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Raymond Cheng

Javad Mashreghi

William T. Ross

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RAYMOND CHENG, JAVAD MASHREGHI, AND WILLIAM T. ROSS

Dedicated to Thomas Ransford on the occasion of his sixtieth birthday.

ABSTRACT. In this paper we survey and bring together several approaches to obtaining inner functions for Toeplitz operators. These approaches include the classical definition, the Wold decomposition, the operator-valued Poisson Integral, and Clark measures. We then extend these notions somewhat to inner functions on model spaces. Along the way we present some novel examples.

1. Introduction

For $\varphi \in H^\infty$, the bounded analytic functions on the open unit disk $\mathbb{D}$, let

$$T_\varphi : H^2 \to H^2, \quad T_\varphi f = \varphi f, \tag{1.1}$$

denote the analytic *Toeplitz operator* on the classical Hardy space H^2. In this paper we survey, continue, and synthesize some discussions begun in [4, 10, 11] dealing with the notion of an "inner vector" for T_φ along with the general notion of an inner vector for a contraction on a Hilbert space. We connect these results with the operator-valued Poisson kernel and some work from [2, 3] concerning "factoring an L^1 function through a contraction". Along the way we also produce some interesting examples and reformulations of these connections.

2. Basic definitions and facts

We begin with the definition of an inner vector for a Toeplitz operator from [10]. Recall that the inner product on the Hardy space H^2 is

$$\langle f, g \rangle := \int_{\mathbb{T}} f \overline{g} \, dm, \tag{2.1}$$

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where m is normalized Lebesgue measure on the unit circle $\mathbb{T}$. As is tradition, we equate an $f \in H^2$ with its $L^2 = L^2(\mathbb{T}, m)$ radial boundary function, i.e.,

$$f(\zeta) = \lim_{r \to 1^-} f(r\zeta)$$

for almost every $\zeta \in \mathbb{T}$. We will also use the term *inner* function (without any qualifiers like in Definition 2.2 below) to denote an H^∞ function that has unimodular boundary values almost everywhere. Classical theory [6] says that an inner function I can be factored uniquely as $I = \xi B S_\mu$, where ξ is a unimodular constant, B is a Blaschke product, and S_μ is a singular inner function associated with a positive measure μ on $\mathbb{T}$ that is singular with respect to m. We say the *degree* of I is equal to d if I is a finite Blaschke product of order d, and equal to infinity otherwise. Furthermore, any function $f \in H^2$ can be factored, uniquely up to multiplicative constants, as $f = IG$, where I is an inner function and $G \in H^2$ is an outer function.

For $\varphi \in H^\infty$ the analytic Toeplitz operator T_φ from (1.1) is a bounded operator on H^2 whose norm $\|T_\varphi\|$ satisfies

$$\|T_\varphi\| = \|\varphi\|_\infty := \text{ess-sup}\{\, |\varphi(\xi)| : \xi \in \mathbb{T}\}.$$

Also recall that the adjoint T_φ^* of T_φ satisfies $T_\varphi^* = T_{\overline{\varphi}}$, where $T_{\overline{\varphi}} f = P(\overline{\varphi} f)$ and P is the Riesz projection of L^2 onto H^2. When φ is an inner function, observe from (2.1) that T_φ is an isometry. See [8, Ch. 4] for the details of these basic Toeplitz operator facts and [1] for a definitive treatise.

Definition 2.2. For $\varphi \in H^\infty$ we say a unit vector $f \in H^2$ is T_φ-*inner* if $\langle T_\varphi^n f, f\rangle = 0$ for all $n \geqslant 1$.

When $\varphi(z) = z$, one can see from Fourier analysis that the T_z-inner vectors are precisely the inner functions. Also observe that replacing φ with $c\varphi$, where $c > 0$, in Definition 2.2 does not change whether or not a function f is T_φ-inner. Thus we can always assume, by scaling φ, that

$$\varphi \in b(H^\infty) := \{g \in H^\infty : \|g\|_\infty \leqslant 1\},$$

the closed unit ball of H^∞. This normalization will be important when we need T_φ to be a contraction operator since in this case $\|T_\varphi\| = \|\varphi\|_\infty \leqslant 1$. Immediate from Definition 2.2 and the inner product formula from (2.1) are the following facts.

Proposition 2.3. *Let* $\varphi \in b(H^\infty)$.

(1) If $f \in H^2$ is T_φ-inner and I is any inner function, then If is T_φ-inner.

(2) If $f \in H^2$ is T_φ-inner and Θ is any inner divisor of f, i.e., $f/\Theta \in H^2$, then f/Θ is T_φ-inner.

(3) Any unit vector belonging to $\ker T_{\overline{\varphi}}$ is T_φ-inner.

If u denotes the inner factor of φ, it is known [8, p. 108] that

$$\ker T_{\overline{\varphi}} = \mathcal{K}_u := (uH^2)^\perp,$$

the *model space* corresponding to u. Thus we have the simple corollary.

Corollary 2.4. *If I is any inner function and u is the inner factor of $\varphi \in b(H^\infty)$, then any unit vector from $I\mathcal{K}_u$ is T_φ-inner.*

This corollary gives us many specific examples of T_φ-inner vectors. For example, if $\lambda \in \mathbb{D}$, the reproducing kernel functions

$$k_\lambda(z) := \frac{1 - \overline{u(\lambda)}u(z)}{1 - \overline{\lambda}z}$$

belong to $\mathcal{K}_u$. In fact, finite linear combinations of these functions are dense in $\mathcal{K}_u$ [8, Ch. 5]. Since

$$\|k_\lambda\| = \sqrt{k_\lambda(\lambda)} = \sqrt{\frac{1 - |u(\lambda)|^2}{1 - |\lambda|^2}},$$

then

$$I\sqrt{\frac{1 - |\lambda|^2}{1 - |u(\lambda)|^2}}\frac{1 - \overline{u(\lambda)}u(z)}{1 - \overline{\lambda}z}, \quad \lambda \in \mathbb{D}, \quad I \text{ inner},$$

are T_φ-inner functions.

When $\varphi = u$ is a finite Blaschke product, then the model space $\mathcal{K}_u$ is a certain finite dimensional space of rational functions that are analytic in a neighborhood of $\overline{\mathbb{D}}$ [8, p. 117]. Furthermore, as we will see in a moment in Theorem 3.12, every T_u-inner function is bounded. However, when u is not a finite Blaschke product then $\mathcal{K}_u$ is infinite dimensional [8, p. 117] and, since multiplication by an inner function I is an isometry on H^2 (see (2.1)), $I\mathcal{K}_u$ is a closed infinite dimensional subspace of L^2. By a theorem of Grothendieck, it will contain an unbounded function. Putting this all together, we obtain the following.

Corollary 2.5. *If the inner factor of $\varphi \in b(H^\infty)$ is not a finite Blaschke product, then there are unbounded T_φ-inner functions.*

A specific version of this was pointed out in [10, p. 103].

