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The Dual of the Compressed Shift

M. C. Câmara and W. T. Ross

Abstract. For an inner function u, we discuss the dual operator for the compressed shift $P_u S|_{\mathcal{K}_u}$, where $\mathcal{K}_u$ is the model space for u. We describe the unitary equivalence/similarity classes for these duals as well as their invariant subspaces.

1 Introduction

This paper deals with the unitary equivalence classes and the invariant subspaces of the dual operators for the well-known compressed shift operator on a model space. The main tool to explore these results is to connect these dual operators to the bilateral shift on L^2 as well as a direct sum of the unilateral shift and its adjoint.

For an inner function u on $\mathbb{D} := \{|z| < 1\}$, consider the *model space* [11]

$$\mathcal{K}_u := H^2 \cap (uH^2)^{\perp},$$

where H^2 is the Hardy space [10]. By Beurling's theorem, the subspaces uH^2 are the non-zero invariant subspaces of the shift operator

$$(Sf)(z) = zf(z) \tag{1.1}$$

on H^2, and thus, via annihilators, the spaces $\mathcal{K}_u$ are the non-trivial S^*-invariant subspaces of H^2. The operator S^* can be realized as the backward shift

$$(S^* f)(z) = \frac{f(z) - f(0)}{z}. \tag{1.2}$$

As H^2 is a closed subspace of $L^2(\mathbb{T}, d\theta/2\pi)$, one denotes by P_u the orthogonal projection of L^2 onto $\mathcal{K}_u$. The operator

$$S_u := P_u S|_{\mathcal{K}_u},$$

is called the *compressed shift* and plays an important role in operator theory [11, p. 195].

Related to S_u are the *truncated Toeplitz operators* $A_{\varphi}^u := P_u M_{\varphi}|_{\mathcal{K}_u}$, where $\varphi \in L^{\infty}$ and $M_{\varphi} f = \varphi f$ is multiplication by φ on L^2. Note that $A_z^u = S_u$. These truncated

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Toeplitz operators have received considerable attention since their initial introduction in [15] (see also [4, 12]).

The recent papers [3, 7, 8, 9] began an interesting study of the *dual truncated Toeplitz operators* D^u_φ, $\varphi \in L^\infty$, defined on $\mathcal{K}_u^\perp$ by

$$D^u_\varphi := (I - P_u)M_\varphi|_{\mathcal{K}_u^\perp}.$$

Notice that $I - P_u$ is the orthogonal projection of L^2 onto $\mathcal{K}_u^\perp$. Decomposing L^2 as $L^2 = \mathcal{K}_u \oplus \mathcal{K}_u^\perp$, one can think of A^u_φ and its associated dual D^u_φ as parts of the multiplication operator

$$M_\varphi : L^2 = \mathcal{K}_u \oplus \mathcal{K}_u^\perp \longrightarrow L^2, \quad M_\varphi f = \varphi \cdot f,$$

by means of its matrix decomposition

$$M_\varphi = \begin{bmatrix} A^u_\varphi & * \\ * & D^u_\varphi \end{bmatrix}. \tag{1.3}$$

In this paper, we focus on the *dual of the compressed shift* S_u, denoted by

$$D_u := (I - P_u)S|_{\mathcal{K}_u^\perp}. \tag{1.4}$$

By (3), we can understand D_u in terms of matrices as

$$M := \begin{bmatrix} S_u & * \\ * & D_u \end{bmatrix},$$

where $M := M_z$ on L^2 and the matrix above is, with respect to the orthogonal decomposition, $L^2 = \mathcal{K}_u \oplus \mathcal{K}_u^\perp$. There are other contexts of dual operators defined for Toeplitz and subnormal operators [1, 5, 6, 16], and thus these duals enjoy a tradition in operator theory.

Along with a discussion of some basic properties of D_u, we will describe the D_u invariant subspaces of $\mathcal{K}_u^\perp$ as well as the similarity and unitary equivalence properties of D_u and D_v for inner u and v. We will show that when $u(0) = 0$, D_u is unitarily equivalent to $S \oplus S^*$ on $H^2 \oplus H^2$, and thus D_u and D_v are unitarily equivalent whenever $u(0) = v(0) = 0$. When $u(0) \neq 0$, D_u turns out to be similar to M on L^2, and thus D_u is similar to D_v whenever $u(0) \neq 0$, $v(0) \neq 0$. Finally, we show that D_u is unitarily equivalent to D_v precisely when $|u(0)| = |v(0)|$. These results become important when describing the invariant subspaces of D_u (Sections 6 and 7) and have connections to results from [14, 17]. After this paper was completed, we learned of the paper [17], which approaches the D_u-invariant subspaces of $\mathcal{K}_u^\perp$ in a different way.

2 Some Basics

The space $L^2 = L^2(\mathbb{T}, dm)$, where $\mathbb{T}$ is the unit circle and $m = d\theta/2\pi$ on $\mathbb{T}$, is a Hilbert space with inner product $\langle f, g\rangle := \langle f, g\rangle_{L^2}$. The Fourier coefficients of f will be denoted by $\hat{f}(j) = \langle f, \xi^j\rangle$. Viewing the Hardy space H^2 as $\{f \in L^2 : \hat{f}(n) = 0 \ \forall n < 0\}$ and $\overline{H_0^2}$ as $\{\overline{zf} : f \in H^2\}$, note that $L^2 = H^2 \oplus \overline{H_0^2}$. Let P_+ and P_- denote the standard orthogonal projections from L^2 onto H^2 and $\overline{H_0^2}$, respectively.

