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THE SURFACE DIFFUSION AND THE WILLMORE FLOW FOR UNIFORMLY REGULAR HYPERSURFACES

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ABSTRACT. We consider the surface diffusion and Willmore flows acting on a general class of (possibly non–compact) hypersurfaces parameterized over a uniformly regular reference manifold possessing a tubular neighborhood with uniform radius. The surface diffusion and Willmore flows each give rise to a fourth–order quasilinear parabolic equation with nonlinear terms satisfying a specific singular structure. We establish well–posedness of both flows for initial surfaces that are $C^{1+\alpha}$–regular and parameterized over a uniformly regular hypersurface. For the Willmore flow, we also show long–term existence for initial surfaces which are $C^{1+\alpha}$–close to a sphere, and we prove that these solutions become spherical as time goes to infinity.

1. INTRODUCTION

The surface diffusion and Willmore flows are geometric evolution equations that describe the motion of hypersurfaces in Euclidean space (or, more generally, in an ambient Riemannian manifold). The normal velocity of evolving surfaces is determined by purely geometric quantities. For both flows, the mean curvature is involved in the evolution equations, while the Willmore flow additionally depends upon Gauss curvature.

These flows have been studied by several authors for compact (closed) hypersurfaces. In this setting, existence, regularity, and qualitative behavior of solutions have been analyzed in [13, 14, 20, 27, 33, 36, 37] for the surface diffusion flow, and in [9, 17, 18, 19, 24, 25, 26, 32, 35] for the Willmore flow, to mention just a few publications.

In this paper, we consider uniformly regular hypersurfaces. It should be emphasized that these surfaces may be non-compact. The concept of uniformly regular Riemannian manifolds was introduced by Amann [3, 4] and it contains the class of compact Riemannian manifolds as a special case. The study of geometric flows on non–compact manifolds is an active research topic, both from the point of view of PDE theory and in relation to its applications in geometry and topology. To the best of our knowledge, the current literature on the surface diffusion and Willmore flows for non–compact manifolds all concern surfaces defined over an infinite cylinder or entire graphs over $\mathbb{R}^m$, or the Willmore flow with small initial energy, cf. [8, 16, 17, 21, 22]. Our work generalizes the study of these two flows to a larger class of manifolds.

In our main result we establish well–posedness for initial surfaces that are $C^{1+\alpha}$–regular and parameterized over a uniformly regular hypersurface. Moreover, we show that solutions instantaneously regularize and become smooth, and even analytic in case Σ is analytic. In order to obtain our results, we show that the pertinent underlying evolution equations can be formulated as parabolic quasilinear equations of fourth order over the reference surface Σ. Our analysis relies on the theory of continuous maximal regularity and the results and techniques developed in [22, 33, 34].

The results in Theorem 4.3 and Theorem 5.1 are new. However, we note that in case Σ is an infinitely long cylinder embedded in $\mathbb{R}^3$, an analogous result to Theorem 4.3 was obtained in [22] for the surface diffusion flow.

For the Willmore flow, Theorem 5.1 is also new even if Σ is a compact (smooth, closed) surface. Previous results impose more regularity on the initial surface, for instance $C^{2+\alpha}$ in [35].

Theorem 5.2, where global existence and convergence to a sphere is shown for surfaces that are $C^{1+\alpha}$–close to a sphere, also seems to be new. A corresponding result was obtained in [35] for surfaces close to a sphere in the $C^{2+\alpha}$–topology. The authors in [18] showed the existence of a lower bound on the lifespan of a smooth solution, which depends only on how much the curvature of the initial surface is concentrated in space. In [17, 19], the authors proved convergence to round spheres under suitable smallness assumptions on the total energy of the surface. Here we note that the energy used in [17, 19] involves second–order derivatives, whereas we only need smallness in the $C^{1+\alpha}$–topology. In particular, we obtain global existence and convergence for non–convex initial surfaces.

The organization of the paper is as follows:

In Sections 2.1 and 2.2, we introduce the concept of uniformly regular manifolds and define the function spaces used in this paper. In Sections 2.3 and 2.4, we review continuous maximal regularity theory and its applications to quasilinear parabolic equations with singular nonlinearity. These results form the theoretic basis for the study of the surface diffusion and Willmore flows.

In Section 3, we introduce the concept of uniformly regular hypersurfaces with a uniform tubular neighborhood (called (URT)–hypersurfaces) and work out several examples. We utilize these concepts to parameterize the evolving hypersurfaces driven by surface diffusion and Willmore flows as normal graphs over a (URT)-reference hypersurface.

In Section 4, we establish our main results regarding existence, uniqueness, regularity, and semiflow properties for solutions to the surface diffusion flow over (URT)–hypersurfaces in $\mathbb{R}^{m+1}$. In Section 5, we likewise establish well–posedness properties for solutions to the Willmore flow over (URT)–hypersurfaces in $\mathbb{R}^3$. Additionally, we show stability of Euclidean spheres under perturbations in the $C^{1+\alpha}$–topology.

We conclude the paper with an appendix where we state and prove some additional properties of normal graphs over (URT)-hypersurfaces.

Notation: For two Banach spaces X and Y, $X \doteq Y$ means that they are equal in the sense of equivalent norms. $\mathcal{L}(X, Y)$ denotes the set of all bounded linear maps from X to Y and $\mathcal{L}\text{is}(X, Y)$ is the subset of $\mathcal{L}(X, Y)$ consisting of all bounded linear isomorphisms from X to Y. For $x \in X$, $\mathbb{B}_X(x, r)$ denotes the (open) ball in X with radius r and center x. We sometimes write $\mathbb{B}(x, r)$, in lieu of $\mathbb{B}_X(x, r)$, in case the setting is clear, and we write $\mathbb{B}^m(x, r)$ when $X = \mathbb{R}^m$. We denote by

g_m the Euclidean metric in $\mathbb{R}^m$. Given an embedded hypersurface M in $\mathbb{R}^m$, $g_m|_{\mathsf{M}}$ means the metric on M induced by g_m. Finally, we set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

2. Preliminaries

2.1. **Uniformly regular manifolds.** The concept of *uniformly regular (Riemannian) manifolds* was introduced by H. Amann in [3] and [4]. Loosely speaking, an m–dimensional Riemannian manifold (M, g) is uniformly regular if its differentiable structure is induced by an atlas such that all its local patches are of approximately the same size, all derivatives of the transition maps are bounded, and the pull-back metric of g in every local coordinate is comparable to the Euclidean metric g_m.

We will now state some structural properties of uniformly regular manifolds which will be used in the analysis of the the surface diffusion flow and and the Willmore flow in subsequent sections.

An oriented C^∞–manifold (M, g) of dimension m and without boundary is *uniformly regular* if it admits an orientation-preserving atlas $\mathfrak{A} := \{(\mathsf{O}_\kappa, \varphi_\kappa) : \kappa \in \mathfrak{K}\}$, with a countable index set $\mathfrak{K}$, satisfying the following conditions.

- (R1) There exists $K \in \mathbb{N}$ such that any intersection of more than K coordinate patches is empty.
- (R2) $\varphi_\kappa(\mathsf{O}_\kappa) = \mathbb{B}^m$, where $\mathbb{B}^m$ is the unit Euclidean ball centered at the origin in $\mathbb{R}^m$. Moreover, $\mathfrak{A}$ is uniformly shrinkable; by which we mean that there exists some $r \in (0, 1)$ such that $\{\psi_\kappa(r\mathbb{B}^m) : \kappa \in \mathfrak{K}\}$ forms a cover for M, where $\psi_\kappa := \varphi_\kappa^{-1}$.
- (R3) $\|\varphi_\eta \circ \psi_\kappa\|_{k,\infty} \leq c(k)$ for all $\kappa \in \mathfrak{K}$, $k \in \mathbb{N}_0$ and $\eta \in \mathfrak{K}$ such that $\mathsf{O}_\eta \cap \mathsf{O}_\kappa \neq \emptyset$.
- (R4) $\|\psi_\kappa^* g\|_{k,\infty} \leq c(k)$ for all $\kappa \in \mathfrak{K}$ and $k \in \mathbb{N}_0$.
- (R5) $\psi_\kappa^* g \sim g_m$ for all $\kappa \in \mathfrak{K}$. Here g_m is the Euclidean metric in $\mathbb{R}^m$ and $\psi_\kappa^* g$ denotes the pull-back metric of g by ψ_κ.

