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An Introduction to Obstacle Problems

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April 30, 2021

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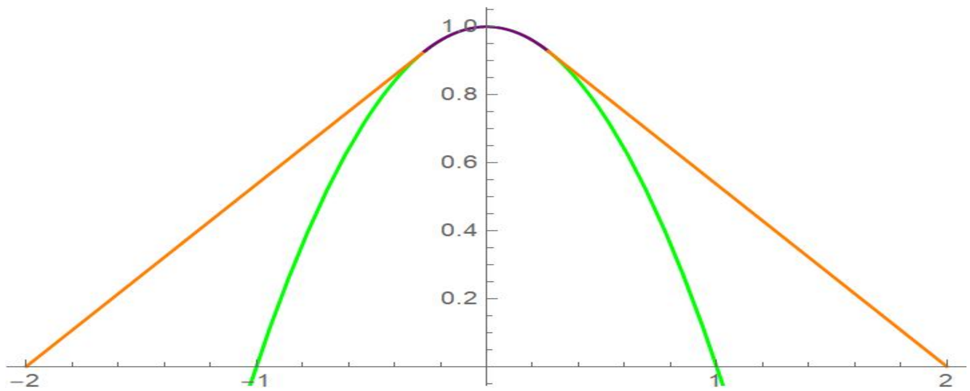
Abstract

The *obstacle problem* can be used to predict the shape of an elastic membrane lying over an obstacle in a domain Ω. In this paper we introduce and motivate a mathematical formulation for this problem, and give an example to demonstrate the need to search for solutions in non-classical settings. We then introduce Sobolev spaces as the proper setting for solutions, and prove that unique solutions exist in $W^{1,2}(\Omega)$.

1 Description of Problem

Acknowledgement: The description given of the formulation of the obstacle problem and the outlines for the proofs of Theorem 2.1 and Theorem 5.1 are drawn from a survey paper by Figalli [1].

The obstacle problem seeks to describe the shape of an elastic membrane lying over an obstacle. Mathematically speaking, we will have a domain $\Omega \subset \mathbb{R}^n$, and the obstacle will be a function $\phi : \Omega \to \mathbb{R}$. The membrane will be described by some function v satisfying $v \geq \phi$ inside Ω and the boundary condition $v|_{\partial\Omega} = 0$. (To make this possible, we will assume that $\phi|_{\partial\Omega} \leq 0$). For example, in the image below, the membrane is in orange and the obstacle is in green, with the contact set between the two colored purple. In particular, the domain is $\Omega = (-2, 2) \subset \mathbb{R}$, the obstacle is $\phi(x) = 1 - x^2$, and the membrane must satisfy the boundary conditions $v(-2) = v(2) = 0$.



Because the membrane is assumed to be perfectly elastic, it will assume the shape that minimizes its total surface area, subject to the constraint that it must lie above the obstacle. (Note that we disregard the effects of gravity, so the membrane does not simply fall onto the obstacle and adhere to it.) The suface area of the membrane is given by $\int_\Omega \sqrt{|\nabla v|^2 + 1}\ dA$. If we assume that there is little variation in the membrane's surface (in particular, if we assume $|\nabla v|$ is small), then we can approximate $\sqrt{|\nabla v|^2 + 1}$ by inserting $|\nabla v|^2$ into the first two terms of the Maclaurin series for $\sqrt{1 + x}$, which gives $\sqrt{|\nabla v|^2 + 1} \approx 1 + \frac{1}{2}|\nabla v|^2$. Thus we are led to minimize the Dirichlet integral $\int_\Omega \frac{1}{2}|\nabla v|^2$ over all candidate functions v which lie above the obstacle and meet the boundary conditions.

2 Properties of Solutions

Consider the example given above, where Ω is $(-2, 2)$, the boundary conditions are $v(2) = v(-2) = 0$, the obstacle is $\phi(x) = 1 - x^2$, and we wish to minimize the Dirichlet integral $\int_{-2}^{2} \frac{1}{2}|v'(x)|^2 dx$. For the moment, we assume that a continuous minimizer function u exists.

