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THE SURFACE DIFFUSION AND THE WILLMORE FLOW FOR UNIFORMLY REGULAR HYPERSURFACES

JEREMY LECRONE, YUANZHEN SHAO, AND GIERI SIMONETT

ABSTRACT. We consider the surface diffusion and Willmore flows acting on a general class of (possibly non–compact) hypersurfaces parameterized over a uniformly regular reference manifold possessing a tubular neighborhood with uniform radius. The surface diffusion and Willmore flows each give rise to a fourth–order quasilinear parabolic equation with nonlinear terms satisfying a specific singular structure. We establish well–posedness of both flows for initial surfaces that are $C^{1+\alpha}$–regular and parameterized over a uniformly regular hypersurface. For the Willmore flow, we also show long–term existence for initial surfaces which are $C^{1+\alpha}$–close to a sphere, and we prove that these solutions become spherical as time goes to infinity.

1. Introduction

The surface diffusion and Willmore flows are geometric evolution equations that describe the motion of hypersurfaces in Euclidean space (or, more generally, in an ambient Riemannian manifold). The normal velocity of evolving surfaces is determined by purely geometric quantities. For both flows, the mean curvature is involved in the evolution equations, while the Willmore flow additionally depends upon Gauss curvature.

These flows have been studied by several authors for compact (closed) hypersurfaces. In this setting, existence, regularity, and qualitative behavior of solutions have been analyzed in [13, 14, 20, 27, 33, 36, 37] for the surface diffusion flow, and in [17, 18, 24, 25, 26, 32, 35] for the Willmore flow, to mention just a few publications.

In this paper, we consider uniformly regular hypersurfaces. It should be emphasized that these surfaces may be non-compact. The concept of uniformly regular Riemannian manifolds was introduced by Amann [3, 4] and it contains the class of compact Riemannian manifolds as a special case. The study of geometric flows on non–compact manifolds is an active research topic, both from the point of view of PDE theory and in relation to its applications in geometry and topology. To the best of our knowledge, the current literature on the surface diffusion and Willmore flows for non–compact manifolds all concern surfaces defined over an infinite cylinder or entire graphs over $\mathbb{R}^m$, or the Willmore flow with small initial energy, cf. [8, 16, 17, 21, 22]. Our work generalizes the study of these two flows to a larger class of manifolds.

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In our main result we establish well-posedness for initial surfaces that are $C^{1+\alpha}$-regular and parameterized over a uniformly regular hypersurface. Moreover, we show that solutions instantaneously regularize and become smooth, and even analytic in case $\Sigma$ is analytic. In order to obtain our results, we show that the pertinent underlying evolution equations can be formulated as parabolic quasilinear equations of fourth order over the reference surface $\Sigma$. Our analysis relies on the theory of continuous maximal regularity and the results and techniques developed in [22, 33, 34].

The results in Theorem 4.3 and Theorem 5.1 are new. However, we note that in case $\Sigma$ is an infinitely long cylinder embedded in $\mathbb{R}^3$, an analogous result to Theorem 4.3 was obtained in [22] for the surface diffusion flow.

For the Willmore flow, Theorem 5.1 is also new even if $\Sigma$ is a compact (smooth, closed) surface. Previous results impose more regularity on the initial surface, for instance $C^{2+\alpha}$ in [35].

Theorem 5.2, where global existence and convergence to a sphere is shown for surfaces that are $C^{1+\alpha}$-close to a sphere, also seems to be new. A corresponding result was obtained in [35] for surfaces close to a sphere in the $C^{2+\alpha}$-topology. The authors in [18] showed the existence of a lower bound on the lifespan of a smooth solution, which depends only on how much the curvature of the initial surface is concentrated in space. In [17, 19], the authors proved convergence to round spheres under suitable smallness assumptions on the total energy of the surface. Here we note that the energy used in [17, 19] involves second-order derivatives, whereas we only need smallness in the $C^{1+\alpha}$-topology. In particular, we obtain global existence and convergence for non-convex initial surfaces.

The organization of the paper is as follows:

In Sections 2.1 and 2.2, we introduce the concept of uniformly regular manifolds and define the function spaces used in this paper. In Sections 2.3 and 2.4, we review continuous maximal regularity theory and its applications to quasilinear parabolic equations with singular nonlinearity. These results form the theoretic basis for the study of the surface diffusion and Willmore flows.

In Section 3, we introduce the concept of uniformly regular hypersurfaces with a uniform tubular neighborhood (called (URT)-hypersurfaces) and work out several examples. We utilize these concepts to parameterize the evolving hypersurfaces driven by surface diffusion and Willmore flows as normal graphs over a (URT)-reference hypersurface.

In Section 4, we establish our main results regarding existence, uniqueness, regularity, and semiflow properties for solutions to the surface diffusion flow over (URT)-hypersurfaces in $\mathbb{R}^{m+1}$. In Section 5, we likewise establish well-posedness properties for solutions to the Willmore flow over (URT)-hypersurfaces in $\mathbb{R}^3$. Additionally, we show stability of Euclidean spheres under perturbations in the $C^{1+\alpha}$-topology.

We conclude the paper with an appendix where we state and prove some additional properties of normal graphs over (URT)-hypersurfaces.

**Notation:** For two Banach spaces $X$ and $Y$, $X \cong Y$ means that they are equal in the sense of equivalent norms. $\mathcal{L}(X,Y)$ denotes the set of all bounded linear maps from $X$ to $Y$ and $\text{Lin}(X,Y)$ is the subset of $\mathcal{L}(X,Y)$ consisting of all bounded linear isomorphisms from $X$ to $Y$. For $x \in X$, $\mathbb{B}_X(x,r)$ denotes the (open) ball in $X$ with radius $r$ and center $x$. We sometimes write $\mathbb{B}(x,r)$, in lieu of $\mathbb{B}_X(x,r)$, in case the setting is clear, and we write $\mathbb{B}^m(x,r)$ when $X = \mathbb{R}^m$. We denote by
2. Preliminaries

2.1. Uniformly regular manifolds. The concept of uniformly regular (Riemannian) manifolds was introduced by H. Amann in [3] and [4]. Loosely speaking, an \( m \)-dimensional Riemannian manifold \((M, g)\) is uniformly regular if its differentiable structure is induced by an atlas such that all its local patches are of approximately the same size, all derivatives of the transition maps are bounded, and the pull-back metric of \( g \) in every local coordinate is comparable to the Euclidean metric \( g_m \).

We will now state some structural properties of uniformly regular manifolds which will be used in the analysis of the the surface diffusion flow and the Willmore flow in subsequent sections.

An oriented \( C^\infty \)-manifold \((M, g)\) of dimension \( m \) and without boundary is uniformly regular if it admits an orientation-preserving atlas \( \mathcal{A} := \{ (\mathcal{O}_\kappa, \varphi_\kappa) : \kappa \in \mathcal{R} \} \), with a countable index set \( \mathcal{R} \), satisfying the following conditions.

(R1) There exists \( K \in \mathbb{N} \) such that any intersection of more than \( K \) coordinate patches is empty.

(R2) \( \varphi_\kappa(\mathcal{O}_\kappa) = B^m \), where \( B^m \) is the unit Euclidean ball centered at the origin in \( \mathbb{R}^m \). Moreover, \( \mathcal{A} \) is uniformly shrinkable; by which we mean that there exists some \( r \in (0, 1) \) such that \( \{ \psi_\kappa(rB^m) : \kappa \in \mathcal{R} \} \) forms a cover for \( M \), where \( \psi_\kappa := \varphi_\kappa^{-1} \).

(R3) \( \| \varphi_\kappa \circ \psi_\kappa \|_{k, \infty} \leq c(k) \) for all \( \kappa \in \mathcal{R} \), \( k \in \mathbb{N}_0 \) and \( \eta \in \mathcal{R} \) such that \( \mathcal{O}_\eta \cap \mathcal{O}_\kappa \neq \emptyset \).

(R4) \( \| \psi_\kappa \circ \varphi_\kappa \|_{k, \infty} \leq c(k) \) for all \( \kappa \in \mathcal{R} \) and \( k \in \mathbb{N}_0 \).

(R5) \( \psi_\kappa \circ g \sim g_m \) for all \( \kappa \in \mathcal{R} \). Here \( g_m \) is the Euclidean metric in \( \mathbb{R}^m \) and \( \psi_\kappa \circ g \) denotes the pull-back metric of \( g \) by \( \psi_\kappa \).

Here (R5) means that there exists some number \( c \geq 1 \) such that

\[
(1/c)|\xi|^2 \leq \psi_\kappa^* g(x)(\xi, \xi) \leq c|\xi|^2, \quad x \in B^m, \quad \xi \in \mathbb{R}^m, \quad \kappa \in \mathcal{R}.
\]

Given an open subset \( U \subset \mathbb{R}^m \), a Banach space \( X \), and a mapping \( u : U \to X \),

\[
\|u\|_{k, \infty} := \max_{|\alpha| \leq k} \|\partial^\alpha u\|_{\infty}
\]

is the norm of the space \( BC^k(U, X) \), which consists of all functions \( u \in C^k(U, X) \) such that \( \|u\|_{k, \infty} < \infty \).

Any uniformly regular manifold \((M, g)\) possesses a localization system subordinate to \( \mathcal{A} \), by which we mean a family \( \{ (\pi_\kappa, \zeta_\kappa) : \kappa \in \mathcal{R} \} \) satisfying:

(L1) \( \pi_\kappa \in D(\mathcal{O}_\kappa, [0, 1]) \) and \( \{ \pi_\kappa^2 : \kappa \in \mathcal{R} \} \) is a partition of unity subordinate to the cover \( \{ \mathcal{O}_\kappa : \kappa \in \mathcal{R} \} \).

(L2) \( \zeta_\kappa := \varphi_\kappa^* \zeta \) with \( \zeta \in D(B^m, [0, 1]) \) satisfying \( \zeta|_{\text{supp}(\varphi_\kappa \pi_\kappa)} \equiv 1, \kappa \in \mathcal{R} \).

(L3) \( \| \pi_\kappa \|_{k, \infty} + \| \zeta_\kappa \|_{k, \infty} \leq c(k), \kappa \in \mathcal{R}, k \in \mathbb{N}_0 \).

Given \( k \in \mathbb{N} \cup \{ \omega \} \), the concept of \( C^k \)-uniformly regular manifold is defined by modifying (R3), (R4), (L1)-(L3) in an obvious way, where \( \omega \) is the symbol for real analyticity.

Remark 2.1. In [12], the authors showed that a \( C^\infty \)-manifold without boundary is uniformly regular if it is of bounded geometry, i.e. it is geodesically complete, of positive injectivity radius and all covariant derivatives of the curvature tensor

\( g_m \) the Euclidean metric in \( \mathbb{R}^m \). Given an embedded hypersurface \( M \) in \( \mathbb{R}^m \), \( g_m \vert_M \) means the metric on \( M \) induced by \( g_m \). Finally, we set \( \mathbb{N}_0 = \mathbb{N} \cup \{ 0 \} \).
are bounded. In particular, every compact manifold without boundary is uniformly regular and the manifolds considered in [20, 21] are all uniformly regular.