Of course one needs to discuss the case when φ is an outer function. Since φH^2 is dense in H^2 [8, p. 86], we see that $\ker T_{\overline{\varphi}} = \{0\}$. In this case, it is not clear that there are *any* T_φ-inner functions. Indeed, we do not see any obvious ones like $I \ker T_{\overline{\varphi}}$ since, in this case, $\ker T_{\overline{\varphi}} = \{0\}$.

Example 2.6. Suppose that φ is the outer function $\varphi(z) = 1 + z$ and that $f \in H^2$ is T_φ-inner, i.e.,

$$\langle T_\varphi^n f, f\rangle = 0, \ \ \forall n \geqslant 1.$$

In other words,

$$\int_{\mathbb{T}} (1+\xi)^n |f(\xi)|^2 \, dm(\xi) = 0, \ \ \forall n \geqslant 1. \tag{2.7}$$

Then the L^1 function $|f|^2$ annihilates $(1+z)^n$ for all $n \geqslant 1$, along with all their linear combinations. In particular, $|f|^2$ annihilates

$$(1+z)^2 - (1+z) = 1 + 2z + z^2 - 1 - z = z(1+z).$$

The above observation will be the first step in a proof by induction. Next, suppose that $|f|^2$ annihilates $z^k(1+z)$ for all $1 \leqslant k \leqslant n$. Then

$$z^{n+1}(1+z) = (1+z)^{n+2} - \big[(1+z)^{n+1} - z^{n+1}\big](1+z).$$

By the T_φ-inner property of f notice that $|f|^2$ annihilates the first term on the right. It also annihilates the subtracted expression, by the induction hypothesis (the expression in square brackets is a polynomial of degree n). Thus we have shown by induction that $|f|^2$ annihilates $\{z^n(1+z)\}_{n \geqslant 0}$ (the $n = 0$ case follows from (2.7)). This means that

$$\int_{\mathbb{T}} \xi^n (1+\xi)|f(\xi)|^2 dm(\xi) = 0, \quad n \geqslant 0, \tag{2.8}$$

and by complex conjugation,

$$\int_{\mathbb{T}} \overline{\xi}^n (1+\overline{\xi})|f(\xi)|^2 dm(\xi) = 0, \quad n \geqslant 0.$$

A little algebra yields

$$\int_{\mathbb{T}} \overline{\xi}^{n+1} (1+\xi)|f(\xi)|^2 dm(\xi), \quad n \geqslant 0. \tag{2.9}$$

Equations (2.8) and (2.9) say that all of the Fourier coefficients of $(1+\xi)|f(\xi)|^2$ vanish and so $(1+\xi)|f(\xi)|^2$ is zero. Conclusion: there are no T_φ-inner functions when $\varphi(z) = 1 + z$.

3. Inner vectors via the Wold decomposition

Using some ideas from [10], we can use the Wold decomposition [9] to explore the inner vectors for certain Toeplitz operators. Observe that when u is an inner function the Toeplitz operator T_u is an isometry on H^2. Thus the Wold decomposition of H^2 with respect to T_u becomes

$$H^2 = X_0 \oplus X_1 \oplus T_u X_1 \oplus T_u^2 X_1 \oplus \cdots,$$

where

$$X_0 := \bigcap_{n=1}^{\infty} T_u^n H^2 = \{0\}, \quad X_1 := H^2 \ominus T_u H^2 = \mathcal{K}_u.$$

Thus

$$H^2 = \mathcal{K}_u \oplus u\mathcal{K}_u \oplus u^2\mathcal{K}_u \oplus \cdots.$$

The above decomposition says that every $f \in H^2$ has a unique expansion as

$$f = F_0 + uF_1 + u^2 F_2 + \cdots, \quad F_j \in \mathcal{K}_u. \tag{3.1}$$

Furthermore, for each integer $N \geqslant 1$,

$$\begin{aligned} \langle u^N f, f\rangle &= \Big\langle u^N \sum_{k\geqslant 0} u^k F_k, \sum_{l\geqslant 0} u^l F_l \Big\rangle \\ &= \sum_{k,l\geqslant 0} \langle u^{N+k-l} F_k, F_l\rangle \\ &= \sum_{l-k=N} \langle F_k, F_l\rangle. \end{aligned}$$

This leads us to the following.

Proposition 3.2. *A unit vector $f \in H^2$ with expansion*

$$f = F_0 + uF_1 + u^2 F_2 + \cdots, \quad F_j \in \mathcal{K}_u,$$

as in (3.1) *is T_u-inner if and only if*

$$\sum_{k=0}^{\infty} \langle F_k, F_{N+k}\rangle = 0, \quad N \geqslant 1. \tag{3.3}$$

Though this is just a restatement of the condition for f to be T_u-inner, it is useful for producing more tangible examples of T_u-inner functions.

Example 3.4. Choose orthogonal vectors $F_j, j \geqslant 0$ from $\mathcal{K}_u$ so that $\sum_{j\geqslant 0} \|F_j\|^2 = 1$. Then the condition (3.3) is easily satisfied and thus the unit vector $f = \sum_{j\geqslant 0} u^j F_j$ is a T_u-inner function (as is any inner function times this vector).

Example 3.5. If $u(z) = z^n$, then $\mathcal{K}_u = \operatorname{span}\{1, z, z^2, \ldots z^{n-1}\}$ and the vectors

$$F_j = \frac{z^j}{\sqrt{n}}, \quad 0 \leqslant j \leqslant n-1,$$

satisfy the conditions of the previous example. Thus

$$f = \sum_{j=0}^{n-1} u^j F_j = \frac{1}{\sqrt{n}} + \frac{z^{n+1}}{\sqrt{n}} + \frac{z^{2n+2}}{\sqrt{n}} + \frac{z^{3n+3}}{\sqrt{n}} + \cdots + \frac{z^{(n-1)(n+1)}}{\sqrt{n}}$$

is a T_{z^n}-inner vector.

Example 3.6. The previous example can be generalized to a finite Blaschke product

$$u(z) = \prod_{j=1}^{n} \frac{z - a_j}{1 - \overline{a_j} z}, \quad a_j \in \mathbb{D}.$$

If we define

$$F_0(z) = \frac{\sqrt{1 - |a_1|^2}}{1 - \overline{a_1} z},$$

$$F_1(z) = \frac{\sqrt{1 - |a_2|^2}}{1 - \overline{a_2} z} \frac{z - a_1}{1 - \overline{a_1} z},$$

$$F_2(z) = \frac{\sqrt{1 - |a_3|^2}}{1 - \overline{a_3} z} \frac{z - a_1}{1 - \overline{a_1} z} \frac{z - a_2}{1 - \overline{a_2} z},$$

$$\vdots$$

$$F_{n-1}(z) = \frac{\sqrt{1 - |a_n|^2}}{1 - \overline{a_n} z} \prod_{j=1}^{n-1} \frac{z - a_j}{1 - \overline{a_j} z},$$

one can show that $\{F_0, \ldots, F_{n-1}\}$ is an orthonormal basis for $\mathcal{K}_u$. Now choose $\alpha_0, \ldots, \alpha_{n-1} \in \mathbb{C}$ such that $\sum_{j=0}^{n=1} |\alpha_j|^2 = 1$. Then

$$f = \sum_{j=0}^{n-1} \alpha_j u^j F_j$$

is T_u-inner.