Theorem 2.1. *Let u be a sufficiently smooth minimizer of the Dirichlet integral for our example problem. Then $u'' = 0$ on the non-contact set $\{u > \phi\} = \{x \in \Omega | u(x) > \phi(x)\}$.*

Proof. If we assume that u is continuous (and we know that $\phi(x) = 1 - x^2$ is continuous), it follows that the set $\{u > \phi\}$ is open. If $\{u > \phi\}$ is non-empty, then there exists a point $x_0 \in \Omega$ and a radius $r > 0$ such that the interval $B_r(x_0) = (x_0 - r, x_0 + r)$ is contained in $\{u > \phi\}$. (Note that since x_0 is an arbitrary point in $\{u > \phi\}$, we wish to show that $u''(x_0) = 0$.) Fix a *test function* $\psi \in \mathcal{D}(B_r(x_0))$, where $\mathcal{D}(B_r(x_0))$ is the set of infinitely differentiable functions whose support is contained in a compact subset of $B_r(x_0)$; that is, their value is 0 outside a compact subset of $B_r(x_0)$. Take an arbitrary $\epsilon > 0$, and consider the function $u_\epsilon = u + \epsilon\psi$. (Intuitively, this function simply adds a small perturbation to the function u inside some strict subset of $B_r(x_0)$ while maintaining the smoothness of u).

We first wish to show that u_ϵ is a candidate for our minimization problem; in other words, that $u_\epsilon \geq \phi$ on Ω. Outside of $B_r(x_0)$, $u_\epsilon = u$ due to the compact support of ψ, and we are already assuming u to be a valid minimizer, so $u_\epsilon \geq \phi$ outside $B_r(x_0)$ and boundary conditions are met. Let M be the maximum of $|\psi|$ on $\overline{B_r(x_0)}$, and let $c > 0$ be the minimum of $u - \phi$ on $\overline{B_r(x_0)}$ (these both exist on account of the continuity of ψ, u, and ϕ). Then we can make ϵ small enough that $c - \epsilon M \geq 0$, and for any $x \in B_r(x_0)$, $u(x) \geq \phi(x) + c$. Thus we have

$$u_\epsilon(x) \geq u(x) - \epsilon M \geq \phi(x) + c - \epsilon M \geq \phi(x),$$

so $u_\epsilon(x) \geq \phi(x)$ for all $x \in \Omega$, as desired.

Now note that because u is a minimizer and u_ϵ is a candidate function for the minimization problem, we have

$$\begin{aligned}\int_{-2}^{2} \frac{1}{2}|u'(x)|^2 dx \leq \int_{-2}^{2} \frac{1}{2}|u_\epsilon'(x)|^2 dx &= \int_{-2}^{2} \frac{1}{2}|u'(x) + \epsilon\psi'(x)|^2 dx \\ &= \int_{-2}^{2} \frac{1}{2}|u'(x)|^2 dx + \epsilon \int_{-2}^{2} u'(x)\psi'(x) dx + \epsilon^2 \int_{-2}^{2} \frac{1}{2}|\psi'(x)|^2 dx.\end{aligned}$$

This yields

$$\int_{-2}^{2} u'(x)\psi'(x) dx + \epsilon \int_{-2}^{2} \frac{1}{2}|\psi'(x)|^2 dx \geq 0.$$

Since ϵ can be arbitrarily small, this implies that $\int_{-2}^{2} u'(x)\psi'(x)dx \geq 0$. Integration by parts gives

$$\int_{-2}^{2} u'(x)\psi'(x) dx = u'(x)\psi(x)\Big|_{-2}^{2} - \int_{-2}^{2} u''(x)\psi(x) dx = -\int_{-2}^{2} u''(x)\psi(x) dx,$$

noting that $u'(x)\psi(x)\big|_{-2}^{2}$ is 0 because of the compact support of ψ, so we have $-\int_{-2}^{2} u''(x)\psi(x)dx \geq 0$. Then, because ψ is an arbitrary test function, we can replace it with $-\psi$ to obtain $-\int_{-2}^{2} u''(x)\psi(x)dx \leq 0$ as well, so $\int_{-2}^{2} u''(x)\psi(x)dx = 0$. We wish to conclude from this property holding for every $\psi \in \mathcal{D}(B_r(x_0))$ that $u''(x_0) = 0$.

