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The square roots of some classical operators

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THE SQUARE ROOTS OF SOME CLASSICAL OPERATORS

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Abstract. In this paper we give complete descriptions of the set of square roots of certain classical operators, often providing specific formulas. The classical operators included in this discussion are the square of the unilateral shift, the Volterra operator, certain compressed shifts, the unilateral shift plus its adjoint, the Hilbert matrix, and the Cesàro operator.

1. Introduction

If $\mathcal{H}$ is a complex Hilbert space and $A \in \mathcal{B}(\mathcal{H})$, the bounded linear operators on $\mathcal{H}$, when does A have a square root? By this we mean, does there exist a $B \in \mathcal{B}(\mathcal{H})$ such that $B^2 = A$? If A has a square root, can we describe $\{B \in \mathcal{B}(\mathcal{H}) : B^2 = A\}$, the set of all the square roots of A?

First let us make the, perhaps unexpected, observation that not every operator has a square root. For example, Halmos showed that the unilateral shift $Sf = zf$ on the Hardy space H^2 [5] does not have a square root [7]. Other examples of operators constructed with the shift S and its adjoint S^*, for example $S \oplus S^*$ and $S \otimes S^*$, also do not have square roots [3]. See the papers [9, 13, 14, 19, 20] for some general results concerning square roots of operators.

Second, many operators have an abundance of square roots. For example, any nilpotent operator of order two is a square root of the zero operator. Moreover, to highlight their abundance, Lebow proved (see [8, Prob. 111]) that when $\dim \mathcal{H} = \infty$, the set $\{A \in \mathcal{B}(\mathcal{H}) : A^2 = 0\}$ is dense in $\mathcal{B}(\mathcal{H})$ in the strong operator topology.

Much of the work on square roots has focused on the general topic of which operators have square roots and the prevalence of types of square roots (pth roots and logarithms) in $\mathcal{B}(\mathcal{H})$. Previous papers also have results

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which explore the relationship between the type of square root as related to the type of operator. In this paper, we focus on a collection of some well-known classical operators and proceed to characterize *all* of their square roots. The classical operators included in this discussion are the square of the unilateral shift (Theorem 2.4), the Volterra operator (Theorem 3.2), certain compressed shifts (Theorem 4.1), the unilateral shift plus its adjoint (Theorem 5.2), the Hilbert matrix (Theorem 6.2), and the Cesàro operator (Theorem 7.2 and Theorem 7.4). Our work on the Cesàro operator answers a question posed in [13] and stems from a question posed by Halmos.

2. Square roots of S^2

Suppose that $(Sf)(z) = zf(z)$ denotes the unilateral shift on the Hardy space H^2 [12]. In this section we explore the square roots of S^2. One square root of S^2 is, of course, S itself. Our characterization of *all* of the square roots of S^2, requires a few preliminaries.

For $g \in H^2$, let

$$g_e(z) := \tfrac{1}{2}(g(z) + g(-z)) \quad \text{and} \quad g_o(z) := \tfrac{1}{2}(g(z) - g(-z))$$

and observe that $g(z) = g_e(z) + g_o(z)$. If $g(z) = \sum_{n=0}^{\infty} a_n z^n$, then

$$g_e(z) = \sum_{k=0}^{\infty} a_{2k} z^{2k} \quad \text{and} \quad g_o(z) = \sum_{k=0}^{\infty} a_{2k+1} z^{2k+1}.$$

This is the "even" and "odd" decomposition of g since $g_e(-z) = g_e(z)$ and $g_o(-z) = -g_o(z)$. Finally, let

$$(Wg)(z) = \begin{bmatrix} \displaystyle\sum_{k=0}^{\infty} a_{2k} z^k \\ \displaystyle\sum_{k=0}^{\infty} a_{2k+1} z^k \end{bmatrix}$$

and note that W is a unitary operator from H^2 onto

$$H^2 \oplus H^2 = \Big\{ \begin{bmatrix} f \\ g \end{bmatrix} : f, g \in H^2 \Big\},$$

with

$$W^* \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = g_1(z^2) + z g_2(z^2).$$

Our last bit of notation is the vector-valued shift

$$S \oplus S : H^2 \oplus H^2 \to H^2 \oplus H^2, \quad (S \oplus S) \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} S g_1 \\ S g_2 \end{bmatrix}.$$

It is traditional to think of $S \oplus S$ in matrix form as

$$S \oplus S = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix}.$$

The above formulas yield the following well-known fact.

Proposition 2.1. $W^*(S \oplus S)W = S^2$.

Some other well-known facts used in the section involve the commutants of S and $S \oplus S$. For $\varphi \in H^\infty$, the bounded analytic functions on $\mathbb{D}$, the Toeplitz (Laurent) operator $T_\varphi g = \varphi g$ is bounded on H^2 and $ST_\varphi = T_\varphi S$. Let

$$\{S\}' = \{A \in \mathcal{B}(H^2) : SA = AS\}$$

denote the commutant of S. The following fact is standard [21, Thm. 3.4].

Proposition 2.2. $\{S\}' = \{T_\varphi : \varphi \in H^\infty\}$.

In a similar way, let

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix},$$

where $\Phi_{ij} \in H^\infty$. We use the notation $M_2(H^\infty)$ for the 2×2 matrices above with H^∞ entries. Define $T_\Phi : H^2 \oplus H^2 \to H^2 \oplus H^2$ by

$$T_\Phi \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} \Phi_{11} g_1 + \Phi_{12} g_2 \\ \Phi_{21} g_1 + \Phi_{22} g_2 \end{bmatrix}.$$

A calculation shows that $(S \oplus S)T_\Phi = T_\Phi(S \oplus S)$. Similar to the above, we have the following [21, Cor. 3.20].

Proposition 2.3. $\{S \oplus S\}' = \{T_\Phi : \Phi \in M_2(H^\infty)\}$.

Here is the main theorem of this section describing all of the square roots of S^2.

Theorem 2.4. *For $Q \in \mathcal{B}(H^2)$ the following are equivalent.*

(i) $Q^2 = S^2$.

(ii) There is a 2×2 constant unitary matrix U and functions $a, b, c \in H^\infty$ satisfying

$$za^2 + bc = 1 \tag{2.5}$$

such that

$$Q = W^* U^* \begin{bmatrix} za & b \\ zc & -za \end{bmatrix} UW. \tag{2.6}$$

Proposition 2.1 shows that to prove Theorem 2.4, it suffices to prove the following.

Theorem 2.7. *For $A \in \mathcal{B}(H^2 \oplus H^2)$ the following are equivalent.*

(i) $A^2 = S \oplus S$.

(ii) There is a 2×2 constant unitary matrix U and functions $a, b, c \in H^\infty$ satisfying

$$za^2 + bc = 1 \tag{2.8}$$

such that

$$A = U^* \begin{bmatrix} za & b \\ zc & -za \end{bmatrix} U. \tag{2.9}$$

A matrix calculation shows that the operator A from (2.9) satisfies

$$\begin{aligned} A^2 &= U^* \begin{bmatrix} za & b \\ zc & -za \end{bmatrix} UU^* \begin{bmatrix} za & b \\ zc & -za \end{bmatrix} U \\ &= U^* \begin{bmatrix} z^2a^2 + zbc & 0 \\ 0 & z^2a^2 + zbc \end{bmatrix} U \\ &= U^* \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} U \\ &= U^*(S \oplus S)U \\ &= S^2. \end{aligned}$$

In the above, we use the fact that any constant matrix commutes with $S \oplus S$. Thus every operator of the form (2.9) is a square root of $S \oplus S$. The rest of this section will be devoted to proving the converse – and providing some instances of this characterization.