Here (R5) means that there exists some number $c \geq 1$ such that

$$(1/c)|\xi|^2 \leq \psi_\kappa^* g(x)(\xi, \xi) \leq c|\xi|^2, \quad x \in \mathbb{B}^m, \ \xi \in \mathbb{R}^m, \ \kappa \in \mathfrak{K}.$$

Given an open subset $U \subset \mathbb{R}^m$, a Banach space X, and a mapping $u : U \to X$,

$$\|u\|_{k,\infty} := \max_{|\alpha| \leq k} \|\partial^\alpha u\|_\infty$$

is the norm of the space $BC^k(U, X)$, which consists of all functions $u \in C^k(U, X)$ such that $\|u\|_{k,\infty} < \infty$.

Any uniformly regular manifold (M, g) possesses a *localization system subordinate to $\mathfrak{A}$*, by which we mean a family $\{(\pi_\kappa, \zeta_\kappa) : \kappa \in \mathfrak{K}\}$ satisfying:

- (L1) $\pi_\kappa \in \mathcal{D}(\mathsf{O}_\kappa, [0, 1])$ and $\{\pi_\kappa^2 : \kappa \in \mathfrak{K}\}$ is a partition of unity subordinate to the cover $\{\mathsf{O}_\kappa : \kappa \in \mathfrak{K}\}$.
- (L2) $\zeta_\kappa := \varphi_\kappa^* \zeta$ with $\zeta \in \mathcal{D}(\mathbb{B}^m, [0, 1])$ satisfying $\zeta|_{\mathrm{supp}(\psi_\kappa^* \pi_\kappa)} \equiv 1$, $\kappa \in \mathfrak{K}$.
- (L3) $\|\psi_\kappa^* \pi_\kappa\|_{k,\infty} + \|\zeta\|_{k,\infty} \leq c(k)$, for $\kappa \in \mathfrak{K}$, $k \in \mathbb{N}_0$.

Given $k \in \mathbb{N} \cup \{\omega\}$, the concept of C^k–*uniformly regular manifold* is defined by modifying (R3), (R4), (L1)-(L3) in an obvious way, where ω is the symbol for real analyticity.

Remark 2.1. *In [12], the authors showed that a C^∞–manifold without boundary is uniformly regular iff it is of bounded geometry, i.e. it is geodesically complete, of positive injectivity radius and all covariant derivatives of the curvature tensor*

are bounded. In particular, every compact manifold without boundary is uniformly regular and the manifolds considered in [20, 21] *are all uniformly regular.*

Given $\sigma, \tau \in \mathbb{N}_0$, we define the (σ, τ)–tensor bundle of M as

$$T^\sigma_\tau \mathsf{M} := T\mathsf{M}^{\otimes \sigma} \otimes T^* \mathsf{M}^{\otimes \tau},$$

where $T\mathsf{M}$ and $T^* \mathsf{M}$ are the tangent and the cotangent bundle of M, respectively. Let $\mathcal{T}^\sigma_\tau \mathsf{M}$ denote the $C^\infty(\mathsf{M})$–module of all smooth sections of $T^\sigma_\tau \mathsf{M}$.

Throughout the rest of this paper, we will adopt the following convention.

- p always denotes a point on a uniformly regular manifold.
- $k \in \mathbb{N}_0$ and $s \geq 0$.
- $\sigma, \tau \in \mathbb{N}_0$, $V = V^\sigma_\tau := \{T^\sigma_\tau \mathsf{M}, (\cdot|\cdot)_g\}$, $E = E^\sigma_\tau := \{\mathbb{R}^{m^\sigma \times m^\tau}, (\cdot|\cdot)\}$.

Setting $\mathbb{R}^m_\kappa = \mathbb{R}^m$ for $\kappa \in \mathfrak{K}$, we define $\boldsymbol{L}_{1,loc}(\mathbb{R}^m, E) := \prod_\kappa L_{1,loc}(\mathbb{R}^m_\kappa, E)$,

$$\mathcal{R}^c_\kappa : L_{1,loc}(\mathsf{M}, V) \to L_{1,loc}(\mathbb{R}^m_\kappa, E), \quad u \mapsto \psi^*_\kappa(\pi_\kappa u),$$

$$\mathcal{R}_\kappa : L_{1,loc}(\mathbb{R}^m_\kappa, E) \to L_{1,loc}(\mathsf{M}, V), \quad v_\kappa \mapsto \pi_\kappa \varphi^*_\kappa v_\kappa.$$

Here, and in the following, it is understood that a partially defined and compactly supported tensor field is automatically extended over the whole base manifold by identifying it to be zero outside its original domain. We further introduce two maps:

$$\mathcal{R}^c : L_{1,loc}(\mathsf{M}, V) \to \boldsymbol{L}_{1,loc}(\mathbb{R}^m, E), \quad u \mapsto (\mathcal{R}^c_\kappa u)_{\kappa \in \mathfrak{K}},$$

$$\mathcal{R} : \boldsymbol{L}_{1,loc}(\mathbb{R}^m, E) \to L_{1,loc}(\mathsf{M}, V), \quad (v_\kappa)_{\kappa \in \mathfrak{K}} \mapsto \sum_{\kappa \in \mathfrak{K}} \mathcal{R}_\kappa v_\kappa.$$

2.2. **Hölder and little Hölder spaces on uniformly regular manifolds.** In this subsection we follow Amann [4, 3], see also [34]. We define

$$BC^k(\mathsf{M}, V) := (\{u \in C^k(\mathsf{M}, V) : \|u\|^{\mathsf{M}}_{k,\infty} < \infty\}, \|\cdot\|_{k,\infty}),$$

where $\|u\|_{k,\infty} := \max_{0 \leq i \leq k} \||\nabla^i u|_g\|_\infty$. Set

$$BC^\infty(\mathsf{M}, V) := \bigcap_k BC^k(\mathsf{M}, V)$$

endowed with the conventional projective topology. Then

$$bc^k(\mathsf{M}, V) := \text{the closure of } BC^\infty(\mathsf{M}, V) \text{ in } BC^k(\mathsf{M}, V).$$

Letting $k < s < k+1$, the Hölder space $BC^s(\mathsf{M}, V)$ is defined by

$$BC^s(\mathsf{M}, V) := (bc^k(\mathsf{M}, V), bc^{k+1}(\mathsf{M}, V))_{s-k,\infty}.$$

Here $(\cdot, \cdot)_{\theta, \infty}$ is the real interpolation method, see [1, Example I.2.4.1]. For $s \geq 0$, we define the little Hölder spaces by

$$bc^s(\mathsf{M}, V) := \text{the closure of } BC^\infty(\mathsf{M}, V) \text{ in } BC^s(\mathsf{M}, V).$$

The spaces $BC^s(\mathbb{R}^m, E)$ and $bc^s(\mathbb{R}^m, E)$ are defined in a similar manner. When $s \notin \mathbb{N}_0$, we can give an alternative characterization of these spaces on $\mathbb{R}^m$. For $0 < s < 1$ and $0 < \delta \leq \infty$, we define a seminorm by

$$[u]^\delta_{s,\infty} := \sup_{h \in (0,\delta)^m} \frac{\|u(\cdot + h) - u(\cdot)\|_\infty}{|h|^s}, \quad [\cdot]_{s,\infty} := [\cdot]^\infty_{s,\infty}.$$

For $k < s < k+1$, the space $BC^s(\mathbb{R}^m, E)$ can be equivalently defined as

$$BC^s(\mathbb{R}^m, E) = (\{u \in BC^k(\mathbb{R}^m, E) : \|u\|_{s,\infty} < \infty\}, \|\cdot\|_{s,\infty}),$$

where $\|u\|_{s,\infty} := \|u\|_{k,\infty} + \max_{|\alpha|=k} [\partial^\alpha u]_{s-k,\infty}$; and

$u \in BC^s(\mathbb{R}^m, E)$ belongs to $bc^s(\mathbb{R}^m, E)$ iff $\lim_{\delta \to 0} [\partial^\alpha u]^\delta_{s-[s],\infty} = 0$, $|\alpha| = [s]$.

