus $\mathcal{F}$ is isometric on $\mathcal{S}$. The polarization identity yields

$$\langle \mathcal{F}f, \mathcal{F}g \rangle = \langle f, g \rangle$$

for all $f, g \in \mathcal{S}$.

Consider the linear transformation $\mathcal{G} : \mathcal{S} \to \mathcal{S}$ defined by

$$\mathcal{G}(\sum_{n=0}^{N} \langle f, h_n \rangle h_n) = \sum_{n=0}^{N} \langle f, h_n \rangle i^n h_n.$$

Then

$$\begin{aligned}
\mathcal{G}\mathcal{F}(\sum_{n=0}^{N} \langle f, h_n \rangle h_n) &= \mathcal{G}(\sum_{n=0}^{N} \langle f, h_n \rangle (-i)^n h_n) \\
&= \sum_{n=0}^{N} \langle f, h_n \rangle i^n (-i)^n h_n \\
&= \sum_{n=0}^{N} \langle f, h_n \rangle h_n.
\end{aligned}$$

Additionally,

$$\begin{aligned}
\mathcal{F}\mathcal{G}(\sum_{n=0}^{N} \langle f, h_n \rangle h_n) &= \mathcal{F}(\sum_{n=0}^{N} \langle f, h_n \rangle i^n h_n) \\
&= \sum_{n=0}^{N} \langle f, h_n \rangle (-i)^n i^n h_n
\end{aligned}$$

$$= \sum_{n=0}^{N} \langle f, h_n \rangle h_n.$$

Thus $\mathcal{G} = \mathcal{F}^{-1}$ on $\mathcal{S}$.

We now want to know how $\mathcal{G}$ acts on any given basis element h_n, as all $f \in \mathcal{S}$ can be written as linear combinations of these basis elements. We claim that

$$(\mathcal{G}h_n)(t) = \int_{-\infty}^{\infty} h_n(x) e^{2\pi ixt} dx.$$

To show this, we will use u-substitution. Let $u = -x$ and $du = -dx$. Then

$$\begin{aligned}
\int_{-\infty}^{\infty} h_n(-u) e^{-2\pi iut} du &= (-1)^n \int_{-\infty}^{\infty} h_n(u) e^{-2\pi iut} du \\
&= (-1)^n (-i)^n h_n(t) \\
&= i^n h_n(t) \\
&= (Gh_n)(t).
\end{aligned}$$

Note the identity $h_n(-x) = (-1)^n h_n(x)$ is used in the above computation. To see this, let σ_n be the normalizing constant for a given h_n so that

$$h_n(-x) = \sigma_n \frac{(-1)^n}{n!} e^{\pi x^2} \left(\frac{d}{d(-x)} \right)^n e^{-2\pi x^2}.$$

Then via the chain rule, we have

$$\begin{aligned}
\frac{d}{d(-x)} &= \frac{d}{dx} \frac{dx}{d(-x)} \\
&= \frac{d}{dx}(-1).
\end{aligned}$$

Thus

$$\begin{aligned}
\sigma_n \frac{(-1)^n}{n!} e^{\pi x^2} \left(\frac{d}{d(-x)} \right)^n e^{-2\pi x^2} &= \sigma_n \frac{(-1)^n}{n!} e^{\pi x^2} \left(\frac{d}{dx}(-1) \right)^n e^{-2\pi x^2} \\
&= (-1)^n \sigma_n \frac{(-1)^n}{n!} e^{\pi x^2} \left(\frac{d}{dx} \right)^n e^{-2\pi x^2} \\
&= (-1)^n h_n(x).
\end{aligned}$$

We then have

$$(\mathcal{G}f)(t) = \int_{-\infty}^{\infty} f(x) e^{2\pi ixt} dx$$

for all $f \in \mathcal{S}$

We now have one crucial fact to show. Observe

$$\langle \mathcal{F}(\sum_{n=0}^{N} a_n h_n), \sum_{m=0}^{M} b_m h_m \rangle = \langle \sum_{n=0}^{N} (-i)^n a_n h_n, \sum_{m=0}^{M} b_m h_m \rangle$$

$$
\begin{aligned}
&= \sum_{n=0}^{N}\sum_{m=0}^{M} a_n(-i)^n\overline{b_m}\langle h_n, h_m\rangle \\
&= \sum_{n=0}^{N} a_n(-i)^n\overline{b_n} \\
&= \sum_{n=0}^{N} a_n\overline{(i)^n b_n} \\
&= \sum_{n=0}^{N}\sum_{m=0}^{M} a_n\overline{(i)^m b_m}\langle h_n, h_m\rangle \\
&= \langle\sum_{n=0}^{N} a_n h_n, \sum_{m=0}^{M}(i)^m b_m h_m\rangle \\
&= \langle\sum_{n=0}^{N} a_n h_n, \mathcal{G}(\sum_{m=0}^{M} b_m h_m)\rangle.
\end{aligned}
$$

Then $\mathcal{F}^* = \mathcal{G}$ on $\mathcal{S}$. What we have accomplished in this section is the first definition of the Fourier transform we will consider. Additionally, we have shown that it is unitary and defined its inverse.

Here is how we extend $\mathcal{F}$ to all of $L^2(\mathbb{R})$. If $f \in L^2(\mathbb{R})$,

$$f_N = \sum_{n=0}^{N}\langle f, h_n\rangle h_n$$

and

$$f_M = \sum_{n=0}^{M}\langle f, h_n\rangle h_n$$

for $N \geqslant M$, then

$$
\begin{aligned}
\|\mathcal{F}f_N - \mathcal{F}f_M\| &= \|\mathcal{F}(f_N - f_M)\| \\
&= \sum_{n=M}^{N} |\langle f, h_n\rangle|^2 \to 0.
\end{aligned}
$$

Then $(\mathcal{F}f_N)$ is a Cauchy sequence and thus $\mathcal{F}f_N$ converges to some function we will denote by $\mathcal{F}f$. By this we mean

$$\|\mathcal{F}f_N - \mathcal{F}f\| \to 0.$$

Then

$$\mathcal{F}(\sum_{n=0}^{N}\langle f, h_n\rangle h_n) = \sum_{n=0}^{N}\langle f, h_n\rangle(-i)^n h_n.$$

One can show that $\mathcal{F}f + \mathcal{F}g = \mathcal{F}(f+g)$ and $\mathcal{F}(\alpha f) = \alpha\mathcal{F}(f)$ for all $\alpha \in \mathbb{C}$ and $f \in L^2(\mathbb{R})$.

3.2. Eigenbasis

We will now show that the Hermite functions $(h_n)_{n\geqslant 0}$ form an eigenbasis for $\mathcal{F}$. As with Chapter 2, some of the computations from this chapter come from Hsu **[8]**.

Theorem 3.2.1. *For each $n \in \mathbb{N}_0$, $\mathcal{F}h_n = (-i)^n h_n$.*

To prove Theorem 3.2.1, we must first present some identities involving the Fourier transform.

Lemma 3.2.2. *For $f \in \mathcal{S}$, the following identities are true:*

(i) $(\mathcal{F}f')(t) = (2\pi i t)(\mathcal{F}f)(t)$,

(ii) $(\mathcal{F}(-2\pi i x)f)(t) = (\mathcal{F}f)'(t)$.

Proof. Let $f \in \mathcal{S}$. We will begin with the proof of part (i). Recall that

$$(\mathcal{F}f')(t) = \int_{-\infty}^{\infty} f'(x)e^{-2\pi i x t}dx.$$

We will proceed with integration by parts. Let $u = e^{-2\pi i x t}$, $v = f(x)$, $du = -2\pi i t e^{-2\pi i x t}dx$ and $dv = f'(x)dx$. Then

$$\begin{aligned}\int_{-\infty}^{\infty} f'(x)e^{-2\pi i x t}dx &= e^{-2\pi i x t}f(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)(-2\pi i t)e^{-2\pi i x t}dx \\ &= (2\pi i t)\int_{-\infty}^{\infty} f(x)e^{-2\pi i x t}dx \\ &= (2\pi i t)(\mathcal{F}f)(t).\end{aligned}$$

Note that $f \in \mathcal{S}$, and so $f(-\infty) = f(\infty) = 0$. Thus $(\mathcal{F}f')(t) = (2\pi i t)(\mathcal{F}f)(t)$.

We will now prove part (ii). Observe

$$\begin{aligned}(\mathcal{F}(-2\pi i t)f)(t) &= \int_{-\infty}^{\infty} -2\pi i x f(x)e^{-2\pi i x t}dx \\ &= \int_{-\infty}^{\infty} f(x)\frac{d}{dt}e^{-2\pi i x t}dx \\ &= \frac{d}{dt}\int_{-\infty}^{\infty} f(x)e^{-2\pi i x t}dx \\ &= (\mathcal{F}f)'(t).\end{aligned}$$

Thus $(\mathcal{F}(-2\pi i x)f)(t) = (\mathcal{F}f)'(t)$ □

For the use of Lemma 3.2.2 for $f \in L^2(\mathbb{R})$, refer to Theorem 2.1.7. We are now ready to prove Theorem 3.2.1.

Proof of Theorem 3.2.1. Recall the identity

$$h_n'(x) - 2\pi x h_n(x) = -(n+1)h_{n+1}(x)$$

from Lemma 2.1.10 (i). Taking the Fourier transform of this identity gives

$$\begin{aligned}
\mathcal{F}(h_n'(x)) + \frac{1}{i}\mathcal{F}(-2\pi i x h_n(x)) &= -(n+1)\mathcal{F}(h_n(x)) \\
2\pi i x(\mathcal{F}h_n)(x) + \frac{1}{i}(\mathcal{F}h_n)'(x) &= -(n+1)(\mathcal{F}h_{n+1})(x) \\
(\mathcal{F}h_n)'(x) - 2\pi x(\mathcal{F}h_n)(x) &= -(n+1)i(\mathcal{F}h_{n+1})(x) \\
i^n(\mathcal{F}h_n)'(x) - 2\pi x i^n(\mathcal{F}h_n)(x) &= -(n+1)i^{n+1}(\mathcal{F}h_{n+1})(x)
\end{aligned}$$

by Lemma 3.2.2. We will proceed by induction to show $(h_n)_{n\geqslant 0}$ are eigenfunctions of $\mathcal{F}$. In the case where $n = 0$, we have $h_0(x) = e^{-\pi x^2}$ and

$$(\mathcal{F}h_0)(t) = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x t} dx = e^{-\pi t^2}.$$

Then $h_0(x) = i^0(\mathcal{F}h_0)(x)$. Let $k \geqslant 0$ and suppose $h_k(x) = i^k(\mathcal{F}h_k)(x)$. Then from the computation above, we have

$$\begin{aligned}
i^k(\mathcal{F}h_k)'(x) - 2\pi x i^k(\mathcal{F}h_k)(x) &= -(k+1)i^{k+1}(\mathcal{F}h_{k+1})(x) \\
h_k'(x) - 2\pi x h_k(x) &= -(k+1)i^{k+1}(\mathcal{F}h_{k+1})(x) \\
-(k+1)h_{k+1}(x) &= -(k+1)i^{k+1}(\mathcal{F}h_{k+1})(x) \\
h_{k+1}(x) &= i^{k+1}(\mathcal{F}h_{k+1})(x)
\end{aligned}$$

by the induction hypothesis and another application of Lemma 2.1.10. Then $h_n(x) = i^n(\mathcal{F}h_n)(x)$ for all $n \geqslant 0$. Dividing by i^n gives $(\mathcal{F}h_n)(x) = (-i)^n h_n(x)$. Thus, the nth Hermite function is an eigenfunction of the Fourier transform with corresponding eigenvalue $(-i)^n$. □

This brings us to the most important theorem of this section.

Theorem 3.2.3. *Let $\mathcal{F}$ be the Fourier transform considered as a bounded operator on $L^2(\mathbb{R})$. Then $(h_n)_{n\geqslant 0}$ is an eigenbasis for $\mathcal{F}$.*

Proof of Theorem 3.2.3. We have by definition that

$$\operatorname{span}\{h_n : n \geqslant 0\} = \mathcal{S}$$

and by Theorem 2.1.7, $\mathcal{S}$ is dense in $L^2(\mathbb{R})$. We have already shown that $(h_n)_{n\geqslant 0}$ are eigenfunctions for $\mathcal{F}$, and the theorem follows. □

An immediate result of Theorem 3.2.3 is the following, which will play an important role later on.

Corollary 3.2.4. $\mathcal{F}^4 = I$.

Proof. Let $f \in L^2(\mathbb{R})$. To see how $\mathcal{F}^4$ acts on f, it is enough to learn how $\mathcal{F}^4$ acts on basis elements $(h_n)_{n\geqslant 0}$. Observe

$$\mathcal{F}^4 h_n = \mathcal{F}^3((-i)^n h_n) = \mathcal{F}^2((-i)^{2n} h_n) = \mathcal{F}((-i)^{3n} h_n) = (-i)^{4n} h_n = h_n.$$

Then $\mathcal{F}^4$ is the identity operator for all basis elements, and thus $\mathcal{F}^4 f = f$ for all $f \in L^2(\mathbb{R})$. Hence, $\mathcal{F}^4 = I$. □

3.3. Matrix Representation

In this section, we will consider matrix representations of linear operators on $L^2(\mathbb{R})$. As discussed in Chapter 2, the set of all Hermite functions form an orthonormal basis for $L^2(\mathbb{R})$. Then for a bounded linear operator $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, we wish to know how T acts on the basis elements h_n.