Our proof involves a few more preliminaries. The first is a simple fact about square roots of bounded Hilbert space operators.

Lemma 2.10. *If $B \in \mathcal{B}(\mathcal{H})$ and $A^2 = B$, then $A \in \{B\}'$.*

Proof. Note that $AB = AA^2 = A^2A = BA$. □

Combining this with the discussion above, we see that if $Q \in \mathcal{B}(H^2)$ with $Q^2 = S^2$, then $WQW^* \in \mathcal{B}(H^2 \oplus H^2)$ with $(WQW^*)^2 = S \oplus S$. It follows from Lemma 2.10 and Proposition 2.3 that

$$WQW^* = A, \quad A \in M_2(H^\infty).$$

To identify A, let us start with a lemma about 2×2 matrices $M_2(\mathbb{C})$ of complex numbers. For $X, Y \in M_2(\mathbb{C})$ let

$$[X, Y]^+ := XY + YX.$$

One can quickly verify the following useful facts about the subspace

$$\mathcal{S} = \left\{ \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix} : \alpha, \beta, \gamma \in \mathbb{C} \right\}. \tag{2.11}$$

Lemma 2.12. *Let $\alpha, \beta, \gamma, \lambda \in \mathbb{C}$.*

(i) If

$$X = \begin{bmatrix} \alpha & \beta \\ 0 & \gamma \end{bmatrix}$$

and $X^2 = 0$, then $\alpha = \gamma = 0$, in other words, $X^2 = 0$ if and only if

$$X = \begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix}.$$

(ii) If

$$X = \begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix}$$

and $Y \in M_2(\mathbb{C})$ with $[X, Y]^+ = \lambda I_2$ with $\lambda \neq 0$, then $\beta \neq 0$ and

$$Y = \begin{bmatrix} \alpha & \eta \\ \lambda/\beta & -\alpha \end{bmatrix},$$

where $\alpha, \eta \in \mathbb{C}$ are arbitrary.

(iii) If

$$X = \begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix}$$

with $\beta \neq 0$ and and $Y \in M_2(\mathbb{C})$ with $[X, Y]^+ = 0$, then

$$Y = \begin{bmatrix} \alpha & \eta \\ 0 & -\alpha \end{bmatrix},$$

where $\alpha, \eta \in \mathbb{C}$ are arbitrary.

(iv) If $X, Y \in \mathcal{S}$, then X^2 and $[X, Y]^+ \in \mathbb{C}I_2$.

For a sequence $(A_k)_{k=0}^{\infty}$, where $A_k \in M_2(\mathbb{C})$ for all $k \geqslant 0$, consider the formal sum

$$A = \sum_{k=0}^{\infty} A_k (S \oplus S)^k.$$

Each term $A_k(S \oplus S)^k$ belongs to $\mathcal{B}(H^2 \oplus H^2)$ as does each partial sum of the series above. If we suppose that the series above converges in the strong operator topology, then $A \in \mathcal{B}(H^2 \oplus H^2)$. Suppose $U \in M_2(\mathbb{C})$ is a constant unitary matrix. A simple 2×2 matrix calculation shows that

$$U(S \oplus S)^k = (S \oplus S)^k U \quad \text{for all } k \geqslant 0.$$

This yields the important identity

$$UAU^* = \sum_{k=0}^{\infty} UA_kU^*(S \oplus S)^k. \tag{2.13}$$

Proof of Theorem 2.4. We will prove Theorem 2.7. Proposition 2.1 will then imply Theorem 2.4.

Let $A \in \mathcal{B}(H^2 \oplus H^2)$ with $A^2 = S \oplus S$. Lemma 2.10 and Proposition 2.3 together show that

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

where a, b, c, $d \in H^\infty$. Let a_k, b_k, c_k, d_k denote the Taylor coefficients of a, b, c, d respectively and define

$$A_k = \begin{bmatrix} a_k & b_k \\ c_k & d_k \end{bmatrix} \in M_2(\mathbb{C}), \quad k \geqslant 0.$$

Notice that

$$A = \sum_{k=0}^{\infty} A_k(S \oplus S)^k. \tag{2.14}$$

For the matrix A_0, Schur's theorem provides us with a unitary matrix U such that UA_0U^* is upper triangular. By (2.13) (and unitary equivalence) we can always assume that A is a square root of $(S \oplus S)$ with A_0 being upper triangular. As a reminder, the convergence of the series above is in the strong operator topology.

Since $A^2 = S \oplus S$ then

$$\begin{aligned} S \oplus S &= A^2 \\ &= \Big(\sum_{k=0}^{\infty} A_k (S\oplus S)^k\Big)^2 \\ &= \sum_{k=0}^{\infty} \Big(\sum_{m=0}^{k} A_m\, A_{k-m}\Big)(S\oplus S)^k \\ &= A_0^2 + [A_0, A_1]^+ (S\oplus S) \\ &\quad + \sum_{k=3,k \text{ odd}}^{\infty} \Big([A_0, A_k]^+ + \sum_{m=1}^{\left[\frac{k}{2}\right]} [A_m\,,\, A_{k-m}]^+\Big)(S\oplus S)^k \\ &\quad + \sum_{k=2,k \text{ even}}^{\infty} \Big([A_0, A_k]^+ + \sum_{m=1}^{\left[\frac{k}{2}\right]-1} [A_m\,,\, A_{k-m}]^+ + A_{\frac{k}{2}} A_{\frac{k}{2}}\Big)(S\oplus S)^k. \end{aligned}$$

The expansion in (2.14) is unique since $S \oplus S$ is diagonal with the same entires along the diagonal. Moreover, the Cauchy product (and gathering up like terms) is also justified since $S \oplus S$ commutes with each of the A_k. Comparing the operator coefficients in front of each $(S \oplus S)^k$ we have

$$A_0^2 = 0, \tag{2.15}$$

$$[A_0, A_1]^+ = I_2 \quad (2 \times 2 \text{ identity matrix}), \tag{2.16}$$

$$[A_0, A_k]^+ = -\sum_{m=1}^{\left[\frac{k}{2}\right]} [A_m\,,\, A_{k-m}]^+, \quad \text{for } k \geqslant 2, k \text{ odd}, \tag{2.17}$$

$$[A_0, A_k]^+ = -\sum_{m=1}^{\left[\frac{k}{2}\right]-1} [A_m\,,\, A_{k-m}]^+ - A_{\frac{k}{2}} A_{\frac{k}{2}} \quad \text{for } k \geqslant 2, k \text{ even}. \tag{2.18}$$

Now we will inductively find a formula for A.

The matrix A_0 is upper triangular. By (2.15) and Lemma 2.12,

$$A_0 = \begin{bmatrix} 0 & b_0 \\ 0 & 0 \end{bmatrix}.$$

By (2.16) and Lemma 2.12 we get $b_0 \neq 0$ and

$$A_1 = \begin{bmatrix} a_1 & b_1 \\ 1/\alpha & -a_1 \end{bmatrix}$$

for arbitrary a_1, b_1.

We will now use induction to prove that $A_k \in \mathcal{S}$. The base cases A_0, A_1 belong to $\mathcal{S}$. By Lemma 2.12, right hand side of (2.17) or (2.18) are constant multiples of the identity operator I on $H^2 \oplus H^2$. Thus, by Lemma 2.12, $A_k \in \mathcal{S}$.