Given \( \sigma, \tau \in \mathbb{N}_0 \), we define the \((\sigma, \tau)\)-tensor bundle of \( M \) as
\[
T^\sigma_\tau M := TM^{\otimes \sigma} \otimes T^*M^{\otimes \tau},
\]
where \( TM \) and \( T^*M \) are the tangent and the cotangent bundle of \( M \), respectively. Let \( T^\sigma_\tau M \) denote the \( C^\infty(M) \)-module of all smooth sections of \( T^\sigma_\tau M \).

Throughout the rest of this paper, we will adopt the following convention:

- \( p \) always denotes a point on a uniformly regular manifold.
- \( k \in \mathbb{N}_0 \) and \( s \geq 0 \).
- \( \sigma, \tau \in \mathbb{N}_0 \), \( V = V^\sigma : = \{ T^\sigma_\tau M, (|\cdot|)_g \} \), \( E = E^\tau : = \{ \mathbb{R}^{m^* \times m} \}, (\cdot, \cdot) \} \).

Setting \( \mathbb{R}^\infty_\kappa = \mathbb{R}^m \) for \( \kappa \in \mathfrak{r} \), we define \( L_{1,loc}(\mathbb{R}^m, E) := \prod_{\kappa} L_{1,loc}(\mathbb{R}^m_\kappa, E) \), \( R^c : L_{1,loc}(M, V) \rightarrow L_{1,loc}(\mathbb{R}^m, E) \), \( u \mapsto \psi^*_\kappa(\pi^*_\kappa u) \), \( R^\kappa : L_{1,loc}(\mathbb{R}^m_\kappa, E) \rightarrow L_{1,loc}(M, V) \), \( v_\kappa \mapsto \pi^*_\kappa \phi^*_\kappa v_\kappa \).

Here, and in the following, it is understood that a partially defined and compactly supported tensor field is automatically extended over the whole base manifold by identifying it to be zero outside its original domain. We further introduce two maps:
\[
R^c : L_{1,loc}(M, V) \rightarrow L_{1,loc}(\mathbb{R}^m, E), \quad u \mapsto (R^c u)_\kappa \in \mathbb{R}^m,
\]
\[
R : L_{1,loc}(\mathbb{R}^m, E) \rightarrow L_{1,loc}(M, V), \quad (v_\kappa)_\kappa \in \mathbb{R} \rightarrow \sum_{\kappa \in \mathbb{R}} R^\kappa v_\kappa.
\]

2.2. Hölder and little Hölder spaces on uniformly regular manifolds. In this subsection we follow Amann [1, 3], see also [34]. We define
\[
BC^k(M, V) := \{ u \in C^k(M, V) : \|u\|^M_{k, \infty} < \infty, \|\cdot\|_{k, \infty} \},
\]
where \( \|u\|_{k, \infty} := \max_{0 \leq i \leq k} \|\nabla^i u\|_{\infty} \). Set
\[
BC^\infty(M, V) := \bigcap_k BC^k(M, V)
\]
endowed with the conventional projective topology. Then
\[
bc^k(M, V) := \text{the closure of } BC^\infty(M, V) \text{ in } BC^k(M, V).
\]
Letting \( k < s < k + 1 \), the Hölder space \( BC^s(M, V) \) is defined by
\[
BC^s(M, V) := (bc^k(M, V), bc^{k+1}(M, V))_{s-k, \infty}.
\]
Here \((\cdot, \cdot)_{s, \infty}\) is the real interpolation method, see [1] Example 1.2.4.1. For \( s \geq 0 \), we define the little Hölder spaces by
\[
bc^s(M, V) := \text{the closure of } BC^\infty(M, V) \text{ in } BC^s(M, V).
\]
The spaces \( BC^s(\mathbb{R}^m, E) \) and \( bc^s(\mathbb{R}^m, E) \) are defined in a similar manner. When \( s \notin \mathbb{N}_0 \), we can give an alternative characterization of these spaces on \( \mathbb{R}^m \). For \( 0 < s < 1 \) and \( 0 < \delta \leq \infty \), we define a seminorm by
\[
[u]_s^\delta := \sup_{h \in (0, \delta)^m} \frac{\|u(\cdot + h) - u(\cdot)\|_{\infty}}{|h|^s}, \quad [\cdot]_s^\delta := [\cdot]_s^\infty.
\]
For \( k < s < k + 1 \), the space \( BC^s(\mathbb{R}^m, E) \) can be equivalently defined as
\[
BC^s(\mathbb{R}^m, E) = \{ u \in BC^k(\mathbb{R}^m, E) : \|u\|_{s, \infty} < \infty, \|\cdot\|_{s, \infty} \},
\]
where \( \|u\|_{s, \infty} := \|u\|_{k, \infty} + \max_{|\alpha| = k} [\partial^\alpha u]_{s-k, \infty} \); and
For $\mathcal{F}$ in \{bc, BC\}, we put $\mathfrak{F}^\kappa := \prod_{\kappa} \mathfrak{F}_\kappa$ with $\mathfrak{F}^\kappa := \mathfrak{F}^{x}(\mathbb{R}^m, E)$. We denote by $l_\infty(\mathfrak{F}^\kappa)$ the linear subspace of $\mathfrak{F}^\kappa$ consisting of all $\mathfrak{F} = (\mathfrak{F}_\kappa)_{\kappa \in \mathcal{R}}$ such that
\[
\|\mathfrak{F}\|_{l_\infty(\mathfrak{F}^\kappa)} := \sup_{\kappa} \|\mathfrak{F}_\kappa\|_{\mathfrak{F}_\kappa} < \infty.
\]
We define $l_{\infty, \text{unif}}(bc^k)$ as the linear subspace of $l_\infty(bc^k)$ consisting of all $\mathfrak{F} = (\mathfrak{F}_\kappa)_{\kappa \in \mathcal{R}}$ such that $(\partial^\alpha \mathfrak{F}_\kappa)_{\kappa \in \mathcal{R}}$ is uniformly continuous on $\mathbb{R}^m_\kappa$ for $|\alpha| \leq k$, uniformly with respect to $\kappa \in \mathcal{R}$. For $k < s < k + 1$, we define $l_{\infty, \text{unif}}(bc^s)$ as the linear subspace of $l_{\infty, \text{unif}}(bc^k)$ of all $\mathfrak{F} = (\mathfrak{F}_\kappa)_{\kappa \in \mathcal{R}}$ such that
\[
\lim_{\delta \to 0} [\partial^\alpha \mathfrak{F}_\kappa]_{s-k, \infty} = 0 \quad \text{uniformly with respect to } \kappa \in \mathcal{R}.
\]

The following properties of little Hölder spaces were first established in [3, 4]. We also refer to [33, Theorem 2.1 and Proposition 2.2].

**Proposition 2.2.** Let $s \geq 0$. Then $\mathcal{R}$ is a retraction from $l_{\infty, \text{unif}}(bc^s)$ onto $bc^s(M, V)$ with $\mathcal{R}$ as a coretraction. Similarly,$$
\left[ u \mapsto (\psi^s_\kappa(\mathfrak{G}, u))_{\kappa \in \mathcal{R}} \right] \in \mathcal{L}(bc^s(M, V), l_{\infty, \text{unif}}(bc^s)).
\]
Let $(\cdot, \cdot)_0^0$ denote the continuous interpolation method, c.f. [11 Example I.2.4.4].

**Proposition 2.3.** Suppose that $0 < \theta < 1$, $0 \leq s_0 < s_1$ and $s = (1 - \theta)s_0 + \theta s_1$ with $s_1, s_2, s \notin \mathbb{N}_0$. Then
\[
(bc^s(M, V), bc^{s_1}(M, V))_{\theta, \infty}^0 \cong bc^s(M, V).
\]

2.3. **Continuous maximal regularity.** For a fixed interval $I = [0, T]$ and a given Banach space $X$, we define
\[
BC_{1-\mu}(I, X) := \{ u \in C(I, X) : [t \mapsto t^{1-\mu}u] \in C(I, X), \lim_{t \to 0^+} t^{1-\mu}\|u(t)\|_X = 0 \},
\]
\[
\|u\|_{C_{1-\mu}} := \sup_{t \in I} t^{1-\mu}\|u(t)\|_X,
\]
where $I = I \setminus \{0\}$; and
\[
BC_{1-\mu}^1(I, X) := \{ u \in C^1(I, X) : u, \dot{u} \in BC_{1-\mu}(I, X) \}.
\]
If $I = [0, T)$ is a half open interval, then
\[
C_{1-\mu}(I, X) := \{ v \in C(I, X) : v \in BC_{1-\mu}([0, t], X), \ t < T \},
\]
\[
C_{1-\mu}^1(I, X) := \{ v \in C^1(I, X) : v, \dot{v} \in C_{1-\mu}(I, X) \}.
\]
We equip these two spaces with the natural Fréchet topology induced by the topology of $BC_{1-\mu}([0, t], X)$ and $BC_{1-\mu}^1([0, t], X)$, respectively.

Assume that $E_1 \xrightarrow{d} E_0$ is a pair of densely embedded Banach spaces. An operator $A$ is said to belong to the class $\mathcal{H}(E_1, E_0)$, if $-A$ generates a strongly continuous analytic semigroup on $E_0$ with $\text{dom}(A) = E_1$. We define
\[
E_{0, \mu}(I) := BC_{1-\mu}(I, E_0), \quad E_{1, \mu}(I) := BC_{1-\mu}(I, E_1) \cap BC_{1-\mu}^1(I, E_0),
\]
which are themselves Banach spaces when equipped with the norms
\[
\|u\|_{E_{0, \mu}(I)} := \sup_{t \in I} t^{1-\mu}\|u(t)\|_{E_0},
\]
\[
\|u\|_{E_{1, \mu}(I)} := \sup_{t \in I} t^{1-\mu}(\|\dot{u}(t)\|_{E_0} + \|u(t)\|_{E_1}),
\]
respectively. For \( A \in \mathcal{H}(E_1, E_0) \), we say \((E_{0,\mu}(I), E_{1,\mu}(I))\) is a pair of maximal regularity of \( A \) if
\[
\left( \frac{d}{dt} + A, \gamma_0 \right) \in \mathcal{L}(E_{1,\mu}(I), E_{0,\mu}(I) \times E_\mu)
\]
where \( \gamma_0 \) is the evaluation map at 0, i.e., \( \gamma_0(u) = u(0) \), and \( E_\mu := (E_0, E_1)_{\mu,\infty}^0 \). In this case, we use the notation
\[
A \in \mathcal{M}_\mu(E_1, E_0).
\]

2.4. **Quasilinear equations with singular nonlinearity.** Consider the following abstract quasilinear parabolic evolution equation
\[
\begin{cases}
\frac{d}{dt} u + A(u) = F_1(u) + F_2(u), & t > 0, \\
u(0) = x.
\end{cases}
\tag{2.3}
\]
We assume that \( V_\mu \subset E_\mu \) is an open subset of the continuous interpolation space \( E_\mu := (E_0, E_1)_{\mu,\infty}^0 \) and the operators \((A, F_1, F_2)\) satisfy the following conditions.