Let us first consider Th_0. We can write this vector as some linear combination of basis vectors, so that

$$Th_0 = \sum_{j=0}^{\infty} c_j h_j$$

for some coefficients c_j. Then via the orthonormality of h_j, any one c_j is equal to

$$c_j = \langle Th_0, h_j \rangle.$$

Similarly, for

$$Th_1 = \sum_{j=0}^{\infty} d_j h_j,$$

we have that $d_j = \langle Th_1, h_j \rangle$. Continuing to perform this computation for all basis elements, the matrix representation $\mathbf{T}$ of T is

$$\mathbf{T} = \begin{bmatrix} \langle Th_0, h_0 \rangle & \langle Th_1, h_0 \rangle & \langle Th_2, h_0 \rangle & \dots \\ \langle Th_0, h_1 \rangle & \langle Th_1, h_1 \rangle & \langle Th_2, h_1 \rangle & \dots \\ \langle Th_0, h_2 \rangle & \langle Th_1, h_2 \rangle & \langle Th_2, h_2 \rangle & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

To observe how T acts on some $f \in L^2$, we can write f as a linear combination of basis elements

$$f = \sum_{j=0}^{\infty} \langle f, h_j \rangle h_j.$$

Then Tf is given in matrix form by

$$\mathbf{T}f = \begin{bmatrix} \langle Th_0, h_0\rangle & \langle Th_1, h_0\rangle & \langle Th_2, h_0\rangle & \dots \\ \langle Th_0, h_1\rangle & \langle Th_1, h_1\rangle & \langle Th_2, h_1\rangle & \dots \\ \langle Th_0, h_2\rangle & \langle Th_1, h_2\rangle & \langle Th_2, h_2\rangle & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \langle f, h_0\rangle \\ \langle f, h_1\rangle \\ \langle f, h_2\rangle \\ \vdots \end{bmatrix}.$$

Then

$$T(\sum_{j=0}^{\infty}\langle f, h_j\rangle h_j) = \sum_{j=0}^{\infty}\langle f, h_j\rangle Th_j.$$

Let us compute the matrix representation $\mathbf{F}$ of the Fourier transform $\mathcal{F}$. Using the matrix form of any linear operator on $L^2(\mathbb{R})$ as given above, we have that

$$\mathbf{F} = \begin{bmatrix} \langle \mathcal{F}h_0, h_0\rangle & \langle \mathcal{F}h_1, h_0\rangle & \langle \mathcal{F}h_2, h_0\rangle & \langle \mathcal{F}h_3, h_0\rangle & \dots \\ \langle \mathcal{F}h_0, h_1\rangle & \langle \mathcal{F}h_1, h_1\rangle & \langle \mathcal{F}h_2, h_1\rangle & \langle \mathcal{F}h_3, h_1\rangle & \dots \\ \langle \mathcal{F}h_0, h_2\rangle & \langle \mathcal{F}h_1, h_2\rangle & \langle \mathcal{F}h_2, h_2\rangle & \langle \mathcal{F}h_3, h_2\rangle & \dots \\ \langle \mathcal{F}h_0, h_3\rangle & \langle \mathcal{F}h_1, h_3\rangle & \langle \mathcal{F}h_2, h_3\rangle & \langle \mathcal{F}h_3, h_3\rangle & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$= \begin{bmatrix} \langle h_0, h_0\rangle & -i\langle h_1, h_0\rangle & -\langle h_2, h_0\rangle & i\langle h_3, h_0\rangle & \dots \\ \langle h_0, h_1\rangle & -i\langle h_1, h_1\rangle & -\langle h_2, h_1\rangle & i\langle h_3, h_1\rangle & \dots \\ \langle h_0, h_2\rangle & -i\langle h_1, h_2\rangle & -\langle h_2, h_2\rangle & i\langle h_3, h_2\rangle & \dots \\ \langle h_0, h_3\rangle & -i\langle h_1, h_3\rangle & -\langle h_2, h_3\rangle & i\langle h_3, h_3\rangle & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Then by the orthonormality of the Hermite functions, we have

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & -i & 0 & 0 & \dots \\ 0 & 0 & -1 & 0 & \dots \\ 0 & 0 & 0 & i & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

so that $\mathbf{F}$ is an infinite diagonal matrix. Because the eigenvalue of $\mathcal{F}$ for some h_n is $(-i)^n$, this 4×4 pattern will repeat itself on the diagonal infinitely. Let

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & i \end{bmatrix}.$$

Then in block form,

$$\mathbf{F} = \begin{bmatrix} \mathbf{D} & & & \\ & \mathbf{D} & & \\ & & \mathbf{D} & \\ & & & \ddots \end{bmatrix}$$

omitting zeros. This representation of $\mathcal{F}$ as a matrix will be used in later chapters. We will also see another matrix representation which will be more useful in discussing various properties of the Fourier transform. We presented this matrix representation first since it is the most natural one.

Chapter 4

The Commutant

In linear algebra, we have often asked what matrices commute with a given matrix. We now ask this question for the Fourier transform. In this chapter, we fully characterize the set of bounded linear operators in the commutant of $\mathcal{F}$ in their matrix representations, considering $\mathbf{F}$ as given in Chapter 3.

4.1. Basic Facts

Definition 4.1.1. The *commutant* of $\mathcal{F}$ is defined by

$$\{\mathcal{F}\}' = \{T \in \mathcal{B}(L^2(\mathbb{R})) : T\mathcal{F} = \mathcal{F}T\}.$$

These are the bounded operators on $L^2(\mathbb{R})$ that commute with the Fourier transform. Note that $\{\mathcal{F}\}'$ is closed under addition and scalar multiplication. Indeed, for $T \in \{\mathcal{F}\}'$ and $c \in \mathbb{C}$, we have

$$\mathcal{F}(cT) = c(\mathcal{F}T) = c(T\mathcal{F}) = (cT)\mathcal{F}$$

so $cT \in \{F\}'$. Additionally, for $T, T' \in \{\mathcal{F}\}'$, we have

$$\mathcal{F}(T + T') = \mathcal{F}T + \mathcal{F}T' = T\mathcal{F} + T'\mathcal{F} = (T + T')\mathcal{F}$$

so that $(T + T') \in \{\mathcal{F}\}'$.

We also have that that $\{\mathcal{F}\}'$ is closed under operator composition. For instance, for $S, T \in \{\mathcal{F}\}'$, we have

$$ST\mathcal{F} = S\mathcal{F}T = \mathcal{F}ST$$

and thus $ST \in \{\mathcal{F}\}'$.

Finally, $\{\mathcal{F}\}'$ is closed under adjoints. Indeed, if $T \in \{\mathcal{F}\}'$, then

$$T\mathcal{F} = \mathcal{F}T \Rightarrow (T\mathcal{F})^* = (\mathcal{F}T)^*$$

$$\begin{aligned}
&\Rightarrow \mathcal{F}^* T^* = T^* \mathcal{F}^* \\
&\Rightarrow T^* = F T^* \mathcal{F}^* \\
&\Rightarrow T^* \mathcal{F} = \mathcal{F} T^*.
\end{aligned}$$

Some obvious operators in $\{\mathcal{F}\}'$ are $p(\mathcal{F})q(\mathcal{F}^*)$ for $p, q \in \mathbb{C}[x]$. A somewhat less obvious one is in the following.

Example 4.1.2. Consider the operator $A : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined by $(Af)(x) = f(-x)$. Then

$$\begin{aligned}
(A\mathcal{F}f)(x) &= A \int_{-\infty}^{\infty} f(t) e^{-2\pi i x t} dt \\
&= \int_{-\infty}^{\infty} f(t) e^{2\pi i x t} dt.
\end{aligned}$$

Let $u = -t$ and $du = -dt$. Then

$$\begin{aligned}
\int_{-\infty}^{\infty} f(t) e^{2\pi i x t} dt &= -\int_{\infty}^{-\infty} f(-u) e^{-2\pi i x u} du \\
&= \int_{-\infty}^{\infty} f(-u) e^{-2\pi i x u} du \\
&= (\mathcal{F}Af)(x).
\end{aligned}$$

In the other direction, we have

$$\begin{aligned}
(\mathcal{F}Af)(x) &= (\mathcal{F}f)(-x) \\
&= \int_{-\infty}^{\infty} f(-t) e^{-2\pi i x t} dt \\
&= (A\mathcal{F}f)(x).
\end{aligned}$$

Thus $A \in \{\mathcal{F}\}'$.

4.2. Kronecker Product

We now define an important operation for the section that follows, the *Kronecker product*.

Definition 4.2.1. Let $\mathbf{A} = [a_{ij}]_{i,j=1}^{\infty}$ be an infinite matrix and $\mathbf{B} \in M_{n\times n}$. The *Kronecker product* $\mathbf{A} \otimes \mathbf{B}$ is the infinite matrix

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & a_{13}\mathbf{B} & \dots \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & a_{23}\mathbf{B} & \dots \\ a_{31}\mathbf{B} & a_{32}\mathbf{B} & a_{33}\mathbf{B} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Taking a more in depth look at an arbitrary block $a_{ij}\mathbf{B}$, we have that

$$a_{ij}\mathbf{B} = \begin{bmatrix} a_{ij}b_{11} & a_{ij}b_{12} & a_{ij}b_{13} & \dots & a_{ij}b_{1n} \\ a_{ij}b_{21} & a_{ij}b_{22} & a_{ij}b_{23} & \dots & a_{ij}b_{2n} \\ a_{ij}b_{31} & a_{ij}b_{32} & a_{ij}b_{33} & \dots & a_{ij}b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \\ a_{ij}b_{n1} & a_{ij}b_{n2} & a_{ij}b_{3n} & & a_{ij}b_{nn} \end{bmatrix}.$$

It is sometimes the case that the interesting matrix to compute is the product of two Kronecker products. This is called the *mixed product*, defined below.

Theorem 4.2.2. *Let* $\mathbf{A}$ *and* $\mathbf{C}$ *be infinite matrices such that* $\mathbf{AC}$ *exists and* $\mathbf{B}, \mathbf{D} \in M_{n\times n}$*. The mixed product of* $\mathbf{A}$*,* $\mathbf{B}$*,* $\mathbf{C}$*, and* $\mathbf{D}$ *is*

$$(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = \mathbf{AC} \otimes \mathbf{BD}.$$

Proof. Let $\mathbf{A}$ and $\mathbf{C}$ be infinite matrices and $\mathbf{B}, \mathbf{D} \in M_{n\times n}$. Then

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & a_{13}\mathbf{B} & a_{14}\mathbf{B} & \dots \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & a_{23}\mathbf{B} & a_{24}\mathbf{B} & \dots \\ a_{31}\mathbf{B} & a_{32}\mathbf{B} & a_{33}\mathbf{B} & a_{34}\mathbf{B} & \dots \\ a_{41}\mathbf{B} & a_{42}\mathbf{B} & a_{43}\mathbf{B} & a_{44}\mathbf{B} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and

$$\mathbf{C} \otimes \mathbf{D} = \begin{bmatrix} c_{11}\mathbf{D} & c_{12}\mathbf{D} & c_{13}\mathbf{D} & c_{14}\mathbf{D} & \dots \\ c_{21}\mathbf{D} & c_{22}\mathbf{D} & c_{23}\mathbf{D} & c_{24}\mathbf{D} & \dots \\ c_{31}\mathbf{D} & c_{32}\mathbf{D} & c_{33}\mathbf{D} & c_{34}\mathbf{D} & \dots \\ c_{41}\mathbf{D} & c_{42}\mathbf{D} & c_{43}\mathbf{D} & c_{44}\mathbf{D} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Multiplying these two matrices together gives

$$(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = \begin{bmatrix} \sum_{j=1}^{\infty} a_{1j}c_{j1}\mathbf{BD} & \sum_{j=1}^{\infty} a_{1j}c_{j2}\mathbf{BD} & \sum_{j=1}^{\infty} a_{1j}c_{j3}\mathbf{BD} & \dots \\ \sum_{j=1}^{\infty} a_{2j}c_{j1}\mathbf{BD} & \sum_{j=1}^{\infty} a_{2j}c_{j2}\mathbf{BD} & \sum_{j=1}^{\infty} a_{2j}c_{j3}\mathbf{BD} & \dots \\ \sum_{j=1}^{\infty} a_{3j}c_{j1}\mathbf{BD} & \sum_{j=1}^{\infty} a_{3j}c_{j2}\mathbf{BD} & \sum_{j=1}^{\infty} a_{3j}c_{j3}\mathbf{BD} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Note that the product $\mathbf{AC}$ in summation notation is given by

$$\mathbf{AC} = \left[\sum_{n=1}^{\infty} a_{in}c_{nj} \right]_{i,j=0}^{\infty}.$$

Then the above matrix is exactly $\mathbf{AC} \otimes \mathbf{BD}$ by Definition 4.2.1. □

4.3. Main Theorem

In the matrix representation of $\mathcal{F}$ as described in Chapter 3, we claim the following theorem.

Theorem 4.3.1 (Commutant of $\mathcal{F}$). *A bounded linear operator T on $L^2(\mathbb{R})$ commutes with $\mathcal{F}$ if and only if the matrix representation $\mathbf{T}$ of T with respect to the Hermite basis takes the form $\mathbf{T} = [\mathbf{T}_{ij}]_{i,j=1}^{\infty}$ where each $\mathbf{T}_{ij}$ is a 4×4 diagonal matrix.*

Proof. Let $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be some bounded linear operator. Suppose $T\mathcal{F} = \mathcal{F}T$. Consider $\mathbf{T}$ as being composed of an infinite number of 4×4 matrices in its matrix representation, so that we can write $\mathbf{T} = [\mathbf{T}_{ij}]_{i=1,j=1}^{\infty}$ for $\mathbf{T}_{ij} \in \mathrm{M}_{4\times 4}$. Then, in terms of block matrix multiplication,

$$\mathbf{TF} = \begin{bmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} & \mathbf{T}_{13} & \dots \\ \mathbf{T}_{21} & \mathbf{T}_{22} & \mathbf{T}_{23} & \dots \\ \mathbf{T}_{31} & \mathbf{T}_{32} & \mathbf{T}_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \mathbf{D} & & & \\ & \mathbf{D} & & \\ & & \mathbf{D} & \\ & & & \ddots \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{T}_{11}\mathbf{D} & \mathbf{T}_{12}\mathbf{D} & \mathbf{T}_{13}\mathbf{D} & \dots \\ \mathbf{T}_{21}\mathbf{D} & \mathbf{T}_{22}\mathbf{D} & \mathbf{T}_{23}\mathbf{D} & \dots \\ \mathbf{T}_{31}\mathbf{D} & \mathbf{T}_{32}\mathbf{D} & \mathbf{T}_{33}\mathbf{D} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Similarly,

$$\mathbf{FT} = \begin{bmatrix} \mathbf{DT}_{11} & \mathbf{DT}_{12} & \mathbf{DT}_{13} & \dots \\ \mathbf{DT}_{21} & \mathbf{DT}_{22} & \mathbf{DT}_{23} & \dots \\ \mathbf{DT}_{31} & \mathbf{DT}_{32} & \mathbf{DT}_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Comparing (block) entries says that $\mathbf{T}_{jk}\mathbf{D} = \mathbf{DT}_{jk}$ for all j, k.