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For an inner function u, define the model space $\mathcal{K}_u = H^2 \cap (uH^2)^\perp$. Elementary facts about annihilators will verify that

$$\mathcal{K}_u^\perp = \overline{H_0^2} \oplus uH^2.$$

As $\mathcal{K}_u$ is a closed subspace of L^2, we have an orthogonal projection P_u from L^2 onto $\mathcal{K}_u$. A result from [11, p. 124] relates $P_u, I - P_u, P^+$, and P^-.

Lemma 2.1 *If u is inner, then $P_u = P^+ - M_u P^+ M_{\overline{u}} = M_u P^- M_{\overline{u}} P^+$ and $I - P_u = P^- + M_u P^+ M_{\overline{u}}$.*

Any $f \in L^2 = H^2 \oplus \overline{H_0^2}$ can be written uniquely as

$$f = f_+ + f_-, \quad f_+ \in H^2, \quad f_- \in \overline{H_0^2};$$

that is, $f_+ = P^+ f$ and $f_- = P^- f$.

We will also use the notation

$$\varphi_f := \overline{z \overline{f_-}}, \quad f \in L^2. \tag{2.1}$$

Observe that $\varphi_f \in H^2$, and hence is analytic on $\mathbb{D}$, and so we can utilize the quantity $\varphi_f(0)$. A Fourier series argument will show that

$$\varphi_f(0) = \int_{\mathbb{T}} \overline{z} \overline{f_-} \, dm = \overline{\widehat{f_-}(-1)}. \tag{2.2}$$

Any $f \in \mathcal{K}_u^\perp = \overline{H_0^2} \oplus uH^2$ can be written uniquely as

$$f = f_- + u\widetilde{f}_+, \quad f_- \in \overline{H_0^2}, \quad \widetilde{f}_+ \in H^2. \tag{2.3}$$

Lemma 2.1 shows that $f_- = P^- f$ and $\widetilde{f}_+ = P^+(\overline{u} f)$ and a Fourier series argument will verify the following identities.

Lemma 2.2 *For $f \in L^2$, we have*

(i) $P^-(z f_-) = \overline{\varphi_f} - \overline{\varphi_f(0)}$;
(ii) $P^-(\overline{z} f_+) = f_+(0)\overline{z}$;
(iii) $P^+(\overline{z} f_+) = (f_+ - f_+(0))\overline{z}$;
(iv) $P^+(z f_-) = \overline{\varphi_f(0)}$.

Regarding $\mathcal{K}_u$ as a subspace of L^2, we have the following useful result.

Proposition 2.3 *If u is inner, then $\overline{u}\mathcal{K}_u = \overline{z \mathcal{K}_u}$.*

Proof It is a standard fact [11] that the conjugate-linear operator

$$C_u : L^2 \longrightarrow L^2, \quad C_u f = u \overline{z f}, \tag{2.4}$$

is an involutive isometry on L^2 with $C_u \mathcal{K}_u = \mathcal{K}_u$ and $C_u \mathcal{K}_u^\perp = \mathcal{K}_u^\perp$. Thus, $\overline{u}\mathcal{K}_u = \overline{u} C_u \mathcal{K}_u = \overline{z \mathcal{K}_u}$. ■

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The model space $\mathcal{K}_u$ is a reproducing kernel Hilbert space on $\mathbb{D}$ with kernel

$$k_\lambda^u(z) = \frac{1 - \overline{u(\lambda)}u(z)}{1 - \overline{\lambda}z}, \quad \lambda, z \in \mathbb{D},$$

meaning that $f(\lambda) = \langle f, k_\lambda^u \rangle$ for $f \in \mathcal{K}_u$ and $\lambda \in \mathbb{D}$ [11, p. 111].

3 Some Basic Facts About the Dual

In this section, we will develop some useful facts about D_u. We start with a more useful formula for D_u than the one in (1.4).

Proposition 3.1 *If u is inner, then*

$$D_u f = zf - \overline{\varphi_f(0)}k_0^u, \quad f \in \mathcal{K}_u^\perp.$$

Proof For $f = f_- + u\widetilde{f_+} \in \mathcal{K}_u^\perp$, use Lemma 2.1 to see that

$$\begin{aligned} D_u f &= (I - P_u)(zf) = (P^- + uP^+\overline{u})(zf_- + zu\widetilde{f_+}) \\ &= zf_- - \overline{\varphi_f(0)} + uP^+z\overline{u}f_- + zu\widetilde{f_+} \\ &= zf_- + zu\widetilde{f_+} - \overline{\varphi_f(0)} + u\overline{u(0)\varphi_f(0)} \\ &= zf - \overline{\varphi_f(0)}(1 - \overline{u(0)}u) = zf - \overline{\varphi_f(0)}k_0^u. \end{aligned}$$

Note the use of Lemma 2.2 and $\overline{u}f_- = \overline{zu\varphi_f}$ and $\varphi_{\overline{u}f_-} = u\varphi_f$. ■

Corollary 3.2 *If u is inner, then $D_u|_{uH^2} = S|_{uH^2}$, and thus $D_u(uH^2) \subset uH^2$. When $u(0) = 0$, we have $D_u\overline{H_0^2} = \overline{H_0^2}$.*

The definition of D_u from (4) shows that $D_u^* = D_{\overline{z}}^u$. In fact, via the conjugation operator C_u from (4), we have $C_u D_u C_u = D_u^*$ [7]. Proposition 3.1 and the above conjugation identity yield the following proposition.