**(H1)** Local Lipschitz continuity of \((A, F_1)\):
\[
(A, F_1) \in C^{1-}(V_\mu, \mathcal{M}_\mu(E_1, E_0) \times E_0).
\]

**(H2)** Structural regularity of \( F_2 \):
There exists a number \( \gamma \in (\mu, 1) \) such that \( F_2 : V_\mu \cap E_\gamma \to E_0 \). Moreover, there are numbers \( \gamma_j \in [\mu, \gamma], \gamma_j \geq 0 \), and \( m \in \mathbb{N} \) with
\[
\gamma_j(\gamma - \mu) + (\gamma_j - \mu) \leq 1, \quad \text{for all } j = 1, 2, \ldots, m,
\tag{2.4}
\]
so that for each \( x_0 \in V_\mu \) and \( R > 0 \) there is a constant \( C_R = C_R(x_0) > 0 \) for which the estimate
\[
|F_2(x_1) - F_2(x_2)|_{E_0} \leq C_R \sum_{j=1}^{m} (1 + |x_1|_{E_{\gamma_j}}^{\gamma_j} + |x_2|_{E_{\gamma_j}}^{\gamma_j})|x_1 - x_2|_{E_{\gamma_j}}
\tag{2.5}
\]
holds for all \( x_1, x_2 \in \overline{B}_{E_\mu}(x_0, R) \cap (V_\mu \cap E_\gamma) \).

Following the convention in Prüss, Wilke \[30\] and \[22\], we call the index \( j \) sub-critical if \((2.5)\) is a strict inequality and critical in case equality holds in \((2.5)\).

**Theorem 2.4.** \[22\] Theorem 2.2] Suppose \((A, F_1, F_2)\) satisfies \((H1)-(H2)\).

(a) Given any \( x_0 \in V_\mu \), there exist positive constants \( \tau = \tau(x_0), \varepsilon = \varepsilon(x_0) \), and \( \sigma = \sigma(x_0) \) such that \((2.3)\) has a unique solution
\[
u(\cdot, x) \in E_{1,\mu}([0, \tau])
\]
for all initial values \( x \in \overline{B}_{E_\mu}(x_0, \varepsilon) \). Moreover,
\[
\|\nu(\cdot, x_1) - \nu(\cdot, x_2)\|_{E_{1,\mu}([0, \tau])} \leq \sigma\|x_1 - x_2\|_{E_\mu}, \quad x_1, x_2 \in \overline{B}_{E_\mu}(x_0, \varepsilon).
\]
(b) Each solution with initial value \( x_0 \in V_\mu \) exists on a maximal interval \( J(x_0) := [0, t^+) = [0, t^+(x_0)] \) and enjoys the regularity
\[
u(\cdot, x_0) \in C([0, t^+), E_\mu) \cap C([0, t^+), E_1).
\]
(c) If the solution \( \nu(\cdot, x_0) \) satisfies the conditions:
(i) \( \nu(\cdot, x_0) \in UC(J(x_0), E_\mu) \) and
(ii) there exists \( \eta > 0 \) so that \( \text{dist}_{E_\mu}(\nu(t,x_0), \partial V_\mu) > \eta \) for all \( t \in J(x_0) \),
then it holds that \( t^+(x_0) = \infty \) and so \( u(\cdot, x_0) \) is a global solution of (2.3).
Moreover, if the embedding \( E_1 \hookrightarrow E_0 \) is compact, then condition (i) may be replaced by the assumption:

(i.a) the orbit \( \{ u(t, x_0) : t \in [\tau, t^+(x_0)] \} \) is bounded in \( E_0 \) for some \( \delta \in (\mu, 1] \)
and some \( \tau \in (0, t^+(x_0)) \).

3. URT–hypersurfaces

Suppose \( \Sigma \) is an oriented smooth hypersurface without boundary which is embedded in \( \mathbb{R}^{m+1} \). Let \( a > 0 \). Then \( \Sigma \) is said to have a \textit{tubular neighborhood of radius} \( a \) if the map

\[
X : \Sigma \times (-a, a) \to \mathbb{R}^{m+1} : \ [(p, r) \mapsto p + r\nu_\Sigma(p)]
\]

(3.1)
is a diffeomorphism onto its image \( U_a := X((-a, a) \times \Sigma) \). Here \( \nu_\Sigma \) is the normal unit vector field compatible with the orientation of \( \Sigma \). We refer to \( U_a \) as the tubular neighborhood of \( \Sigma \) of width \( 2a \) and note that \( U_a = \{ x \in \mathbb{R}^{m+1} : \text{dist}(x, \Sigma) < a \} \).

Finally, we say that \( \Sigma \) has a \textit{tubular neighborhood} if there exists a number \( a > 0 \) such that the above property holds.

Remarks 3.1. (a) We lose no generality in assuming \( \Sigma \) is oriented, as any smooth embedded hypersurface without boundary is orientable, cf. [31].
(b) Any smooth (in fact, \( C^2 \)) compact embedded hypersurface without boundary has a tubular neighborhood, see for instance [15 Exercise 2.11].
(c) Suppose \( \Sigma \) is a smooth (oriented) embedded hypersurface with unit normal field \( \nu_\Sigma \). Then \( \Sigma \) is said to satisfy the uniform ball condition of radius \( a > 0 \) if at each point \( p \in \Sigma \), the open balls \( B(p \pm a\nu_\Sigma(p), a) \) do not intersect \( \Sigma \).

The following assertions are equivalent:

(i) \( \Sigma \) has a tubular neighborhood of radius \( a \).
(ii) \( \Sigma \) satisfies the uniform ball condition of radius \( a \).

For the reader’s convenience, we include a proof of this equivalence.

Proof. (i) \( \Rightarrow \) (ii). Suppose there exists \( p \in \Sigma \) such that \( \mathbb{B}(x_0, a) \cap \Sigma \neq \emptyset \), where \( x_0 := p \pm a\nu_\Sigma(p) \). Then \( s := \text{dist}(x_0, \mathbb{B}(x_0, a) \cap \Sigma) < a \) and there exists \( q \in \mathbb{B}(x_0, a) \cap \Sigma \) such that \( x_0 = q + s\nu_\Sigma(q) \). Hence \( x_0 = X(p, a) = X(q, s) \), with \( (p, a) \neq (q, s) \), contradicting the assumption that \( X \) is bijective. The case \( x_0 = p - a\nu_\Sigma(p) \) is treated in the same way.

(ii) \( \Rightarrow \) (i). We only need to prove the injectivity of \( X \). Suppose, by contradiction, that \( X(p_1, r_1) = X(p_2, r_2) = x \) for \( (p_1, r_1) \neq (p_2, r_2) \). Without loss of generality we may assume that \( r_1 \in (0, a) \), as we can otherwise replace \( \nu_\Sigma(p_1) \) by \( -\nu_\Sigma(p_1) \).
Moreover, we may assume that \( |r_2| \leq r_1 \). Let \( s \in (r_1, a) \) and set

\[
y := X(p_1, s) = x + (s - r_1)\nu_\Sigma(p_1).
\]

Then we have \( |y - p_2| \leq |y - x| + |x - p_2| = s - r_1 + |r_2| \leq s \), showing that \( p_2 \in \mathbb{B}(y, s) = \mathbb{B}(p_1 + s\nu_\Sigma(p_1), s) \subset \mathbb{B}(p_1 + a\nu_\Sigma(p_1), a) \).
Therefore, \( \mathbb{B}(p_1 + a\nu_\Sigma(p_1), a) \cap \Sigma \neq \emptyset \), contradicting the assumption in (ii).

(d) Suppose \( \Sigma \) has a tubular neighborhood of radius \( a \). Let \( \{ \kappa_1, \ldots, \kappa_m \} \) be the principal curvatures of \( \Sigma \), and \( L_\Sigma \) the Weingarten tensor.

Then it follows from part (c) that \( |\kappa_1|, \ldots, |\kappa_m| \leq 1/a \) and \( |L_\Sigma| \leq 1/a \).
In the following, we say that $\Sigma$ is a (URT)–hypersurface in $\mathbb{R}^{m+1}$ if

- (T1) $\Sigma$ is a smooth oriented hypersurface without boundary embedded in $\mathbb{R}^{m+1}$.
- (T2) $(\Sigma, g)$ is uniformly regular, where $g = g_{m+1}|\Sigma$ denotes the metric induced by the Euclidean metric $g_{m+1}$.
- (T3) $\Sigma$ has a tubular neighborhood.

**Examples 3.2.** (a) Every smooth compact hypersurface without boundary embedded in $\mathbb{R}^{m+1}$ is a (URT)–hypersurface.

(b) All of the manifolds considered in [20, 21] are (URT)–hypersurfaces. In particular, the infinite cylinder with radius $r > 0$,

$$C_r = \{(x, y, z) \in \mathbb{R}^3 : y^2 + z^2 = r^2, \quad x \in \mathbb{R}\},$$

is a (URT)–hypersurface with tubular neighborhood of radius $a = r$.

(c) Assume that $f : \mathbb{R}^m \rightarrow \mathbb{R}$ belongs to $BC^2(\mathbb{R}^m)$. Then the graph of $f$ has a tubular neighborhood of radius $a$ for some $a > 0$.

**Proof.** By the inverse function theorem, there exist uniform constants $\eta > 0$ and $\varepsilon > 0$ such that, at every point $x \in \mathbb{R}^m$, $f|_{B^m(x, \eta)}$ can be expressed as the graph of a $BC^2$–function $h_x$ over $T_x\text{gr}(f)$, the tangent space to the graph of $f$ at the point $(x, f(x))$, such that the set $\{(y, h_x(y)) : y \in B_{T_x\text{gr}(f)}(0, \varepsilon)\}$ is contained in $\{(z, f(z)) : z \in B^m(\mathbb{x}, \eta)\}$. Moreover, there exists a uniform constant $c$, independent of $x$, such that

$$\|h_x\|_{2, \infty} \leq c$$

where the supremum is taken over the ball $B_{T_x\text{gr}(f)}(0, \varepsilon)$. We refer to the proof of Claim 1 in Proposition A.1(b) in the Appendix for a more general situation.

Further, we have $h_x(0) = 0$ and $\nabla h_x(0) = 0$. Due to (3.2), after Taylor expansion of $h_x$ around $0 \in T_x\text{gr}(f)$, we have

$$|h_x(y)| \leq \|\nabla_y^2 h_x\|_{\infty} \|y\|^2, \quad y \in B_{T_x\text{gr}(f)}(0, \varepsilon),$$

for sufficiently small $\varepsilon$. Choosing $C \geq \|\nabla_y^2 h_x\|_{\infty}$ such that $1/2C \leq \varepsilon$, we define $a := 1/2C$. It follows that the ball $B^{m+1}(a\nu_x, a)$ lies above the graph $\{(y, h_x(y)) : y \in T_x\text{gr}(f)(0, \varepsilon)\}$, where $\nu_x$ is the upwards pointing unit normal of $\text{gr}(f)$ at the point $(x, f(x))$. An analogous argument shows that the ball $B^{m+1}(-a\nu_x, a)$ lies below the graph.

Since the constants $\varepsilon$ and $a$ are independent of $x$, combining with Remark 3.1(c), this proves that $\text{gr}(f)$ has a tubular neighborhood of radius $a$. □

(d) We refer to [3, 4, 5] for additional examples of uniformly regular manifolds. In particular, embedded hypersurfaces with tame ends, considered in [5] Theorem 1.2, are (URT)–hypersurfaces. More precisely, given a compact hypersurface without boundary $B$, embedded in $\mathbb{R}^m$, and $0 \leq \alpha \leq 1$, we define

$$F_\alpha(B) := \{(t, t^\alpha y) : t > 1, y \in B\},$$

which we endow with the metric $g_{F_\alpha(B)}$ induced by its embedding into $\mathbb{R}^{m+1}$. An embedded hypersurface $\Sigma \subset \mathbb{R}^{m+1}$ is said to have tame ends if

$$\Sigma = V_0 \cup \bigcup_{i=1}^n V_i,$$
where \((V_0, g_{m+1}|_{V_0})\) is compact and \((V_1, g_{m+1}|_{V_1})\) is isometric to \((F_\alpha(B), g_{F_\alpha(B)})\). Then, \((\Sigma, g_{m+1}|_\Sigma)\) is a (URT)-hypersurface.

