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[Introduction to] Finite Blaschke Products and Their Connections

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Finite Blaschke Products and Their Connections

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Preface

This is a book about a beautiful subject that begins with the topic of Möbius transformations. Indeed, Möbius transformations

$$z \mapsto \frac{az+b}{cz+d}$$

are studied in complex analysis since their mapping properties demonstrate wonderful connections with geometry. These transformations map extended circles to extended circles, enjoy the symmetry principle, come in several types yielding different behavior depending on their fixed point(s), and, through an identification with 2×2 matrices, make connections to group theory and projective geometry. Finite Blaschke products, the focus of this book, are products of certain types of Möbius transformations, the automorphisms of the open unit disk $\mathbb{D}$, namely

$$z \mapsto \xi \frac{w-z}{1-\overline{w}z},$$

where $|w| < 1$ and $|\xi| = 1$ are fixed. These products have an uncanny way of appearing in many areas of mathematics such as complex analysis, linear algebra, group theory, operator theory, and systems theory. This book covers finite Blaschke products and is designed for advanced undergraduate students, graduate students, and researchers who are familiar with complex analysis but who want to see more of its connections to other fields of mathematics. Much of the material in this book is scattered throughout mathematical history, often only appearing in its original language, and some of it has never seen a modern exposition. We gather up these gems and put them together as a cohesive whole, taking a leisurely pace through the subject and leaving plenty of time for exposition and examples. There are plenty of exercises for the reader who not only wants to appreciate the beauty of the subject but to gain a working knowledge of it as well.

In the early twentieth century, the study of infinite products of the form

$$B(z) = \prod_{k \geqslant 1} \frac{|z_k|}{z_k} \frac{z_k - z}{1 - \overline{z_k} z},$$

in which $z_1, z_2, \ldots$ is a sequence in $\mathbb{D}$, was initiated in 1915 by Wilhelm Blaschke (1885–1962). This product converges uniformly on compact subsets of $\mathbb{D}$ if and only if the zero sequence z_k satisfies $\sum_{k \geqslant 1}(1 - |z_k|) < \infty$. These *Blaschke products* are analytic on $\mathbb{D}$ and have the additional property that the radial limit $\lim_{r \to 1^-} B(re^{i\theta})$ exists and is of unit modulus for almost every $\theta \in [0, 2\pi)$. In other words, B is an inner function. Blaschke products have been studied intensely since they were first introduced and they appear in many contexts throughout complex analysis and operator theory.

This book is concerned with *finite Blaschke products*, in which the zero sequence $z_1, z_2, \ldots, z_n$ is finite and the product terminates. Although the skeptical reader might think this focus is too narrow, there are many fascinating connections with geometry, complex analysis, and operator theory that demand attention.

There are already some excellent texts that cover infinite Blaschke products and, more generally, inner functions [38, 61]. However, as the reader will see, there are many beautiful theorems involving finite Blaschke products that have no clear analogues in the infinite case. Finite Blaschke products are not often discussed in the standard texts on function spaces or complex variables since the focus there is often on inner functions as part of the broader theory of Hardy spaces. This book focuses on finite Blaschke products and the many results that pertain only to the finite case.

The book begins with an exposition of the *Schur class* $\mathscr{S}$, the set of analytic functions from $\mathbb{D}$ to $\mathbb{D}^-$, the closure of $\mathbb{D}$, and an introduction to hyperbolic geometry. We develop this material from scratch, assuming only that the reader has had a basic course in complex variables. We characterize the finite Blaschke products in several different ways. First, a rational function is a finite Blaschke product if and only if it is of the form

$$\frac{\alpha_0 + \alpha_1 z + \cdots + \alpha_n z^n}{\overline{\alpha_n} + \overline{\alpha_{n-1}} z^{n-1} + \cdots + \overline{\alpha_0} z^n},$$

in which the numerator is a polynomial whose n roots lie in $\mathbb{D}$. Second, a finite Blaschke product maps $\mathbb{D}$ onto $\mathbb{D}$ (and the unit circle $\mathbb{T}$ onto itself) precisely n times and a theorem of Fatou confirms that these are the only functions that are continuous on $\mathbb{D}^-$ and analytic on $\mathbb{D}$ with this property. Third, each finite Blaschke product B satisfies

$$\lim_{|z| \to 1^-} |B(z)| = 1$$

and another result of Fatou shows that the finite Blaschke products are the only analytic functions on $\mathbb{D}$ that do this. Whether as rational functions whose defining polynomials enjoy certain symmetries, as n-to-1 analytic functions on $\mathbb{D}$, or as analytic functions with unimodular boundary values, the finite Blaschke products distinguish themselves as special elements of the Schur class.

The approximation of a given analytic function by well-understood functions from a fixed class is a standard technique in complex analysis. For example, there are the well-known approximation theorems of Runge, Mergelyan, and Weierstrass. We examine a few results of this type that involve finite Blaschke products. More specifically, a celebrated theorem of Carathéodory ensures that any function in the Schur class $\mathscr{S}$ can be approximated, uniformly on compact subsets of $\mathbb{D}$, by a sequence of finite Blaschke products. In fact, one can even take the approximating Blaschke products to have simple zeros. After Carathéodory's theorem, we discuss Fisher's theorem, which says that any function in $\mathscr{S}$ that extends continuously to $\mathbb{D}^-$ can be approximated uniformly on $\mathbb{D}^-$ by convex combinations of finite Blaschke products. As another example, a theorem of Helson and Sarason states that any continuous function from $\mathbb{T}$ to $\mathbb{T}$ can be uniformly approximated by a sequence of quotients of finite Blaschke products.