Let us examine this relationship for an arbitrary selection of j, k,

$$\begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ t_{31} & t_{32} & t_{33} & t_{34} \\ t_{41} & t_{42} & t_{43} & t_{44} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & i \end{bmatrix} \begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ t_{31} & t_{32} & t_{33} & t_{34} \\ t_{41} & t_{42} & t_{43} & t_{44} \end{bmatrix}.$$

Matrix multiplication yields

$$\begin{bmatrix} t_{11} & (-i)t_{12} & -t_{13} & (i)t_{14} \\ t_{21} & (-i)t_{22} & -t_{23} & (i)t_{24} \\ t_{31} & (-i)t_{32} & -t_{33} & (i)t_{34} \\ t_{41} & (-i)t_{42} & -t_{43} & (i)t_{44} \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ (-i)t_{21} & (-i)t_{22} & (-i)t_{23} & (-i)t_{24} \\ -t_{31} & -t_{32} & -t_{33} & -t_{34} \\ (i)t_{41} & (i)t_{42} & (i)t_{43} & (i)t_{44} \end{bmatrix}.$$

It is clear from the above equation that all but the diagonal elements of $\mathbf{T}_{jk}$ must be 0 to produce equality. Thus, each block of $\mathbf{T}$ must be a 4×4 diagonal matrix.

Suppose $\mathbf{T} = [\mathbf{T}_{jk}]_{j=1,k=1}^{\infty}$ so that each block $\mathbf{T}_{jk}$ is a 4×4 diagonal matrix. Consider an arbitrary block $\mathbf{T}_{jk}$. Then $\mathbf{T}_{jk}\mathbf{D} = \mathbf{DT}_{jk}$, as diagonal matrices commute. Thus

$$\mathbf{TF} = \begin{bmatrix} \mathbf{T}_{11}\mathbf{D} & \mathbf{T}_{12}\mathbf{D} & \mathbf{T}_{13}\mathbf{D} & \dots \\ \mathbf{T}_{21}\mathbf{D} & \mathbf{T}_{22}\mathbf{D} & \mathbf{T}_{23}\mathbf{D} & \dots \\ \mathbf{T}_{31}\mathbf{D} & \mathbf{T}_{32}\mathbf{D} & \mathbf{T}_{33}\mathbf{D} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} \mathbf{DT}_{11} & \mathbf{DT}_{12} & \mathbf{DT}_{13} & \dots \\ \mathbf{DT}_{21} & \mathbf{DT}_{22} & \mathbf{DT}_{23} & \dots \\ \mathbf{DT}_{31} & \mathbf{DT}_{32} & \mathbf{DT}_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \mathbf{FT},$$

and $\mathbf{TF} = \mathbf{FT}$. □

A powerful consequence of this theorem is that producing a $T \in \{\mathcal{F}\}'$ is not at all difficult. Take any known bounded operator $\mathbf{A}$ represented as a matrix on ℓ^2 and any 4×4 diagonal matrix Λ and compute $\mathbf{A} \otimes \Lambda$. A technical result from operator theory will show that this defines a bounded operator. This product will commute with the Fourier transform.

Corollary 4.3.2. *Let* $\mathbf{A}$ *be any bounded linear operator on* ℓ^2 *and let* Λ *be any* 4×4 *diagonal matrix. Then* $\mathbf{A} \otimes \Lambda$ *commutes with* $\mathbf{F}$*.*

Proof. Let $\mathbf{A}$ be a bounded operator and Λ be a 4×4 diagonal matrix. Note that one can write $\mathbf{F} = \mathbf{I} \otimes \mathbf{D}$ where $\mathbf{I}$ is the infinite identity operator. Then, by Theorem 4.2.2,

$$(\mathbf{A} \otimes \Lambda)(\mathbf{I} \otimes \mathbf{D}) = \mathbf{AI} \otimes \Lambda\mathbf{D} \quad \text{and} \quad (\mathbf{I} \otimes \mathbf{D})(\mathbf{A} \otimes \Lambda) = \mathbf{IA} \otimes \mathbf{D}\Lambda.$$

Thus, $(\mathbf{A} \otimes \Lambda)(\mathbf{I} \otimes \mathbf{D}) = (\mathbf{I} \otimes \mathbf{D})(\mathbf{A} \otimes \Lambda)$, and the product $\mathbf{A} \otimes \Lambda$ commutes with $\mathbf{F}$. □

The remainder of this section is composed of several interesting examples of Theorem 4.3.1.

Example 4.3.3. Any polynomial in the Fourier transform is in the commutant of $\mathcal{F}$. Let

$$T = a_0 I + a_1 \mathcal{F} + a_2 \mathcal{F}^2 + \ldots + a_n \mathcal{F}^n.$$

Let $p(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$. Then

$$\mathbf{T} = \begin{bmatrix} p(1) & & & & & \\ & p(-i) & & & & \\ & & p(-1) & & & \\ & & & p(i) & & \\ & & & & p(1) & \\ & & & & & \ddots \end{bmatrix}$$

is a diagonal matrix. Thus, $\mathbf{T}$ is in the commutant of $\mathcal{F}$.

Example 4.3.4. The integral example of this was shown at the beginning of this chapter, with $(Tf)(x) = f(-x)$. In the matrix representation, note that $(Th_n)(x) = h_n(-x) = (-1)^n h_n(x)$ (this result was shown in Chapter

3). Then

$$\mathbf{T} = \left[\begin{array}{cccc|cccc|c} 1 & & & & & & & & \\ & -1 & & & & & & & \\ & & 1 & & & & & & \\ & & & -1 & & & & & \\ \hline & & & & 1 & & & & \\ & & & & & -1 & & & \\ & & & & & & 1 & & \\ & & & & & & & -1 & \\ \hline & & & & & & & & \ddots \end{array}\right]$$

is in the commutant of $\mathcal{F}$.

Example 4.3.5. Let $\mathbf{H}$ be the *Hilbert matrix*

$$\mathbf{H} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \cdots \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

A result of Schur **[15]** says that $\mathbf{Hx} \in \ell^2$ for all $\mathbf{x} \in \ell^2$. Additionally, we have

$$\|\mathbf{H}\| = \sup_{\mathbf{x} \in \ell^2; \|x\|=1} \|\mathbf{Hx}\| = \pi.$$

Thus $\mathbf{H}$ is a bounded operator on ℓ^2. Let

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix}.$$

Then the matrix

$$\mathbf{H} \otimes \Lambda = \left[\begin{array}{cccc|cccc|c} \lambda_1 & & & & \frac{1}{2}\lambda_1 & & & & \cdots \\ & \lambda_2 & & & & \frac{1}{2}\lambda_2 & & & \cdots \\ & & \lambda_3 & & & & \frac{1}{2}\lambda_3 & & \cdots \\ & & & \lambda_4 & & & & \frac{1}{2}\lambda_4 & \cdots \\ \hline \frac{1}{2}\lambda_1 & & & & \frac{1}{3}\lambda_1 & & & & \cdots \\ & \frac{1}{2}\lambda_2 & & & & \frac{1}{3}\lambda_2 & & & \cdots \\ & & \frac{1}{2}\lambda_3 & & & & \frac{1}{3}\lambda_3 & & \cdots \\ & & & \frac{1}{2}\lambda_4 & & & & \frac{1}{3}\lambda_4 & \cdots \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right]$$

is in the commutant of $\mathcal{F}$.

Example 4.3.6. Let $\mathbf{T_a}$ be the *Toeplitz matrix*

$$\mathbf{T_a} = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & a_{-4} & \cdots \\ a_1 & a_0 & a_{-1} & a_{-2} & a_{-3} & \cdots \\ a_2 & a_1 & a_0 & a_{-1} & a_{-2} & \cdots \\ a_3 & a_2 & a_1 & a_0 & a_{-1} & \cdots \\ a_4 & a_3 & a_2 & a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

for some sequence $\mathbf{a} = (a_n)_{n=-\infty}^{\infty}$. As first explored by Toeplitz **[16, 17]** but then more rigorously verified by Hartman and Wintner **[6]**, $\mathbf{T_a}$ defines a bounded operator on ℓ^2 if and only if

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} \phi(\theta)e^{-in\theta}d\theta$$

for some bounded measurable function ϕ on $[0, 2\pi]$. In other words, if $\mathbf{a}$ is the sequence of Fourier coefficients of ϕ. The norm of this matrix is given by

$$\|\mathbf{T_a}\| = \sup_\theta |\phi(e^{i\theta})|.$$

Then for any 4×4 diagonal matrix

$$\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \lambda_4 \end{bmatrix},$$

$$\mathbf{T_a} \otimes \Lambda = \left[\begin{array}{cccc|cccc|c} a_0\lambda_1 & & & & a_{-1}\lambda_1 & & & & \cdots \\ & a_0\lambda_2 & & & & a_{-1}\lambda_2 & & & \cdots \\ & & a_0\lambda_3 & & & & a_{-1}\lambda_3 & & \cdots \\ & & & a_0\lambda_4 & & & & a_{-1}\lambda_4 & \cdots \\ \hline a_1\lambda_1 & & & & a_0\lambda_1 & & & & \cdots \\ & a_1\lambda_2 & & & & a_0\lambda_2 & & & \cdots \\ & & a_1\lambda_3 & & & & a_0\lambda_3 & & \cdots \\ & & & a_1\lambda_4 & & & & a_0\lambda_4 & \cdots \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right]$$

is an element of $\{\mathcal{F}\}'$.

Example 4.3.7. Let $\mathbf{C}$ be the *Cesàro matrix*

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \cdots \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

A well known result of Brown, Halmos, and Shields **[3]** says that $\mathbf{C}\ell^2 \subset \ell^2$ and

$$\|\mathbf{C}\| = \sup_{\mathbf{x} \in \ell^2, \|x\|=1} \|\mathbf{Cx}\| = 2.$$

Then

$$
\mathbf{C} \otimes \Lambda = \left[\begin{array}{cccc|cccc|c}
\lambda_1 & & & & & & & & \cdots \\
& \lambda_2 & & & & & & & \cdots \\
& & \lambda_3 & & & & & & \cdots \\
& & & \lambda_4 & & & & & \cdots \\
\hline
\frac{1}{2}\lambda_1 & & & & \frac{1}{2}\lambda_1 & & & & \cdots \\
& \frac{1}{2}\lambda_2 & & & & \frac{1}{2}\lambda_2 & & & \cdots \\
& & \frac{1}{2}\lambda_3 & & & & \frac{1}{2}\lambda_3 & & \cdots \\
& & & \frac{1}{2}\lambda_4 & & & & \frac{1}{2}\lambda_4 & \cdots \\
\hline
\frac{1}{3}\lambda_1 & & & & \frac{1}{3}\lambda_1 & & & & \cdots \\
& \frac{1}{3}\lambda_2 & & & & \frac{1}{3}\lambda_2 & & & \cdots \\
& & \frac{1}{3}\lambda_3 & & & & \frac{1}{3}\lambda_3 & & \cdots \\
& & & \frac{1}{3}\lambda_4 & & & & \frac{1}{3}\lambda_4 & \cdots \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right]
$$

is in the commutant of $\mathcal{F}$.

4.4. A Better Approach

Though the matrix representation of $\mathcal{F}$ from the previous section seems the most natural since the indexing corresponds to the natural indexing of $(h_n)_{n\geqslant 0}$, one can see that the description of the commutant of $\mathcal{F}$ is a bit cumbersome, and one might argue, not all that useful. In this section, we give another matrix representation of $\mathcal{F}$ that yields a more useful description of $\{\mathcal{F}\}'$. To begin, decompose $L^2(\mathbb{R})$ into the closed linear span

$$
\begin{aligned}
L^2(\mathbb{R}) = \bigvee\{h_0, h_4, h_8, \ldots\} \oplus \bigvee\{h_1, h_5, h_9, \ldots\} \oplus \\
\bigvee\{h_2, h_6, h_{10}, \ldots\} \oplus \bigvee\{h_3, h_7, h_{11}, \ldots\}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
\bigvee\{h_0, h_4, h_8, \ldots\} &= \ker(\mathcal{F} - I) \\
\bigvee\{h_1, h_5, h_9, \ldots\} &= \ker(\mathcal{F} - (-i)I)
\end{aligned}
$$

$$\bigvee\{h_2, h_6, h_{10}, \ldots\} = \ker(\mathcal{F} - (-1)I)$$
$$\bigvee\{h_3, h_7, h_{11}, \ldots\} = \ker(\mathcal{F} - iI)$$

and these eigenspaces are orthogonal. Then we can rewrite the matrix representation of $\mathcal{F}$ so that

$$\mathbf{F} = \begin{bmatrix} \mathbf{I} & & & \\ & -i\mathbf{I} & & \\ & & -\mathbf{I} & \\ & & & i\mathbf{I} \end{bmatrix} \tag{4.4.1}$$

where each $j\mathbf{I}$ for $j = \pm 1, \pm i$ is an infinite diagonal matrix with j on the main diagonal, omitting zeros. For $f \in L^2(\mathbb{R})$, define

$$f = f_1 \oplus f_{-i} \oplus f_{-1} \oplus f_1,$$

where $f_j \in \ker(\mathcal{F} - jI)$ for $j = \pm 1, \pm i$. We now have

$$\mathbf{F}f = \begin{bmatrix} \mathbf{I} & & & \\ & -i\mathbf{I} & & \\ & & -\mathbf{I} & \\ & & & i\mathbf{I} \end{bmatrix} \begin{bmatrix} f_1 \\ \hdashline f_{-i} \\ \hdashline f_{-1} \\ \hdashline f_i \end{bmatrix} = \begin{bmatrix} f_1 \\ \hdashline -if_{-i} \\ \hdashline -f_{-1} \\ \hdashline if_i \end{bmatrix}$$

Then for $T \in \mathcal{B}(L^2(\mathbb{R}))$,

$$\mathbf{T}f = \begin{bmatrix} \mathbf{T}_{00} & \mathbf{T}_{01} & \mathbf{T}_{02} & \mathbf{T}_{03} \\ \mathbf{T}_{10} & \mathbf{T}_{11} & \mathbf{T}_{12} & \mathbf{T}_{13} \\ \mathbf{T}_{20} & \mathbf{T}_{21} & \mathbf{T}_{22} & \mathbf{T}_{23} \\ \mathbf{T}_{30} & \mathbf{T}_{31} & \mathbf{T}_{32} & \mathbf{T}_{33} \end{bmatrix} \begin{bmatrix} f_1 \\ \hdashline f_{-i} \\ \hdashline f_{-1} \\ \hdashline f_i \end{bmatrix}.$$

One can see from the above equation that each $\mathbf{T}_{jk}$ is a linear transformation such that $T_{jk} : \ker(\mathcal{F} - (-i)^k I) \to \ker(\mathcal{F} - (-i)^j I)$. We can now characterize the elements of $\{\mathcal{F}\}'$ using this representation of $\mathcal{F}$.