In particular, when \(\alpha = 0\), \((\Sigma, g_{m+1}|_\Sigma)\) has finitely many cylinder ends; when \(\alpha = 1\), \((\Sigma, g_{m+1}|_\Sigma)\) has finitely many (blunt) cone ends.

(c) Let 
\[
  C_k = \{(x, y, z) \in \mathbb{R}^3 : y^2 + z^2 = 1 + 1/k, \ x \in \mathbb{R}\}, \quad k \in \mathbb{N}.
\]
Based on part (b), the manifold \(\Sigma = \bigcup_k C_k\), endowed with the metric induced by \(g_3\), is uniformly regular. But it is obvious that \((\Sigma, g)\) does not have a tubular neighborhood.

(f) There also exist connected uniformly regular hypersurfaces that are not (URT). For instance, we can construct a smooth connected curve \(C\) in \(\{(x, y) : y > 0\}\) such that \(C \cap \{(x, y) : x \geq 0\}\) is compact and
\[
  C \cap \{(x, y) : x < 0\} = \{(x, y) : y = 1\} \cup \{(x, y) : y = 1 + e^x\}.
\]
Then \((C, g^2|_C)\) is a uniformly regular hypersurface that is not (URT). One can take the product of \(C\) with \(\mathbb{R}^m\) to produce higher dimensional examples.

Additionally, one can rotate the curve \(C\) around the \(x\)-axis to obtain a connected rotationally symmetric uniformly regular hypersurface which is not (URT).

4. The surface diffusion flow

In solving the surface diffusion flow, one seeks to find a family of (oriented) closed hypersurfaces \(\{\Gamma(t) : t \geq 0\}\) satisfying the evolution equation
\[
  \begin{align*}
  V(t) &= -\Delta_{\Gamma(t)} H_{\Gamma(t)}, \quad t > 0, \\
  \Gamma(0) &= \Gamma_0,
  \end{align*}
\]
for an initial hypersurface \(\Gamma_0\).

Here, \(V(t)\) denotes the velocity in the normal direction of \(\Gamma\) at time \(t\), \(H_{\Gamma(t)}\) is the mean curvature of \(\Gamma(t)\) (i.e., the average of the principal curvatures), and \(\Delta_{\Gamma(t)}\) is the Laplace-Beltrami operator on \(\Gamma(t)\). We use the convention that a sphere has negative mean curvature. We note that this convention is in agreement with \([29, 32, 33]\), but differs from \([13, 20, 22]\).

In the following, we assume that \(\Sigma\) is a (URT)–hypersurface in \(\mathbb{R}^{m+1}\) with tubular neighborhood \(U_a\) and with an orientation-preserving atlas \(\mathfrak{A} := \{(O_\kappa, \varphi_\kappa) : \kappa \in \mathfrak{K}\}\) with \(\varphi_\kappa = \varphi_\kappa^{-1}\) satisfying (R1)–(R5). In the following, we assume that \(\Sigma\) carries the metric induced by the Euclidean metric \(g_{m+1}\). Finally, we assume that \(\Gamma_0\) lies in \(U_a\).

For \(\alpha \in (0, 1)\) a fixed parameter, we define
\[
  E_0 := bc^\alpha(\Sigma) \quad \text{and} \quad E_1 := bc^{4+\alpha}(\Sigma).
\]
For \(\theta \in (0, 1)\), let \(E_\theta := (E_0, E_1)^\theta_0\). Taking \(\mu = 1/4\) and \(\gamma = 3/4\), it follows from Proposition \([2, 3]\) that
\[
  E_\mu = bc^{1+\alpha}(\Sigma) \quad \text{and} \quad E_\gamma = bc^{3+\alpha}(\Sigma).
\]
Given \(\rho \in E_\mu\) with \(\|\rho\|_\infty < a\), it follows, by assumption that \(\Sigma\) is (URT) with tubular neighborhood \(U_a\), that
\[
  \Psi_\rho : \Sigma \to \mathbb{R}^{m+1}, \quad \Psi_\rho(p) = p + \rho(p)\nu_\Sigma(p), \quad (4.2)
\]
is a diffeomorphism from $\Sigma$ onto the $C^1$–manifold $\Gamma_{\rho} := \text{im}(\Psi_{\rho})$; see also Proposition \[1.1\] for additional properties of $\Gamma_{\rho}$.

When the temporal variable $t$ is included in $\rho$, i.e.,
$$
\rho : [0, T) \times \Sigma \to (-a, a),
$$
we can also extend $\Psi_{\rho}$ to $\Psi_{\rho} : [0, T) \times \Sigma \to \mathbb{R}^{m+1}$. In the sequel, we will omit the temporal variable $t$ in $\rho$, $\Psi_{\rho}$ and $\Gamma_{\rho}$ when the dependence on $t$ is clear from context.

Let us fix some notation. We denote by $g_{m+1}|_{\Gamma_{\rho}}$ the metric induced on $\Gamma_{\rho}$ by the Euclidean metric $g_{m+1}$ of $\mathbb{R}^{m+1}$. Let $g_{\rho} := \Psi_{\rho}^* (g_{m+1}|_{\Gamma_{\rho}})$ be the pull-back metric of $g_{m+1}|_{\Gamma_{\rho}}$ on $\Sigma$.

The following expression for $g_{\rho}$ was derived in \cite{28} Formula (23):
$$
g_{ij}(\rho) = g_{ij} - 2\rho l^i_j + \rho^2 l^l_i l^j_l + \partial_i \rho \partial_j \rho, \quad (4.3)
$$
where $l^i_j$ and $l_{ij}$ are the components of the Weingarten tensor $L_{\Sigma}$ and the second fundamental form with respect to $g := g_{m+1}|_{\Sigma}$; i.e.,
$$
l_{ij} = -(\tau_i \partial_j \nu_{\Sigma}), \quad l^i_j = g^{ik} l_{kj}, \quad L_{\Sigma} = l_{ij} \tau^i \otimes \tau^j = l^i_j \tau_i \otimes \tau_j,
$$
where $\{\tau_1, \ldots, \tau_m\} = \{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^m}\}$ is a local basis of $T\Sigma$ at $p$ and $\{\tau^1, \ldots, \tau^m\} = \{dx^1, \ldots, dx^m\}$ is the corresponding dual basis, characterized by $(\tau^i|\tau_j) = \delta^i_j$.

We introduce an open subset of $E_\mu$ defined by
$$
V_\mu := \{\rho \in E_\mu : \|\rho\|_\infty < a\}.
$$
By Remark \[3.1(d)\], the functions
$$
a(\rho) := (I - \rho L_{\Sigma})^{-1} \nabla_{\Sigma} \rho, \quad \beta(\rho) := [1 + |a(\rho)|^2]^{-1/2}, \quad (4.4)
$$
are well-defined for all $\rho \in V_\mu$, where $\nabla_{\Sigma} \rho$ is the gradient vector and $I = \tau_i \otimes \tau^i$.

It is easy to verify that $g_{ij}(\rho) = (\tau_i|K(\rho)\tau_j)$, where
$$
K(\rho) = (I - \rho L_{\Sigma})^2 + \nabla_{\Sigma} \rho \otimes \nabla_{\Sigma} \rho = (I - \rho L_{\Sigma})[I + a(\rho) \otimes a(\rho)](I - \rho L_{\Sigma}).
$$
Hence, we obtain
$$
g_{ij}(\rho) = \left((I - \rho L_{\Sigma})\tau_i \bigg[I + a(\rho) \otimes a(\rho)\bigg] (I - \rho L_{\Sigma})\tau_j\right). \quad (4.5)
$$
It follows from the well-known relation
$$
[I + a \otimes a]^{-1} = I - \frac{a \otimes a}{1 + |a|^2}, \quad a \in \mathbb{R}^{m+1},
$$
that $K(\rho)$ is invertible for every $\rho \in C^1(\Sigma)$ with $\|\rho\|_\infty < a$, with inverse given by
$$
K^{-1}(\rho) = M_0(\rho)(I - \beta^2(\rho)a(\rho) \otimes a(\rho)) M_0(\rho),
$$
where $M_0(\rho) := (I - \rho L_{\Sigma})^{-1}$. We then have $g^{ij}(\rho) = (\tau^i|K^{-1}(\rho)\tau^j)$ for the components of the cotangent metric $g^*\rho$ on $T^*\Sigma$ induced by $g(\rho)$, and hence
$$
g^{ij}(\rho) = \left(M_0(\rho)\tau^i \bigg[I - \beta^2(\rho)a(\rho) \otimes a(\rho)\bigg] M_0(\rho)\tau^j\right), \quad (4.6)
$$
see also \cite{29} Section 2.2).

When parameterizing the evolving hypersurface $\Gamma(t) = \Gamma_{\rho(t)}$ by means of a height function $\rho(t) \in V_1 = V_\mu \cap E_1$, it holds that \[4.1\] is equivalent to
$$
\partial_t \rho = -\frac{1}{\beta(\rho)} \Psi_{\rho}^* (\Delta_{g_{m+1}|_{\Gamma_{\rho}}} H_{\Gamma_{\rho}}) = -\frac{1}{\beta(\rho)} \Delta_{\rho} H_{\rho}. \quad (4.7)
$$
Here, $\Delta_{g_{m+1}|r}$ and $\Delta_\rho$ denote Laplace-Beltrami operators on $(\Gamma_\rho, g_{m+1}|_r)$ and $(\Sigma, g(\rho))$, respectively. It was shown in [33, Section 5] that $H_\rho := \Psi_\rho^* H_{\Gamma_\rho}$ in each local patch $(Q_\kappa, \varphi_\kappa)$ reads as

$$H_\rho = \frac{\beta(\rho)}{m} \left\{ g^{ij}(\rho) \partial_i \partial_j \rho + g^{ij}(\rho)(l_i^h \partial_p - \Gamma^k_{ij} \partial_k \rho \right\} \partial \rho$$

$$+ g^{ij}(\rho) \left[ r^{ij}_1(\rho) + r^{ij}_2(\rho) (\partial_3 l_i^k + \Gamma^h_{ij} l_i^k - \Gamma^h_{ij} l_i^k \partial_h \rho) \right] \partial \rho$$

$$+ \frac{\beta(\rho)}{m} g^{ij}(\rho) (l_{ij} - l_{ik} l^k_\rho),$$

with $g^{ij}(\rho)$ given in (4.6). Here, $\Gamma^k_{ij}(\rho)$ are the components of the Christoffel symbols of $\Sigma$ associated with the metric $g = g_{m+1}|_\Sigma$, and $r^{ij}_1(\rho) = p^{ij}_1(\rho)/q^{ij}_1(\rho)$, where $p^{ij}_1(\rho)$ and $q^{ij}_1(\rho)$ are polynomials of $\rho$ with $BC^{\infty}$-coefficients. Note that although (4.8) was derived for compact hypersurfaces in [33], this expression still holds true for our problem as it is purely local.