For $\mathfrak{F} \in \{bc, BC\}$, we put $\boldsymbol{\mathfrak{F}}^s := \prod_\kappa \mathfrak{F}^s_\kappa$ with $\mathfrak{F}^s_\kappa := \mathfrak{F}^s(\mathbb{R}^m_\kappa, E)$. We denote by $l_\infty(\boldsymbol{\mathfrak{F}}^s)$ the linear subspace of $\boldsymbol{\mathfrak{F}}^s$ consisting of all $\boldsymbol{x} = (x_\kappa)_{\kappa \in \mathfrak{K}}$ such that

$$\|\boldsymbol{x}\|_{l_\infty(\boldsymbol{\mathfrak{F}}^s)} := \sup_\kappa \|x_\kappa\|_{\mathfrak{F}^s_\kappa} < \infty.$$

We define $l_{\infty,\mathrm{unif}}(\boldsymbol{bc}^k)$ as the linear subspace of $l_\infty(\boldsymbol{bc}^k)$ consisting of all $\boldsymbol{u} = (u_\kappa)_{\kappa \in \mathfrak{K}}$ such that $(\partial^\alpha u_\kappa)_{\kappa \in \mathfrak{K}}$ is uniformly continuous on $\mathbb{R}^m_\kappa$ for $|\alpha| \leq k$, uniformly with respect to $\kappa \in \mathfrak{K}$. For $k < s < k+1$, we define $l_{\infty,\mathrm{unif}}(\boldsymbol{bc}^s)$ as the linear subspace of $l_{\infty,\mathrm{unif}}(\boldsymbol{bc}^k)$ of all $\boldsymbol{u} = (u_\kappa)_{\kappa \in \mathfrak{K}}$ such that

$$\lim_{\delta \to 0} \max_{|\alpha|=k} [\partial^\alpha u_\kappa]^\delta_{s-k,\infty} = 0 \quad \text{uniformly with respect to } \kappa \in \mathfrak{K}. \qquad (2.1)$$

The following properties of little Hölder spaces were first established in [3, 4]. We also refer to [34, Theorem 2.1 and Proposition 2.2].

Proposition 2.2. *Let $s \geq 0$. Then $\mathcal{R}$ is a retraction from $l_{\infty,\mathrm{unif}}(\boldsymbol{bc}^s)$ onto $bc^s(\mathsf{M}, V)$ with $\mathcal{R}^c$ as a coretraction. Similarly,*

$$[u \mapsto (\psi^*_\kappa(\zeta_\kappa u))_{\kappa \in \mathfrak{K}}] \in \mathcal{L}(bc^s(\mathsf{M}, V), l_{\infty,\mathrm{unif}}(\boldsymbol{bc}^s)).$$

Let $(\cdot, \cdot)^0_{\theta,\infty}$ denote the continuous interpolation method, c.f. [1, Example I.2.4.4].

Proposition 2.3. *Suppose that $0 < \theta < 1$, $0 \leq s_0 < s_1$ and $s = (1-\theta)s_0 + \theta s_1$ with $s_1, s_2, s \notin \mathbb{N}_0$. Then*

$$(bc^{s_0}(\mathsf{M}, V), bc^{s_1}(\mathsf{M}, V))^0_{\theta,\infty} \doteq bc^s(\mathsf{M}, V).$$

2.3. Continuous maximal regularity.
For a fixed interval $I = [0, T]$, $\mu \in (0, 1)$, and a given Banach space X, we define

$$BC_{1-\mu}(I, X) := \{u \in C(\dot{I}, X) : [t \mapsto t^{1-\mu} u] \in C(\dot{I}, X), \lim_{t \to 0^+} t^{1-\mu} \|u(t)\|_X = 0\},$$

$$\|u\|_{C_{1-\mu}} := \sup_{t \in \dot{I}} t^{1-\mu} \|u(t)\|_X,$$

where $\dot{I} = I \setminus \{0\}$; and

$$BC^1_{1-\mu}(I, X) := \{u \in C^1(\dot{I}, X) : u, \dot{u} \in BC_{1-\mu}(I, X)\}.$$

If $I = [0, T)$ is a half open interval, then

$$C_{1-\mu}(I, X) := \{v \in C(\dot{I}, X) : v \in BC_{1-\mu}([0, t], X), \quad t < T\},$$

$$C^1_{1-\mu}(I, X) := \{v \in C^1(\dot{I}, X) : v, \dot{v} \in C_{1-\mu}(I, X)\}.$$

We equip these two spaces with the natural Fréchet topology induced by the topology of $BC_{1-\mu}([0, t], X)$ and $BC^1_{1-\mu}([0, t], X)$, respectively.

Assume that $E_1 \overset{d}{\hookrightarrow} E_0$ is a pair of densely embedded Banach spaces. An operator A is said to belong to the class $\mathcal{H}(E_1, E_0)$, if $-A$ generates a strongly continuous analytic semigroup on E_0 with $dom(A) = E_1$. We define

$$\mathbb{E}_{0,\mu}(I) := BC_{1-\mu}(I, E_0), \quad \mathbb{E}_{1,\mu}(I) := BC_{1-\mu}(I, E_1) \cap BC^1_{1-\mu}(I, E_0), \qquad (2.2)$$

which are themselves Banach spaces when equipped with the norms

$$\|v\|_{\mathbb{E}_{0,\mu}(I)} := \sup_{t \in I} t^{1-\mu} \|v(t)\|_{E_0},$$

$$\|v\|_{\mathbb{E}_{1,\mu}(I)} := \sup_{t \in I} t^{1-\mu} \big(\|\dot{v}(t)\|_{E_0} + \|v(t)\|_{E_1}\big),$$

respectively. For $A \in \mathcal{H}(E_1, E_0)$, we say $(\mathbb{E}_{0,\mu}(I), \mathbb{E}_{1,\mu}(I))$ is a pair of maximal regularity of A if

$$\left(\frac{d}{dt} + A, \gamma_0 \right) \in \mathcal{L}\mathrm{is}(\mathbb{E}_{1,\mu}(I), \mathbb{E}_{0,\mu}(I) \times E_\mu)$$

where γ_0 is the evaluation map at 0, i.e., $\gamma_0(u) = u(0)$, and $E_\mu := (E_0, E_1)^0_{\mu,\infty}$. In this case, we use the notation

$$A \in \mathcal{M}_\mu(E_1, E_0).$$

2.4. **Quasilinear equations with singular nonlinearity.** Consider the following abstract quasilinear parabolic evolution equation

$$\begin{cases} \dfrac{d}{dt} u + A(u)u = F_1(u) + F_2(u), & t > 0, \\ \quad u(0) = x. \end{cases} \tag{2.3}$$

We assume that $V_\mu \subset E_\mu$ is an open subset of the continuous interpolation space $E_\mu := (E_0, E_1)^0_{\mu,\infty}$ and the operators (A, F_1, F_2) satisfy the following conditions.

(H1) Local Lipschitz continuity of (A, F_1):

$$(A, F_1) \in C^{1-}(V_\mu, \mathcal{M}_\mu(E_1, E_0) \times E_0).$$

(H2) Structural regularity of F_2:
There exists a number $\gamma \in (\mu, 1)$ such that $F_2 : V_\mu \cap E_\gamma \to E_0$. Moreover, there are numbers $\gamma_j \in [\mu, \gamma]$, $\varrho_j \geq 0$, and $m \in \mathbb{N}$ with

$$\frac{\varrho_j(\gamma - \mu) + (\gamma_j - \mu)}{1 - \mu} \leq 1, \quad \text{for all } j = 1, 2, \cdots, m, \tag{2.4}$$

so that for each $x_0 \in V_\mu$ and $R > 0$ there is a constant $C_R = C_R(x_0) > 0$ for which the estimate

$$|F_2(x_1) - F_2(x_2)|_{E_0} \leq C_R \sum_{j=1}^m (1 + |x_1|_{E_\gamma}^{\varrho_j} + |x_2|_{E_\gamma}^{\varrho_j})|x_1 - x_2|_{E_{\gamma_j}} \tag{2.5}$$

holds for all $x_1, x_2 \in \bar{\mathbb{B}}_{E_\mu}(x_0, R) \cap (V_\mu \cap E_\gamma)$.