From Corollary 2.4 we know, for an inner function I, that any unit vector from the set $\{I \ker T_{\overline{u}} : I \text{ is inner}\}$ is a $T_{\overline{u}}$-inner vector. Perhaps one might think we have equality here. Indeed, sometimes we do. For example, if $u(z) = z$, then $\ker T_{\overline{z}} = \mathbb{C}$ and, as discussed earlier, the T_z-inner vectors are precisely the inner functions. Here is another positive example of when the unit vectors from $\{I \ker T_{\overline{u}} : I \text{ is inner}\}$ constitute the complete set of T_u-inner vectors.

Example 3.7. If the inner function u is the single Blaschke factor

$$u(z) = \frac{z-a}{1-\overline{a}z}, \quad a \in \mathbb{D},$$

one can show [8, Ch. 5] that

$$\ker T_{\overline{u}} = \mathcal{K}_u = \mathbb{C}\frac{1}{1-\overline{a}z}.$$

As shown in [4], the T_u-inner vectors are

$$I\frac{\sqrt{1-|a|^2}}{1-\overline{a}z}, \quad I \text{ inner}.$$

However, in general, the unit vectors from $\{I \ker T_{\overline{u}} : I \text{ is inner}\}$ form a proper subset of the T_u-inner vectors. One can see this with the following example.

Example 3.8. Using the technique from Example 3.5, we see that when $u(z) = z^n$ the vector

$$f = \frac{1}{\sqrt{2}} + \frac{z^{n+1}}{\sqrt{2}}$$

is T_u-inner. However, f is not of the form Ig, where I is inner and $g \in \mathcal{K}_u$. This follows from the fact that f is outer and does not belong to $\mathcal{K}_u = \operatorname{span}\{1, z, z^2, \ldots, z^{n-1}\}$.

The papers [10, 11] yield a description of the T_u-inner vectors. From the Wold decomposition (3.1) we see that any $f \in H^2$ can be written as

$$f = \sum_{k=0}^{\infty} F_k u^k.$$

If $\{v_j\}_{j\geqslant 1}$ is an orthonormal basis for $\mathcal{K}_u$, then we can expand things a bit further and write

$$\begin{aligned} f &= \sum_{k=0}^{\infty} F_k u^k \\ &= \sum_{k=0}^{\infty} u^k \Big(\sum_{j\geqslant 1} c_{j,k} v_j\Big) \\ &= \sum_{j\geqslant 1} v_j \Big(\sum_{k=0}^{\infty} c_{j,k} u^k\Big). \end{aligned}$$

Observe that

$$\sum_{j\geqslant 1}|c_{j,k}|^2 = \|F_k\|^2$$

and that

$$\begin{aligned}\|f\|^2 &= \sum_{k=0}^{\infty}\|F_k\|^2\\ &= \sum_{k=0}^{\infty}\sum_{j\geqslant 1}|c_{j,k}|^2\\ &= \sum_{j\geqslant 1}\sum_{k=0}^{\infty}|c_{j,k}|^2.\end{aligned}$$

Thus for each j, $\sum_{k\geqslant 0}|c_{j,k}|^2 < \infty$ and so

$$f_j(z) = \sum_{k=0}^{\infty} c_{j,k}z^k$$

defines a function in H^2 (square summable power series). By the Littlewood subordination principle [8, p. 126], $f_j \circ u$ also belongs to H^2.

Thus every unit vector $f \in H^2$ has the unique representation

$$f(z) = \sum_{j\geqslant 1} v_j(z) f_j(u(z)), \tag{3.9}$$

where $f_j \in H^2$ with $\sum_{j\geqslant 1}\|f_j\|^2 < \infty$, and $\{v_j\}_{j\geqslant 1}$ is an orthonormal basis for $\mathcal{K}_u$. Furthermore, as observed in [10, Prop. 1] (and can be proved using the above calculation), if

$$f = \sum_{j\geqslant 1} v_j f_j(u), \quad g = \sum_{j\geqslant 1} v_j g_j(u), \tag{3.10}$$

as in (3.9), then

$$\langle f, g\rangle = \sum_{j\geqslant 1}\langle f_j, g_j\rangle. \tag{3.11}$$

Theorem 3.12. *A unit vector f written as in* (3.9) *is T_u-inner if and only if*

$$\sum_{j\geqslant 1}^{\infty}|f_j(\xi)|^2 = 1$$

for almost every $\xi \in \mathbb{T}$.

Proof. Here is the original proof from [10]. With

$$f = \sum_{j \geqslant 1} v_j f_j(u),$$

and $n \geqslant 1$, (3.11) yields

$$\begin{aligned}
\langle T_u^n f, f \rangle &= \langle f u^n, f \rangle \\
&= \Big\langle \sum_j v_j u^n f_j(u), \sum_k v_k f_k(u) \Big\rangle \\
&= \sum_{j \geqslant 1} \langle z^n f_j, f_j \rangle \\
&= \sum_{j \geqslant 1} \int_{\mathbb{T}} \xi^n |f_j(\xi)|^2 dm(\xi) \\
&= \int_{\mathbb{T}} \xi^n \Big(\sum_{j \geqslant 1} |f_j(\xi)|^2 \Big) dm(\xi). \qquad (3.13)
\end{aligned}$$

Then $\langle T_u^n f, f \rangle = 0$ for all $n = 1, 2, \ldots$ if and only if, by Fourier analysis, $\sum_{j \geqslant 1} |f_j|^2$ is constant almost everywhere. But since we assuming that f is a unit vector, we see, by putting $n = 0$ in (3.13), that $\sum_{j \geqslant 1} |f_j|^2 = 1$ almost everywhere. □

When u is a finite Blaschke product, then $\mathcal{K}_u$ is finite dimensional. In this case (3.9) is finite and each basis vector v_j is a rational function that is analytic in a neighborhood of $\overline{\mathbb{D}}$ [8, Ch. 5]. From here it follows that every T_u-inner vector is a bounded function. Contrast this with Corollary 2.5 which says that when u is not a finite Blaschke product there are always T_u-inner vectors that are unbounded functions.

The two papers [10, 11] go further and discuss an “inner-outer” factorization of any $f \in H^2$ in terms of T_u-inner and T_u-outer vectors. They also discuss the concept of T_u-inner in H^p, for $p > 1$, along with some properties of the norms of T_u-inner vectors as well as their growth near $\mathbb{T}$.