By the expansion

$$A = \sum_{k=0}^{\infty} A_k (S \oplus S)^k,$$

and the fact that each $A_k \in \mathcal{S}$, yields $a, b, c \in H^\infty$ such that

$$A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$$

with $a(0) = 0$, $c(0) = 0$ and $b(0) \neq 0$. Since

$$S \oplus S = \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}^2 = \begin{bmatrix} a^2 + bc & 0 \\ 0 & a^2 + bc \end{bmatrix},$$

it follows that $a^2 + bc = z$. Equivalently, by relabeling a, b, c, we can write

$$A = \begin{bmatrix} za & b \\ zc & -za \end{bmatrix},$$

where $a, b, c \in H^\infty$ with $za^2 + bc = 1$.

The converse was shown earlier. □

Remark 2.19. (i) Since unitary operators preserve determinants, every square root A of $S \oplus S$ will satisfy $\det A = -z$.

(ii) It follows from Proposition 2.3 and Proposition 2.1 that every $B \in \{S^2\}'$ is of the form $(Bg)(z) = \varphi(z) g_e(z) + \psi(z) g_o(z)$ for some $\varphi, \psi \in H^\infty$. This is an interesting (and known) fact.

(iii) Taking $U = I_2$ (the 2×2 identity matrix in $M_2(\mathbb{C})$) in Theorem 2.4 yields the following class of square roots Q of S^2:

$$(Qg)(z) = \big(z^2 a(z^2) + z^3 c(z^2)\big) g_e(z) + \big(b(z^2) - z^3 a(z^2)\big) \frac{g_o(z)}{z}, \tag{2.20}$$

where $za^2 + bc = 1$. Setting $a \equiv 0, b \equiv 1, c \equiv 1$,we get

$$(Qg)(z) = z^3 g_e(z) + \frac{g_o(z)}{z}.$$

With respect to the standard basis $(z^n)_{n=0}^{\infty}$ for H^2, the operator Q has the matrix representation

$$[Q] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

(iv) Taking

$$U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

in Theorem 2.4, yields another class of square roots Q of S^2:

$$(Qg)(z) = (zb(z^2) - z^2a(z^2))g_e(z) + (z^2a(z^2) + zc(z^2))g_o(z). \tag{2.21}$$

Setting $a \equiv 1$, $b(z) = \sqrt{1-z}$, $c(z) = \sqrt{1-z}$, this becomes

$$(Qg)(z) = (z\sqrt{1-z^2} - z^2)g_e(z) + (z^2 + z\sqrt{1-z^2})g_e(z).$$

With respect to the standard basis, Q has the matrix representation,

$$[Q] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -\frac{1}{2} & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -\frac{1}{2} & -1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -\frac{1}{8} & 0 & -\frac{1}{2} & 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -\frac{1}{8} & 0 & -\frac{1}{2} & -1 & 1 & 0 & 0 & 0 & \cdots \\ -\frac{1}{16} & 0 & -\frac{1}{8} & 0 & -\frac{1}{2} & 1 & 1 & 0 & 0 & \cdots \\ 0 & -\frac{1}{16} & 0 & -\frac{1}{8} & 0 & -\frac{1}{2} & -1 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

(v) In (2.21) take $a \equiv 0$ and $b = c \equiv 1$ to get

$$(Qg)(z) = zg_e(z) + zg_0(z) = zg(z)$$

which is just the "obvious" square root of S^2, namely S.

(vi) If $a(\mathbb{D}) \subseteq \mathbb{D}$ (a analytic self map of $\mathbb{D}$), then $1 - za(z)^2$ is outer and thus $\sqrt{1 - za(z)^2}$ is a bounded analytic function on $\mathbb{D}$. With $b(z) = c(z) = \sqrt{1 - za(z)^2}$ (and e.g., $U = I_2$), we can produce an infinite class of square roots Q from (2.20) and (2.21) as

$$(Qg)(z) = \left(z^2a(z^2) + z^3\sqrt{1 - z^2a(z^2)^2}\right)g_e(z) + \left(\sqrt{1 - z^2a(z^2)^2} - z^2a(z^2)\right)\frac{g_o(z)}{z}.$$

This brings us to a brief comment as to when the square root of S^2 is a (analytic) Toeplitz operator. Here is a general fact concerning Toeplitz operators. For $\varphi \in L^\infty(\mathbb{T})$, define the Toeplitz operator T_φ on H^2 by $T_\varphi f = P_+(\varphi f)$, where P_+ is the orthogonal projection (the Riesz projection) of $L^2(\mathbb{T})$ onto H^2. See [5, Ch. 4] for the basics of Toeplitz operators on H^2. A well-known result of Brown and Halmos [1] says that $T_f T_g$ is a Toeplitz operator if and only if either $g \in H^\infty$ or $f \in \overline{H^\infty}$. This yields the following.

Theorem 2.22. *For $\varphi \in L^\infty(\mathbb{T})$, the following are equivalent.*

(i) There is a Toeplitz operator T such that $T^2 = T_\varphi$.

(ii) $\varphi = \psi^2$ for some $\psi \in H^\infty$. or $\varphi = \overline{\psi^2}$ for some $\psi \in H^\infty$.

The previous theorem, along with the standard inner-outer factorization of H^∞ functions yields the following corollary.

Corollary 2.23. *For $\varphi \in H^\infty$, the analytic Toeplitz operator T_φ has a square root in the Toeplitz operators if and only if all zeros of φ inside the open unit disc $\mathbb{D}$ are of even degrees.*

We end this section with the remark that S^{2n} has infinitely many square roots since S^{2n} is unitarily equivalent to $(S \oplus S)^{(n)}$, and we already know that $S \oplus S$ has infinitely many square roots. However S^{2n+1} does not have any square roots. We will discuss these results and some others in a forthcoming paper.

3. Square roots of the Volterra operator

The Volterra operator

$$(Vf)(x) = \int_0^x f(t)\,dt$$

is a well-known bounded operator on $L^2[0,1]$ with a known square root [21, p. 81]

$$(Yf)(x) = \frac{1}{\sqrt{\pi}}\int_0^x \frac{f(t)}{\sqrt{x-t}}dt. \tag{3.1}$$

One can prove this using the Laplace transform and convolutions. Thus, V has *at least* two square roots, Y and $-Y$. As it turns out, these are the *only* two.

Theorem 3.2 (Sarason [25]). *The operators Y and $-Y$ are the only two square roots of the Volterra operator V.*

For the sake of completeness, and since the ideas of the proof will be used in the next section, we give an exposition, with a different proof, of this result. The proof of this will involve expressing the Volterra operator as the compression of the shift S to a model space. We will divide up the proof into several parts.

If Θ is the atomic inner function

$$\Theta(z) = \exp\Big(\frac{z+1}{z-1}\Big),$$

a result of Sarason [24] (see also [21, Ch. 4]), shows that for $g \in L^2[0,1]$, the function

$$(Jg)(z) = \frac{i\sqrt{2}}{z-1}\int_0^1 g(t)\Theta(z)^t\,dt, \quad z \in \mathbb{D},$$

belongs to the model space $K_\Theta = H^2 \cap (\Theta H^2)^\perp$ and the operator $J : L^2[0,1] \to \mathcal{K}_\Theta$ is unitary. Since $\sigma(V) = \{0\}$, it follows that $(I-V)(I+V)^{-1}$ is a bounded operator on $L^2[0,1]$. The same paper says that

$$J(I-V)(I+V)^{-1}J^* = S_\Theta, \tag{3.3}$$

where S_Θ is the compression of S to $\mathcal{K}_\Theta$, that is $S_\Theta = P_\Theta S|_{\mathcal{K}_\Theta}$, where P_Θ is the orthogonal projection of H^2 onto $\mathcal{K}_\Theta$. It follows that $\sigma(S_\Theta) = \{1\}$ and thus $(I-S_\Theta)(I+S_\Theta)^{-1}$ is a bounded operator on $\mathcal{K}_\Theta$. The compressed shift S_Θ has an H^∞ functional calculus in that $\varphi(S_\varphi)$ is a well-defined bounded operator on $\mathcal{K}_\Theta$ for any $\varphi \in H^\infty$ [5, Ch. 9].