Following the convention in Prüss, Wilke [30] and [22], we call the index j *subcritical* if (2.5) is a strict inequality and *critical* in case equality holds in (2.5).

Theorem 2.4. [22, Theorem 2.2] *Suppose (A, F_1, F_2) satisfies* **(H1)**–**(H2)**.

(a) *Given any $x_0 \in V_\mu$, there exist positive constants $\tau = \tau(x_0)$, $\varepsilon = \varepsilon(x_0)$, and $\sigma = \sigma(x_0)$ such that (2.3) has a unique solution*

$$u(\cdot, x) \in \mathbb{E}_{1,\mu}([0, \tau])$$

for all initial values $x \in \bar{\mathbb{B}}_{E_\mu}(x_0, \varepsilon)$. Moreover,

$$\|u(\cdot, x_1) - u(\cdot, x_2)\|_{\mathbb{E}_{1,\mu}([0,\tau])} \leq \sigma \|x_1 - x_2\|_{E_\mu}, \quad x_1, x_2 \in \bar{\mathbb{B}}_{E_\mu}(x_0, \varepsilon).$$

(b) *Each solution with initial value $x_0 \in V_\mu$ exists on a maximal interval $J(x_0) := [0, t^+) = [0, t^+(x_0))$ and enjoys the regularity*

$$u(\cdot, x_0) \in C([0, t^+), E_\mu) \cap C((0, t^+), E_1).$$

(c) *If the solution $u(\cdot, x_0)$ satisfies the conditions:*
   (i) *$u(\cdot, x_0) \in UC(J(x_0), E_\mu)$ and*
   (ii) *there exists $\eta > 0$ so that $\mathrm{dist}_{E_\mu}(u(t, x_0), \partial V_\mu) > \eta$ for all $t \in J(x_0)$,*

> then it holds that $t^+(x_0) = \infty$ and so $u(\cdot, x_0)$ is a global solution of (2.3)
> Moreover, if the embedding $E_1 \hookrightarrow E_0$ is compact, then condition (i) may be
> replaced by the assumption:
> (i.a) the orbit $\{u(t, x_0) : t \in [\tau, t^+(x_0))\}$ is bounded in E_δ for some $\delta \in (\mu, 1]$
> and some $\tau \in (0, t^+(x_0))$.

3. URT–HYPERSURFACES

Suppose Σ is an oriented smooth hypersurface without boundary which is embedded in $\mathbb{R}^{m+1}$. Let $\mathsf{a} > 0$. Then Σ is said to have a *tubular neighborhood of radius* a if the map

$$X : \Sigma \times (-\mathsf{a}, \mathsf{a}) \to \mathbb{R}^{m+1} : \quad [(\mathsf{p}, r) \mapsto \mathsf{p} + r\nu_\Sigma(\mathsf{p})] \qquad (3.1)$$

is a diffeomorphism onto its image $U_\mathsf{a} := X((-\mathsf{a}, \mathsf{a}) \times \Sigma)$. Here ν_Σ is the normal unit vector field compatible with the orientation of Σ. We refer to U_a as the tubular neighborhood of Σ of width $2\mathsf{a}$ and note that $U_\mathsf{a} = \{x \in \mathbb{R}^{m+1} : \operatorname{dist}(x, \Sigma) < \mathsf{a}\}$.

Finally, we say that Σ has *a tubular neighborhood* if there exists a number $\mathsf{a} > 0$ such that the above property holds.

Remarks 3.1. *(a) We lose no generality in assuming Σ is oriented, as any smooth embedded hypersurface without boundary is orientable, cf. [31].*

(b) Any smooth (in fact, C^2) compact embedded hypersurface without boundary has a tubular neighborhood, see for instance [15, Exercise 2.11].

(c) Suppose Σ is a smooth (oriented) embedded hypersurface with unit normal field ν_Σ. Then Σ is said to satisfy the uniform ball condition of radius $\mathsf{a} > 0$ *if at each point $\mathsf{p} \in \Sigma$, the open balls $\mathbb{B}(\mathsf{p} \pm \mathsf{a}\nu_\Sigma(\mathsf{p}), \mathsf{a})$ do not intersect Σ.*

The following assertions are equivalent:

 (i) Σ *has a tubular neighborhood of radius* a.
 (ii) Σ *satisfies the uniform ball condition of radius* a.

For the reader's convenience, we include a proof of this equivalence.

Proof. (i) $\Rightarrow$ (ii). Suppose there exists $\mathsf{p} \in \Sigma$ such that $\mathbb{B}(x_0, \mathsf{a}) \cap \Sigma \neq \emptyset$, where $x_0 := \mathsf{p} + \mathsf{a}\nu_\Sigma(\mathsf{p})$. Then $s := \operatorname{dist}(x_0, \bar{\mathbb{B}}(x_0, \mathsf{a}) \cap \Sigma) < \mathsf{a}$ and there exists $\mathsf{q} \in \bar{\mathbb{B}}(x_0, \mathsf{a}) \cap \Sigma$ such that $x_0 = \mathsf{q} + s\nu_\Sigma(\mathsf{q})$. Hence $x_0 = X(\mathsf{p}, \mathsf{a}) = X(\mathsf{q}, s)$, with $(\mathsf{p}, \mathsf{a}) \neq (\mathsf{q}, s)$, contradicting the assumption that X is bijective. The case $x_0 = \mathsf{p} - \mathsf{a}\nu_\Sigma(\mathsf{p})$ is treated in the same way.

(ii) $\Rightarrow$ (i). We only need to prove the injectivity of X. Suppose, by contradiction, that $X(\mathsf{p}_1, r_1) = X(\mathsf{p}_2, r_2) = x$ for $(\mathsf{p}_1, r_1) \neq (\mathsf{p}_2, r_2)$. Without loss of generality we may assume that $r_1 \in (0, \mathsf{a})$, as we can otherwise replace $\nu_\Sigma(\mathsf{p}_i)$ by $-\nu_\Sigma(\mathsf{p}_i)$. Moreover, we may assume that $|r_2| \leq r_1$. Let $s \in (r_1, \mathsf{a})$ and set

$$y := X(\mathsf{p}_1, s) = x + (s - r_1)\nu_\Sigma(\mathsf{p}_1).$$

Then we have $|y - \mathsf{p}_2| \leq |y - x| + |x - \mathsf{p}_2| = s - r_1 + |r_2| \leq s$, showing that

$$\mathsf{p}_2 \in \bar{\mathbb{B}}(y, s) = \bar{\mathbb{B}}(\mathsf{p}_1 + s\nu_\Sigma(\mathsf{p}_1), s) \subset \mathbb{B}(\mathsf{p}_1 + \mathsf{a}\nu_\Sigma(\mathsf{p}_1), \mathsf{a}).$$

Therefore, $\mathbb{B}(\mathsf{p}_1 + \mathsf{a}\nu_\Sigma(\mathsf{p}_1), \mathsf{a}) \cap \Sigma \neq \emptyset$, contradicting the assumption in (ii). $\qquad \square$

(d) Suppose Σ has a tubular neighborhood of radius a. Let $\{\kappa_1, \ldots, \kappa_m\}$ be the principal curvatures of Σ, and L_Σ the Weingarten tensor.

Then it follows from part (c) that $|\kappa_1|, \ldots, |\kappa_m| \leq 1/\mathsf{a}$ and $|L_\Sigma| \leq 1/\mathsf{a}$.

In the following, we say that Σ is a (URT)–hypersurface in $\mathbb{R}^{m+1}$ if

(T1) Σ is a smooth oriented hypersurface without boundary embedded in $\mathbb{R}^{m+1}$.

(T2) (Σ, g) is uniformly regular, where $g = g_{m+1}|_\Sigma$ denotes the metric induced by the Euclidean metric g_{m+1}.

(T3) Σ has a tubular neighborhood.