4. Inner vectors via the operator-valued Poisson kernel

We can rephrase the language of inner vectors for Toeplitz operators in terms of operator-valued Poisson kernels [2]. Moreover, using this new language, we can extend our discussion to inner vectors for contractions on Hilbert spaces. For $\lambda \in \mathbb{D}$ and $\xi \in \mathbb{T}$, define

$$P_\lambda(\xi) := \frac{1}{1 - \overline{\lambda}\xi} + \frac{1}{1 - \lambda\overline{\xi}} - 1 \tag{4.1}$$

and observe that this can be written as

$$P_\lambda(\xi) = \frac{1 - |\lambda|^2}{|\xi - \lambda|^2},$$

which is the standard Poisson kernel. Classical theory says that for any $g \in L^1 = L^1(\mathbb{T}, m)$ the function

$$\int_{\mathbb{T}} P_\lambda(\xi) f(\xi) dm(\xi)$$

is harmonic on $\mathbb{D}$ with

$$\lim_{r \to 1^-} \int_{\mathbb{T}} P_{r\zeta}(\xi) f(\xi) dm(\xi) = f(\zeta) \tag{4.2}$$

for almost every $\zeta \in \mathbb{T}$. Furthermore, if μ is a finite complex measure on $\mathbb{T}$, we have

$$\int_{\mathbb{T}} P_\lambda(\xi) d\mu(\xi) = \widehat{\mu}(0) + \sum_{n \geqslant 1} \widehat{\mu}(n)\lambda^n + \sum_{n \geqslant 1} \widehat{\mu}(-n)\overline{\lambda}^n, \tag{4.3}$$

where

$$\widehat{\mu}(n) := \int_{\mathbb{T}} \overline{\xi}^n d\mu(\xi), \quad n \in \mathbb{Z},$$

are the Fourier coefficients of μ. We will now discuss an operator version of the Poisson kernel.

For a contraction T on a Hilbert space $\mathcal{H}$, we imitate the formula in (4.1) and define, for $\lambda \in \mathbb{D}$, the *operator-valued Poisson kernel* $K_\lambda(T)$ as

$$K_\lambda(T) := (I - \lambda T^*)^{-1} + (I - \overline{\lambda} T)^{-1} - I.$$

By the spectral radius formula, notice how $\sigma(T) \subseteq \overline{\mathbb{D}}$ and thus the formula for $K_\lambda(T)$ above makes sense. A computation with Neumann series will show that for $r \in [0, 1)$ and $\theta \in [0, 2\pi)$

$$K_{re^{i\theta}}(T) = \sum_{n=0}^{\infty} r^n e^{in\theta} T^{*n} + \sum_{n=0}^{\infty} r^n e^{-in\theta} T^n - I. \tag{4.4}$$

The operator identity

$$K_\lambda(T) = (I - \overline{\lambda}T)^{-1}(I - |\lambda|^2 TT^*)(I - \lambda T^*)^{-1}$$

from [2, Lemma 2.4] shows that for each $\mathbf{x} \in \mathcal{H}$

$$\langle K_\lambda(T)\mathbf{x}, \mathbf{x} \rangle \geqslant 0, \quad \lambda \in \mathbb{D}.$$

Moreover, the function

$$\lambda \mapsto \langle K_\lambda(T)\mathbf{x}, \mathbf{x}\rangle$$

is harmonic on $\mathbb{D}$. Hence, a classical harmonic analysis result of Herglotz ([6, p. 10] or [8, p. 17]) produces a unique positive finite Borel measure $\mu_{T,\mathbf{x}}$ on $\mathbb{T}$ such that

$$\langle K_\lambda(T)\mathbf{x}, \mathbf{x}\rangle = \int_{\mathbb{T}} P_\lambda(\zeta) d\mu_{T,\mathbf{x}}(\zeta). \tag{4.5}$$

Since $K_0(T) = I$ we have

$$1 = \langle \mathbf{x}, \mathbf{x}\rangle = \langle K_0(T)\mathbf{x}, \mathbf{x}\rangle = \int_{\mathbb{T}} d\mu_{T,\mathbf{x}}$$

and so $\mu_{T,\mathbf{x}}$ is a probability measure.

As we defined for Toeplitz operators earlier in Definition 2.2, we say that a unit vector $\mathbf{x}$ is T*-inner* if

$$\langle T^n\mathbf{x}, \mathbf{x}\rangle = 0, \quad n \geqslant 1.$$

Note that $\mathbf{x}$ is T-inner if and only if $\mathbf{x}$ is T^*-inner. From (4.4) we see that $\mathbf{x}$ is T-inner if and only if $\langle K_\lambda(T)\mathbf{x}, \mathbf{x}\rangle = 1$ for all $\lambda \in \mathbb{D}$, or equivalently,

$$1 = \int_{\mathbb{T}} P_\lambda(\zeta) d\mu_{T,\mathbf{x}}(\zeta), \quad \lambda \in \mathbb{D}.$$

By (4.3) this is equivalent to the condition $\mu_{T,\mathbf{x}} = m$. This gives us the following.

Proposition 4.6. *Suppose that T is a contraction on a Hilbert space $\mathcal{H}$ and $\mathbf{x}$ is unit vector in $\mathcal{H}$. Then $\mathbf{x}$ is T-inner if and only if $\mu_{T,\mathbf{x}} = m$, where $\mu_{T,\mathbf{x}}$ is defined as in* (4.5)*.*

For an inner function u, note that T_u is an isometry, hence a contraction. Thus we can apply the above analysis to $\mu_{T_u,f}$.

Proposition 4.7. *If*

$$f = \sum_{j \geqslant 1} v_j f_j(u)$$

is a vector from H^2 as in (3.9)*, then*

$$d\mu_{T_u,f} = \sum_{j \geqslant 1} |f_j|^2 \, dm. \tag{4.8}$$

Proof. If

$$f = \sum_{j \geqslant 1} v_j f_j(u),$$

then

$$\|f\|^2 = \sum_{j \geqslant 1} \|f_j\|^2 = \sum_{j \geqslant 1} \int_{\mathbb{T}} |f_j|^2 dm = \int_{\mathbb{T}} \sum_{j \geqslant 1} |f_j|^2 dm$$

and the calculation used to prove Theorem 3.12 yields

$$\langle T_u^n f, f\rangle = \int_{\mathbb{T}} \xi^n \Big(\sum_{j \geqslant 1} |f_j(\xi)|^2 \Big) dm(\xi),$$

$$\langle T_u^{*n} f, f\rangle = \int_{\mathbb{T}} \overline{\xi}^n \Big(\sum_{j \geqslant 1} |f_j(\xi)|^2 \Big) dm(\xi).$$