One example of a test function in $\mathcal{D}(B_r(x_0))$ is $\psi_{x_0,\epsilon}$, which for an arbitrary $0 < \epsilon < r$ is defined by

$$\psi_{x_0,\epsilon}(x) = \begin{cases} \exp\left(-\frac{\epsilon^2}{\epsilon^2 - |x-x_0|^2}\right) & |x - x_0| < \epsilon \\ 0 & \text{otherwise.} \end{cases}$$

This function takes on positive values in the interval $(x_0 - \epsilon, x_0 + \epsilon)$ and 0 values elsewhere. While the proof requires more advanced arguments than we can present here, it turns out that a minimizer u will have continuous second derivatives on the non-contact set. Since u'' is continuous at x_0, then if $u''(x_0) \neq 0$, there exists an open interval $(x_0 - \epsilon, x_0 + \epsilon)$ on which $|u''(x)| > 0$ and $\psi_{x_0,\epsilon}(x) > 0$, so we would have $\int_{-2}^{2} u''(x)\psi_{x_0,\epsilon}(x)dx \neq 0$. Since we must have $\int_{-2}^{2} u''(x)\psi_{x_0,\epsilon}(x)dx = 0$, we conclude that $u''(x_0) = 0$, as desired. □

It is worth noting, however, that while solutions have continuous second derivatives on the non-contact set, are not guaranteed to have continuous second derivatives on the whole domain, or even to have second derivatives at every point. Consider the example problem we are working with. We have concluded that a minimizer u must have $u''(x) = 0$ for any x where $u(x) > \phi(x)$; in other words, that outside the set where $u = \phi$ (where the membrane contacts the obstacle), u must be linear. The only single line that gives $u(-2) = u(2) = 0$ is the flat line $u = 0$, which lies underneath the obstacle and is therefore not a candidate. This means somewhere in Ω the solution u must deviate from being a single line, at which point its second derivative will be undefined. In any case, we can conclude that a minimizer does not exist in $C^2(\Omega)$.

3 Sobolev Spaces

At this point there is an apparent tension in our approach: we can make meaningful statements about the second derivatives of solutions at some points, but we have no guarantee that second derivatives of solutions exist at all points. As mentioned before, it turns out that u has continuous second derivatives on the non-contact set, but this does not give us a general setting in which to look for solutions. We cannot search for solutions among functions that are smooth on the non-contact set, because we do not know what the non-contact set will be until we know what the solution is. We will resolve this tension in the following way: the absence of solutions in $C^2(\Omega)$ motivates a search for solutions outside of classical settings. In particular, we will show that unique solutions exist for obstacle problems in the Sobolev space $H^1(\Omega)$, where derivatives exist not in a classical sense but rather in a *distributional* sense. To make this argument, we must first lay out some preliminary concepts, including Banach spaces, Hilbert spaces, Lebesgue spaces, and Sobolev spaces themselves.

Acknowledgement: The material in this section and the following section is drawn largely from a text by Renardy and Rogers [2].

Definition: A **Banach space** is a complete normed vector space. A **norm** is a mapping from the elements of the vector space to the non-negative real numbers which satisfies the triangle inequality and two other properties, and generalizes the notions of the size of an element and the distance between two elements (we denote the norm of an element x by $||x||$). A complete normed vector space is one in which every Cauchy sequence converges to an element of the space; recall that a Cauchy sequence $\{x_j\}$ is one where for any $\epsilon > 0$, there exists an N such that $m, n \geq N$ implies $||x_m - x_n|| < \epsilon$. In a Banach space, for every such sequence, there exists an element x such that $\lim_{j\to\infty} ||x_j - x|| = 0$.

Definition: A **Hilbert space** is a complete inner product space. An inner product space H derives its norm from an inner product, a mapping from $H \times H$ to $\mathbb{R}$ denoted by (x, y) for $x, y \in H$. The norm of an inner product space is given by $||x|| = \sqrt{(x, x)}$, and a Hilbert space is an inner product space that is complete as a normed vector space (meaning that every Hilbert space is also a Banach space).

Definition: The **Lebesgue space** $L^p(\Omega)$ is the set of equivalence classes of functions $f : \Omega \to \mathbb{R}$ for which $|f|^p$ is Lebesgue integrable. Two functions are considered equivalent (and hence correspond to the same member of the L^p space) if they agree except on a set of Lebesgue measure zero. L^p space is endowed with the norm $||f||_p = \left(\int_\Omega |f|^p\right)^{1/p}$, and under this norm every L^p space is a Banach space. Of the L^p spaces, only L^2 space is a Hilbert space, with the inner product $(f, g) = \int_\Omega f(\mathbf{x})g(\mathbf{x})d\mathbf{x}$.