Examples 3.2. *(a) Every smooth compact hypersurface without boundary embedded in $\mathbb{R}^{m+1}$ is a (URT)–hypersurface.*

(b) All of the manifolds considered in [20, 21] are (URT)–hypersurfaces. In particular, the infinite cylinder with radius $r > 0$,

$$\mathcal{C}_r = \{(x, y, z) \in \mathbb{R}^3 : y^2 + z^2 = r^2, \ x \in \mathbb{R}\},$$

is a (URT)–hypersurface with tubular neighborhood of radius $\mathsf{a} = r$.

(c) Assume that $f : \mathbb{R}^m \to \mathbb{R}$ belongs to $BC^2(\mathbb{R}^m)$. Then the graph of f has a tubular neighborhood of radius a for some $\mathsf{a} > 0$.

Proof. By the inverse function theorem, there exist uniform constants $\eta > 0$ and $\varepsilon > 0$ such that, at every point $x \in \mathbb{R}^m$, $f|_{\mathbb{B}^m(x,\eta)}$ can be expressed as the graph of a BC^2–function h_x over $T_x\mathrm{gr}(f)$, the tangent space to the graph of f at the point $(x, f(x))$, such that the set $\{(y, h_x(y)) : y \in \mathbb{B}_{T_x\mathrm{gr}(f)}(0, \varepsilon)\}$ is contained in $\{(z, f(z)) : z \in \mathbb{B}^m(x, \eta)\}$. Moreover, there exists a uniform constant c, independent of x, such that

$$\|h_x\|_{2,\infty} \leq c \tag{3.2}$$

where the supremum is taken over the ball $\mathbb{B}_{T_x\mathrm{gr}(f)}(0, \varepsilon)$. We refer to the proof of Claim 1 in Proposition A.1(b) in the Appendix for a more general situation. Further, we have $h_x(0) = 0$ and $\nabla h_x(0) = 0$. Due to (3.2), after Taylor expansion of h_x around $0 \in T_x\mathrm{gr}(f)$, we have

$$|h_x(y)| \leq \|\nabla_y^2 h_x\|_\infty |y|^2, \quad y \in \mathbb{B}_{T_x\mathrm{gr}(f)}(0, \varepsilon),$$

for sufficiently small ε. Choosing $C \geq \|\nabla_y^2 h_x\|_\infty$ such that $1/2C \leq \varepsilon$, we define $\mathsf{a} := 1/2C$. It follows that the ball $\mathbb{B}^{m+1}(\mathsf{a}\nu_x, \mathsf{a})$ lies above the graph

$$\{(y, h_x(y)) : y \in T_x\mathrm{gr}(f)(0, \varepsilon)\},$$

where ν_x is the upwards pointing unit normal of $\mathrm{gr}(f)$ at the point $(x, f(x))$. An analogous argument shows that the ball $\mathbb{B}^{m+1}(-\mathsf{a}\nu_x, \mathsf{a})$ lies below the graph.

Since the constants ε and a are independent of x, combining with Remark 3.1(c), this proves that $\mathrm{gr}(f)$ has a tubular neighborhood of radius a. □

(d) We refer to [3, 4, 5] for additional examples of uniformly regular manifolds. In particular, embedded hypersurfaces with tame ends, considered in [5, Theorem 1.2], are (URT)–hypersurfaces. More precisely, given a compact hypersurface without boundary B, embedded in $\mathbb{R}^m$, and $0 \leq \alpha \leq 1$, we define

$$F_\alpha(B) := \{(t, t^\alpha y) : t > 1, y \in B\},$$

which we endow with the metric $g_{F_\alpha(B)}$ induced by its embedding into $\mathbb{R}^{m+1}$. An embedded hypersurface $\Sigma \subset \mathbb{R}^{m+1}$ is said to have tame ends if

$$\Sigma = V_0 \cup \bigcup_{i=1}^n V_i,$$

where $(V_0, g_{m+1}|_{V_0})$ is compact and $(V_i, g_{m+1}|_{V_i})$ is isometric to $(F_\alpha(B), g_{F_\alpha(B)})$. Then, $(\Sigma, g_{m+1}|_\Sigma)$ is a (URT)–hypersurface.

In particular, when $\alpha = 0$, $(\Sigma, g_{m+1}|_\Sigma)$ has finitely many cylinder ends; when $\alpha = 1$, $(\Sigma, g_{m+1}|_\Sigma)$ has finitely many (blunt) cone ends.

(e) Let

$$\mathcal{C}_k = \{(x, y, z) \in \mathbb{R}^3 : y^2 + z^2 = 1 + 1/k, \ x \in \mathbb{R}\}, \quad k \in \mathbb{N}.$$

Based on part (b), the manifold $\Sigma = \bigcup_k \mathcal{C}_k$, endowed with the metric induced by g_3, is uniformly regular. But it is obvious that (Σ, g) does not have a tubular neighborhood.

(f) There also exist connected uniformly regular hypersurfaces that are not (URT). For instance, we can construct a smooth connected curve C in $\{(x, y) : y > 0\}$ such that $C \cap \{(x, y) : x \geq 0\}$ is compact and

$$C \cap \{(x, y) : x < 0\} = \{(x, y) : y = 1\} \cup \{(x, y) : y = 1 + e^x\}.$$

Then $(C, g_2|_C)$ is a uniformly regular hypersurface that is not (URT). One can take the product of C with $\mathbb{R}^m$ to produce higher dimensional examples.

Additionally, one can rotate the curve C around the x–axis to obtain a connected rotationally symmetric uniformly regular hypersurface which is not (URT).

4. The surface diffusion flow

In solving the surface diffusion flow, one seeks to find a family of (oriented) closed hypersurfaces $\{\Gamma(t) : t \geq 0\}$ satisfying the evolution equation

$$\begin{cases} V(t) = -\Delta_{\Gamma(t)} H_{\Gamma(t)}, & t > 0, \\ \Gamma(0) = \Gamma_0, \end{cases} \tag{4.1}$$

for an initial hypersurface Γ_0.

Here, $V(t)$ denotes the velocity in the normal direction of Γ at time t, $H_{\Gamma(t)}$ is the mean curvature of $\Gamma(t)$ (i.e., the average of the principal curvatures), and $\Delta_{\Gamma(t)}$ is the Laplace-Beltrami operator on $\Gamma(t)$. We use the convention that a sphere has *negative* mean curvature. We note that this convention is in agreement with [29, 32, 33], but differs from [13, 20, 22].

In the following, we assume that Σ is a (URT)–hypersurface in $\mathbb{R}^{m+1}$ with tubular neighborhood U_{a} and with an orientation-preserving atlas $\mathfrak{A} := \{(\mathsf{O}_\kappa, \varphi_\kappa) : \kappa \in \mathfrak{K}\}$ with $\psi_\kappa = \varphi_\kappa^{-1}$ satisfying (R1)–(R5). In the following, we assume that Σ carries the metric induced by the Euclidean metric g_{m+1}. Finally, we assume that Γ_0 lies in U_{a}.

For $\alpha \in (0, 1)$ a fixed parameter, we define

$$E_0 := bc^\alpha(\Sigma) \quad \text{and} \quad E_1 := bc^{4+\alpha}(\Sigma).$$

For $\theta \in (0, 1)$, let $E_\theta := (E_0, E_1)_{\theta, \infty}^0$. Taking $\mu = 1/4$ and $\gamma = 3/4$, it follows from Proposition 2.3 that

$$E_\mu = bc^{1+\alpha}(\Sigma) \quad \text{and} \quad E_\gamma = bc^{3+\alpha}(\Sigma).$$

Given $\rho \in E_\mu$ with $\|\rho\|_\infty < \mathsf{a}$, it follows, by assumption that Σ is (URT) with tubular neighborhood U_{a}, that

$$\Psi_\rho : \Sigma \to \mathbb{R}^{m+1}, \quad \Psi_\rho(\mathsf{p}) = \mathsf{p} + \rho(\mathsf{p})\nu_\Sigma(\mathsf{p}), \tag{4.2}$$

is a diffeomorphism from Σ onto the C^1–manifold $\Gamma_\rho := \mathrm{im}(\Psi_\rho)$; see also Proposition A.1 for additional properties of Γ_ρ.