For $\psi \in H^\infty$, the operator $\psi(S_\Theta)$ can be written as a truncated Toeplitz operator. Indeed, for any $\psi \in L^\infty(\mathbb{T})$, define the operator A_ψ on $\mathcal{K}_\Theta$ by $A_\psi f = P_\Theta(\psi f)$, where P_Θ denotes orthogonal projection of $L^2(\mathbb{T})$ onto $\mathcal{K}_\Theta$ (where we regard $\mathcal{K}_\Theta$, via radial boundary values, as a subspace of $L^2(\mathbb{T})$). Let us record some facts about truncated Toeplitz operators that will be used below. One can find their proofs in [5] or [25].

Proposition 3.4. *Let $\varphi \in H^\infty$ and $\psi \in L^\infty(\mathbb{T})$.*

(i) $A_z = S_\Theta$.

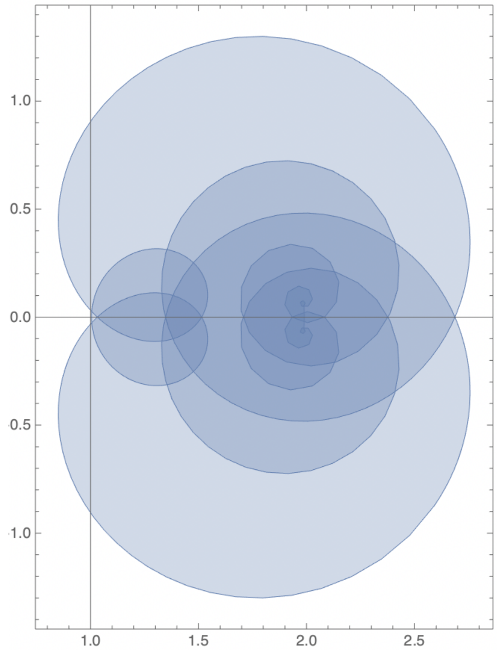

FIGURE 1. The above is the the image of $z \mapsto 1+z+\Theta(z)$ for $z \in \mathbb{D}$. Notice how the closure of this image does not contain the origin and thus $\inf_{z\in\mathbb{D}} |1 + z + \Theta(z)| > 0$.

(ii) $A_\psi = 0$ *if and only if* $\psi \in \Theta H^2 + \overline{\Theta H^2}$.

(iii) $\varphi(S_\Theta) = A_\varphi$.

(iv) $\{S_\Theta\}' = \{A_\varphi : \varphi \in H^\infty\}$.

Though the operator $(I - S_\Theta)(I + S_\Theta)^{-1}$ is well defined, we need to represent it as a truncated Toeplitz operator with an H^∞ symbol. This is accomplished with the following.

Proposition 3.5. *If*

$$\varphi(z) = \frac{1 - z}{1 + z + \Theta(z)},$$

then $\varphi \in H^\infty$*, is outer, and* $A_\varphi = (I - S_\Theta)(1 + S_\Theta)^{-1}$.

Proof. We first argue that $f(z) = 1 + z + \Theta(z)$ is bounded away from zero on $\mathbb{D}$ (see Figure 1) and thus is an invertible element of H^∞. Thus $\varphi \in H^\infty$. Notice that

$$\Re f(e^{i\theta}) = 1 + \cos\theta + \cos(\cot\theta/2).$$

The function $\cot\theta/2$ is strictly decreasing on $(0, \pi)$ as it moves from $+\infty$ to zero, and at $\theta = \pi/2$ its value is 1. Hence there is a unique $\theta_0 \in (0, \pi/2)$

such that $\cot\theta_0/2 = \pi/2$. Fix any $\theta' \in (\theta_0, \pi/2)$ and consider the partition $(0,\pi] = (0,\theta') \cup [\theta', \pi]$. On $(0,\theta')$,

$$\Re f(e^{i\theta}) = \cos\theta + \Big(1 + \cos(\cot\theta/2)\Big) \geqslant \cos\theta',$$

and, on $[\theta', \pi]$,

$$\Re f(e^{i\theta}) = \Big(1 + \cos\theta\Big) + \cos(\cot\theta/2) \geqslant \cos(\cot\theta'/2).$$

Therefore, $\Re f(e^{i\theta}) \geqslant m$ on $\mathbb{T} \setminus \{1\}$, where

$$m = \min\{\cos\theta',\, \cos(\cot\theta'/2)\} > 0.$$

By the Poisson integral formula, we conclude that

$$\Re f(z) = \int_0^{2\pi} \Re f(e^{i\theta}) \frac{1-|z|^2}{|z-e^{i\theta}|^2} \frac{d\theta}{2\pi} \geqslant m, \quad z \in \mathbb{D}.$$

A well known fact says that if $\Re f > 0$ then f is an outer function and thus has no zeros in $\mathbb{D}$ [6, p. 65].

If $\psi(z) = 1 + z + \Theta(z)$, notice that $\varphi(z)\psi(z) = 1 - z$ and hence the functional calculus yields $A_\varphi A_\psi = A_{1-z}$. Proposition 3.4 implies that

$$A_\psi = A_{1+z+\Theta} = A_{1+z} + A_\Theta = I + S_\Theta + 0.$$

Since $A_{1-z} = I - S_\Theta$, it follows that $A_\varphi = (I - S_\Theta)(I + S_\Theta)^{-1}$. □

Corollary 3.6. $V = J^* A_\varphi J$.

Proof of Theorem 3.2. Now let $A \in \mathcal{B}(L^2[0,1])$ such that $A^2 = V$. Lemma 2.10 yields $A \in \{V\}'$. Since

$$(I - V)(I + V)^{-1} = I + 2\sum_{n=1}^{\infty} (-1)^n V^n,$$

then $A \in \{(I-V)(I+V)^{-1}\}'$. Note that the series above converges in norm since V is quasinilpotent and thus $\|V^n\|^{1/n} \to 0$. From (3.3) we see that $JAJ^* \in \{S_\Theta\}'$. Thus $JAJ^* = A_\psi$ for some $\psi \in H^\infty$ (Proposition 3.4). Since

$$A_\psi^2 = (JAJ^*)^2 = JA^2J^* = JVJ^* = A_\varphi,$$

Proposition 3.4 also implies that $\psi^2 - \varphi \in \Theta H^2 + \overline{\Theta H^2}$. Since $\psi^2 - \varphi$ belongs to H^∞ and must also belong to $\Theta H^\infty + \overline{\Theta H^\infty}$, it follows from $\overline{\Theta H^\infty} \cap H^\infty = \{0\}$ that $\psi^2 - \varphi \in \Theta H^2$. This implies that $\psi^2 = \varphi + \Theta h$ for some $h \in H^\infty$.

Recall that φ is an outer function (and hence is zero free in $\mathbb{D}$) and so there are indeed $\psi \in H^\infty$ with $\psi^2 = \varphi$. This says that $A = J^* A_\psi J$ for some $\psi \in H^\infty$ with $\psi^2 = \varphi + \Theta h$ for some $h \in H^\infty$.