Notation: For a domain $\Omega \subset \mathbb{R}^n$, a function $f(x_1, \ldots, x_n)$ mapping Ω to $\mathbb{R}$, and an n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of non-negative integers (this is called a **multi-index**), define $|\alpha|$ to be the sum of all the entries of α, and define $D^\alpha f$ to be the result of taking α_i derivatives of f with respect to each x_i (so that a total of $|\alpha|$ derivatives are taken).

As noted before, the derivatives we are considering will not necessarily correspond to the classical sense of derivatives. Instead, members of L^p space can be identified as distributions, which have derivatives in a weaker sense.

Definition: Before defining distributions, recall from our proof of Theorem 2.1 that a **test function** is an infinitely differentiable real-valued function on a domain Ω with compact support, meaning that its values on Ω are 0 outside of some closed and bounded set. We denote the set of test functions on Ω by $\mathcal{D}(\Omega)$.

Definition: A **distribution** is a continuous linear mapping $T : \mathcal{D}(\Omega) \to \mathbb{R}$, denoted by $T(\psi) = (T, \psi)$

for test functions ψ. For a distribution T and a test function ψ, we define distributional derivatives as follows: $\left(\frac{\partial T}{\partial x_i}, \psi\right) = -\left(T, \frac{\partial \psi}{\partial x_i}\right)$. Note that while distributions may not even correspond to pointwise-defined functions, much less differentiable ones, distributional derivatives of arbitrary orders always exist because test functions are infinitely differentiable. An integrable function f on Ω can be identified as a distribution through the mapping $(f, \psi) = \int_\Omega f(\mathbf{x})\psi(\mathbf{x})d\mathbf{x}$ for $\psi \in \mathcal{D}(\Omega)$. It is worth noting, though, that while some distributions have a representation as a function (that is, are equal to the distribution given by a function f as described above), others do not. An example of a distribution that does not have an identification with a pointwise function is the Dirac delta, defined by $(\delta, \psi) = \psi(\mathbf{x}_0)$, which would require a "function" with infinite value at the point $\mathbf{x}_0$ and 0 values elsewhere.

Now that we have defined Lebesgue spaces and the sense in which we are taking derivatives, we can define Sobolev spaces, the setting in which we will show existence and uniqueness of solutions for the obstacle problem.

Definition: The **Sobolev space** $W^{k,p}(\Omega)$ is the set of all elements $f \in L^p(\Omega)$ for which $D^\alpha f \in L^p(\Omega)$ for all multi-indices α satisfying $|\alpha| \leq k$. In other words, if we consider f as a distribution and take up to k distributional derivatives, the result will be a distribution which has a representation as some member of $L^p(\Omega)$. Recalling the definition of the p-norm $||f||_p$ from L^p space, the norm in a Sobolev space extends this norm to also include the p-norms of derivatives:

$$||f||_{k,p} = \left(\sum_{|\alpha| \leq k} (||D^\alpha f||_p)^p\right)^{1/p}.$$

All Sobolev spaces are Banach spaces; for $p = 2$, the Sobolev spaces are Hilbert spaces (denoted by $H^k(\Omega)$), with the inner product

$$(f, g)_k = \sum_{|\alpha| \leq k} \int_\Omega D^\alpha f(\mathbf{x}) D^\alpha g(\mathbf{x}) d\mathbf{x}.$$

Specifically, we will show that unique solutions exist for the obstacle problem in the Sobolev space $W^{1,2}(\Omega) = H^1(\Omega)$.

4 Further Preliminaries

In order to prove this result, there are several concepts and theorems that need to be laid out first. In particular, we must define dual spaces of Banach spaces and note some of their properties, and we must also state (without proof) several theorems that will be used in the argument.

Definition: A **linear functional** on a Banach space X is a continuous linear mapping from X to $\mathbb{R}$. For such an operator L, we assign the norm $||L|| = \sup_{||x|| \neq 0} \frac{|Lx|}{||x||}$. Under this norm, the vector space of linear functionals on X forms a Banach space, called the **dual space** of X and denoted by X^*.

These spaces take on a special significance with regard to Hilbert spaces, for which we have the following result:

Theorem [Riesz Representation]: A Hilbert space H is isometric to its dual space, meaning that there is a linear bijection L from H to H^* such that $||Lx|| = ||x||$ for every $x \in H$. (This bijection exists in Hilbert spaces because it is based on the inner product: an element $x \in H$ is identified with the functional $l_x(y) = (x, y)$.) Note that this also means H^* is isometric to $(H^*)^*$, H is isometric to $(H^*)^*$, and so on.