Theorem 4.4.2. *Let $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be some bounded linear operator. Then the matrix representation $\mathbf{T}$ commutes with $\mathcal{F}$, both represented according to equation* (4.4.1)*, if and only if*

$$\mathbf{T} = \begin{bmatrix} \mathbf{T}_0 & & & \\ & \mathbf{T}_1 & & \\ & & \mathbf{T}_2 & \\ & & & \mathbf{T}_3 \end{bmatrix},$$

where each $T_j \in \mathcal{B}(\ell^2)$ for $0 \leqslant j \leqslant 3$.

This proof is carried out in much the same way as the proof for Theorem 4.3.1; however, one can also achieve this result by noting that this new definition of $\mathcal{F}$ is simply a new notation, gathering all terms in a given eigenspace together. Then re-notating $T \in \{\mathcal{F}\}'$ in the same way will yield the matrix given in the above theorem. We will use this notation throughout the remainder of this thesis unless otherwise noted.

If not already apparent, the convenience of this notation will be further demonstrated in the following chapters on square roots and invariant subspaces of $\mathcal{F}$.

We now return to some of the examples in Section 4.3 and express them in their new notation.

Example 4.4.3. The transformation of Examples 4.3.3 and 4.3.4 are analogous to the way in which we renotate $\mathcal{F}$, as these examples involve diagonal matrices. For a polynomial $p(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$, we have that

$$\mathbf{T} = \begin{bmatrix} \mathbf{P_1} & & & \\ & \mathbf{P}_{-i} & & \\ & & \mathbf{P}_{-1} & \\ & & & \mathbf{P}_i \end{bmatrix}$$

where each $\mathbf{P}_j$ is an infinite diagonal matrix with $p(j)$ on the main diagonal. Additionally, for $(T'f)(x) = f(-x)$, we have

$$\mathbf{T}' = \begin{bmatrix} \mathbf{I} & & & \\ & -\mathbf{I} & & \\ & & \mathbf{I} & \\ & & & -\mathbf{I} \end{bmatrix}.$$

Example 4.4.4. We now return to Example 4.3.5 involving the Hilbert matrix. For $\mathbf{H} \otimes \Lambda \in \{\mathcal{F}\}'$, we have

$$\begin{bmatrix} \lambda_1 \mathbf{H} & & & \\ & \lambda_2 \mathbf{H} & & \\ & & \lambda_3 \mathbf{H} & \\ & & & \lambda_4 \mathbf{H} \end{bmatrix}$$

in the new notation. Note that this is exactly $\Lambda \otimes \mathbf{H}$. For the Toeplitz and Cesàro operators, the result from the examples in the new notation is also given by $\Lambda \otimes \mathbf{T}_\alpha$ and $\Lambda \otimes \mathbf{C}$, respectively.

4.5. The Fourier Cosine and Sine Transforms

We now turn our attention to two forms of the Fourier transform: the *Fourier cosine transform* and *Fourier sine transform*.

Definition 4.5.1. The *Fourier cosine transform* is

$$\mathcal{F}_{\cos} = \frac{1}{2}(\mathcal{F} + \mathcal{F}^*).$$

Similarly, the *Fourier sine transform* is

$$\mathcal{F}_{\sin} = \frac{1}{2i}(\mathcal{F} - \mathcal{F}^*).$$

To find the matrix representation of these operators, we will again observe how they act on the basis $(h_n)_{n \geqslant 0}$. Let us begin with $\mathcal{F}_{\cos}$. We have

$$\mathcal{F}_{\cos} h_{4k} = \frac{1}{2}(\mathcal{F} + \mathcal{F}^*) h_{4k} = \frac{1}{2}(\mathcal{F} h_{4k} + \mathcal{F}^* h_{4k}) = h_{4k}$$

for $k \in \mathbb{N}_0$. A similar computation shows

$$\begin{aligned} \mathcal{F}_{\cos} h_{4k+1} &= 0 \\ \mathcal{F}_{\cos} h_{4k+2} &= -h_{4k+2} \\ \mathcal{F}_{\cos} h_{4k+3} &= 0 \end{aligned}$$

so that the eigenspaces of $\mathcal{F}_{\cos}$ are $\ker(\mathcal{F} - I)$ and $\ker(\mathcal{F} + I)$, where $\ker(\mathcal{F}_{\cos} - I) = \bigvee \{h_{4k} : k \in \mathbb{N}_o\}$ and $\ker(\mathcal{F}_{\cos} + I) = \bigvee \{h_{4k+2} : k \in \mathbb{N}_0\}$. Then the matrix representation of $\mathcal{F}_{\cos}$ is

$$\mathcal{F}_{\cos} = \begin{bmatrix} \mathbf{I} & & & \\ & \mathbf{0} & & \\ & & -\mathbf{I} & \\ & & & \mathbf{0} \end{bmatrix}.$$

One can go through similar computations to show that the eigenspaces for $\mathcal{F}_{\sin}$ are $\ker(\mathcal{F}+iI)$ and $\ker(\mathcal{F}-iI)$ so that the matrix representation is

$$\mathcal{F}_{\sin} = \begin{bmatrix} \mathbf{0} & & & \\ & -\mathbf{I} & & \\ & & \mathbf{0} & \\ & & & \mathbf{I} \end{bmatrix}.$$

As with the Fourier transform, we wish to characterize the commutant of $\mathcal{F}_{\sin}$ and $\mathcal{F}_{\cos}$. This is demonstrated in the following theorem.

Theorem 4.5.2.

$$\{\mathcal{F}_{\cos}\}' = \left\{ \begin{bmatrix} \mathbf{T}_{00} & & & \\ & \mathbf{T}_{11} & & \mathbf{T}_{13} \\ & & \mathbf{T}_{22} & \\ & \mathbf{T}_{31} & & \mathbf{T}_{33} \end{bmatrix} \right\}$$

and

$$\{\mathcal{F}_{\sin}\}' = \left\{ \begin{bmatrix} \mathbf{T}_{00} & & \mathbf{T}_{02} & \\ & \mathbf{T}_{11} & & \\ \mathbf{T}_{20} & & \mathbf{T}_{22} & \\ & & & \mathbf{T}_{33} \end{bmatrix} \right\}$$

such that $T_{jk} \in \mathcal{B}(\ell^2)$ *where* $T_{jk} : \ker(\mathcal{F}-(-i)^k I) \to \ker(\mathcal{F}-(-i)^j I)$*, omitting zeros.*

Proof. We will begin with $\{\mathcal{F}_{\cos}\}'$. Let $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be represented by the (second) matrix representation with respect to the decomposition of $L^2(\mathbb{R})$ as $\ker(\mathcal{F}-I) \oplus \ker(\mathcal{F}+I) \oplus \ker(\mathcal{F}-iI) \oplus \ker(\mathcal{F}+iI)$,

$$\mathbf{T} = \begin{bmatrix} \mathbf{T}_{00} & \mathbf{T}_{01} & \mathbf{T}_{02} & \mathbf{T}_{03} \\ \mathbf{T}_{10} & \mathbf{T}_{11} & \mathbf{T}_{12} & \mathbf{T}_{13} \\ \mathbf{T}_{20} & \mathbf{T}_{21} & \mathbf{T}_{22} & \mathbf{T}_{23} \\ \mathbf{T}_{30} & \mathbf{T}_{31} & \mathbf{T}_{32} & \mathbf{T}_{33} \end{bmatrix},$$

where each $\mathbf{T}_{jk}$ is a bounded operator on ℓ^2. Suppose $T\mathcal{F}_{\cos} = \mathcal{F}_{\cos}T$. Then, in terms of block matrix multiplication,

$$\mathbf{T}\mathcal{F}_{\cos} = \begin{bmatrix} \mathbf{T}_{00} & \mathbf{T}_{01} & \mathbf{T}_{02} & \mathbf{T}_{03} \\ \mathbf{T}_{10} & \mathbf{T}_{11} & \mathbf{T}_{12} & \mathbf{T}_{13} \\ \mathbf{T}_{20} & \mathbf{T}_{21} & \mathbf{T}_{22} & \mathbf{T}_{23} \\ \mathbf{T}_{30} & \mathbf{T}_{31} & \mathbf{T}_{32} & \mathbf{T}_{33} \end{bmatrix} \begin{bmatrix} \mathbf{I} & & & \\ & \mathbf{0} & & \\ & & -\mathbf{I} & \\ & & & \mathbf{0} \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{T}_{00} & \mathbf{0} & -\mathbf{T}_{02} & \mathbf{0} \\ \mathbf{T}_{10} & \mathbf{0} & -\mathbf{T}_{12} & \mathbf{0} \\ \mathbf{T}_{20} & \mathbf{0} & -\mathbf{T}_{22} & \mathbf{0} \\ \mathbf{T}_{30} & \mathbf{0} & -\mathbf{T}_{32} & \mathbf{0} \end{bmatrix}.$$

Similarly,

$$\mathcal{F}_{\cos}\mathbf{T} = \begin{bmatrix} \mathbf{T}_{00} & \mathbf{T}_{01} & \mathbf{T}_{02} & \mathbf{T}_{03} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{T}_{20} & -\mathbf{T}_{21} & -\mathbf{T}_{22} & -\mathbf{T}_{23} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

We see here that all but the diagonal elements and the $\mathbf{T}_{13}$, $\mathbf{T}_{11}$, $\mathbf{T}_{32}$, and $\mathbf{T}_{33}$ are non-zero. Thus,

$$\mathbf{T} = \begin{bmatrix} \mathbf{T}_{00} & & & \\ & \mathbf{T}_{11} & & \mathbf{T}_{13} \\ & & \mathbf{T}_{22} & \\ & \mathbf{T}_{31} & & \mathbf{T}_{33} \end{bmatrix},$$

omitting zeros.

Suppose

$$\mathbf{T} = \begin{bmatrix} \mathbf{T}_{00} & & & \\ & \mathbf{T}_{11} & & \mathbf{T}_{13} \\ & & \mathbf{T}_{22} & \\ & \mathbf{T}_{31} & & \mathbf{T}_{33} \end{bmatrix},$$

omitting zeros. Observe

$$
\begin{aligned}
\mathbf{T}\mathcal{F}_{\cos} &= \begin{bmatrix} \mathbf{T}_{00} & & & \\ & \mathbf{T}_{11} & & \mathbf{T}_{13} \\ & & \mathbf{T}_{22} & \\ & \mathbf{T}_{31} & & \mathbf{T}_{33} \end{bmatrix} \begin{bmatrix} \mathbf{I} & & & \\ & \mathbf{0} & & \\ & & -\mathbf{I} & \\ & & & \mathbf{0} \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{T}_{00} & & & \\ & \mathbf{0} & & \\ & & -\mathbf{T}_{22} & \\ & & & \mathbf{0} \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{I} & & & \\ & \mathbf{0} & & \\ & & -\mathbf{I} & \\ & & & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{T}_{00} & & & \\ & \mathbf{T}_{11} & & \mathbf{T}_{13} \\ & & \mathbf{T}_{22} & \\ & \mathbf{T}_{31} & & \mathbf{T}_{33} \end{bmatrix} \\
&= \mathcal{F}_{\cos}\mathbf{T}.
\end{aligned}
$$

Thus, $\mathbf{T} \in \{\mathcal{F}_{\cos}\}'$. A similar proof will show $\{\mathcal{F}_{\sin}\}'$ is as described. □

We will revisit the Fourier cosine and sine transforms in Chapter 6 when discussing their relative invariant subspaces.

4.6. von Neumann's Double Commutant Theorem

We have characterized $\{\mathcal{F}\}'$; what about $\{\mathcal{F}\}''$, $\{\mathcal{F}\}'''$, $\{\mathcal{F}\}''''$, etc.? Thankfully, this process terminates (quickly) via the double commutant theorem.

First, let us mention that $\{\mathcal{F}\}''' = \{\mathcal{F}\}'$, $\{\mathcal{F}\}'''' = \{\mathcal{F}\}''$, etc. The proof of this claim will come after we have determined $\{\mathcal{F}\}''$.

We now state von Neumann's double commutant theorem, and in a moment we will prove it in the special case of the Fourier transform. Note that all matrix representations of operators will be in the notation of Section 4.4.

Theorem 4.6.1 (von Neumann's Double Commutant Theorem **[18]**). *Let $\mathcal{A}$ denote a subset of $\mathcal{B}(\mathcal{H})$ such that $\mathcal{A}$ is closed under addition, scalar multiplication, operator composition, adjoints, and contains the identity operator. Then*

$$\mathcal{A}'' = \{T \in \mathcal{B}(\mathcal{H}) : TR = RT \quad \forall R \in \{A\}'\}$$

is equal to the strong operator closure of $\mathcal{A}$.

The set $\mathcal{A}''$ is known as the *double commutant* of $\mathcal{A}$. We will not get into the meaning of the term "strong operator closure" since we will not need it for the discussion below.

In the following, we formulate and prove a version of the double commutant theorem for the Fourier transform. Consider the set

$$\mathcal{A}(\mathcal{F}) := \{a_0 I + a_1\mathcal{F} + a_2\mathcal{F}^2 + a_3\mathcal{F}^3 : a_j \in \mathbb{C}, 0 \leqslant j \leqslant 3\}.$$

Using Corollary 3.2.4 and the fact that $\mathcal{F}\mathcal{F}^* = \mathcal{F}^*\mathcal{F} = I$, it follows that $\mathcal{A}(\mathcal{F})$ is closed under addition, scalar multiplication, operator composition, and adjoints. Since $\mathcal{A}(\mathcal{F})$ is a finite dimensional vector space, it is also closed in the strong operator topology.

Before stating the double commutant of $\mathcal{F}$, we establish the following helpful lemma.