Proposition 3.3 *If u is inner, then $D_u^* f = \overline{z}f - \widetilde{f_+}(0)C_u k_0^u$. Furthermore, $D_u^*|_{\overline{H_0^2}} = M^*|_{\overline{H_0^2}}$, where $M^* f = \overline{z}f$, and thus $D_u^*\overline{H_0^2} \subset \overline{H_0^2}$. When $u(0) = 0$, we have $D_u^*(uH^2) = uH^2$.*

Here are some interesting facts from [2, 7] about D_u.

Proposition 3.4 *For an inner function u we have the following:*

(i) $\|D_u\| = 1$;
(ii) $\sigma(D_u) = \overline{\mathbb{D}}$ *when* $u(0) = 0$ *while* $\sigma(D_u) = \mathbb{T}$ *when* $u(0) \neq 0$;
(iii) $D_u D_u^* = I - (1 - |u(0)|^2)u \otimes u$.

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4 Unitary Equivalence and Similarity

For two compressed shifts S_u and S_v we know that S_u is unitarily equivalent to S_v if and only if u is a constant unimodular multiple of v. For their duals, they are often unitarily equivalent and even more often similar. This will be an important part of our analysis of their invariant subspaces.

For an inner function u, the authors in [8] define the onto isometry

$$U : L^2 = H^2 \oplus \overline{H_0^2} \longrightarrow \mathcal{K}_u^{\perp} = uH^2 \oplus \overline{H_0^2}, \quad U = \begin{bmatrix} M_u & 0 \\ 0 & I \end{bmatrix}, \tag{4.1}$$

where recall that $M_u f = u \cdot f$ on L^2. A computation in that paper yields the following lemma. For any $\varphi \in L^{\infty}$, recall the definition of the Hankel operator $H_{\varphi} : H^2 \to \overline{H_0^2}$, $H_{\varphi} f = P^-(\varphi f)$ as well as the following formula for its adjoint: $H_{\varphi}^* : \overline{H_0^2} \to H^2$, $H_{\varphi}^* f = P^+(\overline{\varphi} f)$.

Lemma 4.1 *For an inner function u, we have*

$$U^* D_u U = \begin{bmatrix} S & H_{u\bar{z}}^* \\ 0 & Q \end{bmatrix},$$

where S is the shift on H^2 from (1) and $Q : \overline{H_0^2} \to \overline{H_0^2}$, $Qg = P^-(zg)$.

One of the main theorems of this section is the following.

Theorem 4.2 *Let u be an inner function.*

(i) *If $u(0) = 0$, then D_u is unitarily equivalent to the operator*

$$\begin{bmatrix} S & 0 \\ 0 & Q \end{bmatrix} : H^2 \oplus \overline{H_0^2} \longrightarrow H^2 \oplus \overline{H_0^2},$$

and thus for any two inner functions u and v that vanish at 0, the operators D_u and D_v are unitarily equivalent.

(ii) *If $u(0) \neq 0$, then D_u is unitarily equivalent to the operator*

$$\begin{bmatrix} S & \overline{u(0)}(1 \otimes \bar{z}) \\ 0 & Q \end{bmatrix} : H^2 \oplus \overline{H_0^2} \longrightarrow H^2 \oplus \overline{H_0^2}.$$

Proof If $u(0) = 0$, then $H_{u\bar{z}}^* \equiv 0$. Indeed, for $g \in \overline{H_0^2}$,

$$H_{u\bar{z}}^* g = P^+(z\overline{u}g) = P^+\Big(\frac{\overline{u}}{\overline{z}} \cdot g\Big) = 0,$$

since $\overline{u}/\overline{z} \in \overline{H^2}$, and thus $(\overline{u}/\overline{z})g \in \overline{H_0^2}$.

When $u(0) \neq 0$ and $g \in \overline{H_0^2}$, we can use Lemma 2.2(iv) and (2.2) to get

$$H_{u\bar{z}}^* g = P^+(\overline{u}zg) = \overline{\varphi_{\overline{u}g}(0)} = \overline{u(0)\varphi_g(0)} = \overline{u(0)}\hat{g}(-1).$$

But this is the rank one operator $\overline{u(0)}(1 \otimes \bar{z}) : \overline{H_0^2} \to H^2$. ∎

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We can refine this a bit further. Recall S and S^* from (1.1) and (1.2).

Corollary 4.3 *If u is inner and $u(0) = 0$, then D_u is unitarily equivalent to $S \oplus S^*$ on $H^2 \oplus H^2$.*

Proof Via the unitary operator U from (4.1), we see from Theorem 4.2 that D_u is unitarily equivalent to $S \oplus Q$ on $H^2 \oplus \overline{H_0^2}$, where $Qg = P^-(zg)$, $g \in \overline{H_0^2}$. One can quickly check that $W : \overline{H_0^2} \to H^2$, $(Wg)(z) = \overline{g(\overline{z})/z}$ is unitary with $S^* W = WQ$. Thus, the unitary operator $L = I \oplus W : H^2 \oplus \overline{H_0^2} \to H^2 \oplus H^2$ will satisfy $(S \oplus S^*)L = L(S \oplus Q)$. ∎

We will refine this unitarily equivalence result further in Theorem 4.6 below.