Using the relationship between Banach spaces and their dual spaces, we can develop weaker notions of convergence for both a Banach space and its dual space:

Definition: Let X be a Banach space, and let $\{x_n\}$ be a sequence of vectors in X. We say that $\{x_n\}$ **converges weakly** to $x \in X$ (denoted by $x_n \rightharpoonup x$) if $f(x_n) \to f(x)$ for all $f \in X^*$. Along similar lines, we say that a sequence of functionals $\{f_n\}$ in X^* **converges weakly-$*$** to f (denoted by $f_n \overset{*}{\rightharpoonup} f$) if $f_n(x) \to f(x)$ for all $x \in X$.

In our existence and uniqueness argument, we will also use the following theorem of Alaoglu:

Theorem [Alaoglu]: Let X be a Banach space that contains a countable dense subset and let $\{f_n\}$ be a bounded sequence in X^*. Then $\{f_n\}$ has a weakly-$*$ convergent subsequence.

5 Existence and Uniqueness of Solutions in $H^1(\Omega)$

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a continuous boundary, and let $\phi : \overline{\Omega} \to \mathbb{R}$ be a member of $C^1(\overline{\Omega})$ satisfying $\phi|_{\partial\Omega} \leq 0$. Among elements $v \in H^1(\Omega)$ satisfying $v|_{\partial\Omega} = 0$ and $v \geq \phi$, there exists a unique minimizer for the Dirichlet integral $\int_\Omega \frac{|\nabla v|^2}{2}$.*

Note: There is some difficulty in defining the relations $v|_{\partial\Omega} = 0$ and $v \geq \phi$ in a Sobolev space: elements of Sobolev spaces are equivalence classes of functions that only have to agree up to a measure zero set. Therefore neither of these relations can be intended in a pointwise sense, because there is not a unique pointwise representation for $v \in H^1(\Omega)$. We can interpret the relation $v \geq \phi$ in a distributional sense: we say $v \geq \phi$ if $(v, \psi) \geq (\phi, \psi)$ for all $\psi \in \mathcal{D}(\Omega)$.

There is a particular difficulty in defining boundary conditions, because $\partial\Omega$ is a measure zero set in $\mathbb{R}^n$, so representatives for the same Sobolev function v can differ substantially on $\partial\Omega$. To resolve this difficulty, we define the set $W_0^{k,p}(\Omega)$ to be the closure of the set of test functions $\mathcal{D}(\Omega)$ in the $W^{k,p}(\Omega)$ topology. (Note that by definition $W_0^{k,p}(\Omega)$ is a closed set, and it also turns out to be a subspace of $W^{k,p}(\Omega)$.) On the boundary of Ω, we say that two Sobolev functions u and v are **equal in the sense of traces of Sobolev functions** if $u - v \in W_0^{k,p}(\Omega)$. Thus $v|_{\partial\Omega} = 0$ if $v \in W_0^{k,p}(\Omega)$. Intuitively, it makes sense to say that a function is 0 on $\partial\Omega$ if it can be approximated by a sequence of test functions, because test functions are compactly supported, so they are always 0 near the boundary.

Having defined these relations, there are two other theorems worth noting before we begin the proof of Theorem 5.1: the Poincare inequality and the lower semicontinuity of the L^2 norm.

Poincare Inequality: Let $\Omega \subset \mathbb{R}^n$ be contained in the strip $|x_1| \leq d < \infty$ (i.e. Ω is bounded in one direction). Then there exists a constant c, depending only on k and d, such that for all $u \in H_0^k(\Omega)$,

$$||u||_{k,2}^2 \leq c \sum_{|\alpha|=k} ||D^\alpha u||_2^2.$$

Semicontinuity: Let $\{v_n\}$ be a sequence in $L^2(\Omega)$ converging weakly to $v \in L^2(\Omega)$. Then $||v||_2 \leq \liminf_{n\to\infty} ||v_n||_2$.

Proof of Theorem 5.1: The remainder of this section will consist of a proof of Theorem 5.1.

Note that we are seeking to minimize the Dirichlet integral over the set

$$K_\phi = \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0, v \geq \phi\}.$$

Proposition 5.2: The set K_ϕ defined above is closed in $H^1(\Omega)$; that is, any converging sequence $\{v_j\}$ in K_ϕ has its limit in K_ϕ.