In local coordinates with respect to the atlas $\mathcal{A}$, $\Delta_\rho$ is given by

$$\Delta_\rho = g^{ij}(\rho) (\partial_i \partial_j - \Gamma^k_{ij}(\rho) \partial_k),$$

where $\Gamma^k_{ij}(\rho)$ are the Christoffel symbols of $(\Sigma, g(\rho))$. Here we note that the terms $\Gamma^k_{ij}(\rho)$ depend on $\rho$ and up to its second–order derivatives. More precisely,

$$\Gamma^k_{ij}(\rho) = \frac{p^{ij}_k(\rho, \partial \rho, \partial^2 \rho)}{q^{ij}_k(\rho, \partial \rho)},$$

where $p^{ij}_k$ is a polynomial of $\rho$ and its derivatives up to second order and $q^{ij}_k$ is a polynomial of $\rho$ and its first–order derivatives (both polynomials having $BC^{\infty}$-coefficients).

By the expression above, we obtain

$$H_\rho = \frac{\beta(\rho)}{m} C(g^*(\rho), \nabla^2 \rho) + \text{lower order terms}$$

and

$$\Delta_\rho H_\rho = \frac{1}{m} C(g^*(\rho) \otimes g^*(\rho), \nabla^4 \rho) + \text{lower order terms},$$

where $C(\cdot, \cdot)$ denotes the complete contraction and $\nabla$ is the covariant derivative with respect to $(\Sigma, g)$. Here and in the sequel, we will still use $\nabla$ to denote its extension to $T^*_\Sigma$. Note that for $u \in C^4(\Sigma)$, the tensor $\nabla^4 u \in C(\Sigma, T^*_\Sigma \otimes^4)$ can be expressed in local coordinates by

$$\nabla^4 u = \partial_{(j)} u \tau^{(j)} + \sum a_{\beta, (j)} \partial^\beta u \tau^{(j)},$$

with coefficients $a_{\beta, (j)} \in BC^{\infty}(\mathbb{B}^m)$, where the summation runs over all multi-indices $(j) = (j_1, \cdots, j_4) \in \{1, \cdots, m\}^4$ and all $\beta \in \mathbb{N}^m$ with $|\beta| \leq 3$. Here we are also using

$$\partial_{(j)} = \partial_{j_1} \partial_{j_2} \partial_{j_3} \partial_{j_4}, \quad \tau^{(j)} = \tau^{j_1} \otimes \tau^{j_2} \otimes \tau^{j_3} \otimes \tau^{j_4},$$

and $\partial^\beta := \partial^\beta_{(i_1} \cdots \partial^\beta_{(i_4)}$; see for instance [33, page 444]. Hence we obtain

$$C(g^*(\rho) \otimes g^*(\rho), \nabla^4 u) = g^{ij}(\rho) g^{lm}(\rho) \partial_{(j} \partial_{k} \partial_{m} u + \sum_{0<|\beta|\leq3} b_{\beta}(\rho, \partial \rho) \partial^{\beta} u$$

(4.10)

for each $\rho \in BC^1(\Sigma)$ with $\|\rho\|_{\infty} < a$ and $u \in C^4(\Sigma)$. 

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By defining
\[ A(\rho) := \frac{1}{m} C(g^* (\rho) \otimes g^* (\rho), \nabla^4 \rho), \quad F(\rho) := A(\rho) \rho - \frac{1}{\beta(\rho)} \Delta_\rho H_\rho, \quad (4.11) \]
we obtain an equivalent formulation of (4.1) as
\[
\begin{cases}
\partial_t \rho + A(\rho) \rho = F(\rho) & \text{in } (0, \infty) \times \Sigma, \\
\rho(0) = \rho_0 & \text{in } \Sigma. 
\end{cases}
\quad (4.12)
\]
We note that for each \( \rho \in C^1(\Sigma, \mathbb{R}) \) with \( \| \rho \|_\infty < a \), the mapping
\[
A(\rho) : C^4(\Sigma, \mathbb{R}) \to C(\Sigma, \mathbb{R}) : [u \mapsto \frac{1}{m} C(g^* (\rho) \otimes g^* (\rho), \nabla^4 u)]
\]
gives rise to a differential operator of order 4.

A linear operator
\[
\mathcal{A} := \sum_{i=0}^l C(a_i, \nabla^i \cdot), \quad u \mapsto \mathcal{A} u = \sum_{i=0}^l C(a_i, \nabla^i u),
\]
of order \( l \), acting on scalar functions, is said to be uniformly strongly elliptic if there exist positive constants \( r, R > 0 \) such that the principal symbol of \( \mathcal{A} \),
\[
\hat{\sigma} \mathcal{A}^* (p, \xi) := C(a_i, (\xi^\tau)^\odot_i)(p) \in \mathbb{R}, \quad (p, \xi) \in \Sigma \times T_p^* \Sigma,
\]
satisfies
\[
r \leq \text{Re} \hat{\sigma} \mathcal{A}^* (p, \xi) \leq R, \quad \text{for all } (p, \xi) \in \Sigma \times T_p^* \Sigma \text{ with } |\xi|_{g^* (p)} = 1. \quad (4.13)
\]

**Remark 4.1.** It is not difficult to see that in the scalar case, the notion of uniformly strongly elliptic is equivalent to the notion of uniformly normally elliptic introduced in [34] Section 3], see also [6].

In our setting, the principal symbol of \( A(\rho) \) is given by
\[
\hat{\sigma} \mathcal{A}^* (p, \xi) = |\xi|_{g^* (p)}^4, \quad \xi \in T_p^* \Gamma.
\]
It follows from (4.6) that \( g^* (\rho) \sim g^* \) for all \( \rho \in V_\mu \), in the sense that there exists some \( c \geq 1 \) such that
\[
(1/c)|\xi|_{g^* (p)}^2 \leq |\xi|_{g^* (p)}^2 \leq c|\xi|_{g^* (p)}^2 \quad \text{for any } (p, \xi) \in \Sigma \times T_p^* \Sigma.
\]
In fact, note that with \( \xi = \xi^\tau \in T_p^* \Sigma, \quad (4.6) \) implies
\[
|\xi|_{g^* (p)}^2 = g^{ij} (\rho)(\rho)(\xi, \xi) = |M_0(\rho) \xi|^2 (p) - \beta^2 (\rho)(a(\rho)|M_0(\rho) \xi|^2 (p).
\]
Next, observe that
\[
\beta^2 (\rho)|M_0(\rho) \xi|^2 (p) \leq g^{ij} (\rho)(\xi, \xi) \leq |M_0(\rho) \xi|^2 (p), \quad (4.14)
\]
where we employed the Cauchy-Schwarz inequality and \( 1 - \beta^2 (\rho)|a(\rho)|^2 = \beta^2 (\rho) \) for the first estimate. It remains to observe that
\[
\min \left\{ \frac{1}{(1 - \rho \kappa_+ (p))^2} \right\} g^{ij} (\rho)(\xi, \xi) \leq |M_0(\rho) \xi|^2 (p) \leq \max \left\{ \frac{1}{(1 - \rho \kappa_- (p))^2} \right\} g^{ij} (\rho)(\xi, \xi),
\]
where \( \kappa_+ \) are the principal curvatures of \( \Sigma \), which are bounded by \( 1/a \) since \( \Sigma \) satisfies a uniform ball condition of radius \( a \). This shows that \( A(\rho) \) is uniformly strongly elliptic. Remark 1.1 and [34] Proposition 2.7, Theorem 3.7 now imply the following result.

**Proposition 4.2.** \( A \in C^\infty (V_\mu, M_\mu (E_1, E_0)) \).
Next, we will verify that the operator \( F \) satisfies (H2). In each patch \((\Omega_\kappa, \varphi_\kappa)\), we reference [22, Section 4.4] and (4.10) to confirm that the local expression for \( F(\rho) \) is of the form

\[
F(\rho) = \sum_{|\eta|=3,|\tau|\leq 2} c_{\eta,\tau}(\rho, \partial \rho) \partial^\tau \rho \partial^\eta \rho + \sum_{|\eta|,|\sigma|,|\tau|\leq 2} d_{\eta,\sigma,\tau}(\rho, \partial \rho) \partial^\sigma \rho \partial^\eta \rho, \tag{4.15}
\]

where \( \eta, \tau, \sigma \in \mathbb{N}^n \) are multi-indices of length \(|\eta| = \eta_1 + \cdots + \eta_m \) and \( \partial^n := \partial^\eta_{x_1} \cdots \partial^\eta_{x_m} \) is the mixed partial derivative operator in local coordinates. The coefficient functions \( c_{\eta,\tau} \) and \( d_{\eta,\sigma,\tau} \) depend analytically on \( \rho \) and its first-order derivatives. In the sequel, for a function \( u : \Sigma \to \mathbb{R} \), we define \( u_\kappa := \zeta \psi_\kappa u \). Let \( \rho_0 \in V_\mu \).

For \( R > 0 \), we choose \( \rho_1, \rho_2 \in \mathbb{B}_E(\rho_0, R) \cap (V_\mu \cap E_\gamma) \). By Proposition 2.2, we have

\[
\| F(\rho_1) - F(\rho_2) \|_{\alpha, \infty} \leq C \| R^c F(\rho_1) - R^c F(\rho_2) \|_{L^\infty(BC^n)} \leq C \sup_{\kappa \in \mathbb{R}} \| R^c_\kappa F(\rho_1) - R^c_\kappa F(\rho_1) \|_{\alpha, \infty}. \tag{4.16}
\]

In the following computations, \( \tilde{C} \) denotes a generic constant depending only on \( R \) and \( \| \rho_0 \|_{1+\alpha, \infty} \). In every patch \((\Omega_\kappa, \varphi_\kappa)\), by the discussion in [22, Section 4.1], we have the following estimate.

\[
\| R^c_\kappa F(\rho_1) - R^c_\kappa F(\rho_1) \|_{\alpha, \infty} \leq \tilde{C} \| \rho_{1,\kappa} - \rho_{2,\kappa} \|_{3+\alpha, \infty} \tag{4.17}
\]

\[
+ \tilde{C} \left( \| \rho_{1,\kappa} \|_{3+\alpha, \infty} \| \rho_{1,\kappa} - \rho_{2,\kappa} \|_{2+\alpha, \infty} + \| \rho_{2,\kappa} \|_{2+\alpha, \infty} \| \rho_{1,\kappa} - \rho_{2,\kappa} \|_{3+\alpha, \infty} \right) \tag{4.18}
\]

\[
+ \| \rho_{2,\kappa} \|_{2+\alpha, \infty} \| \rho_{1,\kappa} - \rho_{2,\kappa} \|_{1+\alpha, \infty} \tag{4.19}
\]

\[
+ \tilde{C} \left( \| \rho_{1,\kappa} \|_{2+\alpha, \infty} \| \rho_{1,\kappa} - \rho_{2,\kappa} \|_{1+\alpha, \infty} + \| \rho_{1,\kappa} \|_{2+\alpha, \infty} \| \rho_{1,\kappa} - \rho_{2,\kappa} \|_{2+\alpha, \infty} \right) \tag{4.20}
\]

\[
+ \| \rho_{1,\kappa} \|_{2+\alpha, \infty} \| \rho_{2,\kappa} \|_{2+\alpha, \infty} \| \rho_{1,\kappa} - \rho_{2,\kappa} \|_{1+\alpha, \infty} \right) \tag{4.21}
\]

the definitions of the spaces \( E_0, E_\mu, E_\gamma \), and \( E_1 \), we have \( \mu = 1/4 \) and \( \gamma = 3/4 \) in our current setting. Thus, we refer back to (2.4) to see that index \( j \) is subcritical when \((\varrho_j, \gamma_j)\) satisfies \( \varrho_j/2 + \gamma_j < 1 \), and \( j \) is critical when \( \varrho_j/2 + \gamma_j = 1 \). It follows from Proposition 2.2 that (4.17) is bounded by

\[
\| \rho_{1,\kappa} - \rho_{2,\kappa} \|_{3+\alpha, \infty} \leq \sup_{\eta \in \mathbb{R}} \| \rho_{1,\eta} - \rho_{2,\eta} \|_{3+\alpha, \infty} \leq \tilde{C} \| \rho_1 - \rho_2 \|_{E_\gamma}.
\]