On the other hand, if $\psi \in H^\infty$ and $h \in H^\infty$ with $\psi^2 = \varphi + \Theta h$, then the operator $J^* A_\psi J$ on $L^2[0,1]$ satisfies

$$(J^* A_\psi J)^2 = J^* A_\psi^2 J = J^*(A_\psi + A_\Theta A_h) J = J^*(A_\varphi + 0) J = J^* A_\varphi J = V.$$

Note the use of the H^∞ functional calculus for the compressed shift S_Θ as well as the fact that $A_\Theta = 0$ (Proposition 3.4). So far we have shown the following: For $A \in \mathcal{B}(L^2[0,1])$

$$A^2 = V \iff A = J^* A_\psi J \tag{3.7}$$

for some $\psi \in H^\infty$ with $\psi^2 = \varphi + \Theta h$ for some $h \in H^\infty$.

So far we have shown that V has square roots. To show there are only *two* square roots of V, we follow a variation of an argument of Sarason [25]. Notice that one square root of V is $J^* A_{\sqrt{\varphi}} J$. Let us show that the other is $J^* A_{-\sqrt{\varphi}} J$. If B is another square root of V, then $B = J^* A_\psi J$ where $\psi^2 = \varphi + \Theta h$. In other words, $\psi^2 - \varphi = \Theta h$. Write

$$\Theta h = \psi^2 - \varphi = (\psi + \sqrt{\varphi})(\psi - \sqrt{\varphi})$$

and observe that for some $\gamma_j \geqslant 0$, the inner functions

$$q_1(z) = \exp\Big(-\gamma_1 \frac{1+z}{1-z}\Big) \quad \text{and} \quad q_2(z) = \exp\Big(-\gamma_2 \frac{1+z}{1-z}\Big)$$

divide $\psi - \sqrt{\varphi}$ and $\psi + \sqrt{\varphi}$ respectively. Moreover, choose the largest γ_1, γ_2 such that q_1 and q_2 are inner divisors of $\psi - \sqrt{\varphi}$ and $\psi + \sqrt{\varphi}$. Write

$$\psi + \sqrt{\varphi} = q_1 h_1 \quad \text{and} \quad \psi - \sqrt{\varphi} = q_2 h_2, \quad h_1, h_1 \in H^\infty.$$

It follows that $\sqrt{\varphi} = \frac{1}{2}(q_1 h_1 - q_2 h_2)$. Since $\sqrt{\varphi}$ is outer, it must be the case that one of γ_1 or γ_2 must be zero. If $\gamma_1 > 0$ and $\gamma_2 = 0$, then $\gamma_1 \geqslant 1$ and it follows that $\psi + \sqrt{\varphi}$ is divisible by Θ. An application of Proposition 3.4 yields $A_\psi = A_{-\sqrt{\varphi}}$.

4. Square root of a compressed shift

The previous section leads us to a discussion about the square roots of a compressed shift. For *any* inner function u, there is the compressed shift $S_u = P_u S|_{\mathcal{K}_u}$. The proof of (3.7) implies the following theorem.

Theorem 4.1. *For the atomic inner function Θ, the compressed shift S_Θ has exactly two square roots. They are $A_{\sqrt{f}}$ and $-A_{\sqrt{f}}$, where*

$$f(z) = z + \Theta(z)(1-z)^{1/5}.$$

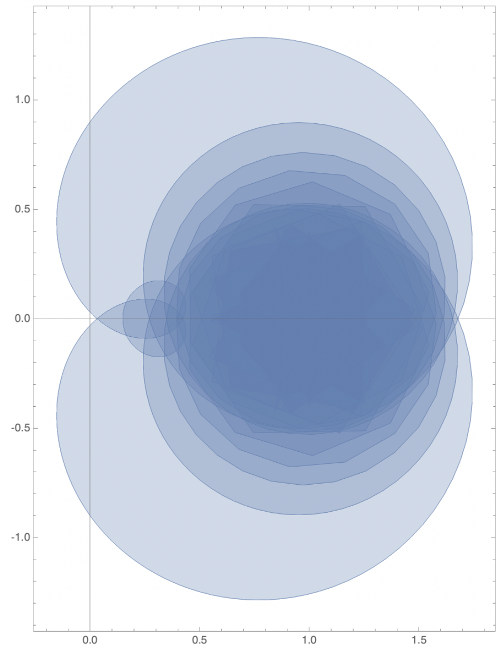

FIGURE 2. The image of $z \mapsto z + \Theta(z)(1-z)^{1/5}$ for $z \in \mathbb{D}$. Notice how the closure of this image does not contain the origin.

Proof. First let us prove that the set of square roots of S_u is nonempty. For this it is enough to check that $z + \Theta(z)(1-z)^{1/5}$ has no zeros in $\mathbb{D}$ (see Figure 2) and thus has an analytic square root. The reasoning is similar to the argument in Proposition 3.5, albeit a bit more complex. In this case

$$f(z) = z + \Theta(z)(1-z)^{1/5}$$

and thus

$$\Re f(e^{i\theta}) = \cos\theta + (2\sin\theta/2)^{1/5}\cos\left(\tfrac{\theta-\pi}{10} - \cot(\theta/2)\right), \quad 0 < \theta < \pi.$$

There is a similar formula for $-\pi < \theta < 0$. Then it is enough to observe that

$$m = \inf_{0<|\theta|\leqslant\pi} \Re f(e^{i\theta}) > 0.$$

By the Poisson integral formula, we conclude that $\Re f(z) \geqslant m$ for all $z \in \mathbb{D}$. Thus f is outer and hence has no zeros in $\mathbb{D}$.

Next we observe that $A_{\sqrt{f}}$ is a square root of S_Θ. Now follow the argument used to prove there are only two square roots of the Volterra operator (following (3.7)) to prove that the other square root of S_Θ is $A_{-\sqrt{f}}$. □

Not every compressed shift has a square root.

Proposition 4.2. *Suppose u is inner and u has a zero at $z = 0$ of order at least two. Then S_u does not have a square root.*

Proof. The discussion used to prove Theorem 4.1 shows that the set of square roots of S_u are $\{A_\psi : \psi \in H^\infty, \psi^2 = z + uh, h \in H^\infty\}$. If u has a zero of order at least two at $z = 0$, then $z + uh = z + z^2 k$ for some $k \in H^\infty$ and thus $z + z^2 k(z)$ has a zero of order one at $z = 0$. Thus, there is no H^∞ function ψ for which $\psi^2(z) = z + uh$. □

5. Square roots of $T_{\cos\theta}$.

The Toeplitz operator with symbol $\cos\theta$, equivalently

$$T_{\cos\theta} = \tfrac{1}{2}(S + S^*),$$

is a self-adjoint operator. Therefore, by the spectral theorem for normal operators, it has a square root and, at least theoretically, can write them all down. However, in the special case of $T_{\cos\theta}$, one can be more specific.