Lemma 5.3: Given a Banach space X, a functional $T \in X^*$, and an $\alpha \in \mathbb{R}$, the set $S = \{x \in X : T(x) \geq \alpha\}$ is closed.

Proof: By definition, linear functionals are continuous, so if a sequence $\{x_n\}$ in S converges to some $x \in X$, then the sequence $L(x_n)$ in $\mathbb{R}$ converges to $L(x)$. Note that the set $\{y \in \mathbb{R} : y \geq \alpha\}$ is closed in the topology for $\mathbb{R}$, so $L(x_n) \geq \alpha$ for all n implies $L(x) \geq \alpha$. Therefore $x \in S$, so the limit of any converging sequence in S is also contained in S. □

Proof of Proposition 5.2: We show this by showing K_ϕ to be the intersection of two closed sets. The set $\{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}$ is closed by definition, because the functions satisfying $v|_{\partial\Omega} = 0$ are members of $H^1_0(\Omega)$, which is defined as the closure of $\mathcal{D}(\Omega)$. We now note that for a given $\psi \in \mathcal{D}(\Omega)$, the mapping $T_\psi(v) = (v - \phi, \psi)$ is a linear functional on $H^1(\Omega)$ and therefore a member of the dual space $H^1(\Omega)^*$. As a result, the set $\{v \in H^1(\Omega) : T_\psi(v) \geq 0\}$ is closed by the preceding lemma, so for each ψ, the set $S_\psi = \{v : (v, \psi) \geq (\phi, \psi)\}$ is closed. The set $S_\phi = \{v \in H^1(\Omega) : v \geq \phi\}$ is the intersection of the closed sets S_ψ over all $\psi \in \mathcal{D}(\Omega)$, so it is a closed set as well. Then K_ϕ is the intersection of S_ϕ with $H^1_0(\Omega)$, so K_ϕ is closed, as desired. □

Proposition 5.4: K_ϕ is convex, which means that for any two elements $u, v \in K_\phi$, the line segment between them, which is all points $tv + (1-t)u$ for $t \in [0, 1]$, is also contained in K_ϕ.

Proof: Note that because $H^1_0(\Omega)$ is a linear subspace of $H^1(\Omega)$, v and u being in $H^1_0(\Omega)$ implies $tv+(1-t)u$ is in $H^1_0(\Omega)$ as well, so boundary conditions are met. Moreover, because distributions are linear mappings, for any $\psi \in \mathcal{D}(\Omega)$, (noting that $(u, \psi) \geq (\phi, \psi)$ and $(v, \psi) \geq (\phi, \psi)$) we have

$$\begin{aligned}
(tv + (1-t)u, \psi) &= (tv, \psi) + ((1-t)u, \psi) \\
&= t(v, \psi) + (1-t)(u, \psi) \\
&\geq t(\phi, \psi) + (1-t)(\phi, \psi) \\
&= (\phi, \psi),
\end{aligned}$$

showing that $tv + (1-t)u \in K_\phi$, as desired. □

It turns out, based on results that we will not prove here, that because K_ϕ is convex and it is closed in the $H^1(\Omega)$ topology, it is also closed in the weak $H^1(\Omega)$ topology, which means that any weakly converging sequence in K_ϕ has its weak limit point in K_ϕ as well.

To show the existence of a minimizer in K_ϕ, we wish to show that the infimum of all possible values of the Dirichlet integral for elements of K_ϕ is attained. To do this, we will consider a minimizing sequence for the Dirichlet integral in K_ϕ, show that the sequence is bounded in $H^1(\Omega)$, and use Alaoglu and Riesz Representation to show that this sequence has a subsequence with a weak limit point in K_ϕ. Then semicontinuity of the L^2 norm will imply that the infimum is attained at this weak limit point.

First, let $\alpha = \inf_{v \in K_\phi} \int_\Omega \frac{|\nabla v|^2}{2}$. Note that α is finite, since, for example, the function $V = \max\{\phi, 0\}$ is a member of K_ϕ (noting that we assume $\phi|_{\partial\Omega} \leq 0$) and satisfies $\int_\Omega \frac{|\nabla V|^2}{2} < \infty$. Then we can consider a sequence $\{v_k\} \subset K_\phi$ such that $\int_\Omega \frac{|\nabla v_k|^2}{2} \to \alpha$.