This corresponds to \((\varrho_j, \gamma_j) = (0, 3/4)\), which is subcritical. In (4.18), similarly it holds

\[
\| \rho_{1,\kappa} \|_{3+\alpha, \infty} \| \rho_{1,\kappa} - \rho_{2,\kappa} \|_{2+\alpha, \infty} \leq \tilde{C} \| \rho_{1,\kappa} \|_{E_\gamma} \| \rho_1 - \rho_2 \|_{2+\alpha, \infty}.
\]
This corresponds to \((\theta_j, \gamma_j) = (1, 1/2)\) (which is again subcritical). We can estimate the remaining terms of (4.18) by using Propositions 2.2 and 2.3:

\[
\|\rho_{2,\kappa}\|_{2+\alpha,\infty} \|\rho_{1,\kappa} - \rho_{2,\kappa}\|_{2+\alpha,\infty} + \|\rho_{2,\kappa}\|_{3+\alpha,\infty} \|\rho_{1,\kappa} - \rho_{2,\kappa}\|_{1+\alpha,\infty}
\]
\[
\leq \tilde{C} \left( \|\rho_{2}\|_{1/2}^{1/2} \|\rho_1 - \rho_2\|_{E_\gamma} + \|\rho_2\|_{E_\gamma}^{3/2} \|\rho_1 - \rho_2\|_{E_{\mu}} \right)
\]

These correspond to \((\theta_j, \gamma_j) = (1/2, 3/4)\) and \((\theta_j, \gamma_j) = (3/2, 1/4)\), which are critical. The remaining terms, i.e. (4.19)-(4.21), can be estimated similarly, cf. [22, Section 4]. We conclude that (4.19)-(4.21) is bounded by (4.16) implies that

\[
F
\]

\[
\text{The indices for those estimates are} \quad (\theta_j, \gamma_j) = (1/2, 1/4), (0, 1/2), (1, 1/4), (1/2, 1/2) \quad \text{(subcritical)}, \quad \text{and} \quad (\theta_j, \gamma_j) = (1, 1/2), (3/2, 1/4) \quad \text{(critical)}, \quad \text{respectively}.
\]

Additionally, it follows from (2.1) that

\[
R^\rho F(\rho) \in \ell_{\infty, \text{uniq}}(bc^{\kappa}) \quad \text{for any} \quad \rho \in V_\mu \cap E_\gamma.
\]

Since the constant \(C\) is independent of \(\kappa\), the above computations together with (4.16) imply that \(F\) satisfies (H2) and

\[
F \in C^{\infty}(V_\mu \cap E_\gamma, E_0).
\]

Combining the above discussions, we apply Theorem 2.4 to produce the following well-posedness result for (4.12). Note that we assume throughout that \(\Sigma\) carries the metric induced by the Euclidean metric \(g_{n+1}\).

**Theorem 4.3.** Let \(\alpha \in (0, 1), \mu = 1/4\) and \(\Sigma\) be a (URT)–hypersurface in \(\mathbb{R}^{n+1}\) with a tubular neighborhood of radius \(a\).

(a) Then for any \(\rho_0 \in V_\mu := \{ \rho \in bc^{1+\alpha}(\Sigma) : \|\rho\|_{\infty} < a \}, (4.12)\) has a unique solution

\[
\rho(\cdot, \rho_0) \in C_{1-\mu}(J, bc^{1+\alpha}(\Sigma)) \cap C_{1-\mu}^1(J, bc^{\alpha}(\Sigma))
\]

on a maximal interval \(J = [0, T) = [0, T(\rho_0))\), with the additional property that \(\rho(\cdot, \rho_0) \in C(J, bc^{1+\alpha}(\Sigma))\).

(b) \(\mathcal{M} := \bigcup_{t \in (0, T)} \{t\} \times \Gamma(t)\)

is a \(C^{\infty}\)–hypersurface in \(\mathbb{R}^{m+2}\). In particular, each manifold \(\Gamma(t)\) is \(C^{\infty}\) for \(t \in (0, T)\). If, in addition, \(\Sigma\) is \(C^{\infty}\)–uniformly regular, then \(\mathcal{M}\) is a \(C^{\infty}\)–hypersurface in \(\mathbb{R}^{m+2}\).

(c) The map \([t, \rho_0) \mapsto \rho(t, \rho_0)\] defines a semiflow on \(V_\mu\) which is analytic for \(t > 0\) and Lipschitz continuous for \(t \geq 0\).

**Proof.** We have already proved part (a) above. Part (b) follows directly from the argument in [34 Sections 3 and 5]. For part (c), we first note that Lipschitz continuity of the semiflow follows from [22 Corollary 2.3]. Regarding additional regularity of the semiflow; for any \(\tau > 0\), we note that

\[
\rho(\tau, \rho_0) \in V_\gamma = bc^{3+\alpha}(\Sigma) \cap \{\|\rho\|_{\infty} < a\},
\]
and so the result holds in \( V \) because of \([10]\) Theorem 6.1 and the mapping properties of \( A(\cdot) \) and \( F(\cdot) \). Regularity of the semiflow in \( V_\mu \) then follows by embedding.

\( \square \)

5. The Willmore Flow

In this section, we take \( \Sigma \) to be a (URT)-hypersurface in \( \mathbb{R}^3 \). For the Willmore flow, we seek a family of hypersurfaces \( \{\Gamma(t): t \geq 0\} \) satisfying the evolution equation

\[
\begin{aligned}
V(t) &= -\Delta_{\Gamma(t)} H_{\Gamma(t)} - 2H_{\Gamma(t)} \left( H_{\Gamma(t)}^2 - K_{\Gamma(t)} \right), \quad t > 0, \\
\Gamma(0) &= \Gamma_0,
\end{aligned}
\]

where the term \( K_{\Gamma(t)} \) denotes Gaussian curvature of \( \Gamma(t) \).

Working in the same setting as Section 4 above, we consider (5.1) acting on \( \Sigma \) via height functions \( \rho(t): \Sigma \to \mathbb{R} \). Assuming that \( \Sigma \) has a tubular neighborhood \( U_a \) of radius \( a > 0 \), we recall that \( E_\mu = bc^{1+\alpha}(\Sigma) \) and \( E_\gamma = bc^{3+\alpha}(\Sigma) \) are interpolation spaces between \( E_0 := bc^\alpha(\Sigma) \) and \( E_1 := bc^{4+\alpha}(\Sigma) \), and we consider initial functions from \( V_\mu := \{\rho \in E_\mu : \|\rho\|_\infty < a\} \).

Treating (5.1) as a lower-order perturbation of (4.1), we again define

\[
A(\rho): E_1 \to E_0 : \ [u \mapsto \frac{1}{m} C(g^*(\rho) \otimes g^*(\rho), \nabla^4 u)]
\]

for all \( \rho \in V_\mu \), and we introduce the mapping \( Q: V_\mu \cap E_\gamma \to E_0 \) defined as

\[
Q(\rho) := A(\rho)\rho - \frac{1}{\beta(\rho)} \Psi^*_\rho \left( \Delta_{\rho,\nabla^4} H_{\rho,\nabla^4} + 2H_{\rho,\nabla^4} \left( H_{\rho,\nabla^4}^2 - K_{\rho,\nabla^4} \right) \right)
\]

\[
= A(\rho)\rho - \frac{1}{\beta(\rho)} \left( \Delta_{\rho} H_{\rho} + 2H_{\rho} \left( H_{\rho}^2 - K_{\rho} \right) \right).
\]

We thus arrive at the following expression for (5.1) in our current setting:

\[
\begin{aligned}
\partial \rho + A(\rho)\rho &= Q(\rho) \quad \text{in} \ (0, \infty) \times \Sigma, \\
\rho(0) &= \rho_0 \quad \text{in} \ \Sigma.
\end{aligned}
\]

By Proposition 4.2, we know that \( A \in C^\omega(V_\mu, M_\mu(E_1, E_0)) \) so we focus on showing regularity and structural properties for \( Q(\rho) \).

By (4.11) and the definition of \( Q(\rho) \), we note that

\[
Q(\rho) = F(\rho) - \frac{2}{\beta(\rho)} H_\rho^3 + \frac{2}{\beta(\rho)} H_\rho K_\rho
\]

and it follows that the local expression for \( Q(\rho) \) is of the form

\[
Q(\rho) = \sum_{|\eta|=1} c_{\eta,\tau}(\rho, \partial \rho) \partial^{\eta} \rho \partial^{\tau} \rho + \sum_{|\eta|, |\sigma|, |\tau| \leq 2} d_{\eta,\sigma,\tau}(\rho, \partial \rho) \partial^{\eta} \rho \partial^{\sigma} \rho \partial^{\tau} \rho.
\]

To confirm this local expression for \( Q(\rho) \), we first note that all third-order derivatives of \( \rho \) appear in \( F(\rho) \), while the terms \( \frac{2}{\beta(\rho)} H_\rho^3 \) and \( \frac{2}{\beta(\rho)} H_\rho K_\rho \) depend only on up to second-order derivatives. With the structure for \( F(\rho) \) already established in (4.15), it suffices to confirm that \( Q(\rho) \) only contributes additional terms of the form

\[
\sum_{|\eta|, |\sigma|, |\tau| \leq 2} d_{\eta,\sigma,\tau}(\rho, \partial \rho) \partial^{\eta} \rho \partial^{\sigma} \rho \partial^{\tau} \rho.
\]
Local expressions for $\beta(\rho)$ and $H_\rho$ are given in (4.4) and (4.8), respectively. Since $\beta(\rho)$ depends on at most first–order derivatives of $\rho$ and $H_\rho$ depends linearly on second–order derivatives, we see that at most cubic powers of $\partial^2 \rho$ appear in $\frac{2}{\beta(\rho)} \partial^2 \rho$.