A result of Hilbert [11] (see [23, Ch. 3] for a modern presentation) shows that if $(u_n)_{n=0}^\infty$ are the Chebyschev polynomials of the second kind [26], then the operator $F : L^2(\rho) \to H^2$, where $\rho = \sqrt{1 - x^2}$ on $[-1, 1]$, defined by

$$Fu_n = \sqrt{\frac{\pi}{2}}\, z^n, \qquad n \geqslant 0, \tag{5.1}$$

is unitary and intertwines M_x on $L^2(\rho)$ and $T_{\cos\theta}$. More explicitly,

$$FM_x = T_{\cos\theta}F.$$

Thus, the matrix representation for $T_{\cos\theta}$ with respect to the orthonormal basis $(z^n)_{n=0}^\infty$ for H^2 is $[a_{mn}]_{m,n=0}^\infty$, where

$$a_{mn} := \langle T_{\cos\theta} z^n, z^m\rangle_{H^2}, \qquad m, n \geqslant 0,$$

which is the Toeplitz matrix

$$\begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \cdots \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & \cdots \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

By means of the unitary operator F, we can also write

$$a_{mn} = \frac{2}{\pi}\int_{-1}^{1} x u_n(x) u_m(x) \rho(x)\, dx, \qquad m, n \geqslant 0.$$

This observation gives us a path to describe *all* of the square roots of $T_{\cos\theta}$ in a very explicit way. Indeed, if φ is any measurable function on $[-1, 1]$

for which $\varphi(x)^2 = x$ for all $x \in [-1,1]$, then M_φ satisfies $M_\varphi^2 = M_x$. For example, one choice of φ can be

$$\varphi(x) = \begin{cases} \sqrt{x} & \text{if } x \geqslant 0, \\ i\sqrt{-x} & \text{if } x < 0. \end{cases}$$

Therefore, $FM_\varphi F^*$ is a square root of $T_{\cos\theta}$. Of course, there are many other φ for which $\varphi^2 = x$. The matrix representations of M_φ (with respect to the Chebyschev basis) and $FM_\varphi F^*$ (with respect to the standard basis for H^2) are

$$\left[\frac{2}{\pi}\int_{-1}^{1} \varphi(x)u_n(x)u_m(x)\rho(x)dx\right]_{m,n=0}^{\infty}.$$

In fact, these are all the square roots of $T_{\cos\theta}$.

Theorem 5.2. *For $B \in \mathcal{B}(H^2)$ the following are equivalent.*

(i) $B^2 = T_{\cos\theta}$.

(ii) With respect to the orthonormal basis $(z^n)_{n=0}^{\infty}$ of H^2, the matrix representation of B is

$$\left[\frac{2}{\pi}\int_{-1}^{1} \varphi(x)u_n(x)u_m(x)\rho(x)dx\right]_{m,n=0}^{\infty}, \tag{5.3}$$

where φ is a measurable function on $[-1,1]$ satisfying $\varphi(x)^2 = x$.

Proof. Let B be any fixed square root of $T_{\cos\theta}$. Since $T_{\cos\theta}$ is unitarily equivalent to M_x on $L^2(\rho)$ via the unitary operator F defined by (5.1), the operator F^*BF is a square root of M_x. By Lemma 2.10, F^*BF commutes with M_x and thus, by a standard characterization of cyclic normal operators, is equal to M_φ for some $\varphi \in L^\infty(\rho)$ This immediately implies

$$M_x = (F^*BF)^2 = M_{\varphi^2}.$$

By the uniqueness of the symbol of a multiplication operator, we must have $\varphi(x)^2 = x$. The matrix representation of $F^*BF = M_\varphi$ with respect to the orthonormal basis $(\sqrt{\frac{2}{\pi}}u_n)_{n=0}^{\infty}$ of $L^2(\rho)$ is the same as the matrix representation of B with respect to the orthonormal basis $(z^n)_{n=0}^{\infty}$ of H^2, and is given by (5.3). □

Notice that all of these square roots are complex symmetric operators, since with respect to the Chebyschev basis, the matrix representation (5.3) is self transpose.

6. Square roots of the Hilbert matrix

The square root of the Hilbert matrix

$$H = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \cdots \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

as an operator on ℓ^2, involves a similar analysis as with the Toeplitz matrix $T_{\cos\theta}$ from the previous section. Indeed, H is selfadjoint and thus, by the spectral theorem, has square roots and one can, at least in theory, write them all down. As with the Toeplitz case, we can express these square roots in a more tangible way. For this we replace the spectral representation theorem of Hilbert with one of Rosenblum [22]. We outline this analysis here.

The Laguerre polynomials $\{L_n(x) : n \geqslant 0\}$ form an orthonormal basis for $L^2((0,\infty), e^{-x}dx)$. A simple integral substitution shows that the map $(Qf)(x) = e^{-x/2}f(x)$ is unitary from $L^2((0,\infty), e^{-x}dx)$ onto $L^2((0,\infty), dx)$. Thus $\{QL_n = e^{-x/2}L_n(x) : n \geqslant 0\}$ is an orthonormal basis for $L^2((0,\infty), dx)$.

Lebedev [17, 18] proved that if

$$K_\nu(z) = \int_0^\infty e^{-z\cosh(t)} \cosh(\nu t)\, dt,$$

the modified Bessel function of the third kind, then the operator

$$(Uf)(\tau) = \int_0^\infty \frac{\sqrt{2\tau \sinh(\pi\tau)}}{\pi\sqrt{x}} K_{i\tau}(\frac{x}{2}) f(x) dx$$

is a unitary operator from $L^2((0,\infty), dx)$ to itself. Thus $\{w_n(x) = UQL_n : n \geqslant 0\}$ is an orthonormal basis for $L^2((0,\infty), dx)$. Rosenblum [22] proves that if

$$h(\tau) = \frac{\pi}{\cosh(\pi\tau)}, \tag{6.1}$$

then

$$\langle M_h w_m, w_n\rangle_{L^2((0,\infty),dx)} = \frac{1}{n+m+1}, \quad m, n \geqslant 0.$$

This last quantity equals $\langle H\mathbf{e}_m, \mathbf{e}_n\rangle_{\ell^2}$ (the entries of the Hilbert matrix). In summary, the linear transformation $W : \ell^2 \to L^2((0,\infty), dx)$ defined by

$$W(\{a_n\}_{n\geqslant 0}) = \sum_{n=0}^\infty a_n w_n$$

is unitary with $WHW^* = M_h$.

As in the previous section, if $g \in L^\infty((0,\infty), dx)$ with $g^2 = h$, then M_g is a square root of M_h and thus $W^* M_g W$ is a square root of H. Conversely, if $T \in \mathcal{B}(\ell^2)$ with $T^2 = H$, then WTW^* is a square root of M_h and hence, as we have seen several times before, WTW^* belongs to the commutant of M_h. Since h is a monotone decreasing function on $(0,\infty)$, h is injective and hence by a well-known fact about multiplication operators, M_h is cyclic. Since the commutant of a cyclic multiplication operator is the set of multiplication operators M_g on $L^2((0,\infty), dx)$ with $g \in L^\infty((0,\infty), dx)$ we see as before that $T = W^* M_g W$, where $g^2 = h$. We therefore arrive at the following theorem. Below, we regard any $T \in \mathcal{B}(\ell^2)$ as an infinite matrix.

Theorem 6.2. *For $T \in \mathcal{B}(\ell^2)$ the following are equivalent.*

(i) $T^2 = H$.

(ii) There is a measurable function g on $(0,\infty)$ with $g^2 = h$, where h is the function from (6.1)*, such that*

$$T = \Big[\int_0^\infty g(x) w_m(x) \overline{w_n(x)} dx\Big]_{m,n=0}^\infty.$$

7. Square roots of the Cesàro operator

The Cesàro operator $C : H^2 \to H^2$ defined by

$$(Cf)(z) = \frac{1}{z}\int_0^z \frac{f(\xi)}{1-\xi} d\xi, \quad z \in \mathbb{D},$$

is bounded on H^2 and a power series computation shows that if $f(z) = \sum_{j=0}^\infty a_j z^j \in H^2$, then

$$(Cf)(z) = \sum_{n=0}^\infty \Big(\frac{1}{n+1}\sum_{j=0}^n a_j\Big) z^n.$$

Some basic facts about the Cesàro operator C are found in [2]. With resect to the standard orthonormal basis $(z^n)_{n=0}^\infty$ for H^2, the matrix representation of C is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \cdots \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \cdots \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{7.1}$$

which is known as the *Cesàro matrix*. Though not quite obvious, C has a square root and in fact, one can write them all down - since there are only two of them. This is the topic of this section. It is important to note here that by the Conway-Olin functional calculus for subnormal operators [4], one can prove that C has *at least* two square roots. The purpose here is to show that C has *exactly* two square roots and to specifically write them down.