Note that because $\int_\Omega \frac{|\nabla v_k|^2}{2} \to \alpha$, there exists a $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, $\int_\Omega \frac{|\nabla v_k|^2}{2} \leq \alpha + 1$.

Now note that for each k, the function $v_k - V$ is in $H_0^1(\Omega)$, so we can apply the Poincare inequality to obtain a constant C_Ω such that

$$||v_k - V||_2 \leq ||v_k - V||_{1,2} \leq C_\Omega ||\nabla v_k - \nabla V||_2.$$

By applying the triangle inequality to $||v_k - V + V||_2$, we can say that

$$||v_k||_{1,2} = ||v_k||_2 + ||\nabla v_k||_2 \leq ||v_k - V||_2 + ||V||_2 + ||\nabla v_k||_2.$$

By the Poincare inequality as described above, we have

$$||v_k||_{1,2} \leq C_\Omega ||\nabla v_k - \nabla V||_2 + ||V||_2 + ||\nabla v_k||_2.$$

Applying the triangle inequality to $||\nabla v_k - \nabla V||_2$ gives

$$||v_k||_{1,2} \leq (C_\Omega + 1)||\nabla v_k||_2 + ||V||_2 + C_\Omega ||\nabla V||_2.$$

Note that for all $k \geq k_0$, $||\nabla v_k||_2 = \sqrt{\int_\Omega \frac{|\nabla v_k|^2}{2}} \leq \sqrt{\alpha + 1}$. Thus, for all k, $||v_k||_{1,2}$ is bounded by the constant

$$\max \left\{ \max_{k<k_0} ||v_k||_{1,2}, (C_\Omega + 1)\sqrt{\alpha + 1} + ||V||_2 + C_\Omega ||\nabla V||_2 \right\},$$

so $\{v_k\}$ is a bounded sequence in $H^1(\Omega)$.

By Riesz representation, the bounded sequence $\{v_k\}$ in $H^1(\Omega)$ can be identified with the sequence of functionals $\{f_k(v)\} = \{(v_k, v)\}$ in the dual space $H^1(\Omega)^*$, which is also bounded because $H^1(\Omega)$ is isometric to its dual space under this mapping.

Then, by Alaoglu, there exists a weakly-$*$ convergent subsequence of the sequence $\{(v_k, v)\}$; that is, a sequence of functionals $\{f_{k_j}(v)\} = \{(v_{k_j}, v)\}$ such that for some $f \in H^1(\Omega)^*$, $f_{k_j}(v) \to f(v)$ for all $v \in H^1(\Omega)$. By Riesz representation, f can be identified as the functional $f(v) = (u, v)$ for some $u \in H^1(\Omega)$, so there exists a $u \in H^1(\Omega)$ such that $(v_{k_j}, v) \to (u, v)$ for all $v \in H^1(\Omega)$.

Equivalently, since inner products are symmetric, we can say that $(v, v_{k_j}) \to (v, u)$ for all $v \in H^1(\Omega)$. Finally, by Riesz representation, this is the same as saying that $l(v_{k_j}) \to l(u)$ for all linear functionals $l \in H^1(\Omega)^*$. Thus there exists a subsequence $\{v_{k_j}\}$ of $\{v_k\}$ converging weakly to $u \in H^1(\Omega)$.

Because $\{v_{k_j}\}$ is a weakly converging sequence in K_ϕ and K_ϕ is closed in the weak $H^1(\Omega)$ topology, the weak limit point u of $\{v_{k_j}\}$ is an element of K_ϕ. Finally, lower semicontinuity of the L^2 norm under weak convergence gives

$$\int_\Omega \frac{|\nabla u|^2}{2} \leq \liminf_{j\to\infty} \int_\Omega \frac{|\nabla v_{k_j}|^2}{2} = \alpha,$$

so $u \in K_\phi$ attains the infimum of the possible values for the Dirichlet integral. Thus a minimizer for the Dirichlet integral in K_ϕ exists.