From here we observe

$$\begin{aligned}
\int_{\mathbb{T}} P_\lambda(\xi) d\mu_{T_u,f}(\xi) &= \langle K_\lambda(T_u) f, f\rangle \\
&= \sum_{n \geqslant 0} \lambda^n \langle T_u^{*n} f, f\rangle + \sum_{n \geqslant 0} \overline{\lambda}^n \langle T_u^n f, f\rangle - \langle f, f\rangle. \\
&= \sum_{n \geqslant 0} \lambda^n \int_{\mathbb{T}} \overline{\xi}^n \Big(\sum_{j \geqslant 1} |f_j(\xi)|^2 \Big) dm(\xi) \\
&\quad + \sum_{n \geqslant 0} \overline{\lambda}^n \int_{\mathbb{T}} \xi^n \Big(\sum_{j \geqslant 1} |f_j(\xi)|^2 \Big) dm(\xi) \\
&\quad - \sum_{j \geqslant 1} \int_{\mathbb{T}} |f_j(\xi)|^2 dm \\
&= \int_{\mathbb{T}} \Big(\frac{1}{1 - \lambda \overline{\xi}} + \frac{1}{1 - \overline{\lambda} \xi} - 1 \Big) \sum_{j \geqslant 1} |f_j(\xi)|^2 dm(\xi) \\
&= \int_{\mathbb{T}} P_\lambda(\xi) \sum_{j \geqslant 1} |f_j(\xi)|^2 dm(\xi)
\end{aligned}$$

Now use the uniqueness of the Fourier coefficients of a measure along with (4.3) to obtain (4.8). □

Notice how this gives us another way of thinking about Theorem 3.12: a unit vector $f \in H^2$ is T_u-inner if and only if $\mu_{T_u,f} = m$.

This brings us to an interesting related question. One can also show that for any $f, g \in H^2$, we can define the harmonic function $\langle K_\lambda(T_u) f, g\rangle$ on $\mathbb{D}$ and prove this function also has bounded integral means. This yields, via Herglotz's theorem, a complex valued measure $\mu_{T_u,f,g}$ on $\mathbb{T}$ for which

$$\langle K_\lambda(T_u)f, g\rangle = \int_{\mathbb{T}} P_\lambda(\xi) d\mu_{T,f,g}(\xi), \quad \lambda \in \mathbb{D}. \tag{4.9}$$

See [2, Prop. 2.6] for details. A similar calculation used to prove Proposition 4.6 shows that

$$d\mu_{T_u,f,g} = \sum_{j \geqslant 1} f_j \overline{g_j}\, dm. \tag{4.10}$$

In the above formula, f_j and g_j come from the representations of f and g from (3.10). A general result from [3] says that given any $F \in L^1$ and a non-constant inner function u that is not an automorphism, there are $f, g \in H^2$ for which

$$F(\zeta) = \frac{d\mu_{T_u,f,g}}{dm}(\zeta) \tag{4.11}$$

m-almost everywhere. In the language of [3] this says that any $F \in L^1$ can be "factored through T_u". Equivalently stated, using (4.10) and (4.11), we have

$$F(\zeta) = \sum_{j \geqslant 1} f_j(\zeta)\overline{g_j(\zeta)}.$$

This is an interesting representation for L^1 functions and a refinement of the one from [3].

Question 4.12. Proposition 4.7 shows that when φ is an inner function and $f, g \in H^2$, then $d\mu_{T_\varphi,f,g}$ is absolutely continuous with respect to m. When $\varphi \in b(H^\infty)$ is this still the case? For this to be true we would need to know that $\langle \varphi^n f, g\rangle, n \geqslant 1$, are the Fourier coefficients of an L^1 function.

5. Inner vectors via Clark measures

For any fixed $\alpha \in \mathbb{T}$ and inner function u, the function

$$z \mapsto \frac{1 - |u(z)|^2}{|\alpha - u(z)|^2} = \Re\Big(\frac{\alpha + u(z)}{\alpha - u(z)}\Big)$$

is a positive harmonic function on $\mathbb{D}$. Thus by Herglotz's theorem, there is a unique positive measure σ_α on $\mathbb{T}$ for which

$$\frac{1 - |u(z)|^2}{|\alpha - u(z)|^2} = \int_{\mathbb{T}} P_\lambda(\xi) d\sigma_\alpha(\xi).$$

The family of measures $\{\sigma_\alpha : \alpha \in \mathbb{T}\}$ is called the family of *Clark measures* corresponding to u. Let us record some important facts about this family of measures. Proofs can be found in [5].

First, one can use the fact that u is an inner function, along with standard harmonic analysis, to prove that each σ_α is singular with respect to m. Second, if E_α is defined to be the set of $\xi \in \mathbb{T}$ for which

$$\lim_{r\to 1^-} u(r\xi) = \alpha,$$

then E_α is a Borel subset of $\mathbb{T}$ with

$$\sigma_\alpha(\mathbb{T} \setminus E_\alpha) = 0. \tag{5.1}$$

In other words, σ_α is "carried" by E_α. From this we also see that the measures $\{\sigma_\alpha : \alpha \in \mathbb{T}\}$ are singular with respect to each other. Third, a beautiful disintegration theorem of Aleksandrov says that if $g \in L^1$ then for m-almost every $\alpha \in \mathbb{T}$, integral

$$\int_{\mathbb{T}} g(\xi) d\sigma_\alpha(\xi)$$

is well defined. Moreover this almost everywhere defined function

$$\alpha \mapsto \int_{\mathbb{T}} g(\xi) d\sigma_\alpha(\xi)$$

is integrable with respect to m and

$$\int_{\mathbb{T}} \Big(\int_{\mathbb{T}} g(\xi) d\sigma_\alpha(\xi)\Big) dm(\alpha) = \int_{\mathbb{T}} g(\zeta) dm(\zeta). \tag{5.2}$$

Using Clark measures, we can use a technique from [11] to compute a formula for $\langle K_\lambda(T_u)f, f\rangle$ along with the measure $d\mu_{T_u,f}/dm$. This gives us another way to think about the formula (4.11). The result here is the following.

Theorem 5.3. *For an inner function u and $f \in H^2$ we have*

$$d\mu_{T_u,f}(\alpha) = \Big(\int_{\mathbb{T}} |f(\xi)|^2 d\sigma_\alpha(\xi)\Big)\, dm(\alpha).$$

Proof. For any $f \in H^2$ use the formulas from (5.1) and (5.2) to obtain

$$\begin{aligned}
\langle T_u^n f, f\rangle &= \int_{\mathbb{T}} |f(\xi)|^2 u(\xi)^n dm(\xi)\\
&= \int_{\mathbb{T}} \Big(\int_{\mathbb{T}} |f(\xi)|^2 u(\xi)^n d\sigma_\alpha(\xi)\Big) dm(\alpha)\\
&= \int_{\mathbb{T}} \Big(\int_{\mathbb{T}} |f(\xi)|^2 \alpha^n d\sigma_\alpha(\xi)\Big) dm(\alpha)
\end{aligned}$$

$$= \int_{\mathbb{T}} \alpha^n \Big(\int_{\mathbb{T}} |f(\xi)|^2 d\sigma_\alpha(\xi) \Big) dm(\alpha).$$