Regarding the term $(2/\beta(\rho))H_\rho K_\rho$, we first express Gaussian curvature $K_\rho = \det[\gamma^{kj}(\rho)l_{ij}(\rho)]$, as derived in [32, Section 2]. Here $l_{ij}(\rho)$ are the components of the pull–back of the second fundamental form of $\Gamma_{\rho}$. It follows from (4.8) that

$$l_{ij}(\rho) = \beta(\rho)\left\{ \partial_i \partial_j \rho + (l_{ij}^k \partial_k \rho - \Gamma_{ij}^k) \partial_k \rho 
+ [r_{ik}(\rho)l_{jk}^k \partial_j \rho + r_{jk}(\rho)l_{ik}^k \partial_i \rho + \Gamma_{ik}^l l_{jl}^k \rho + \Gamma_{jk}^l l_{il}^k \rho \partial_i \rho]ight\} \partial \rho 
+ \beta(\rho)(l_{ij} - l_{ik}^k \rho).$$

Observing that each $l_{ij}(\rho)$ is linear with respect to $\partial^2 \rho$, it follows that $\partial^2 \rho$ appears at most quadratically in $\det[l_{ij}(\rho)]$, since it is a $2 \times 2$ matrix. Therefore, we conclude that $K_\rho$ contains at most quadratic factors of $\partial^2 \rho$ and thus, multiplying with the second–order quasilinear term $H_\rho$, we conclude that the term $(2/\beta(\rho))H_\rho K_\rho$ contains at most cubic powers of $\partial^2 \rho$.

With confirmation that $Q(\rho)$ satisfies the same structural condition (2.5) as $F(\rho)$ in Section 4, we employ the same argument outlined in [4.16]–[4.21] to conclude that $(A, Q)$ satisfies conditions (H1)–(H2). The following well–posedness result for (5.1) then follows from Theorem 2.4.

**Theorem 5.1.** Let $\alpha \in (0, 1)$, $\mu = 1/4$ and $\Sigma$ be a (URT)–hypothenuse in $\mathbb{R}^3$ with tubular neighborhood of radius $a$.

(a) Then for any $\rho_0 \in V_\mu := \{ \rho \in bc^{1+\alpha}(\Sigma) : \|\rho\|_\infty < a \}$, (5.2) has a unique solution $\rho(\cdot, \rho_0) \in C_{1-\mu}^1(J, bc^{1+\alpha}(\Sigma)) \cap C_{1-\mu}^1(J, bc^{\alpha}(\Sigma))$ on a maximal interval $J = [0, T) = [0, T(\rho_0))$, with the additional property that $\rho(\cdot, \rho_0) \in C(J, bc^{1+\alpha}(\Sigma))$.

(b) $\mathcal{M} := \bigcup_{t \in (0, T)} \{ \{t\} \times \Gamma(t) \}$ is a $C^\infty$–hypothenuse in $\mathbb{R}^4$. In particular, each manifold $\Gamma(t)$ is $C^\infty$ for $t \in (0, T)$. If, in addition, $\Sigma$ is $C^\omega$–uniformly regular, then $\mathcal{M}$ is a $C^\omega$–hypothenuse in $\mathbb{R}^{m+2}$.

(c) The map $[t, \rho_0) \mapsto \rho(t, \rho_0)$ defines a semiflow on $V_\mu$ which is analytic for $t > 0$ and Lipschitz continuous for $t \geq 0$.

**Proof.** Part (b) follows from [32] and [33, Section 3]. Part (c) follows exactly as in the proof of Theorem 4.3(c) above. \hfill \Box

### 5.1. **Stability of spheres.**

In the case $\Sigma$ is a Euclidean sphere in $\mathbb{R}^3$, we apply the generalized principle of linearized stability (c.f. [22, Section 3]) to prove the following result regarding stability of spheres under the Willmore flow, with control on only first–order derivatives of perturbations.
Theorem 5.2. Fix $\alpha \in (0, 1)$, $\mu = 1/4$, and $\bar{\mu} \in (0, 1)$, and let $\Sigma$ be a Euclidean sphere in $\mathbb{R}^3$ with radius $r > 0$. There exists a constant $\delta \in (0, r)$ such that, given any admissible perturbation $\Gamma_{\rho_0}$ for
\[
\rho_0 \in V_{\mu, \delta} := \{\rho \in b^{1+\alpha}(\Sigma) : \|\rho\|_\infty < r \text{ and } \|\rho_0\|_{1+\alpha, \infty} < \delta\},
\]the solution $\rho(\cdot, \rho_0)$ of (5.2) exists globally in time and converges to some $\rho \in \mathcal{M}_{sph}$ at an exponential rate, in the topology of $E_{\bar{\mu}}$. Here, $\mathcal{M}_{sph}$ denotes the family of functions $\rho \in C^\infty(\Sigma, \mathbb{R})$ for which $\Gamma_\rho$ is a sphere that is close to $\Sigma$ in $\mathbb{R}^3$.

Proof. It is shown in the proof of [35, Theorem 1.2] that $\rho_0 = 0$ is normally stable under (5.2). The result then follows from [22, Theorem 3.2]. \hfill \Box

Corollary 5.3. There exist non-convex hypersurfaces $\Gamma_0$ such that the solution $\rho(\cdot, \rho_0)$ to (5.2) with $\Gamma(\rho_0) = \Gamma_0$, exists globally in time and converges exponentially fast to a sphere.

We note here that Theorem 5.2 also holds true for the surface diffusion flow, as was shown in [22, Section 4.5].

APPENDIX A.

Suppose $\Sigma$ is a (URT)-hypersurface with tubular neighborhood of radius $a$. Given $\rho \in C(\Sigma)$ with $\|\rho\|_\infty < a$, let $\Gamma_\rho := \Psi_\rho(\Sigma)$, where $\Psi_\rho(p) = p + \rho(p)\nu_\Sigma(p)$ for $p \in \Sigma$. Then $\Gamma_\rho$ enjoys the following properties.

Proposition A.1. Let $k \in \mathbb{N} \cup \{\infty\}$.
(a) Suppose $\rho \in BC^{k+1}(\Sigma)$ and $\|\rho\|_\infty < a$. Then $\Gamma_\rho$ is $C^k$-uniformly regular.
(b) There exists $\epsilon_1 > 0$ such that, for any $\rho \in BC^2(\Sigma)$ with $\|\rho\|_\infty \leq \epsilon_1$, the hypersurface $\Gamma_\rho$ has a tubular neighborhood of radius $a_1$ for some positive number $a_1 = a_1(\epsilon_1, \rho)$.

Proof. (a) We can construct an atlas $\mathfrak{A}_\rho = \{O_{k, \rho}, \varphi_{k, \rho} : \kappa \in \mathbb{R}\}$ for $\Gamma_\rho$ as follows. Define
\[
O_{k, \rho} := \Psi_\rho \circ \varphi_B(\mathbb{B}^m), \quad \varphi_{k, \rho} := \varphi_k \circ \Psi_\rho^{-1}, \quad \varphi_{k, \rho} = \varphi_{k, \rho}^{-1}.
\]
Then $\mathfrak{A}_\rho$ inherits properties (R1)-(R3) from $\mathfrak{A}$. Next we note that
\[
\psi_{k, \rho}(g_{m+1}|_{\Gamma_\rho}) = \psi_k g(\rho),
\]
and that by (4.5), $g(\rho)$ involves first order derivatives of $\rho$. Hence, $\psi_k g(\rho)$ is $C^k$ for $\rho \in C^{k+1}$. It follows readily from (4.5) that
\[
|I - \rho L_\Sigma|\xi|^2(p) \leq g_{ij}(\rho)(p)(\xi, \xi) \leq (1 + |a(\rho)|^2)\|(I - \rho L_\Sigma)\xi|^2(p)
\]
for $p \in \Sigma$ and $\xi = \xi^i \tau_i(p) \in T_p \Sigma$. Properties (R4)-(R5) now follow from
\[
\min\{(1 - (\rho \kappa_\rho))^2\}|\xi|^2 \leq \|(I - \rho L_\Sigma)\xi|^2(p) \leq \max\{(1 - (\rho \kappa_\rho))^2\}|\xi|^2
\]
and Remark 3.1(d).
(b) Let $r_0 \in (0, 1)$ be the constant related to the uniformly shrinkable property.

Claim 1: Let $\tilde{r}_0 := \frac{1+\tilde{r}_0}{2}$. There exists a uniform constant $r_1$ such that for any $\kappa \in \mathbb{R}$ and $p \in \varphi_\kappa(r_0\mathbb{B}^m)$, $\varphi_\kappa(\mathbb{B}^m(x_p, r_1))$ is a graph $f_{\kappa, p}$ over $T_p \Sigma$ with $x_p := \varphi_\kappa(p)$ satisfying
\[
\|f_{\kappa, p}\|_{2, \infty} \leq c_0 \quad \text{(A.1)}
\]
for some $c_0 > 0$ independent of $\kappa$ and $p$. 

Proof of Claim 1. Let \( \kappa \in \mathfrak{K} \) and \( p \in \psi_\kappa(\hat{r}_0 \mathbb{B}^m) \) be given. Then there exists \( x_p \) in \( \hat{r}_0 \mathbb{B}^m \) such that \( p = \psi_\kappa(x_p) \). Let \[
P_p := I - \nu_\Sigma(p) \otimes \nu_\Sigma(p)
\] be the orthogonal projection of \( \mathbb{R}^{m+1} \) onto \( T_p \Sigma \). In the following we will identify \( T_p \Sigma \) with \( \mathbb{R}^m \). It follows from the boundedness of \( \|L_\Sigma\|_\infty \) that there is a universal constant \( b_0 \) such that \( P_p : \psi_\kappa(\mathbb{B}(x_p, b_0)) \to T_p(\Sigma) \cong \mathbb{R}^m \) is injective. Let \[
F_{\kappa, p}(x) := (P_p \circ \psi_\kappa)(x), \quad x \in \mathbb{B}(x_p, b_0).
\]

Then we obtain for the Fréchet derivative of \( F_{\kappa, p} \)
\[
DF_{\kappa, p}(x_p) = P_p D\psi_\kappa(x_p) = D\psi_\kappa(x_p), \quad \tag{A.2}
\]
as \( D\psi_\kappa(x_p) \xi \in T_p \Sigma \) for all \( \xi \in \mathbb{R}^m \). We infer from (R5) that
\[
(1/\gamma_1)|\xi|^2 \leq |D\psi_\kappa(x_p)\xi|^2 = (\psi_\kappa g)(x)(\xi, \xi) \leq \gamma_1^2|\xi|^2, \quad x \in \mathbb{B}^m, \quad \xi \in \mathbb{R}^m, \quad \tag{A.3}
\]
for some uniform constant \( \gamma_1 \geq 1 \). It follows from (A.2) and (A.3) that the spectrum of \( DF_{\kappa, p}(x) \) lies outside the ball \( \mathbb{B}_C(0, 1/\gamma_1) \) for any \( x \in \mathbb{B}^m \). Indeed, suppose \( \mu v = DF_{\kappa, p}(x)v \) for some \( \mu \in \mathbb{C} \) and \( v = \xi + i\eta \in \mathbb{C}^m \) with \( |v| = 1 \). Then
\[
|\mu|^2 = |DF_{\kappa, p}(x)v|^2 = |D\psi_\kappa(x)\xi|^2 + |D\psi_\kappa(x)\eta|^2 \geq (1/\gamma_1)^2.
\]
Lemma 4.1 in [7] implies that \( DF_{\kappa, p}(x) \) is invertible with
\[
\left|\left| DF_{\kappa, p}(x) \right|^{-1} \right| \leq \gamma_2, \quad \tag{A.4}
\]
where the constant \( \gamma_2 \) is independent of \( x \in \mathbb{B}^m \) and \( \kappa, p \). By the inverse function theorem, there exists a uniform constant \( r_1 \) which is independent of \( \kappa \) and \( p \in \psi_\kappa(\hat{r}_0 \mathbb{B}^m) \) such that
\[
F_{\kappa, p} : \mathbb{B}^m(x_p, r_1) \to P_p\psi_\kappa(\mathbb{B}^m(x_p, r_1))
\]
is a diffeomorphism. Next we note that
\[
\begin{align*}
\partial_i F_{\kappa, p}^{-1}(y) &= \left[ DF_{\kappa, p}(F_{\kappa, p}^{-1}(y)) \right]^{-1} e_j \\
= \partial_i \partial_j F_{\kappa, p}^{-1}(y) &= \partial_i [DF_{\kappa, p}(F_{\kappa, p}^{-1}(y))] \left[ DF_{\kappa, p}(F_{\kappa, p}^{-1}(y)) \right]^{-1} e_j
\end{align*}
\]
Recall that \( \partial_i \partial_j \psi_\kappa = \Gamma^k_{ij} \partial_k \psi_\kappa + l_{ij} \nu_\Sigma \). In view of [3, Formula (3.19)], (A.3) and the boundedness of \( \|L_\Sigma\|_\infty \), we conclude that
\[
\|\psi_\kappa\|_{2, \infty} \leq \gamma_3, \quad \text{for all } \kappa \in \mathfrak{K}. \quad \tag{A.5}
\]