As it turns out, all of the operators D_u, when $u(0) \neq 0$, are similar to the bilateral shift $Mf = zf$ on L^2. This important observation will come into play when discussing the invariant subspaces for D_u. To this end, for u inner with $u(0) \neq 0$, define

$$V_u : \mathcal{K}_u^{\perp} \longrightarrow L^2, \quad V_u := P^- + \frac{\overline{u}}{\overline{u(0)}} P^+ \tag{4.2}$$

with inverse

$$V_u^{-1} : L^2 \to \mathcal{K}_u^{\perp}, \quad V_u^{-1} = P^- + u\overline{u(0)}P^+. \tag{4.3}$$

Observe that

$$V_u = P^- + \frac{1}{\overline{u(0)}} P^+ \overline{u}. \tag{4.4}$$

Theorem 4.4 *If u is inner with $u(0) \neq 0$, then $V_u D_u V_u^{-1} = M$ on L^2. Consequently, for and inner u and v with $u(0) \neq 0$, $v(0) \neq 0$, D_u is similar to D_v and $D_u = W^{-1} D_v W$, where $W : \mathcal{K}_u^{\perp} \to \mathcal{K}_v^{\perp}$, $W = P^- + \frac{\overline{v(0)}}{\overline{u(0)}} v P^+ \overline{u}$.*

Proof For $f = f_- + f_+ \in L^2$ use Proposition 3.1 and Lemma 2.2 to obtain

$$\begin{aligned}
V_u D_u V_u^{-1}(f_- + f_+) &= V_u D_u (f_- + \overline{u(0)} u f_+) \\
&= \Big(P^- + \frac{\overline{u}}{\overline{u(0)}} P^+\Big)\big(zf_- + zu\overline{u(0)} f_+ - \overline{\varphi_f(0)} + \overline{\varphi_f(0)u(0)}u\big) \\
&= \overline{\varphi_f} - \overline{\varphi_f(0)} + \frac{\overline{u}}{\overline{u(0)}} \overline{\varphi_f(0)} + zf_+ - \frac{\overline{u}}{\overline{u(0)}} \overline{\varphi_f(0)} + \overline{\varphi_f(0)} \\
&= zf_- + zf_+ = Mf.
\end{aligned}$$

From here it follows that $D_u = W^{-1} D_v W$ with $W = V_v^{-1} V_u$. ∎

Remark 4.5 It is important to point out that although D_u is similar to M when $u(0) \neq 0$, it is not unitarily equivalent to M. This is because M is normal, while D_u is not (Proposition 3.4). It also follows that D_u is not similar to D_v when $u(0) = 0$ and $v(0) \neq 0$ (Proposition 3.4).

We return to the unitary equivalence of D_u and D_v begun in Theorem 4.2.

Theorem 4.6 *If u and v are inner functions, then D_u is unitarily equivalent to D_v if and only if $|u(0)| = |v(0)|$.*

Proof When $u(0) = v(0) = 0$, the result follows from Theorem 4.2. So assume that $u(0)$ and $v(0)$ are both nonzero. Suppose $Z : \mathcal{K}_u^\perp \to \mathcal{K}_v^\perp$ is unitary with $ZD_uZ^* = D_v$. From Proposition 3.4, we have

$$I|_{\mathcal{K}_v^\perp} - (1 - |u(0)|^2)Zu \otimes Zu = ZD_uD_u^*Z^* = D_vD_v^* \\ = I|_{\mathcal{K}_v^\perp} - (1 - |v(0)|^2)v \otimes v,$$

and it follows that

$$(1 - |u(0)|^2)Zu \otimes Zu = (1 - |v(0)|^2)v \otimes v.$$

Apply both sides to the unit vector $v \in vH^2 \subset \mathcal{K}_v^\perp$ and observe that

$$(1 - |u(0)|^2)\langle v, Zu\rangle Zu = (1 - |v(0)|^2)v$$

implying that $Zu = cv$ for some unimodular constant c (because u and v are unit vectors and Z is unitary). The previous equation yields $|u(0)| = |v(0)|$.

Conversely, if $|u(0)| = |v(0)|$, then Theorem 4.4 yields $D_u = W^{-1}D_vW$ where

$$W = P^- + \frac{\overline{v(0)}}{\overline{u(0)}}vP^+\overline{u} \quad \text{and} \quad W^{-1} = P^- + \frac{\overline{u(0)}}{\overline{v(0)}}uP^+\overline{v} = W^*,$$

since $\frac{v(0)}{u(0)} = \overline{\frac{u(0)}{v(0)}}$. Therefore, W is unitary. ∎

5 Invariant Subspaces

We begin our discussion with a few general results.

Proposition 5.1 *Let u be any inner function. A subspace $\mathscr{S} \subset \mathcal{K}_u^\perp$ is D_u-invariant with $\mathscr{S} \subset uH^2$, or equivalently $P^-\mathscr{S} = \{0\}$, if and only if $\mathscr{S} = \gamma uH^2$ for some inner function γ.*

Proof If $\mathscr{S} = \gamma uH^2$, then $\mathscr{S} \subset uH^2$ [11, p. 87] and by Corollary 3.2, $D_u\mathscr{S} = z\mathscr{S} \subset \mathscr{S}$ and so $\mathscr{S}$ is D_u-invariant. Conversely, when $\mathscr{S} \subset uH^2$ is a D_u-invariant, then, again by Corollary 3.2, $S\mathscr{S} \subset \mathscr{S}$. By Beurling's theorem, $\mathscr{S} = \beta H^2$ for some inner β. But $\beta H^2 \subset uH^2$, and so $\beta = \gamma u$.