When the temporal variable t is included in ρ, i.e.

$$\rho : [0, T) \times \Sigma \to (-\mathsf{a}, \mathsf{a}),$$

we can also extend Ψ_ρ to $\Psi_\rho : [0, T) \times \Sigma \to \mathbb{R}^{m+1}$. In the sequel, we will omit the temporal variable t in ρ, Ψ_ρ and Γ_ρ when the dependence on t is clear from context.

Let us fix some notation. We denote by $g_{m+1}|_{\Gamma_\rho}$ the metric induced on Γ_ρ by the Euclidean metric g_{m+1} of $\mathbb{R}^{m+1}$. Let $g(\rho) := \Psi_\rho^*(g_{m+1}|_{\Gamma_\rho})$ be the pull-back metric of $g_{m+1}|_{\Gamma_\rho}$ on Σ.

The following expression for $g(\rho)$ was derived in [28, Formula (23)]:

$$g_{ij}(\rho) = g_{ij} - 2\rho l_{ij} + \rho^2 l_i^r l_{jr} + \partial_i \rho \partial_j \rho, \tag{4.3}$$

where l_j^i and l_{ij} are the components of the Weingarten tensor L_Σ and the second fundamental form with respect to $g := g_{m+1}|_\Sigma$; i.e.,

$$l_{ij} = -(\tau_i | \partial_j \nu_\Sigma), \quad l_j^i = g^{ik} l_{kj}, \quad L_\Sigma = l_{ij} \tau^i \otimes \tau^j = l_j^i \tau_i \otimes \tau^j,$$

where $\{\tau_1, \ldots, \tau_m\} = \{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^m}\}$ is a local basis of $T\Sigma$ at p and $\{\tau^1, \ldots, \tau^m\} = \{dx^1, \ldots, dx^m\}$ is the corresponding dual basis, characterized by $(\tau^i | \tau_j) = \delta_j^i$.

We introduce an open subset of E_μ defined by

$$V_\mu := \{\rho \in E_\mu : \|\rho\|_\infty < \mathsf{a}\}.$$

By Remark 3.1(d), the functions

$$a(\rho) := (I - \rho L_\Sigma)^{-1} \nabla_\Sigma \rho, \quad \beta(\rho) := [1 + |a(\rho)|^2]^{-1/2}, \tag{4.4}$$

are well–defined for all $\rho \in V_\mu$, where $\nabla_\Sigma \rho$ is the gradient vector and $I = \tau_i \otimes \tau^i$.

It is easy to verify that $g_{ij}(\rho) = (\tau_i | K(\rho) \tau_j)$, where

$$K(\rho) = (I - \rho L_\Sigma)^2 + \nabla_\Sigma \rho \otimes \nabla_\Sigma \rho = (I - \rho L_\Sigma)[I + a(\rho) \otimes a(\rho)](I - \rho L_\Sigma).$$

Hence, we obtain

$$g_{ij}(\rho) = \Big((I - \rho L_\Sigma)\tau_i \Big| [I + a(\rho) \otimes a(\rho)](I - \rho L_\Sigma)\tau_j\Big). \tag{4.5}$$

It follows from the well-known relation

$$[I + a \otimes a]^{-1} = I - \frac{a \otimes a}{1 + |a|^2}, \quad a \in \mathbb{R}^{m+1},$$

that $K(\rho)$ is invertible for every $\rho \in C^1(\Sigma)$ with $\|\rho\|_\infty < \mathsf{a}$, with inverse given by

$$K^{-1}(\rho) = M_0(\rho)[I - \beta^2(\rho) a(\rho) \otimes a(\rho)] M_0(\rho),$$

where $M_0(\rho) := (I - \rho L_\Sigma)^{-1}$. We then have $g^{ij}(\rho) = (\tau^i | K^{-1}(\rho)\tau^j)$ for the components of the cotangent metric $g^*(\rho)$ on $T^*\Sigma$ induced by $g(\rho)$, and hence

$$g^{ij}(\rho) = \Big(M_0(\rho)\tau^i \Big| [I - \beta^2(\rho) a(\rho) \otimes a(\rho)] M_0(\rho)\tau^j\Big), \tag{4.6}$$

see also [29, Section 2.2].

When parameterizing the evolving hypersurface $\Gamma(t) = \Gamma_{\rho(t)}$ by means of a height function $\rho(t) \in V_1 = V_\mu \cap E_1$, it holds that (4.1) is equivalent to

$$\partial_t \rho = -\frac{1}{\beta(\rho)} \Psi_\rho^*(\Delta_{g_{m+1}|_{\Gamma_\rho}} H_{\Gamma_\rho}) = -\frac{1}{\beta(\rho)} \Delta_\rho H_\rho. \tag{4.7}$$

Here, $\Delta_{g_{m+1}|_{\Gamma_\rho}}$ and Δ_ρ denote Laplace-Beltrami operators on $(\Gamma_\rho, g_{m+1}|_{\Gamma_\rho})$ and $(\Sigma, g(\rho))$, respectively. It was shown in [33, Section 5] that $H_\rho := \Psi_\rho^* H_{\Gamma_\rho}$ in each local patch $(\mathsf{O}_\kappa, \varphi_\kappa)$ reads as

$$H_\rho = \frac{\beta(\rho)}{m} \Big\{ g^{ij}(\rho)\partial_i\partial_j\rho + g^{ij}(\rho)(l_j^k\partial_i\rho - \Gamma_{ij}^k)\partial_k\rho \tag{4.8}$$
$$+ g^{ij}(\rho)\big[r_k^l(\rho)l_i^k\partial_j\rho + r_k^l(\rho)(\partial_j l_i^k + \Gamma_{jh}^k l_i^h - \Gamma_{ij}^h l_h^k)\rho + r_k^l(\rho)l_j^h l_h^k\rho\partial_i\rho\big]\Big\}\partial_l\rho$$
$$+ \frac{\beta(\rho)}{m}g^{ij}(\rho)(l_{ij} - l_{ik}l_j^k\rho),$$

with $g^{ij}(\rho)$ given in (4.6). Here, Γ_{ij}^k are the components of the Christoffel symbols of Σ associated with the metric $g = g_{m+1}|_\Sigma$, and $r_j^i(\rho) = p_j^i(\rho)/q_j^i(\rho)$, where $p_j^i(\rho)$ and $q_j^i(\rho)$ are polynomials of ρ with BC^∞–coefficients. Note that although (4.8) was derived for compact hypersurfaces in [33], this expression still holds true for our problem as it is purely local.

In local coordinates with respect to the atlas $\mathfrak{A}$, Δ_ρ is given by

$$\Delta_\rho = g^{ij}(\rho)(\partial_i\partial_j - \Gamma_{ij}^k(\rho)\partial_k), \tag{4.9}$$

where $\Gamma_{ij}^k(\rho)$ are the Christoffel symbols of $(\Sigma, g(\rho))$. Here we note that the terms $\Gamma_{ij}^k(\rho)$ depend on ρ and up to its second–order derivatives. More precisely,

$$\Gamma_{ij}^k(\rho) = \frac{p_{ij}^k(\rho, \partial\rho, \partial^2\rho)}{q_{ij}^k(\rho, \partial\rho)},$$

where p_{ij}^k is a polynomial of ρ and its derivatives up to second order and q_{ij}^k is a polynomial of ρ and its first–order derivatives (both polynomials having BC^∞–coefficients).