In a similar way

$$\langle T_u^{*n} f, f \rangle = \int_{\mathbb{T}} \overline{\alpha}^n \Big(\int_{\mathbb{T}} |f(\xi)|^2 d\sigma_\alpha(\xi) \Big) dm(\alpha).$$

Now follow the proof of Proposition 4.7 to get

$$\begin{aligned}
&\int_{\mathbb{T}} P_\lambda(\xi) d\mu_{T_u,f}(\xi) \\
&= \langle K_\lambda(T_u) f, f \rangle \\
&= \sum_{n \geqslant 0} \lambda^n \langle T_u^{*n} f, f \rangle + \sum_{n \geqslant 0} \overline{\lambda}^n \langle T_u^n f, f \rangle - \langle f, f \rangle \\
&= \int_{\mathbb{T}} \Big(\Big(\frac{1}{1 - \lambda \overline{\alpha}} + \frac{1}{1 - \overline{\lambda} \alpha} - 1 \Big) \Big(\int_{\mathbb{T}} |f(\xi)|^2 d\sigma_\alpha(\xi) \Big) \Big) \, dm(\alpha) \\
&= \int_{\mathbb{T}} P_\lambda(\alpha) \Big(\int_{\mathbb{T}} |f(\xi)|^2 d\sigma_\alpha(\xi) \Big) dm(\alpha).
\end{aligned}$$

Use (4.3) along with the uniqueness of Fourier coefficients of a measure to compute the proof. □

Combing Theorem 5.3 and Proposition 4.6 yields the following result from [11].

Corollary 5.4. *A unit vector $f \in H^2$ is T_u-inner if and only if*

$$\int_{\mathbb{T}} |f(\xi)|^2 d\sigma_\alpha(\xi) = 1$$

for m-almost every $\alpha \in \mathbb{T}$.

Recall the notation from (4.9) that for a given inner function u and $f, g \in H^2$

$$\langle K_\lambda(T_u) f, g \rangle = \int_{\mathbb{T}} P_\lambda(\xi) d\mu_{T_u,f,g}(\xi).$$

Moreover, if $\deg(u) \geqslant 2$, any $F \in L^1$ can be written as $d\mu_{T_u,f,g}(\xi)/dm$ for some $f, g \in H^2$. Here is another way of thinking about this via Clark measures. The same argument used to prove Theorem 5.3 shows that

$$d\mu_{T_u,f,g} = \int_{\mathbb{T}} f(\xi) \overline{g(\xi)} d\sigma_\alpha(\xi) \; dm \tag{5.5}$$

Since any $F \in L^1$ is equal to $d\mu_{T_u,f,g}/dm$ for some $f, g \in H^2$ [3], we see that any $F \in L^1$ can be written as

$$F(\alpha) = \int_{\mathbb{T}} f(\xi)\overline{g(\xi)} d\sigma_\alpha(\xi).$$

This Clark measure viewpoint has the additional feature, via Aleksandrov's theorem, that

$$\begin{aligned}\int_{\mathbb{T}} F(\alpha) dm(\alpha) &= \int_{\mathbb{T}} \Big(\int_{\mathbb{T}} f(\xi)\overline{g(\xi)} d\sigma_\alpha(\xi) \Big) dm(\alpha) \\ &= \int_{\mathbb{T}} f(\zeta) g(\zeta) dm(\zeta).\end{aligned}$$

Example 5.6. If u is a finite Blaschke product of degree d and $\alpha \in \mathbb{T}$, then one can compute (see [5, p. 209] for the details) the Clark measure to be

$$d\sigma_\alpha = \sum_{j=1}^{d} \frac{1}{|u'(\zeta_j)|} \delta_{\zeta_j},$$

where $\zeta_1, \ldots, \zeta_d$ are the d distinct solutions to the equation $u(z) = \alpha$ and δ_{ζ_j} is the unit point pass as ζ_j. The denominators in the above expression may look troublesome but at the end of the day we have $u' \neq 0$ on $\mathbb{T}$. By Theorem 5.3 we see that

$$\frac{d\mu_{T_u,f}}{dm}(\alpha) = \int_{\mathbb{T}} |f(\xi)|^2 d\sigma_\alpha(\xi) = \sum_{j=1}^{d} \frac{|f(\zeta_j)|^2}{|u'(\zeta_j)|}.$$

Thus the criterion for a unit vector $f \in H^2$ to be a T_u-inner vector is that the above sum is equal to 1 for m-almost every $\alpha \in \mathbb{T}$.

Furthermore, by (5.5), given $F \in L^1$, there are $f, g \in H^2$ so that

$$F(\alpha) = \sum_{j=1}^{d} \frac{f(\zeta_j)\overline{g(\zeta_j)}}{|u'(\zeta_j)|}$$

for m-almost every $\alpha \in \mathbb{T}$. This formula appears in [3].

Example 5.7. Let us apply this to the simple case where $u(z) = z^2$. Given any $\alpha \in \mathbb{T}$, the two solutions ζ_1, ζ_2 to the equation $z^2 = \alpha$ are

$$\zeta_1 = e^{i \arg \alpha/2}, \quad \zeta_2 = -e^{i \arg \alpha/2}.$$

Thus the condition that a unit f is a T_{z^2}-inner vector becomes

$$|f(e^{i \arg \alpha/2})|^2 + |f(-e^{i \arg \alpha/2})|^2 = 2, \quad m\text{-a.e. } \alpha \in \mathbb{T}.$$

Furthermore, given any $F \in L^1$, there are $f, g \in H^2$ for which

$$F(\alpha) = \tfrac{1}{2} f(e^{i \arg \alpha/2})\overline{g(e^{i \arg \alpha/2})} + \tfrac{1}{2} f(-e^{i \arg \alpha/2})\overline{g(-e^{i \arg \alpha/2})}.$$

This second fact was first observed in [3].

Example 5.8. Consider the atomic inner function

$$u(z) = \exp\Big(\frac{z+1}{z-1}\Big).$$

For a fixed $t \in [0, 2\pi)$, the solutions to $u(z) = e^{it}$ are

$$\zeta_k = \frac{i(t+2\pi k)+1}{i(t+2\pi k)-1}, \quad k \in \mathbb{Z}.$$

Noting that

$$|u'(\zeta_k)| = \frac{2}{|\zeta_k - 1|^2},$$

a similar computation as in Example 5.6 shows that

$$d\sigma_{e^{it}} = \frac{1}{2}\sum_{k\in\mathbb{Z}} \delta_{\zeta_k} |\zeta_k - 1|^2.$$

Thus

$$\begin{aligned}\frac{d\mu_{T_u,f}}{dm}(e^{it}) &= \int_{\mathbb{T}} |f(\xi)|^2 d\sigma_{e^{it}}(\xi)\\ &= \frac{1}{2}\sum_{k\in\mathbb{Z}} |f(\zeta_k)|^2 |\zeta_k - 1|^2\\ &= \sum_{k\in\mathbb{Z}} \Big|f\Big(\frac{i[t+2\pi k]+1}{i[t+2\pi k]-1}\Big)\Big|^2 \frac{2}{|i(t+2\pi k)-1|^2}.\end{aligned}$$

To create a T_u-inner function, we need to find a unit vector $f \in H^2$ so that the above expression is equal to one for almost every t. Let us create a specific example of when this happens. In fact we can even make f unbounded. We already knew we could do this from Corollary 2.5 but our example below will be explicit, while the proof of Corollary 2.5 needed Grothendieck's theorem and is not an explicit construction.