Lemma 4.6.2. *Let* $\mathbf{T} \in \mathcal{B}(\ell^2)$*. If* $\mathbf{TA} = \mathbf{AT}$ *for all* $\mathbf{A} \in \mathcal{B}(\ell^2)$*, then* $\mathbf{T} = a\mathbf{I}$ *for some* $a \in \mathbb{C}$*.*

Proof. Let $\mathbf{T} \in \mathcal{B}(\ell^2)$ for $\mathbf{T} = [t_{ij}]_{i,j=1}^{\infty}$. Suppose $\mathbf{TA} = \mathbf{AT}$ for all $\mathbf{A} \in \mathcal{B}(\ell^2)$. Then $\mathbf{T}$ commutes with

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

We then have

$$\mathbf{TA} = \mathbf{AT}$$

$$\begin{bmatrix} t_{11} & t_{12} & t_{13} & \dots \\ t_{21} & t_{22} & t_{23} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} t_{11} & t_{12} & t_{13} & \dots \\ t_{21} & t_{22} & t_{23} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$\begin{bmatrix} t_{11} & 0 & 0 & \dots \\ t_{21} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} & t_{13} & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

so that all of $t_{1j}, t_{j1} = 0$ for $j > 1$. Similarly, we have that $\mathbf{T}$ commutes with

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Following the computation above, we arrive at

$$\begin{bmatrix} 0 & 0 & 0 & \dots \\ 0 & t_{22} & 0 & \dots \\ 0 & t_{32} & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \dots \\ 0 & t_{22} & t_{23} & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

so that all of $t_{j2}, t_{2j} = 0$ for $j > 2$. Continuing for more basis vectors (e_n) shows that the nonzero entries of $\mathbf{T}$ lie on the main diagonal, such that

$$\mathbf{T} = \begin{bmatrix} t_1 & 0 & 0 & \dots \\ 0 & t_2 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

We now consider

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

For $\mathbf{TA} = \mathbf{AT}$, we arrive at

$$\begin{bmatrix} 0 & t_1 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} 0 & t_2 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

so that $t_1 = t_2$. For

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

we have

$$\begin{bmatrix} 0 & 0 & t_1 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} 0 & 0 & t_3 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

so that $t_1 = t_2 = t_3$. Continuing by shifting e_1 one position to the right, we find that all diagonal entries are equal. Then $\mathbf{T} = a\mathbf{I}$ for some $a \in \mathbb{C}$. $\square$

Theorem 4.6.3 (Double Commutant of $\mathcal{F}$). $\{\mathcal{F}\}'' = \mathcal{A}(\mathcal{F})$.

Proof. By the discussion at the beginning of this chapter, $\{\mathcal{F}\}'$ is a $*$-closed algebra of operators. Additionally, we have that in block matrix form with respect to the decomposition of $L^2(\mathbb{R})$ into the four basic eigenspaces for $\mathcal{F}$, as before, an arbitrary element of $\mathcal{A}(\mathcal{F})$ is equal to the sum of

$$a_0\mathbf{I} = \begin{bmatrix} a_0\mathbf{I} & & & \\ & a_0\mathbf{I} & & \\ & & a_0\mathbf{I} & \\ & & & a_0\mathbf{I} \end{bmatrix}$$

$$a_1\mathcal{F} = \begin{bmatrix} a_1\mathbf{I} & & & \\ & -ia_1\mathbf{I} & & \\ & & -a_1\mathbf{I} & \\ & & & ia_1\mathbf{I} \end{bmatrix}$$

$$a_2\mathcal{F}^2 = \begin{bmatrix} a_2\mathbf{I} & & & \\ & -a_2\mathbf{I} & & \\ & & a_2\mathbf{I} & \\ & & & -a_2\mathbf{I} \end{bmatrix}$$

and

$$a_3\mathcal{F}^3 = \begin{bmatrix} a_3\mathbf{I} & & & \\ & ia_3\mathbf{I} & & \\ & & -a_3\mathbf{I} & \\ & & & -ia_3\mathbf{I} \end{bmatrix}.$$

We now want to show that $\mathcal{A}(\mathcal{F})$ is the set of matrices of the form

$$\mathcal{A}(\mathcal{F}) = \begin{bmatrix} c_0\mathbf{I} & & & \\ & c_1\mathbf{I} & & \\ & & c_2\mathbf{I} & \\ & & & c_3\mathbf{I} \end{bmatrix}$$

for arbitrary $c_0, c_1, c_2, c_3 \in \mathbb{C}$. Equating the 5 above equations yields the 4×4 system

$$\begin{aligned} c_0 &= a_0 + a_1 + a_2 + a_3 \\ c_1 &= a_0 - ia_1 - a_2 + ia_3 \\ c_2 &= a_0 - a_1 + a_2 - a_3 \\ c_3 &= a_0 + ia_1 - a_2 - ia_3 \end{aligned}$$

which we can write in matrix representation by

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

A quick computation shows the determinant of the above complex matrix is $16i$, and thus this system has a unique solution for every c_0, c_1, c_2, c_3.

We claim that $\mathcal{A}(\mathcal{F})' = \{\mathcal{F}\}'$. To see this, first let $T \in \{\mathcal{F}\}'$. Then

$$\mathbf{T} = \begin{bmatrix} \mathbf{T}_1 & & & \\ & \mathbf{T}_2 & & \\ & & \mathbf{T}_3 & \\ & & & \mathbf{T}_4 \end{bmatrix}$$

by Theorem 4.4.2. Then $T \in \mathcal{A}(\mathcal{F})'$ (since every element of $\mathcal{A}(\mathcal{F})$ is a linear combination of powers of $\mathcal{F}$), so $\{\mathcal{F}\}' \subseteq \mathcal{A}(\mathcal{F})'$. Now let $T \in \mathcal{A}(\mathcal{F})'$. Then $T\mathcal{F} = \mathcal{F}T$ as $\mathcal{F} \in \mathcal{A}(\mathcal{F})$ so $T \in \{\mathcal{F}\}'$. Thus $\mathcal{A}(\mathcal{F})' \subseteq \{\mathcal{F}\}'$ and we conclude $\mathcal{A}(\mathcal{F})' = \{\mathcal{F}\}'$.

We now want to examine $\{\mathcal{F}\}''$. Let $T \in \{\mathcal{F}\}''$. Then, since $\mathcal{F} \in \{\mathcal{F}\}'$, T commutes with $\mathcal{F}$ and it follows from Theorem 4.4.2 that T has the form

$$\mathbf{T} = \begin{bmatrix} \mathbf{T}_1 & & & \\ & \mathbf{T}_2 & & \\ & & \mathbf{T}_3 & \\ & & & \mathbf{T}_4 \end{bmatrix}.$$

However, we also have that T commutes with all operators of the form

$$\begin{bmatrix} \mathbf{A} & & & \\ & \mathbf{B} & & \\ & & \mathbf{C} & \\ & & & \mathbf{D} \end{bmatrix}$$

by Theorem 4.4.2 and so $\mathbf{T}_1\mathbf{A} = \mathbf{A}\mathbf{T}_1$ for all $A \in \mathcal{B}(L^2(\mathbb{R}))$. Thus, it must be the case that $T_1 = aI$ for some $a \in \mathbb{C}$ by Lemma 4.6.2. A similar computation shows this result holds for all T_j for $1 \leqslant j \leqslant 4$. In other words,

$$\mathbf{T} = \begin{bmatrix} a\mathbf{I} & & & \\ & b\mathbf{I} & & \\ & & c\mathbf{I} & \\ & & & d\mathbf{I} \end{bmatrix}$$

for some $a, b, c, d \in \mathbb{C}$.

We claim that $\{\mathcal{F}\}'' = \mathcal{A}(\mathcal{F})$. To show this, let $T \in \{\mathcal{F}\}''$. Then for T as described above, we have $T \in \mathcal{A}(\mathcal{F})$. Conversely, suppose that $T \in \mathcal{A}(\mathcal{F})$. Then

$$\mathbf{T} = \begin{bmatrix} a\mathbf{I} & & & \\ & b\mathbf{I} & & \\ & & c\mathbf{I} & \\ & & & d\mathbf{I} \end{bmatrix}$$

for $a, b, c, d \in \mathbb{C}$. If $S \in \{\mathcal{F}\}'$, then by Theorem 4.4.2

$$\mathbf{S} = \begin{bmatrix} \mathbf{A} & & & \\ & \mathbf{B} & & \\ & & \mathbf{C} & \\ & & & \mathbf{D} \end{bmatrix}$$

and thus $\mathbf{TS} = \mathbf{ST}$ via block matrix multiplication. Then $T \in \{\mathcal{F}\}''$. Thus, $\{\mathcal{F}\}'' = \mathcal{A}(\mathcal{F})$ by Theorem 4.6.1. □

We can now prove that $\{\mathcal{F}\}''' = \{\mathcal{F}\}'$, and thus $\{\mathcal{F}\}'''' = \{\mathcal{F}\}''$, etc. To see this, let $T \in \{\mathcal{F}\}'$. Then

$$\mathbf{T} = \begin{bmatrix} \mathbf{T}_1 & & & \\ & \mathbf{T}_2 & & \\ & & \mathbf{T}_3 & \\ & & & \mathbf{T}_4 \end{bmatrix}$$

by Theorem 4.4.2. We wish to show that $T \in \{\mathcal{F}\}'''$. In other words, we must show $TA = AT$ for all $A \in \{\mathcal{F}\}''$. Let $A \in \{\mathcal{F}\}''$. Then by Theorem 4.6.3, we have that the matrix representation of A has the form

$$\mathbf{A} = \sum_{j=0}^{3} c_j \begin{bmatrix} \mathbf{I} & & & \\ & -i\mathbf{I} & & \\ & & -\mathbf{I} & \\ & & & i\mathbf{I} \end{bmatrix}^j .$$

Then

$$\mathbf{TA} = \begin{bmatrix} \mathbf{T}_1 & & & \\ & \mathbf{T}_2 & & \\ & & \mathbf{T}_3 & \\ & & & \mathbf{T}_4 \end{bmatrix} \begin{bmatrix} \mathbf{I} & & & \\ & -i\mathbf{I} & & \\ & & -\mathbf{I} & \\ & & & i\mathbf{I} \end{bmatrix}^j = \mathbf{AT}$$

as diagonal matrices commute. Then $T \in \{\mathcal{F}\}'''$. Now, suppose $T \in \{\mathcal{F}\}'''$. Then $TA = AT$ for all $A \in \{\mathcal{F}\}''$. However, we have that $\mathcal{F} \in \{\mathcal{F}\}''$, so that $T\mathcal{F} = \mathcal{F}T$. Thus, $T \in \{\mathcal{F}\}'$. We can now conclude $\{\mathcal{F}\}''' = \{\mathcal{F}\}'$. Then $\{\mathcal{F}\}''''$ will contain all bounded operators that commute with $\{\mathcal{F}\}''' = \{\mathcal{F}\}'$, which is exactly $\{\mathcal{F}\}''$. Thus, $\{\mathcal{F}\}'''' = \{\mathcal{F}\}'' = \mathcal{A}(\mathcal{F})$.

Chapter 5

Square Roots

We will now use our description of $\{\mathcal{F}\}'$ as in Section 4.4 to discuss the square roots of $\mathcal{F}$.

5.1. Main Theorem

We begin with a definition of the set of square roots of the Fourier transform.

Definition 5.1.1. $\sqrt{\mathcal{F}} = \{T \in \mathcal{B}(L^2(\mathbb{R})) : T^2 = \mathcal{F}\}$.

Of course, as it stands now, this set may be empty. We will establish that this is far from the truth. We can use our knowledge of the commutant of $\mathcal{F}$ to claim the following.

Proposition 5.1.2. *If $T \in \sqrt{\mathcal{F}}$, then $T \in \{\mathcal{F}\}'$.*

Proof. Let $T \in \sqrt{\mathcal{F}}$. Then $T\mathcal{F} = TT^2 = T^2T = \mathcal{F}T$. Thus, $T \in \{\mathcal{F}\}'$. □

An obvious example of an element of $\sqrt{\mathcal{F}}$ is given below. Recall that we represent $\mathcal{F}$ with respect to the decomposition of $L^2(\mathbb{R})$ into the four eigenspaces of $\mathcal{F}$.

Example 5.1.3. If we represent the Fourier transform as

$$\mathbf{F} = \begin{bmatrix} \mathbf{I} & & & \\ & -i\mathbf{I} & & \\ & & -\mathbf{I} & \\ & & & i\mathbf{I} \end{bmatrix},$$

then

$$\mathbf{T} = \begin{bmatrix} \pm\mathbf{I} & & & \\ & \pm\sqrt{-i}\mathbf{I} & & \\ & & \pm i\mathbf{I} & \\ & & & \pm\sqrt{i}\mathbf{I} \end{bmatrix}$$

are elements of $\sqrt{\mathcal{F}}$.

To produce a broader range of examples of square roots, we remind the reader of two important types of operators. For $A \in \mathcal{B}(\mathcal{H})$, A is *involutary* if $A^2 = I$. Additionally, for $P \in \mathcal{B}(\mathcal{H})$, P is *idempotent* if $P^2 = P$. This leads us to the following theorem.

Theorem 5.1.4. *$A \in \mathcal{B}(\mathcal{H})$ is involutary if and only if $A = \pm(I - 2P)$ for some idempotent $P \in \mathcal{B}(\mathcal{H})$.*

Proof. Let P be a bounded linear operator such that $P^2 = P$. Suppose A is a linear operator such that $A = I - 2P$. Observe that

$$\begin{aligned} A^2 &= (I - 2P)^2 \\ &= I^2 - 4P + 4P^2 \\ &= I - 4P + 4P \\ &= I. \end{aligned}$$

Thus, A is involutary.

Suppose $A = -I + 2P$. Then

$$\begin{aligned} A^2 &= (-I + 2P)^2 \\ &= I^2 - 4P + 4P^2 \\ &= I - 4P + 4P \\ &= I. \end{aligned}$$

Thus, A is involutary.

For the other direction, suppose $A^2 = I$. Let $P = \frac{1}{2}(I - A)$. Then

$$P^2 = \frac{1}{4}(I - 2A + A^2) = \frac{1}{4}(2I - 2A) = \frac{1}{2}(I - A) = P,$$

thus P is idempotent and $A = I - 2P$. Let $P = \frac{1}{2}(A + I)$. Then

$$P^2 = \frac{1}{4}(A^2 + 2A + I^2) = \frac{1}{4}(2A + 2I) = \frac{1}{2}(A + I) = P,$$

thus P is idempotent and $A = -I + 2P$. □

In the matrix representation of $\mathcal{F}$, we claim the following theorem.