By the expression above, we obtain

$$H_\rho = \frac{\beta(\rho)}{m}\mathsf{C}(g^*(\rho), \nabla^2\rho) + \text{lower order terms}$$

and

$$\Delta_\rho H_\rho = \frac{1}{m}\mathsf{C}(g^*(\rho) \otimes g^*(\rho), \nabla^4\rho) + \text{lower order terms},$$

where $\mathsf{C}(\cdot, \cdot)$ denotes the complete contraction and ∇ is the covariant derivative with respect to (Σ, g). Here and in the sequel, we will still use ∇ to denote its extension to $\mathcal{T}_\tau^\sigma\Sigma$. Note that for $u \in C^4(\Sigma)$, the tensor $\nabla^4 u \in C(\Sigma, T^*\Sigma^{\otimes 4})$ can be expressed in local coordinates by

$$\nabla^4 u = \partial_{(j)}u\,\tau^{(j)} + \sum_{\beta,(j)} a_{\beta,(j)}\partial^\beta u\,\tau^{(j)},$$

with coefficients $a_{\beta,(j)} \in BC^\infty(\mathbb{B}^m)$, where the summation runs over all multi-indices $(j) = (j_1, \cdots, j_4) \in \{1, \cdots, m\}^4$ and all $\beta \in \mathbb{N}^m$ with $|\beta| \le 3$. Here we are also using

$$\partial_{(j)} = \partial_{j_1}\partial_{j_2}\partial_{j_3}\partial_{j_4}, \quad \tau^{(j)} = \tau^{j_1} \otimes \tau^{j_2} \otimes \tau^{j_3} \otimes \tau^{j_4},$$

and $\partial^\beta := \partial_1^{\beta_1} \cdots \partial_m^{\beta_m}$; see for instance [3, page 444]. Hence we obtain

$$\mathsf{C}(g^*(\rho) \otimes g^*(\rho), \nabla^4 u) = g^{ij}(\rho)g^{lm}(\rho)\partial_i\partial_j\partial_l\partial_m u + \sum_{0<|\beta|\le 3} b_\beta(\rho, \partial\rho)\partial^\beta u \tag{4.10}$$

for each $\rho \in BC^1(\Sigma)$ with $\|\rho\|_\infty < \mathsf{a}$ and $u \in C^4(\Sigma)$.

By defining

$$A(\rho)\rho := \frac{1}{m}\mathsf{C}(g^*(\rho) \otimes g^*(\rho), \nabla^4\rho), \quad F(\rho) := A(\rho)\rho - \frac{1}{\beta(\rho)}\Delta_\rho H_\rho, \qquad (4.11)$$

we obtain an equivalent formulation of (4.1) as

$$\begin{cases} \partial_t\rho + A(\rho)\rho = F(\rho) & \text{in} \quad (0,\infty) \times \Sigma, \\ \rho(0) = \rho_0 & \text{in} \quad \Sigma. \end{cases} \qquad (4.12)$$

We note that for each $\rho \in C^1(\Sigma, \mathbb{R})$ with $\|\rho\|_\infty < \mathsf{a}$, the mapping

$$A(\rho) : C^4(\Sigma, \mathbb{R}) \to C(\Sigma, \mathbb{R}) : \quad [u \mapsto \frac{1}{m}\mathsf{C}(g^*(\rho) \otimes g^*(\rho), \nabla^4 u)]$$

gives rise to a differential operator of order 4.

A linear operator

$$\mathcal{A} := \sum_{i=0}^{l} \mathsf{C}(a_i, \nabla^i \cdot), \quad u \mapsto \mathcal{A}u = \sum_{i=0}^{l} \mathsf{C}(a_i, \nabla^i u),$$

of order l, acting on scalar functions, is said to be *uniformly strongly elliptic* if there exist positive constants $r, R > 0$ such that the principal symbol of $\mathcal{A}$,

$$\hat{\sigma}\mathcal{A}^\pi(\mathsf{p}, \xi) := \mathsf{C}(a_l, (-i\xi)^{\otimes l})(\mathsf{p}) \in \mathbb{R}, \quad (\mathsf{p}, \xi) \in \Sigma \times T_p^*\Sigma,$$

satisfies

$$r \le \operatorname{Re}\hat{\sigma}\mathcal{A}^\pi(\mathsf{p}, \xi) \le R, \quad \text{for all } (\mathsf{p}, \xi) \in \Sigma \times T_\mathsf{p}^*\Sigma \text{ with } |\xi|_{g^*(\mathsf{p})} = 1. \qquad (4.13)$$

Remark 4.1. *It is not difficult to see that in the scalar case, the notion of uniformly strongly elliptic is equivalent to the notion of* uniformly normally elliptic *introduced in* [34, Section 3]*, see also* [6].

In our setting, the principal symbol of $A(\rho)$ is given by

$$\hat{\sigma}A^\pi(\rho)(\mathsf{p}, \xi) = |\xi|^4_{g^*(\rho)(\mathsf{p})}, \quad \xi \in T_\mathsf{p}^*\Gamma.$$

It follows from (4.6) that $g^*(\rho) \sim g^*$ for all $\rho \in V_\mu$, in the sense that there exists some $c \ge 1$ such that

$$(1/c)|\xi|^2_{g^*(\mathsf{p})} \le |\xi|^2_{g^*(\rho)(\mathsf{p})} \le c|\xi|^2_{g^*(\mathsf{p})} \quad \text{for any } (\mathsf{p}, \xi) \in \Sigma \times T_\mathsf{p}^*\Sigma.$$

In fact, note that with $\xi = \xi_i \tau^i \in T_\mathsf{p}^*\Sigma$, (4.6) implies

$$|\xi|^2_{g^*(\rho)(\mathsf{p})} = g^{ij}(\rho)(\mathsf{p})(\xi, \xi) = |M_0(\rho)\xi|^2(\mathsf{p}) - \beta^2(\rho)(a(\rho)|M_0(\rho)\xi)^2(\mathsf{p}).$$

Next, observe that

$$\beta^2(\rho)|M_0(\rho)\xi|^2(\mathsf{p}) \le g^{ij}(\rho)(\mathsf{p})(\xi, \xi) \le |M_0(\rho)\xi|^2(\mathsf{p}), \qquad (4.14)$$

where we employed the Cauchy-Schwarz inequality and $1 - \beta^2(\rho)|a(\rho)|^2 = \beta^2(\rho)$ for the first estimate. It remains to observe that

$$\min\left\{\frac{1}{(1-\rho\kappa_r(\mathsf{p}))^2}\right\}g^{ij}(\mathsf{p})(\xi, \xi) \le |M_0(\rho)\xi|^2(\mathsf{p}) \le \max\left\{\frac{1}{(1-\rho\kappa_r(\mathsf{p}))^2}\right\}g^{ij}(\mathsf{p})(\xi, \xi),$$

where κ_r are the principal curvatures of Σ, which are bounded by $1/\mathsf{a}$ since Σ satisfies a uniform ball condition of radius a. This shows that $A(\rho)$ is uniformly strongly elliptic. Remark 4.1 and [34, Proposition 2.7, Theorem 3.7] now imply the following result.

Proposition 4.2. $A \in C^\omega(V_\mu, \mathcal{M}_\mu(E_1, E_0))$.

Next, we will verify that the operator F satisfies **(H2)**. In each patch $(\mathsf{O}_\kappa, \varphi_\kappa)$, we reference [22, Section 4.4] and (4.10) to confirm that the local expression for $F(\rho)$ is of the form

$$F(\rho) = \sum_{|\eta|=3, |\tau|\leq 2} c_{\eta,\tau}(\rho, \partial\rho)\, \partial^\tau \rho\, \partial^\eta \rho + \sum_{|\eta|,|\sigma|,|\tau|\leq 2} d_{\eta,\sigma,\tau}(\rho,\partial\rho)\, \partial^\eta \rho\, \partial^\sigma \rho\, \partial^\tau \rho, \quad (4.15)$$

where $\eta, \tau, \sigma \in \mathbb{N}^m$ are multi–indices of length $|\eta| := \eta_1 + \cdots + \eta_m$ and $\partial^\eta := \partial_{x_1}^{\eta_1} \cdots \partial_{x_m}^{\eta_m}$ is the mixed partial derivative operator in local coordinates. The coefficient functions $c_{\eta,\tau}$ and $d_{\eta,\tau,\sigma}$ depend analytically on ρ and its first–order derivatives. In the sequel, for a function $u : \Sigma \to \mathbb{R}$, we define $u_\kappa := \zeta\psi_\kappa^* u$. Let $\rho_0 \in V_\mu$. For $R > 0$, we choose $\rho_1, \rho_2 \in \bar{\mathbb{B}}_{E_\mu}(\rho_0, R) \cap (V_\mu \cap E_\gamma)$. By Proposition 2.2, we have

$$\|F(\rho_1) - F(\rho_2)\|_{\alpha,\infty} \leq C\|\mathcal{R}^c F(\rho_1) - \mathcal{R}^c F(\rho_2)\|_{l_\infty(\boldsymbol{BC}^\alpha)}$$
$$= C \sup_{\kappa \in \mathfrak{K}} \|\mathcal{R}_\kappa^c F(\rho_1) - \mathcal{R}_\kappa^c F(\rho_1)\|_{\alpha,\infty}. \qquad (4.16)$$