It remains only to show that the solution we have found in $H^1(\Omega)$ is unique. To show this, we will employ the following proposition:

Proposition 5.5: In a real inner product space H, the square of the norm induced by the inner product is a strictly convex operator. In other words, for any $v, w \in H$ such that $v \neq w$ and any $t \in (0, 1)$,

$$||tv + (1 - t)w||^2 < t||v||^2 + (1 - t)||w||^2.$$

Proof: Given $v, w \in H$ such that $v \neq w$ and given $t \in (0,1)$, we wish to show that

$$t||v||^2 + (1-t)||w||^2 - ||tv + (1-t)w||^2 > 0.$$

Note that

$$||tv + (1-t)w||^2 = (tv + (1-t)w, tv + (1-t)w) = t^2||v||^2 + (1-t)^2||w||^2 + 2t(1-t)(v,w).$$

It follows that we can rewrite the left-hand side of our desired inequality as

$$(t - t^2)||v||^2 + ((1-t) - (1-t)^2)||w||^2 - 2t(1-t)(v,w),$$

or

$$t(1-t)(||v||^2 + ||w||^2 - 2(v,w)),$$

or

$$t(1-t)||v-w||^2.$$

It follows from the assumption that $v \neq w$ that $||v - w||^2 > 0$, which proves our desired inequality. $\square$

Finally, to conclude uniqueness, suppose that $u_1, u_2 \in K_\phi$. By the above proposition, if $\nabla u_1 \neq \nabla u_2$, letting $t = \frac{1}{2}$ yields

$$\left|\left|\frac{1}{2}\nabla u_1 + \frac{1}{2}\nabla u_2\right|\right|_2^2 < \frac{1}{2}(||\nabla u_1||_2^2 + ||\nabla u_2||_2^2).$$

In other words,

$$\int_\Omega \left|\frac{\nabla u_1 + \nabla u_2}{2}\right|^2 < \frac{1}{2}\left(\int_\Omega |\nabla u_1|^2 + \int_\Omega |\nabla u_2|^2\right).$$

As a result, if u_1 and u_2 are both minimizers of the Dirichlet integral over K_ϕ, we must have $\nabla u_1 = \nabla u_2$, or else $\frac{1}{2}(u_1 + u_2)$, which is also a member of K_ϕ because K_ϕ is a convex set, would have a lower value of the Dirichlet integral. Because $\nabla(u_1 - u_2) = 0$, we conclude by the Poincare inequality that $||u_1 - u_2||_2 = 0$. Therefore u_1 and u_2 are equal in an $L^2(\Omega)$ sense, so they must also correspond to the same member of $H^1(\Omega)$. Thus we cannot have two distinct minimizers, so solutions to the obstacle problem are unique in $H^1(\Omega)$, as desired. $\square$

6 Conclusion

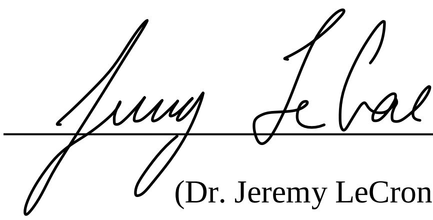
There are several interesting properties of solutions to the obstacle problem that we are not able to discuss in detail here. The advantage of proving existence and uniqueness of solutions in $H^1(\Omega)$ is that with the setting for solutions established, we can begin to characterize them rigorously. For example, we can generalize the results of Section 2 of this paper to say that in a *distributional* sense, if u is the unique solution, $\Delta u \leq 0$ inside Ω, and $\Delta u = 0$ in the non-contact set. (Note that Δu represents the Laplacian operator, defined by $\Delta u = u_{x_1 x_1} + \cdots + u_{x_n x_n}$.) We can also characterize the growth of solutions by saying that the growth of the distance of the solution from the obstacle is bounded by a quadratic function of the distance from the contact set. These results and many others may be explored once the foundation has been laid for the $H^1(\Omega)$ setting.

References

[1] A. Figalli, *Free boundary regularity in obstacle problems.* J. E. D. P. (2018) Exposé n^o II, 24 p.

[2] M. Renardy and R. Rogers, An Introduction to Partial Differential Equations, 2nd edition, Springer-Verlag, 2004.

The signatures below, by the thesis advisor, a departmental reader, and the honors coordinator for mathematics, certify that this thesis, prepared by Calvin Reedy, has been approved, as to style and content.



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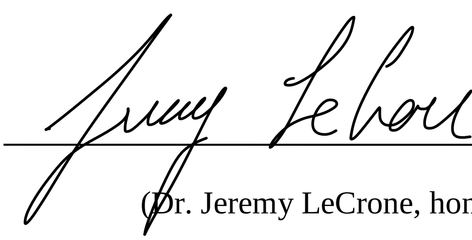

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