Theorem 5.1.5 (Square Roots of $\mathcal{F}$)**.**

$$\sqrt{\mathcal{F}} = \left\{ \begin{bmatrix} \mathbf{T}_1 & & & \\ & \sqrt{i}\mathbf{T}_2 & & \\ & & i\mathbf{T}_3 & \\ & & & \sqrt{-i}\mathbf{T}_4 \end{bmatrix} \right\}$$

such that $\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3, \mathbf{T}_4 \in \mathcal{B}(\ell^2)$ *are involutary.*

Proof. Let

$$\mathbf{T} = \begin{bmatrix} \mathbf{T}_1 & & & \\ & \sqrt{i}\mathbf{T}_2 & & \\ & & i\mathbf{T}_3 & \\ & & & \sqrt{-i}\mathbf{T}_4 \end{bmatrix}.$$

By block multiplication, we have

$$\mathbf{T}^2 = \begin{bmatrix} \mathbf{T}_1^2 & & & \\ & -i\mathbf{T}_2^2 & & \\ & & -\mathbf{T}_3^2 & \\ & & & i\mathbf{T}_4^2 \end{bmatrix} = \begin{bmatrix} \mathbf{I} & & & \\ & -i\mathbf{I} & & \\ & & -\mathbf{I} & \\ & & & i\mathbf{I} \end{bmatrix} = \mathcal{F}.$$

Thus, $T \in \sqrt{F}$.

Suppose $T \in \sqrt{\mathcal{F}}$. Then $T \in \{\mathcal{F}\}'$ by Proposition 5.1.2 and has the form

$$\mathbf{T} = \begin{bmatrix} \mathbf{A} & & & \\ & \mathbf{B} & & \\ & & \mathbf{C} & \\ & & & \mathbf{D} \end{bmatrix}$$

by Theorem 4.4.2. Squaring the above matrix yields

$$\mathbf{T}^2 = \begin{bmatrix} \mathbf{A}^2 & & & \\ & \mathbf{B}^2 & & \\ & & \mathbf{C}^2 & \\ & & & \mathbf{D}^2 \end{bmatrix} = \begin{bmatrix} \mathbf{I} & & & \\ & -i\mathbf{I} & & \\ & & -\mathbf{I} & \\ & & & i\mathbf{I} \end{bmatrix}.$$

Then

$$\mathbf{T} = \begin{bmatrix} \mathbf{A}' & & & \\ & \sqrt{-i}\mathbf{B}' & & \\ & & i\mathbf{C}' & \\ & & & \sqrt{i}\mathbf{D}' \end{bmatrix}$$

where $A'^2, B'^2, C'^2, D'^2 = I$. Matrices with this property are involutary, and the theorem follows. □

The remainder of this section contains a few interesting examples as a result of Theorem 5.1.5.

Example 5.1.6. First, note that for all $z \in \mathbb{C}$,

$$\begin{bmatrix} 0 & z \\ z & 0 \end{bmatrix} \begin{bmatrix} 0 & z \\ z & 0 \end{bmatrix} = \begin{bmatrix} z^2 & 0 \\ 0 & z^2 \end{bmatrix}.$$

Then

$$\mathbf{T} = \begin{bmatrix} \mathbf{0} & \mathbf{I} & & & & & & \\ \mathbf{I} & \mathbf{0} & & & & & & \\ & & \mathbf{0} & \sqrt{-i}\mathbf{I} & & & & \\ & & \sqrt{-i}\mathbf{I} & \mathbf{0} & & & & \\ & & & & \mathbf{0} & i\mathbf{I} & & \\ & & & & i\mathbf{I} & \mathbf{0} & & \\ & & & & & & \mathbf{0} & \sqrt{i}\mathbf{I} \\ & & & & & & \sqrt{i}\mathbf{I} & \mathbf{0} \end{bmatrix}$$

satisfies $\mathbf{T}^2 = \mathcal{F}$.

Example 5.1.7. An easy calculation shows that

$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix}^2 = \mathbf{I}$$

if and only if $a^2 + bc = 1$. Then

$$\mathbf{T} = \begin{bmatrix} \frac{1}{2}\mathbf{I} & \frac{1}{4}\mathbf{I} & & & & & & \\ 3\mathbf{I} & -\frac{1}{2}\mathbf{I} & & & & & & \\ & & \frac{\sqrt{-i}}{2}\mathbf{I} & \frac{\sqrt{-i}}{4}\mathbf{I} & & & & \\ & & 3\sqrt{-i}\mathbf{I} & -\frac{\sqrt{-i}}{2}\mathbf{I} & & & & \\ & & & & \frac{i}{2}\mathbf{I} & \frac{i}{4}\mathbf{I} & & \\ & & & & 3i\mathbf{I} & -\frac{i}{2}\mathbf{I} & & \\ & & & & & & \frac{\sqrt{i}}{2}\mathbf{I} & \frac{\sqrt{i}}{4}\mathbf{I} \\ & & & & & & 3\sqrt{i}\mathbf{I} & -\frac{\sqrt{i}}{2}\mathbf{I} \end{bmatrix}$$

satisfies $\mathbf{T}^2 = \mathcal{F}$.

5.2. Mehler's Formula

So far, the reader might be unsatisfied with the fact that we are using block matrices to describe the square root of an integral operator. In this section, we will find an integral formula for one of the square roots of $\mathcal{F}$, namely

$$\left[\begin{array}{cccc|cccc|c} 1 & & & & & & & & \\ & e^{-i\frac{\pi}{4}} & & & & & & & \\ & & i & & & & & & \\ & & & e^{i\frac{\pi}{4}} & & & & & \\ \hline & & & & 1 & & & & \\ & & & & & e^{-i\frac{\pi}{4}} & & & \\ & & & & & & i & & \\ & & & & & & & e^{i\frac{\pi}{4}} & \\ & & & & & & & & \ddots \end{array}\right].$$

This involves a formula of Mehler. In 1866, Mehler [**10**] developed a formula that later came to be known as *Mehler's Formula*. This formula has been used in quantum physics, harmonic analysis, and probability theory. We note that we have adapted Mehler's formula to the form of the Hermite polynomials used in this thesis.

Definition 5.2.1. For the Hermite polynomials $(H_n)_{n\geqslant 0}$ defined in Chapter 2, define the *Mehler kernel*

$$\sum_{n=0}^{\infty} \frac{\rho^n n!}{(4\pi)^n} H_n(s)H_n(t).$$

We are being a bit vague about the allowable range of ρ. The following theorem will make this range clear.

Theorem 5.2.2 (Mehler 1866)**.**

$$\sum_{n=0}^{\infty} \frac{e^{-\pi(x^2+y^2)}n!}{(4\pi)^n\sqrt{\pi}} \rho^n H_n(x)H_n(y) = \frac{1}{\sqrt{\pi(1-\rho^2)}} e^{\frac{\pi(4\rho xy-(x^2+y^2)(1+\rho^2))}{1-\rho^2}}$$

for $|\rho| \leqslant 1$, $\rho \neq 1, -1$.

Proof. The following proof is taken from G.H. Hardy [**19**]. To begin, we will invoke Proposition 3.1.1 (ii) to obtain

$$e^{-2\pi x^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2+2\sqrt{2\pi}ixu} du. \tag{5.2.3}$$

Differentiating this equation n times gives the known result

$$H_n(x) = \frac{(-2\sqrt{2\pi}i)^n e^{2\pi x^2}}{\sqrt{\pi}n!} \int_{-\infty}^{\infty} u^n e^{-u^2+2\sqrt{2\pi}ixu} du.$$

Substituting this expression for H_n into the left-hand side of Theorem 5.2.2 shows that

$$\sum_{n=0}^{\infty} \frac{e^{-\pi(x^2+y^2)}n!}{(4\pi)^n\sqrt{\pi}} \rho^n H_n(x)H_n(y)$$

is equal to

$$\frac{e^{\pi(x^2+y^2)}}{\pi^{3/2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\sum_{n=0}^{\infty} \frac{(-2\rho uv)^n}{n!}\right] e^{-u^2+2\sqrt{2\pi}ixu-v^2+2\sqrt{2\pi}iyv} dudv.$$

Using the exponential form of the infinite sum gives that the above is equal to

$$\frac{e^{\pi(x^2+y^2)}}{\pi^{3/2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-u^2+2\sqrt{2\pi}ixu-2\rho uv-v^2+2\sqrt{2\pi}iyv} dudv. \tag{5.2.4}$$

Performing the integral in u first, we have

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-u^2+2\sqrt{2\pi}ixu-2\rho uv} du &= \int_{-\infty}^{\infty} e^{-u^2+2\sqrt{2\pi}i(x-\frac{1}{\sqrt{2\pi}i}\rho v)u} du \\ &= \sqrt{\pi} e^{-2\pi(x-\frac{1}{\sqrt{2\pi}i}\rho v)^2} \\ &= \sqrt{\pi} e^{-2\pi(x^2+i\sqrt{\frac{2}{\pi}}x\rho v-\frac{1}{2\pi}\rho^2 v^2)}\end{aligned}$$

by equation (5.2.3). Substituting this into equation (5.2.4) and simplifying gives

$$\frac{e^{\pi(y^2-x^2)}}{\pi}\int_{-\infty}^{\infty} e^{-(1-\rho^2)v^2+2\sqrt{2\pi}i(y-x\rho)v}dv.$$

We will solve this integral using substitution. Let $s^2 = (1-\rho^2)v^2$ such that $ds = \sqrt{1-\rho^2}dv$. We then have

$$\frac{e^{\pi(y^2-x^2)}}{\pi\sqrt{1-\rho^2}}\int_{-\infty}^{\infty} e^{-s^2+2\sqrt{2\pi}i\left(\frac{y-x\rho}{\sqrt{1-\rho^2}}\right)s}dv = \frac{e^{\pi(y^2-x^2)}}{\sqrt{\pi(1-\rho^2)}}e^{-2\pi\left(\frac{y-x\rho}{\sqrt{1-\rho^2}}\right)^2}$$

after another application of equation (5.2.3). Simplifying the above exponentials gives

$$\frac{1}{\sqrt{\pi(1-\rho^2)}}e^{\frac{\pi(4\rho xy-(x^2+y^2)(1+\rho^2))}{1-\rho^2}}$$

and the theorem follows. □

Simplification of Theorem 5.2.2 gives an expression in terms of Definition 5.2.1 so that

$$\sum_{n=0}^{\infty}\frac{\rho^n n!}{(4\pi)^n}H_n(x)H_n(y) = \frac{1}{\sqrt{1-\rho^2}}e^{\frac{\pi(4xy\rho-2(x^2+y^2)\rho^2)}{1-\rho^2}}.$$

Recall

$$h_n(x) = \left(\frac{2^{1/4}\sqrt{n!}}{(4\pi)^{n/2}}\right)H_n(x)e^{-\pi x^2}.$$

Then, expressing Definition 5.2.1 in terms of the Hermite functions yields

$$\sum_{n=0}^{\infty}\frac{\rho^n n!}{(4\pi)^n}H_n(s)H_n(t) = \frac{e^{\pi(s^2+t^2)}}{\sqrt{2}}\sum_{n=0}^{\infty}\rho^n h_n(s)h_n(t).$$

By Theorem 5.2.2, we have

$$\frac{e^{\pi(s^2+t^2)}}{\sqrt{2}}\sum_{n=0}^{\infty}\rho^n h_n(s)h_n(t) = \frac{1}{\sqrt{1-\rho^2}}e^{\frac{\pi(4st\rho-2(s^2+t^2)\rho^2)}{1-\rho^2}}$$

$$\sum_{n=0}^{\infty}\rho^n h_n(s)h_n(t) = \frac{\sqrt{2}}{\sqrt{1-\rho^2}}e^{-\pi(s^2+t^2)}e^{\frac{\pi(4st\rho-2(s^2+t^2)\rho^2)}{1-\rho^2}}.$$

Setting $\rho = -i$ yields

$$\sum_{n=0}^{\infty}(-i)^n h_n(s)h_n(t) = e^{-\pi(s^2+t^2)}e^{\frac{\pi(-4ist+2(s^2+t^2))}{2}} = e^{-2\pi ist}. \tag{5.2.5}$$

The reader may recognize this as the Fourier kernel, as expected.

We can then write
$$\sum_{n=0}^{\infty} \rho^n \langle f, h_n \rangle h_n = \frac{\sqrt{2}}{\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{\frac{\pi(4xy\rho - 2(x^2+y^2)\rho^2)}{1-\rho^2}} e^{-\pi(x^2+y^2)} f(y)dy.$$

Recall that the Fourier transform of a function $f \in L^2(\mathbb{R})$ can be written in its eigenvalue expansion as
$$\mathcal{F}f = \sum_{n=0}^{\infty} (-i)^n \langle f, h_n \rangle h_n.$$
Expressing the inner product in its integral form shows
$$\begin{aligned} \mathcal{F}f &= \int_{-\infty}^{\infty} \left[\sum_{n=0}^{\infty} (-i)^n h_n(x) h_n(y) \right] f(y)dy \\ &= \int_{-\infty}^{\infty} e^{-2\pi ixy} f(y)dy \end{aligned}$$
from equation (5.2.5), as expected.

One natural square root is $\rho = \sqrt{-i} = e^{-i\frac{\pi}{4}}$ such that
$$\sum_{n=0}^{\infty} e^{\frac{-in\pi}{4}} h_n(x) \int_{-\infty}^{\infty} f(y) h_n(y) dy$$
is equal to
$$\frac{\sqrt{2}}{\sqrt{1+i}} \int_{-\infty}^{\infty} e^{\frac{\pi(4xye^{-i(\pi/4)} + 2(x^2+y^2)i)}{1+i}} e^{-\pi(x^2+y^2)} f(y)dy.$$
The previous line is an integral expression for a square root of $\mathcal{F}$. Technically, this integral is not defined for all $f \in L^2(\mathbb{R})$, but certainely for $f \in \mathcal{S}$. This integral expression of a square root was given in its matrix representation at the beginning of this chapter. This natural square root is simply the square root of the entries in
$$\mathbf{F} = \left[\begin{array}{cccc|c} 1 & & & & \\ & -i & & & \\ & & -1 & & \\ & & & i & \\ \hline & & & & \ddots \end{array} \right],$$
where $i = e^{i\frac{\pi}{2}}$ expressed in polar form. This is exactly the result of Example 5.1.3 with all positive entries on the diagonal (note that this example was written in the notation of Chapter 3).