To see how to do this, fix $\beta \in (\frac{1}{2}, 1)$, and let a_k, $k \in \mathbb{Z}$, be the collection of coefficients

$$a_k = \frac{1}{1+|k|^\beta}. \tag{5.9}$$

Note that $\sum_{k\in\mathbb{Z}} |a_k|^2 < \infty$.

Let I_k be the indicator function of the interval $[-\pi + 2\pi k, \pi + 2\pi k)$, $k \in \mathbb{Z}$. Now define F on $\mathbb{T}$ by

$$F(e^{i\theta}) := \sqrt{2}\sum_{k\in\mathbb{Z}} \frac{a_k}{e^{i\theta}-1}\, I_k\Big(i\frac{1+e^{i\theta}}{1-e^{i\theta}}\Big).$$

Then

$$\begin{aligned}
\int_{\mathbb{T}} |F|^2\,dm &= 2\int_{-\pi}^{\pi} |F(e^{i\theta})|^2\,\frac{d\theta}{2\pi}\\
&= 2\sum_{k\in\mathbb{Z}} \int_{-\pi}^{\pi} \frac{|a_k|^2}{|e^{i\theta}-1|^2}\, I_k\Big(i\frac{1+e^{i\theta}}{1-e^{i\theta}}\Big)\,\frac{d\theta}{2\pi}\\
&= 2\sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} |a_k|^2 \frac{|it-1|^2}{2^2}\, I_k(t)\,\frac{2\,dt}{2\pi|it-1|^2}\\
&= \sum_{k\in\mathbb{Z}} \int_{-\pi}^{\pi} |a_k|^2 \big|i[t+2\pi k]-1\big|^2\,\frac{dt}{2\pi|i[t+2\pi k]-1|^2}\\
&= \sum_{k\in\mathbb{Z}} |a_k|^2 < \infty,
\end{aligned}$$

i.e., F is square integrable on $\mathbb{T}$ with

$$\|F\|^2 = \sum_{k\in\mathbb{Z}} |a_k|^2. \tag{5.10}$$

Next we establish that $\log|F|$ is integrable. We'll need the following estimates, which hold for all $k \neq 0$. First note that for $k \neq 0$,

$$\begin{aligned}
|a_k|\big|i(t+2\pi k)-1\big| &= |a_k|\big([\pi+2\pi|k|]^2+1\big)^{1/2}\\
&\geqslant |a_k|\cdot 2\pi|k|\\
&\geqslant \frac{2\pi|k|}{1+|k|^{\beta}}\\
&\geqslant 1.
\end{aligned}$$

Consequently, for $k \neq 0$ and $t \in [-\pi,\pi)$,

$$\begin{aligned}
\Big|\log\Big(|a_k|\big|i(t+2\pi k)-1\Big)\Big| &= \log|a_k||i(t+2\pi k)-1|\\
&\leqslant \log\frac{|i(\pi+2\pi|k|)-1|}{1+|k|^{\beta}}\\
&\leqslant \log\frac{([2\pi(|k|+1/2)]^2+1)^{1/2}}{1+|k|^{\beta}}\\
&\leqslant \log\frac{([2\pi(|k|+|k|/2)]^2+|k|^2)^{1/2}}{|k|^{\beta}}\\
&\leqslant \log(|k|^{1-\beta}\sqrt{9\pi^2+1}).
\end{aligned}$$

We now have

$$\begin{aligned}
&\int_{\mathbb{T}} \big|\log|F|\big|\,dm\\
&= \int_{-\pi}^{\pi} \Big|\log|F(e^{i\theta})|\Big|\,\frac{d\theta}{2\pi}\\
&= \sum_{k\in\mathbb{Z}} \int_{-\pi}^{\pi} \Big|\log\frac{|a_k|\sqrt{2}}{|e^{i\theta}-1|}\Big|\, I_k\Big(i\frac{1+e^{i\theta}}{1-e^{i\theta}}\Big)\,\frac{d\theta}{2\pi}\\
&= \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} \Big|\log\big(|a_k|\big|it-1\big|\sqrt{2}/2\big)\Big|\, I_k(t)\,\frac{dt}{2\pi|it-1|^2}\\
&= \sum_{k\in\mathbb{Z}} \int_{-\pi}^{\pi} \Big|\log\big(6\pi|k|^{1-\beta}\big|i[t+2\pi k]-1\big|/\sqrt{2}\big)\Big|\,\frac{dt}{2\pi|i[t+2\pi k]-1|^2}.
\end{aligned}$$

The series is summable, because the terms behave like $(\log|k|)/|k|^2$.

It follows that there exists an outer function $g \in H^2$ with radial limit function satisfying $|g| = |F|$ almost everywhere on $\mathbb{T}$, namely

$$g(z) := \exp\Big(\int_{\mathbb{T}} \frac{e^{i\theta}+z}{e^{i\theta}-z}\log|F(e^{i\theta})|\,dm(e^{i\theta})\Big).$$

Finally, let J be any classical inner function, and define $f = gJ$. Then

$$\begin{aligned}
\frac{d\mu_{T_u,f}}{dm}(e^{it}) &= \sum_{k\in\mathbb{Z}} \Big|f\Big(\frac{i[t+2\pi k]+1}{i[t+2\pi k]-1}\Big)\Big|^2 \frac{2}{|i(t+2\pi k)-1|^2}\\
&= \sum_{k\in\mathbb{Z}} \Big|F\Big(\frac{i[t+2\pi k]+1}{i[t+2\pi k]-1}\Big)\Big|^2 \frac{2}{|i(t+2\pi k)-1|^2}\\
&= \sum_{k\in\mathbb{Z}} \frac{|a_k(i[t+2\pi k]-1)\sqrt{2}|^2}{2^2}\,\frac{2}{|i(t+2\pi k)-1|^2}\\
&= \sum_{k\in\mathbb{Z}} |a_k|^2.
\end{aligned}$$

Notice from (5.10) that

$$\frac{d\mu_{T_u,f}}{dm}(e^{it}) = \|F\|^2$$

and so one can scale F so that it (and hence f) is a unit vector This also gives us $d\mu_{T_u,f}/dm(e^{it}) = 1$ for almost every t. Any such f will be a T_u-inner function.