Lemma 5.2 *For a non-zero subspace $X \subset \overline{H_0^2}$, we have $X = \overline{z\mathcal{K}_\alpha}$ for some inner α if and only if $P^-(zX) \subset X$ and $X \neq \overline{H_0^2}$.* ∎

Proof Observe that $S^*f = P^+(\overline{z}f)$, $f \in H^2$, and so

$$\begin{aligned} & X = \overline{z\mathcal{K}_\alpha} \ \text{ for some } \alpha \\ \iff & \overline{zX} = \mathcal{K}_\alpha \ \text{ for some } \alpha \\ \iff & P^+(\overline{z}(\overline{zX})) \subset \overline{zX} \ \text{ and } \overline{zX} \neq H^2 \\ \iff & zP^+(\overline{z}(\overline{zX})) \subset \overline{X} \ \text{ and } X \neq \overline{H_0^2}. \end{aligned}$$

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Using the identity $P_-(\overline{f}) = \overline{zP^+(\overline{z}f)}$, we see that

$$zP^+(\overline{z}(\overline{z}\overline{X})) \subset \overline{X} \iff P^-(zX) \subset X \text{ and } X \neq \overline{H_0^2}$$

∎

Lemma 5.3 *Let u be any inner function and let $\mathscr{S} \subset \mathcal{K}_u^\perp$ be a D_u-invariant subspace. If $P^-\mathscr{S} \neq \{0\}$ then there is an $f_- \in P^-\mathscr{S}$ such that $\varphi_{f_-}(0) \neq 0$.*

Proof Suppose that for every $f_- \in P^-\mathscr{S} \setminus \{0\}$, with $f_- = P^-f$, $f \in \mathscr{S} \subset \mathcal{K}_u^\perp$, we have $\varphi_f(0) = 0$. From Proposition 3.1 and (2.2), we have

$$P^-(D_u f) = P^-(zf_-) = zf_-,$$

and so $zf_- \in P^-\mathscr{S}$. Thus, by assumption, $z^2 f_- = \overline{\psi_+}$ with $\psi_+ \in H^2$ and $\psi_+(0) = \varphi_f'(0) = 0$. Therefore,

$$P^-(D_u^2 f) = P^-(z^2 f_-) = z^2 f_- \in P^-\mathscr{S}.$$

Continuing in this manner, we see that

$$D_u^n f_- = z^n f_- \in \overline{H_0^2}, \quad n \geq 0,$$

which is impossible if $f_- \neq 0$. ∎

These next two results further examine $P^+\mathscr{S}$ and $P^-\mathscr{S}$.

Proposition 5.4 *Let u be any inner function and $\mathscr{S} \subset \mathcal{K}_u^\perp$ be a D_u-invariant subspace. Then one of the following three possibilities occurs:*

(i) $P^-\mathscr{S} = \{0\}$;

(ii) $P^-\mathscr{S} = \overline{H_0^2}$;

(iii) *there is a non-constant inner function α such that $P^-\mathscr{S} = \overline{z\mathcal{K}_\alpha}$.*

Proof Let $f_- \in P^-\mathscr{S}$. Then there is an $f = f_- + u\widetilde{f_+} \in \mathscr{S}$. Thus,

$$D_u f = zf_- - \overline{\varphi_f(0)} + u(z\widetilde{f_+} + \overline{u(0)\varphi_f(0)}),$$

and so $P^-(D_u f) = zf_- - \overline{\varphi_f(0)} = P^-(zf_-)$. Since $D_u f \in \mathscr{S}$, we have $P^-(zf_-) \in P^-\mathscr{S}$. Apply Lemma 5.2 to $X = P^-\mathscr{S}$ to obtain the result. ∎

Proposition 5.5 *Let u be any inner function and $\mathscr{S} \subset \mathcal{K}_u^\perp$ be a D_u-invariant subspace. If $u \in P^+\mathscr{S}$, then $P^+\mathscr{S} = uH^2$.*

Proof Let $f \in \mathscr{S}$ with $f = f_- + u$. We have $P^+(D_u f) \in P^+\mathscr{S}$ and

$$\begin{aligned} D_u f &= D_u(f_- + u) = zf_- + zu - \overline{\varphi_f(0)} + \overline{\varphi_f(0)u(0)}u \\ &= (zf_- - \overline{\varphi_f(0)}) + zu + \overline{\varphi_f(0)u(0)}u. \end{aligned}$$

Thus, $P^+D_u f = zu + \overline{\varphi_f(0)u(0)}u$ and so $zu = P^+D_u f - \overline{\varphi_f(0)u(0)}u \in P^+\mathscr{S}$. Now let $f_1 \in \mathscr{S}$ be such that $f_1 = f_{1-} + uz$ with $f_{1-} \in \overline{H_0^2}$. Then $P^+D_u f_1 = uz^2 - \overline{\varphi_{f_1}(0)u(0)}u$,

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and it follows that $uz^2 \in P^+\mathscr{S}$. Analogously, we conclude that $z^j u \in P^+\mathscr{S}$ for all $j \geq 0$, and so, since $P^+\mathscr{S} \subset uH^2$, we have $P^+\mathscr{S} = uH^2$. ∎

6 Invariant Subspaces when $u(0) \neq 0$

Theorem 4.4 says that when $u(0) \neq 0$, D_u is similar to M on L^2. Results of Wiener and Helson [13] together describe the M-invariant subspaces $\mathcal{F}$ of L^2 as follows: If $M\mathcal{F} = \mathcal{F}$, then there is a measurable subset $A \subset \mathbb{T}$ such that $\mathcal{F} = \chi_A L^2$, while if $M\mathcal{F} \neq \mathcal{F}$, then $\mathcal{F} = wH^2$ for some $w \in L^\infty$ with $|w| = 1$ almost everywhere on $\mathbb{T}$. This yields the following theorem.