Chapter 6

Invariant Subspaces

The invariant subspace problem, first posed in the mid-1900s by von Neumann, is an unresolved problem which asks the following: given a $T \in \mathcal{B}(\mathcal{H})$ where $\dim(\mathcal{H}) \geqslant 2$, does there exists a (closed) subspace M of $\mathcal{H}$ where $M \neq \{0\}$ and $M \neq \mathcal{H}$ such that $TM \subseteq M$? For a finite dimensional Hilbert space, we know that for any $T \in \mathcal{B}(\mathcal{H})$, the eigenspace of T, $\mathcal{E}_\lambda = \{x \in \mathcal{H} : Tx = \lambda x\}$ is an invariant subspace. An advanced version of the spectral theorem gives us that, for an infinite dimensional Hilbert space, unitary operators always have invariant subspaces. We thus pose this question of the Fourier transform.

To show the complexity of the invariant subspace problem, Everett Bishop posed a class of bounded operators on $L^2[0,1]$ which are possible candidates for operators *without* non-trivial invariant subspaces. Consider $T_\alpha : L^2[0,1] \to L^2[0,1]$ defined by

$$(T_\alpha f)(x) = xf(\{x + \alpha\})$$

for $\alpha \in \mathbb{R}$. In the above, $\{x + \alpha\}$ is the fractional part of $x + \alpha$. Various authors **[2, 4, 9]** were able to show that for many α, T_α has non-trivial invariant subspaces. For $\alpha \in \mathbb{Q}$, Parrott **[11]** characterized all of the invariant subspaces of T_α. For certain "highly irrational" α, it is unknown whether T_α has non-trivial invariant subspaces.

Throughout this chapter we will be dealing with subspaces of $L^2(\mathbb{R})$. These are vector subspaces of $L^2(\mathbb{R})$ that are also topologically closed. In this chapter we are faced with the opposite of the Bishop problem, where the invariant subspaces seem to be sparse. On the other hand, the Fourier transform has a very rich class of invariant subspaces, and we wish to describe them all.

6.1. Basic Facts

We begin with a definition of an invariant subspace of the Fourier transform.

Definition 6.1.1. A (closed) subspace $M \subseteq L^2(\mathbb{R})$ is *invariant* for $\mathcal{F}$ if $\mathcal{F}M \subseteq M$.

Example 6.1.2. Recall the Hermite functions $(h_n)_{n\geqslant 0}$ defined in Chapter 2. Set

$$M_A = \{f \in L^2(\mathbb{R}) : \langle f, h_n \rangle = 0 \,\forall n \in A\}$$

where $A \subseteq \mathbb{N}_0$.

We will first show that M_A is closed. Suppose $(f_k)_{k\geqslant 0}$ is a Cauchy sequence in M_A. Since $M_A \subseteq L^2(\mathbb{R})$ and $L^2(\mathbb{R})$ is a Hilbert space, then there exists some $f \in L^2(\mathbb{R})$ such that $f_k \to f$. We will now show $f \in M_A$. Observe that for every $n \in A$,

$$\begin{aligned}\langle f, h_n \rangle &= \langle f, h_n \rangle - \langle f_k, h_n \rangle \\ &= \langle f - f_k, h_n \rangle.\end{aligned}$$

Then, by the Cauchy-Schwarz inequality,

$$|\langle f - f_k, h_n \rangle| \leqslant \|f - f_k\| \cdot 1.$$

Note that $\|f - f_k\| \to 0$, so $\langle f, h_n \rangle = 0$ for all $n \in A$, so M_A is closed.

Next we prove that M_A is an invariant subspace for $\mathcal{F}$. For any $f \in M_A$, the fact that $\mathcal{F}$ is unitary gives

$$\begin{aligned}\langle f, h_n \rangle &= \langle \mathcal{F}f, \mathcal{F}h_n \rangle \\ &= \langle \mathcal{F}f, (-i)^n h_n \rangle \\ &= i^n \langle \mathcal{F}f, h_n \rangle.\end{aligned}$$

Then $\langle \mathcal{F}f, h_n \rangle = 0$ for all $n \in A$ (since $\langle f, h_n \rangle = 0$), and $\mathcal{F}M_A \subseteq M_A$. Then M_A is an invariant subspace.

6.2. Invariant versus Reducing

A somewhat stronger criteria of a subspace is called a reducing subspace, defined below.

Definition 6.2.1. A (closed) subspace $M \subseteq L^2(\mathbb{R})$ is *reducing* for $\mathcal{F}$ if $\mathcal{F}M \subseteq M$ and $\mathcal{F}^*M \subseteq M$.

Note how this definition is equivalent to $\mathcal{F}M = M$ as $\mathcal{F}$ is a unitary operator.

Example 6.2.2. Let's revisit the subspace M_A as defined in Example 6.1.2. Let $f \in M_A$ be such that $\langle f, h_n \rangle = 0$ for all $n \in A$. Observe that

$$\begin{aligned}\langle f, h_n \rangle &= \langle \mathcal{F}^* f, \mathcal{F}^* h_n \rangle \\ &= \langle \mathcal{F}^* f, (i)^n h_n \rangle \\ &= (-i)^n \langle \mathcal{F}^* f, h_n \rangle.\end{aligned}$$

Then $\langle \mathcal{F}^* f, h_n \rangle = 0$ and hence $\mathcal{F}^* f \in M_A$. Thus $\mathcal{F}^* M_A \subseteq M_A$. We have previously shown that $\mathcal{F} M_A \subseteq M_A$, therefore M_A is a reducing subspace of $\mathcal{F}$.

The following two theorems show that it is not entirely difficult to find reducing subspaces. To produce more (in fact, *all* of them), we must first define an *orthogonal projection*.

Definition 6.2.3. A bounded operator P on $L^2(\mathbb{R})$ is an *orthogonal projection* if $P^2 = P$ and $P^* = P$.

This is the infinite dimensional analog of a projection matrix in linear algebra defined the same way. The following result is standard in Hilbert space theory, but we give a proof anyway.

Proposition 6.2.4. *If M is a closed subspace of $L^2(\mathbb{R})$, then there exists an orthogonal projection $P : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ such that $M = PL^2(\mathbb{R})$.*

Proof. Let $(m_j)_{j \geqslant 0}$ be an orthonormal basis for M. For $f \in L^2(\mathbb{R})$, define

$$Pf = \sum_{j=0}^{\infty} \langle f, m_j \rangle m_j.$$

This will define a bounded operator on $L^2(\mathbb{R})$. Moreover, for any m_k, we have

$$Pm_k = \sum_{j=0}^{\infty} \langle m_k, m_j \rangle m_j = m_k.$$

Observe that

$$P(Pf) = P(\sum_{j=0}^{\infty} \langle f, m_j \rangle m_j) = \sum_{j=0}^{\infty} \langle f, m_j \rangle Pm_j = \sum_{j=0}^{\infty} \langle f, m_j \rangle m_j = Pf.$$

Thus, $P^2 = P$. Additionally,

$$\begin{aligned}\langle Pf, g \rangle &= \langle \sum_{j=0}^{\infty} \langle f, m_j \rangle m_j, g \rangle \\ &= \sum_{j=0}^{\infty} \langle f, m_j \rangle \langle m_j, g \rangle\end{aligned}$$

and

$$\begin{aligned}\langle f, Pg\rangle &= \langle f, \sum_{j=0}^{\infty}\langle g, m_j\rangle m_j\rangle \\ &= \sum_{j=0}^{\infty}\langle m_j, g\rangle\langle f, m_j\rangle.\end{aligned}$$

Thus, $P^* = P$. Then P is a projection that takes a function in $L^2(\mathbb{R})$ onto the subspace M so that $M = PL^2(\mathbb{R})$. □

This next result relates reducing subspaces with projections onto those subspaces.

Theorem 6.2.5. *Let M be a subspace of $L^2(\mathbb{R})$ and P be the orthogonal projection of $L^2(\mathbb{R})$ onto M. Then the following are equivalent:*

(i) $\mathcal{F}M = M$

(ii) *M is a reducing subspace of $\mathcal{F}$*

(iii) $\mathcal{F}P = P\mathcal{F}$

Proof. The proof for $(i) \iff (ii)$ is given by Definition 6.2.1 and the fact that $\mathcal{F}$ is a unitary operator. We will now show $(i) \Rightarrow (iii)$. Suppose $FM = M$. Let $f \in M$. Then

$$\mathcal{F}Pf = \mathcal{F}f.$$

Note that $\mathcal{F}f \in M$ by assumption. Then we also have

$$P\mathcal{F}f = \mathcal{F}f,$$

and so $P\mathcal{F} = \mathcal{F}P$ for $f \in M$. Let $f \in M^{\perp}$. Then $\mathcal{F}f \in M^{\perp}$ (since $\mathcal{F}$ is unitary) and since $PM^{\perp} = 0$,

$$P\mathcal{F}f = 0 = \mathcal{F}Pf.$$

Thus, $P\mathcal{F} = \mathcal{F}P$ for $f \in L^2(\mathbb{R})$.

We will now show $(iii) \Rightarrow (i)$. Suppose $P\mathcal{F} = \mathcal{F}P$. Let $f \in M$. Then

$$\mathcal{F}f = \mathcal{F}Pf = P(\mathcal{F}f),$$

so $\mathcal{F}f \in M$ and $\mathcal{F}M \subseteq M$. Additionally, operating on both sides of $\mathcal{F}P = P\mathcal{F}$ by $\mathcal{F}^*$ gives $P\mathcal{F}^* = \mathcal{F}^*P$. Then following the same argument as above, we can conclude that $\mathcal{F}^*M \subseteq M$. Then $M \subseteq \mathcal{F}M$, and we have that $\mathcal{F}M = M$. □

Theorem 6.2.6. *Every invariant subspace of $\mathcal{F}$ is also a reducing subspace.*

Proof. Suppose $\mathcal{F}M \subseteq M$. Let $f \in M$. Then $f = f_1 + f_{-i} + f_{-1} + f_i$ where each $f_j \in \ker(\mathcal{F} - jI)$. Observe that

$$\begin{aligned}\mathcal{F}^0 f &= f_1 + f_{-i} + f_{-1} + f_i \in M\\ \mathcal{F} f &= f_1 - i f_{-i} - f_{-1} + i f_i \in M\\ \mathcal{F}^2 f &= f_1 - f_{-i} + f_{-1} - f_i \in M\\ \mathcal{F}^3 f &= f_1 + i f_{-i} - f_{-1} - i f_i \in M.\end{aligned}$$

Then, taking linear combinations of these equations gives

$$\begin{aligned}(\mathcal{F}^0 + \mathcal{F} + \mathcal{F}^2 + \mathcal{F}^3) f &= 4 f_1 \in M\\ (\mathcal{F}^0 + i\mathcal{F} - \mathcal{F}^2 - i\mathcal{F}^3) f &= 4 f_{-i} \in M\\ (\mathcal{F}^0 - \mathcal{F} + \mathcal{F}^2 - \mathcal{F}^3) f &= 4 f_{-1} \in M\\ (\mathcal{F}^0 - i\mathcal{F} - \mathcal{F}^2 + i\mathcal{F}^3) f &= 4 f_i \in M.\end{aligned}$$

Thus, $f_1, f_{-i}, f_{-1}, f_i \in M$. Observe

$$\mathcal{F}^* f = f_1 + i f_{-i} - f_{-1} - i f_i \in M.$$

Therefore $\mathcal{F}^* M \subseteq M$ and M is a reducing subspace for $\mathcal{F}$. □

Theorem 6.2.6 is not a general fact for unitary operators. Consider the following example.

Example 6.2.7. This example will be of a unitary operator with an invariant but *not* reducing subspace to demonstrate the significance of Theorem 6.2.6. Let $M : L^2(\frac{d\theta}{2\pi}) \to L^2(\frac{d\theta}{2\pi})$ be defined by $(Mf)(e^{i\theta}) = e^{i\theta} f(e^{i\theta})$ for all $f \in L^2(\frac{d\theta}{2\pi})$. We will first show that M is unitary. To begin, note $(M^* f)(e^{i\theta}) = e^{-i\theta} f(e^{i\theta})$. Indeed, bserve

$$\begin{aligned}\langle Mf, g\rangle &= \int_0^{2\pi} e^{i\theta} f(e^{i\theta}) \overline{g(e^{i\theta})} \frac{d\theta}{2\pi}\\ &= \int_0^{2\pi} f(e^{i\theta}) \overline{e^{-i\theta} g(e^{i\theta})} \frac{d\theta}{2\pi}\\ &= \langle f, e^{-i\theta} g\rangle\\ &= \langle f, M^* g\rangle.\end{aligned}$$

We also have

$$M^*(Mf) = M^*(e^{i\theta} f) = e^{-i\theta} e^{i\theta} f = f$$

and

$$M(M^* f) = M(e^{-i\theta} f) = e^{i\theta} e^{-i\theta} f = f,$$

thus M is unitary.

Consider the subspace $H = \{f \in L^2(\frac{d\theta}{2\pi}) : f = \sum_{n=0}^{\infty} \widehat{f}(n) e^{in\theta}\}$, the set of functions in $L^2(\frac{d\theta}{2\pi})$ where all negative Fourier coefficients are 0. One can see that H is a linear subspace of $L^2(\frac{d\theta}{2\pi})$ and a revisiting of Example 6.1.2 will show that H is topologically closed. Applying M to H yields

$$MH = \{f \in L^2\Big(\frac{d\theta}{2\pi}\Big) : f = \sum_{n=0}^{\infty} \widehat{f}(n)e^{i(n+1)\theta}\}.$$

From this equation, one can see that M shifts all Fourier coefficients of f one position in the positive n direction. Thus $MH \subseteq H$, as all negative Fourier coefficients remain 0, and H is an invariant subspace of M. However,

$$M^*H = \{f \in L^2\Big(\frac{d\theta}{2\pi}\Big) : f = \sum_{n=0}^{\infty} \widehat{f}(n)e^{i(n-1)\theta}\}.$$

Here, we see that M^* shifts all Fourier coefficients of f one position in the negative n direction, so that $\widehat{f}(0)$ is shifted to $\widehat{f}(-1)$. Thus, $M^*H \nsubseteq H$ as it is possible that $\widehat{f}(-1)$ is nonzero. Therefore, H is not a reducing subspace.

The following theorem allows us to completely characterize the invariant (and consequently the reducing) subspaces of the Fourier transform.

Theorem 6.2.8. *For a closed subspace M of $L^2(\mathbb{R})$, the following are equivalent:*

(i) *M is invariant for $\mathcal{F}$*

(ii) *M is reducing for $\mathcal{F}$*

(iii) *$M = M_1 \oplus M_{-i} \oplus M_{-1} \oplus M_i$ where each M_z for $z = \pm 1, \pm i$ is a closed subspace of $\ker(\mathcal{F} - zI)$.*

We remind the reader that $(h_n)_{n=0}^{\infty}$ is the Hermite basis for $L^2(\mathbb{R})$.