It follows from (A.4) and (A.5) that \( \| F_{\kappa, p}^{-1} \|_{2, \infty} \leq c \) for some \( c \) independent of \( \kappa, p \). Define \( \Phi_{\kappa, p} : P_p\psi_\kappa(\mathbb{B}^m(x_p, r_1)) \to \Sigma \) by \( \Phi_{\kappa, p} := \psi_\kappa \circ F_{\kappa, p}^{-1} \). Note that
\[
\Phi_{\kappa, p}(y) = P_p \circ \Phi_{\kappa, p}(y) + (I - P_p) \circ \Phi_{\kappa, p}(y) \quad = y + (\nu_\Sigma(p) \circ \Phi_{\kappa, p}(y)) \nu_\Sigma(p) = y + f_{\kappa, p}(y) \nu_\Sigma(p).
\]
We can now conclude that (A.1) holds.

In the following, we assume that
\[
r_1 < \min \left\{ \frac{1 - r_0}{2}, \frac{1}{2 \gamma_1 \gamma_3} \right\}. \quad \tag{A.6}
\]
By Claim 1, we can find \( L \in \mathbb{N} \) such that, in every \( O_\kappa \), there exist \( x_{\kappa, i} \in r_0 \mathbb{B}^m \) with \( i = 1, \ldots, L \) such that
\[
\bigcup_{i=1}^L \psi_\kappa(\mathbb{B}^m(x_{\kappa, i}, r_1/4)) \text{ covers } \psi_\kappa(r_0 \mathbb{B}^m).
\]
Taking new local patches $\psi_\kappa(\mathbb{B}^m(x_{\kappa,i}, r_1/2))$, after relabelling, translation and scaling, we obtain a new atlas satisfying (R1)–(R5), still denoted by $\mathfrak{A} = \{ (\mathcal{O}_\kappa, \varphi_\kappa) : \kappa \in \mathcal{K} \}$. Note that for this new atlas, $\mathcal{O}_\kappa$ is the graph of a function $f_{\kappa,p}$ over $T_p\Sigma$ for any $\kappa \in \mathcal{K}$ and $p \in \mathcal{O}_\kappa$. Moreover, (A.1) still holds true. In addition, we can take uniformly shrinkable constant $r_0 = 1/2$. Note also that, by (R5), we can assume that $r_1$ is chosen so small that

$$|p - q| < a/8, \quad p, q \in \mathcal{O}_\kappa. \quad (A.7)$$

Let $\text{dist}(\cdot, \cdot)$ denote the Euclidean distance between two compact subsets in $\mathbb{R}^{m+1}$.

**Claim 2:** There exists $c_1 > 0$ such that $\text{dist}(p, \partial X(\mathcal{O}_\kappa, [-a/2, a/2])) > c_1$ for all $\kappa \in \mathcal{K}$ and $p \in \psi_\kappa(\frac{1}{2}\mathbb{B}^m)$.

**Proof of Claim 2.** We set

$$D_\kappa := X(\mathcal{O}_\kappa \times [-a/2, a/2]),$$

and

$$S_{1,\kappa} := X(\mathcal{O}_\kappa \times \{-a/2\}) \cup X(\mathcal{O}_\kappa \times \{a/2\}), \quad S_{2,\kappa} := \partial D_\kappa \setminus S_{1,\kappa}.$$ We now show that $\text{dist}(p, \partial D_\kappa)$ is uniformly positive.

**Case 1:** $|p-q| = \text{dist}(p, \partial D_\kappa)$ for some $q \in S_{1,\kappa}$

Since $q \in S_{1,\kappa}$, we can find some $q \in \mathcal{O}_\kappa$ such that, without loss of generality, we have $X(q, a/2) = q$. This implies

$$|p-q| \geq a/2 - |p-q| \geq a/4.$$ \hfill (A.8)

**Case 2:** $|p-q| = \text{dist}(p, \partial D_\kappa)$ for some $q \in S_{2,\kappa}$.

We first observe that, for any $p \in \psi_\kappa(\frac{1}{2}\mathbb{B}^m)$, we have by (A.3), (A.5) and Taylor expansion that

$$|p-q| = |\psi_\kappa(x_p) - \psi_\kappa(x_q)| \geq \frac{r_1}{2} \left[ \frac{1}{2\gamma_1} - \frac{r_1}{2\gamma_3} \right],$$

where $p = \psi_\kappa(x_p)$ with $x_p$ in $\frac{1}{2}\mathbb{B}^m$ and $q = \psi_\kappa(x_q)$ with $x_q \in \partial \mathbb{B}^m$. The assumption (A.6) now implies

$$\text{dist}(p, \partial X_\kappa) \geq \eta_0 = r_1/(8\gamma_1).$$ \hfill (A.9)

Moreover, $S_{2,\kappa} = X(\partial X_\kappa \times (-a/2, a/2))$ implies that there exist $q \in \partial X_\kappa$ and $s \in (-a/2, a/2)$ such that $q = X(q, s)$.

Because of (A.7), we can realize $p$ as a point on the graph of $f_{\kappa,q}$, cf. the following figure.
By (A.8), we observe that $|p - q| \geq \eta_0$. Let $d = |p - q|$. Using (A.1), we have

$$d^2 + d^2 c_0^2 \geq \eta_0^2,$$

which implies

$$|p - q| \geq \eta$$

for some uniform constant $\eta_1 > 0$. Thus we can take $c_1 = \min\{a / 4, \eta_1\}$ independent of $p$ and $\kappa$. ■

**Claim 3:** Let $\delta \in (0, \alpha)$ and $M > 0$ be fixed. Then there exists a constant $r_2 > 0$ such that for any $p \in BC^2(\Sigma)$ with $\|\rho\|_{\infty} \leq \delta$, $\|\rho\|_{2, \infty} \leq M$, and any $p \in \psi_\kappa(\frac{1}{2}B^m)$, $\Psi_\rho(\psi_\kappa(x_\rho, r_2))$ is the graph of a $C^2$-function $h_{\kappa, p}$ over $T_\psi_\rho(p) \Gamma_\rho$ satisfying

$$\|h_{\kappa, p}\|_{2, \infty} \leq c_2,$$  \tag{A.9}

for some $c_2 > 0$ independent of $\kappa, p$ and $\rho$.

**Proof of Claim 3.** The proof is basically the same as that of Claim 1, as $\Gamma_\rho$ is a $C^2$-hypersurface and $C^1$-uniformly regular by part (a) of the proposition; and all we need for the proof of Claim 1 is this property. ■

We assume

$$\varepsilon_1 < \min\{a / 8, c_1 / 2\},$$  \tag{A.10}

where $c_1$ is the constant in Claim 2. By our choice of $\varepsilon_1$, following the construction below Claim 1, we can further modify the atlas $\mathfrak{A}$, still with uniform shrinkable parameter $r_0 = 1 / 2$, such that for every $p \in BC^2(\Sigma)$ with $\|\rho\|_{\infty} \leq \varepsilon_1$, $\Psi_\rho(O_\kappa)$ is a graph over $T_\psi_\rho(p) \Gamma_\rho$, for any $\kappa \in \mathfrak{K}$ and $p \in O_\kappa$.

Claim 3 and Example 3.2(c) imply that $\Psi_\rho(O_\kappa)$ has a tubular neighborhood of radius $a_1$, where $a_1$ is independent of $\kappa, p$. In order to prove that $\Gamma_\rho$ has a tubular neighborhood of radius $a_1$, it suffices to show that

$$X_\rho : \Sigma \times (-a_1, a_1) : (p, s) \mapsto \Psi_\rho(p) + s\nu_\rho_\rho(p)$$

is injective.

**Claim 4:** For sufficiently small $a_1 > 0$ and any $p \in BC^2(\Sigma)$ with $\|\rho\|_{\infty} \leq \varepsilon_1$ satisfying (A.10) and $p \in \psi_\kappa(\frac{1}{2}B^m)$, it holds that $B_{\mathbb{R}^{m+1}}(\Psi_\rho(p), 2a_1)$ is contained in $X(O_\kappa, [-a / 2, a / 2])$.

**Proof of Claim 4.** Define $D_\kappa$, $S_1, \kappa$, and $S_2, \kappa$ as in Claim 2. Given any $p \in \psi_\kappa(\frac{1}{2}B^m)$, there exists some $q \in \partial D_\kappa$ such that

$$|\Psi_\rho(p) - q| = \text{dist}(\Psi_\rho(p), \partial D_\kappa).$$

If $q \in S_1, \kappa$, then there exists $q \in O_\kappa$ so that, without loss of generality, $X(q, a / 2) = Q$. By (A.7) and (A.10), we infer that

$$|\Psi_\rho(p) - q| \geq a / 2 - |\Psi_\rho(p) - p| - |p - q| \geq a / 4.$$

If $q \in S_2, \kappa$, by Claim 2 and (A.10)

$$|\Psi_\rho(p) - q| \geq |p - q| - |\rho(p)| \geq c_1 / 2.$$

Therefore, it suffices to take $a_1 \leq \min\{a / 8, c_1 / 4\} = c_1 / 4$. ■
If \( X_\rho(p,s) = X_\rho(q,t) \) for some \( p,q \in \Sigma \) and \( s,t \in (-a_1, a_1) \), we may assume that \( p \in \psi_{\kappa}(\frac{1}{2} B_m) \) for some \( \kappa \in K \). It follows that
\[
|\Psi_\rho(p) - \Psi_\rho(q)| = |s_\Sigma(p) - t_\Sigma(q)| < 2a_1.
\]
We conclude from Claim 4 that \( \Psi_\rho(q) \in X(O_{\kappa}, [-a/2, a/2]) \) and thus \( q \in O_{\kappa} \) as well. However, in this case, we already know that \( \Psi_\rho(O_{\kappa}) \) has an \( a_1 \)-tubular neighborhood, which implies that \( p = q \) and \( s = t \).

Remark A.2. In Proposition A.1(b) it would be desirable to be able to replace the smallness condition \( \|\rho\|_\infty \leq \varepsilon_1 \) by the more natural condition \( \|\rho\|_\infty < a \).

In the special case that \( \Sigma \) is compact, this property holds by Remark 3.1(b), as \( \Gamma_\rho \) is a compact (closed) \( C^2 \)-hypersurface.

References


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