As a bonus, we get that the f we just constructed is unbounded. To see this, note that F is unbounded, since for θ approaching zero, $F(e^{i\theta})$ takes values

$$F\Big(\frac{i[t+2\pi k]+1}{i[t+2\pi k]-1}\Big) = \frac{a_k}{1-\frac{i[t+2\pi k]+1}{i[t+2\pi k]-1}} = \frac{-i[t+2\pi k]+1}{2+2|k|^{\beta}}$$

where $t \in [-\pi, \pi)$. Since $\beta < 1$, this expression is unbounded as $|k| \to \infty$.

6. Inner vectors in model spaces

In this section we depart slightly from Toeplitz operators on H^2 to the related topic of compressions of Toeplitz operators on model spaces. For an inner function Θ, recall the model space $\mathcal{K}_\Theta = (\Theta H^2)^\perp$. An important operator to study here is the *compressed shift operator*

$$S_\Theta : \mathcal{K}_u \to \mathcal{K}_u, \quad S_\Theta f = P_\Theta(zf),$$

where P_Θ is the orthogonal projection of L^2 onto $\mathcal{K}_u$. This operator is used to model a certain class of contraction operators on Hilbert space [8, Ch. 9] – hence the use of the phrase "model space."

As a generalization of our discussion of classifying the T_z-inner vectors in H^2, one can ask for a description of the S_Θ-inner vectors in $\mathcal{K}_\Theta$, i.e., those unit vectors $f \in \mathcal{K}_\Theta$ for which

$$\langle S_\Theta^n f, f\rangle = 0, \quad n \geqslant 1.$$

Before continuing, let us make a few comments about S_Θ. For the proofs, see [8, Ch. 9]. First note that since S_Θ is a compression of T_z to $\mathcal{K}_\Theta$ we have the identity

$$S_\Theta^n = P_\Theta T_{z^n}|_{\mathcal{K}_\Theta}.$$

Furthermore, we have the adjoint formula

$$S_\Theta^* = T_{\overline{z}}|_{\mathcal{K}_\Theta}.$$

For any $\varphi \in H^\infty$ there is the functional calculus for S_Θ which allows us to define

$$\varphi(S_\Theta) = P_\Theta T_\varphi|_{\mathcal{K}_\Theta}$$

along with the adjoint formula

$$\varphi(S_\Theta)^* = P_\Theta T_{\overline{\varphi}}|_{\mathcal{K}_\Theta}.$$

One can actually compute the S_Θ-inner vectors with the following result from [8, p. 177].

Theorem 6.1. *Any S_Θ-inner function is an inner function. Moreover, $\mathcal{K}_\Theta$ contains an inner function if and only if $u(0) = 0$ and the inner functions belonging to $\mathcal{K}_\Theta$ are precisely the inner divisors of $\Theta(z)/z$.*

So now the question becomes the following.

Question 6.2. What are the $\varphi(S_\Theta)$-inner functions?

As we did before with Toeplitz operators, we focus our attention on the case where φ is inner. It is clear that the inner vectors for $\varphi(S_\Theta)$ are the same as those for $\varphi(S_\Theta)^*$. As observed with an analogous result in Proposition 2.3, we see that any (unit) vector in $\ker \varphi(S_\Theta)^*$ is a $\varphi(S_\Theta)^*$-inner vector. It is well-known [8] that (assuming φ is an inner function)

$$\ker \varphi(S_\Theta)^* = \mathcal{K}_\Theta \cap \mathcal{K}_\varphi = \mathcal{K}_{\gcd(\Theta,\varphi)},$$

where $\gcd(\Theta, \varphi)$ is the greatest common inner divisor of the inner functions Θ and φ.

At this point, it might the case that $\gcd(\Theta, \varphi)$ is a unimodular constant function whence $\mathcal{K}_{\gcd(\Theta,\varphi)} = \{0\}$ and it is not clear as to whether or not there are any $\varphi(S_\Theta)$-inner vectors.

Question 6.3. We know that if $\gcd(\Theta, \varphi)$ is non-constant, then there are $\varphi(S_\Theta)$-inner vectors. Is the converse true?

For the special case where $\varphi|\Theta$, let us find a class of $\varphi(S_\Theta)$-inner vectors. Define

$$I := \frac{\Theta}{\varphi}$$

and observe from a result in [7] that an analytic function g on $\mathbb{D}$ multiplies $\mathcal{K}_\varphi$ to $\mathcal{K}_\Theta$ if and only $g \in \mathcal{K}_{zI}$. Recall from Theorem 6.1 that the inner functions in $\mathcal{K}_{zI}$ are precisely the inner divisors of I. Here is our result about some of the $\varphi(S_\Theta)$-inner vectors.

Theorem 6.4. *With the notation above, any unit vector from*

$$\{v\mathcal{K}_\varphi : v|I\}$$

is a $\varphi(S_\Theta)$-inner vector.

Proof. Let f be a unit vector from $\mathcal{K}_\varphi$ and note that $vf \in \mathcal{K}_\Theta$ and hence $P_\Theta(vf) = vf$. Thus for all $n \geqslant 1$ we have

$$\begin{aligned} \langle (\varphi(S_\Theta))^n (vf), vf \rangle &= \langle P_\Theta(\varphi^n f v), vf \rangle \\ &= \langle \varphi^n v f, P_\Theta(vf) \rangle \end{aligned}$$

$$= \langle \varphi^n v f, v f \rangle$$
$$= \langle \varphi^n f, f \rangle$$
$$= \langle f, T_{\overline{\varphi}}^n f \rangle.$$

But since $f \in \mathcal{K}_\varphi = \ker T_{\overline{\varphi}}$, this last quantity is equal to zero. This shows that vf is a $\varphi(S_\Theta)$-inner vector. □

When $\Theta(0) = 0$ and $\varphi(z) = z$, notice how this recovers Theorem 6.1. At the other extreme, notice that when $\varphi = \Theta$ then I is a unimodular constant inner function and the theorem above yields $\mathcal{K}_\Theta$ as the complete set of T_Θ-inner functions. Of course this result is obvious once one realizes that $\langle T_\Theta f, f \rangle = 0$ for any $f \in \mathcal{K}_\Theta$ by the definition of the model space $\mathcal{K}_\Theta = (\Theta H^2)^\perp$.

Also observe that one can relax the assumption that $\varphi | \Theta$ and set $I = u / \gcd(\Theta, \varphi)$ and give a more general version of the theorem above.

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Department of Mathematics and Statistics, Old Dominion University, Norfolk, VA 23529

E-mail address: rcheng@odu.edu

Département de mathématiques et de statistique, Université laval, Québec, QC, Canada, G1V 0A6

E-mail address: javad.mashreghi@mat.ulaval.ca

Department of Mathematics and Computer Science, University of Richmond, Richmond, VA 23173, USA

E-mail address: wross@richmond.edu