Our path to identify the (bounded) square roots of C is through subnormal operators and the work of Kriete and Trutt. Along the way to proving that C is subnormal, a paper of Kriete and Trutt [15] shows that for $w \in \mathbb{D}$, the function

$$v_w(z) = (1-z)^{w/(1-w)}$$

belongs to H^2 and satisfies $(I - C^*)v_w = wv_w$. The space

$$\mathcal{H} = \{F(z) = \langle f, v_{\overline{z}}\rangle_{H^2} : f \in H^2\}$$

defines a vector space of analytic functions on $\mathbb{D}$ that becomes a Hilbert space, in fact a reproducing kernel Hilbert space, when endowed with the norm $\|F\|_{\mathcal{H}} = \|f\|_{H^2}$. This makes the operator $(Uf)(z) = F(z)$ a unitary operator from H^2 to $\mathcal{H}$. Furthermore,

$$\begin{aligned}(U(I-C)f)(z) &= \langle (I-C)f, \varphi_{\overline{z}}\rangle_{H^2}\\ &= \langle f, (I-C^*)v_{\overline{z}}\rangle_{H^2}\\ &= \langle f, \overline{z}v_{\overline{z}}\rangle_{H^2}\\ &= z\langle f, v_{\overline{z}}\rangle_{H^2}\\ &= z(Uf)(z)\end{aligned}$$

for all $f \in H^2$. Thus, $U(I-C) = M_zU$ on $\mathcal{H}$. In summary, C is unitarily equivalent to M_{1-z} on $\mathcal{H}$.

Thus if A is a (bounded) square root of C, then, as we have seen with the other operators covered in this paper, $A \in \{C\}'$ and thus $UAU^* \in \{M_{1-z}\}' = \{M_z\}'$. So now we need to identify $\{M_z\}'$. The Hilbert space $\mathcal{H}$ also contains the polynomials as a dense set [15]. In fact, $\mathcal{H}$ can be identified with the closure of the polynomials in $L^2(\mu)$ for some finite positive Borel measure μ on the closure of $\mathbb{D}$. A well-known, and general fact for reproducing kernel Hilbert spaces for which the polynomials are dense [8, Pr. 147], says that the commutant of M_z is the set of multiplication operators M_φ where φ is a multiplier of $\mathcal{H}$ (i.e., $\varphi\mathcal{H} \subseteq \mathcal{H}$). Another paper of Kriete and Trutt [16] argues that the multipliers of $\mathcal{H}$ are precisely H^∞.

Putting this all together, it follows that if A is a square root of C, then $UAU^* = M_\varphi$ for some $\varphi \in H^\infty$. But

$$M_{\varphi^2} = M_\varphi^2 = (UAU^*)^2 = UCU^* = M_{1-z}.$$

and thus $\varphi^2 = 1 - z$ on $\mathbb{D}$. But since φ is analytic on $\mathbb{D}$, it must be the case that $\varphi(z) = \pm\sqrt{1-z}$. Thus, the Cesàro operator has

$$U^* M_{\sqrt{1-z}} U \quad \text{and} \quad U^* M_{-\sqrt{1-z}} U$$

as its only square roots.

The above formulas for the square roots of C are a bit unsatisfying since they are hidden behind the unitary operator U and a somewhat mysterious Hilbert space $\mathcal{H}$. Our goal in the next two results is to produce a more tangible description of the two square roots of C. Note that

$$\sqrt{1-z} = 1 - \tfrac{1}{2}z - \tfrac{1}{8}z^2 - \tfrac{1}{16}z^3 - \tfrac{5}{128}z^4 - \cdots = 1 - \sum_{k=1}^{\infty} \left|\binom{\frac{1}{2}}{k}\right| z^k,$$

where the branch of the square root is taken so that $\sqrt{1} = 1$. It is well-known that

$$\sum_{k=0}^{\infty} \left|\binom{\frac{1}{2}}{k}\right| < \infty.$$

Theorem 7.2. *The following are equivalent for $A \in \mathcal{B}(H^2)$.*

(i) $A^2 = C$.

(ii) $A = \pm\Big(I - \frac{1}{2}(I-C) - \frac{1}{8}(I-C)^2 - \frac{1}{16}(I-C)^3 + \cdots\Big)$, *where the series above converges in operator norm.*

Proof. From the above discussion,

$$U^* M_{\sqrt{1-z}} U \quad \text{and} \quad U^* M_{-\sqrt{1-z}} U$$

are the only two (bounded) square roots of C. Since $\|M_z\| = \|I - C\| = 1$ [2], the series

$$I - \tfrac{1}{2}M_z - \tfrac{1}{8}M_z^2 - \tfrac{1}{16}M_z^3 - \cdots$$

converges in operator norm to $M_{\sqrt{1-z}}$. But since M_z^k is unitarily equivalent to $(I-C)^k$, we get

$$\begin{aligned} U^* M_{\sqrt{1-z}} U &= U^*\Big(I - \tfrac{1}{2}M_z - \tfrac{1}{8}M_z^2 - \tfrac{1}{16}M_z^3 - \cdots\Big)U \\ &= I - \tfrac{1}{2}U^*M_zU - \tfrac{1}{8}(U^*M_zU)^2 - \tfrac{1}{16}(U^*M_zU)^3 + \cdots \\ &= I - \tfrac{1}{2}(I-C) - \tfrac{1}{8}(I-C)^2 - \tfrac{1}{16}(I-C)^3 + \cdots. \end{aligned}$$

The other square root of C is computed in a similar way. □

Using an idea of Hausdorff [10], the paper [13] produces all of the lower triangular square roots of the Cesàro matrix from (7.1). That paper considers the Cesàro matrix and its resulting square roots as linear transformations on all one-sided sequences (not necessarily ℓ^2 sequences nor any assumption on the linear transformation being bounded). They show that all of the lower triangular square roots of the Cesàro matrix are the matrices $A^\sigma = [A^\sigma_{ij}]_{i,j=0}^\infty$, where

$$A^\sigma_{ij} = \begin{cases} \binom{i}{j} \sum_{\ell=0}^{i-j} (-1)^\ell \sigma(\ell+j+1) \frac{1}{\sqrt{\ell+j+1}} \binom{i-j}{\ell} & i \geqslant j, \\ 0 & i < j, \end{cases} \tag{7.3}$$

and $\sigma : \mathbb{N} \to \{-1, 1\}$. One can work out that A^σ equals

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & -1 & 0 & 0 & 0 & \cdots \\ 1 & -2 & 1 & 0 & 0 & \cdots \\ 1 & -3 & 3 & -1 & 0 & \cdots \\ 1 & -4 & 6 & -4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \pm 1 & 0 & 0 & 0 & \cdots \\ 0 & \pm\sqrt{\frac{1}{2}} & 0 & 0 & \cdots \\ 0 & 0 & \pm\sqrt{\frac{1}{3}} & 0 & \cdots \\ 0 & 0 & 0 & \pm\sqrt{\frac{1}{4}} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & -1 & 0 & 0 & 0 & \cdots \\ 1 & -2 & 1 & 0 & 0 & \cdots \\ 1 & -3 & 3 & -1 & 0 & \cdots \\ 1 & -4 & 6 & -4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where the sign along the diagonal of the middle matrix is determined by the function σ.