Theorem 6.1 *Suppose u is inner, $u(0) \neq 0$, and $\mathscr{S}$ is a D_u-invariant subspace of $\mathcal{K}_u^\perp$. When $D_u\mathscr{S} = \mathscr{S}$, there is a measurable $A \subset \mathbb{T}$ for which*

$$\mathscr{S} = (P^- + u\overline{u(0)}P^+)\chi_A L^2.$$

When $D_u\mathscr{S} \neq \mathscr{S}$,

$$\mathscr{S} = (P^- + u\overline{u(0)}P^+)wH^2,$$

for some $w \in L^\infty$ with $|w| = 1$ almost everywhere on $\mathbb{T}$.

From $P^- + P^+ = I$, we see that any D_u-invariant $\mathscr{S}$ takes the form

$$\{g - k_0^u P^+ g : g \in \mathcal{F}\},$$

where $\mathcal{F}$ is an M-invariant subspace of L^2.

Below are a few examples of

$$(P^- + u\overline{u(0)}P^+)(wH^2) \tag{6.1}$$

for choices of inner u with $u(0) \neq 0$ and $w = \overline{\alpha}\beta$ for inner α and β.

Example 6.2 Let u be inner with $u(0) \neq 0$. If $\alpha \equiv 1$ and β is any inner function, then

$$(P^- + u\overline{u(0)}P^+)(\beta H^2) = u\beta H^2.$$

Observe how this connects to Proposition 5.1.

Example 6.3 Let u be inner with $u(0) \neq 0$. If $\beta \equiv 1$ and α is any inner function, then

$$\begin{aligned}(P^- &+ u\overline{u(0)}P^+)(\overline{\alpha}H^2)\\ &= \{(P^- + u\overline{u(0)}P^+)(\overline{\alpha}f_+) : f_+ \in H^2\}\\ &= \{(P^- + u\overline{u(0)}P^+)(\overline{\alpha}(k + \alpha g_+)) : k \in \mathcal{K}_\alpha, g_+ \in H^2\}\\ &= \{(P^- + u\overline{u(0)}P^+)(\overline{\alpha}k + g_+) : k \in \mathcal{K}_\alpha, g_+ \in H^2\}.\end{aligned}$$

From Proposition 2.3, notice that for any $k \in \mathcal{K}_\alpha$ we have $\overline{\alpha}k \in \overline{H_0^2}$, and so $P^-(\overline{\alpha}k) = \overline{\alpha}k$ and $P^+(\overline{\alpha}k) = 0$. Apply Proposition 2.3 to get

$$\left(P^- + u\overline{u(0)}P^+\right)(\overline{\alpha}H^2) = \overline{\alpha}\mathcal{K}_\alpha \oplus uH^2 = \overline{z\mathcal{K}_\alpha} \oplus uH^2.$$

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Example 6.4 Let $\lambda \in \mathbb{D}\backslash\{0\}$ and

$$u(z) = \alpha(z) = \frac{z-\lambda}{1-\overline{\lambda}z}, \quad \beta(z) = z.$$

Then for any $f_+ \in H^2$,

$$\begin{aligned}
&(P^- + u\overline{u(0)}P^+)(\overline{\alpha}\beta f_+)\\
&= \Big(P^- + \frac{z-\lambda}{1-\overline{\lambda}z}(-\lambda)P^+\Big)\Big(\frac{1-\overline{\lambda}z}{z-\lambda} z f_+\Big)\\
&= P^-\Big(\frac{1-\overline{\lambda}z}{z-\lambda} z f_+\Big) - \lambda\frac{z-\lambda}{1-\overline{\lambda}z}P^+\Big(\frac{1-\overline{\lambda}z}{z-\lambda} z f_+\Big)\\
&= \Big\{\frac{\overline{z}}{1-\lambda\overline{z}}\lambda(1-|\lambda|^2)f_+(\lambda) + \lambda\Big(zf_+ - \frac{\lambda(1-|\lambda|^2)}{1-\overline{\lambda}z}f_+(\lambda)\Big) : f_+ \in H^2\Big\}.
\end{aligned}$$

The above is a proper subspace of $\overline{z\mathcal{K}_\alpha} \oplus uH^2$. Indeed, $z-\lambda \in \overline{z\mathcal{K}_\alpha} \oplus uH^2$ but there is no $f_+ \in H^2$ for which

$$z - \lambda = \frac{\overline{z}}{1-\lambda\overline{z}}\lambda(1-|\lambda|^2)f_+(\lambda) + \lambda\Big(zf_+ - \frac{\lambda(1-|\lambda|^2)}{1-\overline{\lambda}z}f_+(\lambda)\Big).$$

If there were such an f_+, then due to the uniqueness of orthogonal decomposition above, $f_+(\lambda) = 0$. This would mean that $z - \lambda = \lambda z f_+(z)$ for which there is no such $f_+ \in H^2$.

One can only go so far with these types of examples from (6.1) since there are examples of unimodular w which are not the quotient of two inner functions.

Corollary 6.5 *Let u be inner with $u(0) \neq 0$. If $\mathscr{S} \subset \mathcal{K}_u^\perp$, then $P^- V\mathscr{S} = P^-\mathscr{S}$.*

Proof If $g_- \in P^- V\mathscr{S}$, there is an $h \in \mathscr{S}$ such that

$$\begin{aligned}
g_- &= P^-\Big(P^- + \frac{\overline{u}}{\overline{u(0)}}P^+\Big)h = P^-h + P^-\frac{\overline{u}}{\overline{u(0)}}P^+h\\
&= P^-h + P^-\frac{\overline{u}}{\overline{u(0)}}\underbrace{(uh_1)}_{P^+h\in uH^2} = P^-h.
\end{aligned}$$

Thus, $P^-V\mathscr{S} \subset P^-\mathscr{S}$.