Proof. The proof of $(i) \iff (ii)$ was given in the proof of Theorem 6.2.6 and by Definition 6.2.1.

We will now show $(iii) \Rightarrow (ii)$. Suppose $M = M_1 \oplus M_{-i} \oplus M_{-1} \oplus M_i$ where each M_z for $z = \pm 1, \pm i$ are subspaces of $\ker(\mathcal{F} - zI)$. Let $(g_j)_{j \geqslant 1}$ be an orthonormal basis for M_1. Then

$$g_j = \sum_{i=0}^{\infty} c_{ij} h_{4i}.$$

Observe

$$\begin{aligned}\mathcal{F}g_j &= \sum_{i=0}^{\infty} c_{ij} \mathcal{F} h_{4i} \\ &= \sum_{i=0}^{\infty} c_{ij} h_{4i} \\ &= g_j.\end{aligned}$$

Then for $f \in M_1$, $f = \sum_{j=0}^{\infty} a_j g_j$ and $\mathcal{F}f \in M$. Thus $\mathcal{F}\big|_{M_1} = I_{M_1}$ where I_{M_1} is the indentity operator on M_1. A similar computation can be done to show $\mathcal{F}\big|_{M_{-i}} = -iI$, $\mathcal{F}\big|_{M_{-1}} = -I$, and $\mathcal{F}\big|_{M_i} = iI$. Then $\mathcal{F}M = M$ and so M is reducing for $\mathcal{F}$.

We will now show $(ii) \Rightarrow (iii)$. Suppose $M \subseteq L^2(\mathbb{R})$ such that $\mathcal{F}M = M$. Then if $P = P_M$ is an orthonormal projection onto M, by Theorem 6.2.5 we have that $P\mathcal{F} = \mathcal{F}P$, and by Theorem 4.4.2

$$\mathbf{P} = \begin{bmatrix} \mathbf{A_0} & & & \\ & \mathbf{A_1} & & \\ & & \mathbf{A_2} & \\ & & & \mathbf{A_3} \end{bmatrix}$$

where A_n for $0 \leqslant n \leqslant 3$ are bounded operators on the respective ker$(\mathcal{F} - (-i)^n I)$. However, $P^2 = P$ and $P^* = P$ implies that $A_0^2 = A_0, A_1^2 = A_1, A_2^2 = A_2$, and $A_3^2 = A_3$, as well as $A_0^* = A_0, A_1^* = A_1, A_2^* = A_2$, and $A_3^* = A_3$. Then A_n are orthogonal projections onto the subspaces ker$(\mathcal{F} - (-i)^n I)$, respectively. Moreover, each of these subspaces are orthogonal to one another. Then

$$\begin{aligned} M = PM &= (A_0 + A_1 + A_2 + A_3)M \\ &= A_0M \oplus A_1M \oplus A_2M \oplus A_3M \\ &= M_1 \oplus M_{-i} \oplus M_{-1} \oplus M_i. \end{aligned}$$ □

Note that if $z = \pm 1, \pm i$ and $w = \pm 1, \pm i$ such that $z \neq w$, then

$$\ker(\mathcal{F} - zI) \perp \ker(\mathcal{F} - wI).$$

Thus, for $f \in \ker(\mathcal{F} - zI)$ and $g \in \ker(\mathcal{F} - wI)$, we have $\langle f, g \rangle = 0$.

The novel approach described in Section 4.4 can be used here to provide a visual representation of the projection matrices necessary in Theorem 6.2.5. Theorem 6.2.5 (iii) gives that $P \in \{\mathcal{F}\}'$, so that we can invoke Theorem 4.3.1 and the notation of Chapter 4.4 to arrive at

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}_1 & & & \\ & \mathbf{P}_{-i} & & \\ & & \mathbf{P}_{-1} & \\ & & & \mathbf{P}_i \end{bmatrix}$$

in its matrix representation, where each $\mathbf{P}_z$ for $z = \pm 1, \pm i$ is a distinct projection onto the eigenspace of $\mathcal{F}$ corresponding to a given eigenvalue. This makes the classification of the invariant subspaces for $\mathcal{F}$ given in Theorem

6.2.8 (iii) much more intuitive. We can now see how $PL^2(\mathbb{R})$ produces M in the presented form.

6.3. The Fourier Cosine and Sine Transforms

We now revisit the Fourier cosine and sine transforms introduced in Chapter 4.5. Recall

$$\mathcal{F}_{\cos} = \begin{bmatrix} \mathbf{I} & & & \\ & \mathbf{0} & & \\ & & -\mathbf{I} & \\ & & & \mathbf{0} \end{bmatrix} \quad \text{and} \quad \mathcal{F}_{\sin} = \begin{bmatrix} \mathbf{0} & & & \\ & -\mathbf{I} & & \\ & & \mathbf{0} & \\ & & & \mathbf{I} \end{bmatrix}.$$

We can immediately note that $\mathcal{F}_{\cos} = \mathcal{F}_{\cos}^*$ and $\mathcal{F}_{\sin} = \mathcal{F}_{\sin}^*$, so that all invariant subspaces are also reducing automatically. This leads us to the following theorem.

Theorem 6.3.1. *A (closed) subspace $M \subseteq L^2(\mathbb{R})$ is reducing for $\mathcal{F}_{\cos}$ if and only if*

$$M = M_1 \oplus M_{-1} \oplus N$$

where M_1 is any closed subspace of ker$(\mathcal{F} - I)$, M_{-1} is any closed subspace of ker$(\mathcal{F} + I)$, and N is any closed subspace of ker$\mathcal{F}_{\cos} = \ker(\mathcal{F} + iI) \oplus \ker(\mathcal{F} - iI)$.

Proof. Suppose $M \subseteq L^2(\mathbb{R})$ such that $\mathcal{F}_{\cos} M = M$. Then if $P = P_M$ is an orthogonal projection onto M, $P\mathcal{F}_{\cos} = \mathcal{F}_{\cos} P$ by Theorem 6.2.5, and by Theorem 4.5.2,

$$\mathbf{P} = \begin{bmatrix} \mathbf{A}_{00} & & & \\ & \mathbf{A}_{11} & & \mathbf{A}_{13} \\ & & \mathbf{A}_{22} & \\ & \mathbf{A}_{31} & & \mathbf{A}_{33} \end{bmatrix}$$

where $\mathbf{A}_{jk}$ for $0 \leqslant j, k \leqslant 3$ are bounded operators from $\ker(\mathcal{F} - (-i)^k I) \to \ker(\mathcal{F} - (-i)^j I)$. However, $P^* = P$ and $P^2 = P$ implies that

$$\mathbf{P} = \begin{bmatrix} \mathbf{A}_{00}^* & & & \\ & \mathbf{A}_{11}^* & & \mathbf{A}_{31}^* \\ & & \mathbf{A}_{22}^* & \\ & \mathbf{A}_{13}^* & & \mathbf{A}_{33}^* \end{bmatrix}$$

so that $\mathbf{A}_{31} = \mathbf{A}_{13}^*$, and

$$\mathbf{P} = \begin{bmatrix} \mathbf{A}_{00}^2 & & & \\ & \mathbf{A}_{11}^2 + \mathbf{A}_{13}\mathbf{A}_{13}^* & & \mathbf{A}_{11}\mathbf{A}_{13} + \mathbf{A}_{13}\mathbf{A}_{33} \\ & & \mathbf{A}_{22}^2 & \\ & \mathbf{A}_{13}^*\mathbf{A}_{11} + \mathbf{A}_{33}\mathbf{A}_{13}^* & & \mathbf{A}_{13}^*\mathbf{A}_{13} + \mathbf{A}_{33}^2 \end{bmatrix}.$$

We then have that $\mathbf{A}_{00}$ and $\mathbf{A}_{22}$ are orthogonal projections onto $\ker(\mathcal{F} - I)$ and $\ker(\mathcal{F} + I)$, respectively. Additionally, we have

$$\begin{aligned} \mathbf{A}_{11} &= \mathbf{A}_{11}^2 + \mathbf{A}_{13}\mathbf{A}_{13}^* \\ \mathbf{A}_{13} &= \mathbf{A}_{11}\mathbf{A}_{13} + \mathbf{A}_{13}\mathbf{A}_{33} \\ \mathbf{A}_{13}^* &= \mathbf{A}_{13}^*\mathbf{A}_{11} + \mathbf{A}_{33}\mathbf{A}_{13}^* \\ \mathbf{A}_{33} &= \mathbf{A}_{13}^*\mathbf{A}_{13} + \mathbf{A}_{33}^2. \end{aligned}$$

Consider the matrix

$$\mathbf{P}' = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{13} \\ \mathbf{A}_{13}^* & \mathbf{A}_{33} \end{bmatrix}.$$

We have that $(\mathbf{P}')^* = \mathbf{P}'$ by previous results. Observe that

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{13} \\ \mathbf{A}_{13}^* & \mathbf{A}_{33} \end{bmatrix}^2 = \begin{bmatrix} \mathbf{A}_{11}^2 + \mathbf{A}_{13}\mathbf{A}_{13}^* & \mathbf{A}_{11}\mathbf{A}_{13} + \mathbf{A}_{13}\mathbf{A}_{33} \\ \mathbf{A}_{13}^*\mathbf{A}_{11} + \mathbf{A}_{33}\mathbf{A}_{13}^* & \mathbf{A}_{13}^*\mathbf{A}_{13} + \mathbf{A}_{33}^2 \end{bmatrix} = \mathbf{P}'.$$

Thus, $\mathbf{P}'$ is an orthogonal projection onto $\ker(\mathcal{F}_{\cos})$. Then

$$\begin{aligned} M = PM &= (A_{00} + A_{22} + P')M \\ &= A_{00}M \oplus A_{22}M \oplus P'M \\ &= M_1 \oplus M_{-1} \oplus N. \end{aligned}$$

Suppose $M = M_1 \oplus M_{-1} \oplus N$. Let $(g_j)_{j \geqslant 1}$ be an orthonormal basis for M_1. Then

$$g_j = \sum_{i=0}^{\infty} c_{ij} h_{4i}.$$

Observe that

$$\begin{aligned} \mathcal{F}_{\cos} g_j &= \sum_{i=0}^{\infty} c_{ij} \mathcal{F}_{\cos} h_{4i} \\ &= \sum_{i=0}^{\infty} c_{ij} h_{4i} \\ &= g_j. \end{aligned}$$

Then for $f \in M_1$, $f = \sum_{j=0}^{\infty} a_j g_j$ and $\mathcal{F}_{\cos} f \in M$. Thus $\mathcal{F}_{\cos}\big|_{M_1} = I_{M_1}$ where I_{M_1} is the identity operator on M_1. A similar computation can be done to show $\mathcal{F}_{\cos}\big|_{M_{-1}} = I_{M_{-1}}$. Similarly, let $(n_j)_{n \geqslant 0}$ be an orthonormal basis for $N \subseteq \ker(\mathcal{F} + iI) \oplus \ker(\mathcal{F} - iI)$. Then

$$n_j = \sum_{i=0}^{\infty} c_{ij} h_{4i+1} + \sum_{i=0}^{\infty} c'_{ij} h_{4i+3}.$$

Observe that

$$\mathcal{F}_{cos} n_j = \sum_{i=0}^{\infty} c_{ij} \mathcal{F}_{\cos} h_{4i+1} + \sum_{i=0}^{\infty} c'_{ij} \mathcal{F}_{\cos} h_{4i+3} = 0.$$

Then $N \subseteq \ker(\mathcal{F}_{\cos})$, and $\mathcal{F}_{\cos} M = M$. Thus, M is reducing for $\mathcal{F}_{\cos}$. □

Note that this description of an invariant subspace for $\mathcal{F}_{\cos}$ is different than that of $\mathcal{F}$. In fact, there is a relationship between the two that immediately follows from Theorem 6.3.1.

Proposition 6.3.2. *For any $M \subseteq L^2(\mathbb{R})$, if $\mathcal{F} M = M$, then $\mathcal{F}_{\cos} M \subseteq M$.*

Proof. Let $M \subseteq L^2(\mathbb{R})$ be a closed subspace. Suppose $\mathcal{F} M = M$. Observe

$$\mathcal{F}_{\cos} M = \frac{1}{2}(\mathcal{F} + \mathcal{F}^*) M = \mathcal{F} M + \mathcal{F}^* M \subseteq M,$$

thus M is invariant for $\mathcal{F}_{cos}$ by definition. □

The following example demonstrates that the converse is not necessarily true.

Example 6.3.3. Let $M = \text{span}\{h_1 + h_3\}$ where h_1 and h_3 are the first and third Hermite functions. Then

$$\mathcal{F}_{\cos} M = \{0\} \subseteq M,$$

thus M is invariant for $\mathcal{F}_{\cos}$. However, note that for $(h_1 + h_3) \in M$,

$$\mathcal{F}(h_1 + h_3) = -ih_1 + ih_3 \neq c(h_1 + h_3)$$

for any $c \in \mathbb{C}$. To show this, observe

$$\langle -ih_1 + ih_3, h_3 \rangle = -i$$
$$\langle ch_1 + ch_3, h_3 \rangle = c$$

so that $c = -i$. However, $-ih_1 + ih_3 \neq -ih_1 - ih_3$. A similar result is found for $\langle -ih_1 + ih_3, h_1 \rangle$. Thus, M is invariant for $\mathcal{F}_{\cos}$ but not $\mathcal{F}$.

We can extend the claims given in this chapter to $\mathcal{F}_{\sin}$ with the following.

Theorem 6.3.4. *A (closed) subspace $M \subseteq L^2(\mathbb{R})$ is reducing for $\mathcal{F}_{\sin}$ if and only if*

$$M = M_{-i} \oplus M_i \oplus N$$

where M_{-i} is any closed subspace of ker$(\mathcal{F} + iI)$, M_i is any closed subspace of ker$(\mathcal{F} - iI)$, and N is any closed subspace of ker$\mathcal{F}_{\sin} = \ker(\mathcal{F} - I) \oplus \ker(\mathcal{F} + I)$.

Proposition 6.3.5. *For any $M \subseteq L^2(\mathbb{R})$, if $\mathcal{F}M = M$, then $\mathcal{F}_{\sin}M \subseteq M$.*

The proofs of the above two claims follow directly from the proofs of Theorem 6.3.1 and Proposition 6.3.2.

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