They also conjecture that the choices of A^σ, where $\sigma \equiv 1$ or $\sigma \equiv -1$, are the two bounded square roots of the Cesaro matrix (viewed as an operator on ℓ^2). This next theorem verifies this conjecture (thus answering a question posted by Halmos) and also gives an exact description of the square roots from Theorem 7.2.

Theorem 7.4. *The following are equivalent for $A \in \mathcal{B}(H^2)$.*

(i) $A^2 = C$.

(ii) With respect to the orthonormal basis $(z^n)_{n=0}^\infty$ for H^2, the matrix representation of A is either $[A_{ij}]_{i,j=0}^\infty$ or $-[A_{ij}]_{i,j=0}^\infty$, where

$$A_{ij} = \begin{cases} \binom{i}{j} \sum_{\ell=0}^{i-j} (-1)^\ell \frac{1}{\sqrt{\ell+j+1}} \binom{i-j}{\ell} & i \geqslant j \\ 0 & i < j. \end{cases}$$

Proof. By the above discussion of the results from [13], all of the lower triangular square roots of the Cesàro matrix (as viewed as an operator on the space of all sequences) are of the form A^σ for some $\sigma : \mathbb{N} \to \{-1, 1\}$.

From (7.3), notice that

$$A^{\sigma}_{ii} = \sigma(i+1)\frac{1}{\sqrt{i+1}} \tag{7.5}$$

and so the choice of σ is determined by the entries of A^{σ} on its diagonal. If

$$A = \Big(I - \tfrac{1}{2}(I-C) - \tfrac{1}{8}(I-C)^2 - \tfrac{1}{16}(I-C)^3 + \cdots\Big),$$

one of the bounded square roots of the Cesàro matrix from Theorem 7.2, notice that $(I-C)^k$ is lower triangular for all $k \geqslant 0$ and thus so is A. We just need to determine which choice of σ yields $A^{\sigma} = A$.

The (n,n) entry of $I-C$ is $(1-\frac{1}{n+1})$ for $n \geqslant 0$ and since $I-C$ is lower triangular, it follows that the (n,n) entry of $(I-C)^k$ is $(1-\frac{1}{n+1})^k$. Thus, the (n,n) entry of A is

$$1 - \tfrac{1}{2}(1-\tfrac{1}{n+1}) - \tfrac{1}{8}(1-\tfrac{1}{n+1})^2 - \tfrac{1}{16}(1-\tfrac{1}{n+1})^3 - \cdots .$$

But the above is just the Taylor series of $\sqrt{1-z}$ evaluated at $z = 1 - \frac{1}{n+1}$ and this turns out to be $\sqrt{\frac{1}{n+1}}$. By (7.5), this corresponds to A^{σ} with $\sigma \equiv 1$.

When

$$A = -\Big(I - \tfrac{1}{2}(I-C) - \tfrac{1}{8}(I-C)^2 - \tfrac{1}{16}(I-C)^3 + \cdots\Big),$$

a similar analysis shows that corresponds to A^{σ} with $\sigma \equiv -1$. $\square$

Thus the only two bounded square roots of the Cesàro (matrix) operator are

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & -1 & 0 & 0 & 0 & \cdots \\ 1 & -2 & 1 & 0 & 0 & \cdots \\ 1 & -3 & 3 & -1 & 0 & \cdots \\ 1 & -4 & 6 & -4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & \sqrt{\frac{1}{2}} & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{\frac{1}{3}} & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{\frac{1}{4}} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & -1 & 0 & 0 & 0 & \cdots \\ 1 & -2 & 1 & 0 & 0 & \cdots \\ 1 & -3 & 3 & -1 & 0 & \cdots \\ 1 & -4 & 6 & -4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & -1 & 0 & 0 & 0 & \cdots \\ 1 & -2 & 1 & 0 & 0 & \cdots \\ 1 & -3 & 3 & -1 & 0 & \cdots \\ 1 & -4 & 6 & -4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 & \cdots \\ 0 & -\sqrt{\frac{1}{2}} & 0 & 0 & \cdots \\ 0 & 0 & -\sqrt{\frac{1}{3}} & 0 & \cdots \\ 0 & 0 & 0 & -\sqrt{\frac{1}{4}} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & -1 & 0 & 0 & 0 & \cdots \\ 1 & -2 & 1 & 0 & 0 & \cdots \\ 1 & -3 & 3 & -1 & 0 & \cdots \\ 1 & -4 & 6 & -4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Remark 7.6. It is important to note that any other option of sign along the main diagonal of the middle matrix will yield an unbounded operator on ℓ^2. For example,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & -1 & 0 & 0 & 0 & \cdots \\ 1 & -2 & 1 & 0 & 0 & \cdots \\ 1 & -3 & 3 & -1 & 0 & \cdots \\ 1 & -4 & 6 & -4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 & \cdots \\ 0 & \sqrt{\frac{1}{2}} & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{\frac{1}{3}} & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{\frac{1}{4}} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & -1 & 0 & 0 & 0 & \cdots \\ 1 & -2 & 1 & 0 & 0 & \cdots \\ 1 & -3 & 3 & -1 & 0 & \cdots \\ 1 & -4 & 6 & -4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

(notice the minus sign in the first entry of the diagonal matrix and all the other entries are positive) can be written as

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & -1 & 0 & 0 & 0 & \cdots \\ 1 & -2 & 1 & 0 & 0 & \cdots \\ 1 & -3 & 3 & -1 & 0 & \cdots \\ 1 & -4 & 6 & -4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 1-2 & 0 & 0 & 0 & \cdots \\ 0 & \sqrt{\frac{1}{2}} & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{\frac{1}{3}} & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{\frac{1}{4}} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & -1 & 0 & 0 & 0 & \cdots \\ 1 & -2 & 1 & 0 & 0 & \cdots \\ 1 & -3 & 3 & -1 & 0 & \cdots \\ 1 & -4 & 6 & -4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

which is equal to

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & -1 & 0 & 0 & 0 & \cdots \\ 1 & -2 & 1 & 0 & 0 & \cdots \\ 1 & -3 & 3 & -1 & 0 & \cdots \\ 1 & -4 & 6 & -4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & \sqrt{\frac{1}{2}} & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{\frac{1}{3}} & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{\frac{1}{4}} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & -1 & 0 & 0 & 0 & \cdots \\ 1 & -2 & 1 & 0 & 0 & \cdots \\ 1 & -3 & 3 & -1 & 0 & \cdots \\ 1 & -4 & 6 & -4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$-2 \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & -1 & 0 & 0 & 0 & \cdots \\ 1 & -2 & 1 & 0 & 0 & \cdots \\ 1 & -3 & 3 & -1 & 0 & \cdots \\ 1 & -4 & 6 & -4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & -1 & 0 & 0 & 0 & \cdots \\ 1 & -2 & 1 & 0 & 0 & \cdots \\ 1 & -3 & 3 & -1 & 0 & \cdots \\ 1 & -4 & 6 & -4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The first matrix in the sum above is one of the bounded square roots of the Cesàro operator while the second matrix in the sum turns out to be

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

which is clearly an unbounded operator on ℓ^2 (since the vector $(1, 1, 1, \ldots)$ belongs to the range – and is clearly not in ℓ^2).

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