Conversely, if $h_- \in P^-\mathscr{S}$, there exists an $h \in \mathscr{S}$ such that $h_- = P^-h$. Thus for

$$g = \Big(P^- + \frac{\overline{u}}{\overline{u(0)}}P^+\Big)h \in V\mathscr{S},$$

we have $P^-g = h_-$. Thus, $P^-\mathscr{S} \subset P^-V\mathscr{S}$. ∎

Corollary 6.6 *Let u be inner with $u(0) \neq 0$. If $\mathscr{S} \subset \mathcal{K}_u^\perp$ is a D_u-invariant subspace and $\{0\} \subsetneq P^-\mathscr{S} \subsetneq \overline{H_0^2}$, then $\mathscr{S} = V^{-1}(\overline{\alpha}\beta H^2)$ for two coprime inner functions α and β.*

Proof By Proposition 5.4, we have $P^- \mathscr{S} = \overline{z\mathcal{K}_\alpha}$ for some inner function α and by Corollary 6.5, $P^- V\mathscr{S} = P^-\mathscr{S} = \overline{\alpha}\mathcal{K}_\alpha$. Thus,

$$V\mathscr{S} = (P^- + P^+)V\mathscr{S} \subset P^- V\mathscr{S} \oplus P^+ V\mathscr{S} = \overline{\alpha}K_\alpha \oplus P^+ V\mathscr{S}.$$

Thus, $\alpha V\mathscr{S} \subset \mathcal{K}_\alpha \oplus \alpha P^+ V\mathscr{S} \subset H^2$. By Theorem 4.4, $\alpha V\mathscr{S}$ is an S-invariant subspace of H^2 that means that $\alpha V\mathscr{S} = \beta H^2$ for some inner function β. Dividing out by any common inner factors between α and β, we can assume that α and β are coprime. Thus, $\mathscr{S} = V^{-1}(\overline{\alpha}\beta H^2)$. ∎

Corollary 6.7 *Let $\mathcal{F}$ be an M-invariant subspace of L^2 that is not of the form $\overline{\alpha}\beta H^2$ for inner α and β. Then $\mathscr{S} = V^{-1}\mathcal{F}$ is a D_u-invariant subspace with $P^-\mathscr{S} = \overline{H^2_0}$.*

Remark 6.8 (i) The theorems in this section identify $P^-\mathscr{S}$ and $P^+\mathscr{S}$ separately. It is interesting that $\mathscr{S}$ can be a proper subset of $P^-\mathscr{S} \oplus P^+\mathscr{S}$ that seems to create a rich invariant subspace structure.

(ii) If $u(0) \neq 0$ and $\mathscr{S} \neq \{0\}$, we do not have $P^+\mathscr{S} = \{0\}$. Indeed, this would mean that $\mathscr{S} \subset \overline{H^2_0}$. However, for any $f_- \in \mathscr{S}$, we would have

$$D_u f_- = zf_- - \overline{\varphi_f(0)} + \overline{u(0)\varphi_f(0)}u \notin \overline{H^2_0}$$

if $\varphi_f(0) \neq 0$ (Lemma 5.3).

7 Invariant Subspaces when $u(0) = 0$

We characterized the D_u-invariant subspaces of $\mathcal{K}_u^\perp$ when $u(0) \neq 0$. We now discuss the $u(0) = 0$ case.

Proposition 7.1 *Let u be inner with $u(0) = 0$. If α and γ are inner, then $\overline{z\mathcal{K}_\alpha} \oplus \gamma u H^2$ is a D_u-invariant subspace of $\mathcal{K}_u^\perp$.*

Proof Let $f = \overline{zk} + \gamma u h$, where $k \in \mathcal{K}_\alpha$, $h \in H^2$. Proposition 3.1 yields

$$\begin{aligned} D_u(\overline{zk} + \gamma u h) &= (\overline{k} - \overline{k(0)}) + z\gamma u h \\ &= \overline{z} \cdot \frac{\overline{k} - \overline{k(0)}}{\overline{z}} + z\gamma u h \in \overline{z\mathcal{K}_\alpha} + \gamma u H^2, \end{aligned}$$

where we took into account that $k \in \mathcal{K}_\alpha \implies \overline{z}(k - k(0)) \in \mathcal{K}_\alpha$. ∎

Proposition 7.2 *Suppose u is inner with $u(0) = 0$ and $\mathscr{S} \subset \mathcal{K}_u^\perp$ is D_u-invariant. Then either $P^+\mathscr{S} = \{0\}$ or $P^+\mathscr{S} = \gamma u H^2$ where γ is inner.*

Proof Let $P^+\mathscr{S} \neq \{0\}$ and $f = f_- + u\widetilde{f_+} \in \mathscr{S}$. Then

$$P^+(D_u f) = u(z\widetilde{f_+} + \overline{u(0)\varphi_f(0)}) = zu\widetilde{f_+} \in P^+\mathscr{S}.$$

Thus, $P^+\mathscr{S}$ (which is a subspace of uH^2) is a non-zero S-invariant subspace and thus, by Beurling's Theorem, $P^+\mathscr{S} = \gamma u H^2$ for some inner γ. ∎

Proposition 7.1 does not describe all the D_u-invariant subspaces of $\mathcal{K}_u^{\perp}$. To get a better understanding where the complication lies, and since this is an interesting problem in its own right, let us cast this in an equivalent setting. From Corollary 4.3, a description of the D_u-invariant subspaces of $\mathcal{K}_u^{\perp}$ will yield a description of the $S \oplus S^*$-invariant subspaces of $H^2 \oplus H^2$. One can also check that the unitary operator that makes these two operators equivalent takes the D_u-invariant subspace $\gamma u H^2 \oplus \overline{z\mathcal{K}_\alpha}$ to the $S \oplus S^*$-invariant subspace $\gamma u H^2 \oplus \mathcal{K}_\alpha$. However, these are not all of them.

Example 7.3 For $a \in \mathbb{D} \setminus \{0\}$ consider the $S \oplus S^*$-invariant subspace generated by

$$\frac{1}{1-\overline{a}z} \oplus \frac{1}{1-\overline{a}z};$$

that is,

$$\bigvee \Big\{ (S \oplus S^*)^n \Big(\frac{1}{1-\overline{a}z} \oplus \frac{1}{1-\overline{a}z} \Big) : n \geq 0 \Big\}.$$

For any polynomial $p(z)$, we have

$$p(S \oplus S^*) \Big(\frac{1}{1-\overline{a}z} \oplus \frac{1}{1-\overline{a}z} \Big) = \frac{p(z)}{1-\overline{a}z} \oplus \frac{p(\overline{a})}{1-\overline{a}z}.$$

If $\{p_n\}_{n \geq 1}$ is a sequence of polynomials with

$$p_n(S \oplus S^*) \Big(\frac{1}{1-\overline{a}z} \oplus \frac{1}{1-\overline{a}z} \Big) \longrightarrow f \oplus g$$

in $H^2 \oplus H^2$, one can argue that $p_n(z) \to (1-\overline{a}z)f$ in the norm of H^2 and thus $p_n(\overline{a}) \to (1-\overline{a}^2) f(\overline{a})$. Thus,

$$\bigvee \Big\{ (S \oplus S^*)^n \Big(\frac{1}{1-\overline{a}z} \oplus \frac{1}{1-\overline{a}z} \Big) : n \geq 0 \Big\} = \Big\{ f \oplus \frac{f(\overline{a})(1-\overline{a}^2)}{1-\overline{a}z} : f \in H^2 \Big\}.$$

This subspace is contained in $H^2 \oplus K_\alpha$, where

$$\alpha(z) = \frac{z-a}{1-\overline{a}z},$$

but the containment is proper. Indeed, we have

$$1 \oplus \frac{1}{1-\overline{a}z} \in H^2 \oplus \mathcal{K}_\alpha.$$

However,

$$1 \oplus \frac{1}{1-\overline{a}z} \notin \Big\{ f \oplus \frac{f(\overline{a})(1-\overline{a}^2)}{1-\overline{a}z} : f \in H^2 \Big\}.$$

This leads to the question: What are the invariant subspaces of $S \oplus S^*$?

8 Orthogonal Sums

A complicating factor is that for a D_u-invariant subspace $\mathscr{S}$, we canot have $P^{\pm}\mathscr{S} \subset \mathscr{S}$. We always have $\mathscr{S} \subset P^{-}\mathscr{S} \oplus P^{+}\mathscr{S}$, but Example 6.4 shows this containment can be proper. Our main theorem is the following.

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Theorem 8.1 *Let u be a inner and $\mathscr{S}$ be a non-trivial D_u-invariant subspace of the form $\mathscr{S} = X_- \oplus Y_+$, where X_- is a closed subspace of $\overline{H_0^2}$ and Y_+ is a closed subspace of uH^2.*

(i) *If $u(0) \neq 0$, then $\mathscr{S}$ takes one of the forms: γuH^2 or $\overline{z\mathcal{K}_\alpha} \oplus uH^2$, where γ and α are inner.*

(ii) *If $u(0) = 0$, then $\mathscr{S}$ takes one of the following forms: $\overline{H_0^2}$, $\overline{z\mathcal{K}_\alpha}$, γuH^2, $\overline{H_0^2} \oplus \gamma uH^2$, or $\overline{z\mathcal{K}_\alpha} \oplus \gamma uH^2$, where γ and α are inner.*

Proof *Proof of (i).* By Proposition 5.1 we see that if $X_- = \{0\}$, then $Y_+ = \gamma uH^2$. On the other hand, if $X_- \neq \{0\}$, then by Lemma 5.3 there is an $f_- \in X_- \subset \mathscr{S}$ such that for $\varphi_f = \overline{z f_-}$, we have $\varphi_f(0) \neq 0$. Furthermore, $D_u f_- = z f_- - \overline{\varphi_f(0)} + \overline{u(0)\varphi_f(0)}u \in \mathscr{S}$. Therefore,

$$P^-(D_u f_-) = z f_- - \overline{\varphi_f(0)} \in X_- \subset \mathscr{S}.$$

These equations imply that $u \in \mathscr{S}$. Proposition 5.5 implies $Y_+ = uH^2$. Proposition 5.4 says that either $X_- = \overline{H_0^2}$, which yields

$$\mathscr{S} = X_- \oplus Y_+ = \overline{H_0^2} \oplus uH^2 = \mathcal{K}_u^\perp$$

or $X_- = \overline{z\mathcal{K}_\alpha}$, which implies $\mathscr{S} = \overline{z\mathcal{K}_\alpha} \oplus uH^2$.

Proof of (ii). Proposition 7.2 says that either $Y_+ = \{0\}$ or $Y_+ = \gamma uH^2$ for some inner γ. Thus $\mathscr{S} = X_-$ or $\mathscr{S} = X_- \oplus \gamma uH^2$. Proposition 5.4 says that either $X_- = \{0\}$, $X_- = \overline{H_0^2}$, or $X_- = \overline{z\mathcal{K}_\alpha}$. ■

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