In the following computations, $\tilde{C}$ denotes a generic constant depending only on R and $\|\rho_0\|_{1+\alpha,\infty}$. In every patch $(\mathsf{O}_\kappa, \varphi_\kappa)$, by the discussion in [22, Section 4.1], we have the following estimate.

$$\|\mathcal{R}_\kappa^c F(\rho_1) - \mathcal{R}_\kappa^c F(\rho_1)\|_{\alpha,\infty}$$
$$\leq \tilde{C}\|\rho_{1,\kappa} - \rho_{2,\kappa}\|_{3+\alpha,\infty} \qquad (4.17)$$
$$+ \tilde{C}\Big(\|\rho_{1,\kappa}\|_{3+\alpha,\infty}\|\rho_{1,\kappa} - \rho_{2,\kappa}\|_{2+\alpha,\infty} + \|\rho_{2,\kappa}\|_{2+\alpha,\infty}\|\rho_{1,\kappa} - \rho_{2,\kappa}\|_{3+\alpha,\infty}$$
$$\quad + \|\rho_{2,\kappa}\|_{2+\alpha,\infty}\|\rho_{2,\kappa}\|_{3+\alpha,\infty}\|\rho_{1,\kappa} - \rho_{2,\kappa}\|_{1+\alpha,\infty}\Big) \qquad (4.18)$$
$$+ \tilde{C}\Big[(\|\rho_{1,\kappa}\|_{2+\alpha,\infty} + \|\rho_{2,\kappa}\|_{2+\alpha,\infty})\|\rho_{1,\kappa} - \rho_{2,\kappa}\|_{1+\alpha,\infty}$$
$$\quad + \|\rho_{1,\kappa} - \rho_{2,\kappa}\|_{2+\alpha,\infty} + \|\rho_{2,\kappa}\|_{2+\alpha,\infty}\|\rho_{1,\kappa} - \rho_{2,\kappa}\|_{1+\alpha,\infty}\Big] \qquad (4.19)$$
$$+ \tilde{C}\Big(\|\rho_{1,\kappa}\|_{2+\alpha,\infty}^2\|\rho_{1,\kappa} - \rho_{2,\kappa}\|_{1+\alpha,\infty} + \|\rho_{1,\kappa}\|_{2+\alpha,\infty}\|\rho_{1,\kappa} - \rho_{2,\kappa}\|_{2+\alpha,\infty}$$
$$\quad + \|\rho_{2,\kappa}\|_{2+\alpha,\infty}\|\rho_{1,\kappa} - \rho_{2,\kappa}\|_{2+\alpha,\infty} + \|\rho_{2,\kappa}\|_{2+\alpha,\infty}^2\|\rho_{1,\kappa} - \rho_{2,\kappa}\|_{1+\alpha,\infty}\Big) \qquad (4.20)$$
$$+ \tilde{C}\Big[(\|\rho_{1,\kappa}\|_{2+\alpha,\infty} + \|\rho_{2,\kappa}\|_{2+\alpha,\infty})^2\|\rho_{1,\kappa} - \rho_{2,\kappa}\|_{2+\alpha,\infty}$$
$$\quad + \|\rho_{2,\kappa}\|_{2+\alpha,\infty}^3\|\rho_{1,\kappa} - \rho_{2,\kappa}\|_{1+\alpha,\infty}\Big]. \qquad (4.21)$$

the definitions of the spaces E_0, E_μ, E_γ, and E_1, we have $\mu = 1/4$ and $\gamma = 3/4$ in our current setting. Thus, we refer back to (2.4) to see that index j is subcritical when (ϱ_j, γ_j) satisfies $\varrho_j/2 + \gamma_j < 1$, and j is critical when $\varrho_j/2 + \gamma_j = 1$. It follows from Proposition 2.2 that (4.17) is bounded by

$$\|\rho_{1,\kappa} - \rho_{2,\kappa}\|_{3+\alpha,\infty} \leq \sup_{\eta\in\mathfrak{K}} \|\rho_{1,\eta} - \rho_{2,\eta}\|_{3+\alpha,\infty} \leq \tilde{C}\|\rho_1 - \rho_2\|_{E_\gamma}.$$

This corresponds to $(\varrho_j, \gamma_j) = (0, 3/4)$, which is subcritical. In (4.18), similarly it holds

$$\|\rho_{1,\kappa}\|_{3+\alpha,\infty}\|\rho_{1,\kappa} - \rho_{2,\kappa}\|_{2+\alpha,\infty} \leq \tilde{C}\|\rho_1\|_{E_\gamma}\|\rho_1 - \rho_2\|_{2+\alpha,\infty}$$

This corresponds to $(\varrho_j, \gamma_j) = (1, 1/2)$ (which is again subcritical). We can estimate the remaining terms of (4.18) by using Propositions 2.2 and 2.3

$$\|\rho_{2,\kappa}\|_{2+\alpha,\infty}\|\rho_{1,\kappa} - \rho_{2,\kappa}\|_{3+\alpha,\infty} + \|\rho_{2,\kappa}\|_{2+\alpha,\infty}\|\rho_{2,\kappa}\|_{3+\alpha,\infty}\|\rho_{1,\kappa} - \rho_{2,\kappa}\|_{1+\alpha,\infty}$$

$$\leq \tilde{C}\Big(\|\rho_2\|_{E_\gamma}^{1/2}\|\rho_1 - \rho_2\|_{E_\gamma} + \|\rho_2\|_{E_\gamma}^{3/2}\|\rho_1 - \rho_2\|_{E_\mu}\Big)$$

These correspond to $(\varrho_j, \gamma_j) = (1/2, 3/4)$ and $(\varrho_j, \gamma_j) = (3/2, 1/4)$, which are critical. The remaining terms, i.e. (4.19)-(4.21), can be estimated similarly, cf. [22, Section 4]. We conclude that (4.19)-(4.21) is bounded by

$$\tilde{C}\Big[(\|\rho_1\|_{E_\gamma}^{1/2} + \|\rho_2\|_{E_\gamma}^{1/2})\|\rho_1 - \rho_2\|_{E_\mu} + \|\rho_1 - \rho_2\|_{2+\alpha,\infty}$$

$$+ (\|\rho_1\|_{E_\gamma} + \|\rho_2\|_{E_\gamma})\|\rho_1 - \rho_2\|_{E_\mu} + (\|\rho_1\|_{E_\gamma}^{1/2} + \|\rho_2\|_{E_\gamma}^{1/2})\|\rho_1 - \rho_2\|_{2+\alpha,\infty}$$

$$+ (\|\rho_1\|_{E_\gamma} + \|\rho_2\|_{E_\gamma})\|\rho_1 - \rho_2\|_{2+\alpha,\infty} + \|\rho_2\|_{E_\gamma}^{3/2}\|\rho_1 - \rho_2\|_{E_\mu}\Big].$$

The indices for those estimates are $(\varrho_j, \gamma_j) = (1/2, 1/4), (0, 1/2), (1, 1/4), (1/2, 1/2)$ (subcritical), and $(\varrho_j, \gamma_j) = (1, 1/2), (3/2, 1/4)$ (critical), respectively.

Additionally, it follows from (2.1) that $\mathcal{R}^c F(\rho) \in l_{\infty,\text{unif}}(\boldsymbol{bc}^\alpha)$ for any $\rho \in V_\mu \cap E_\gamma$. Since the constant $\tilde{C}$ is independent of κ, the above computations together with (4.16) imply that F satisfies **(H2)** and

$$F \in C^\omega(V_\mu \cap E_\gamma, E_0).$$

Combining the above discussions, we apply Theorem 2.4 to produce the following well–posedness result for (4.12). Note that we assume throughout that Σ carries the metric induced by the Euclidean metric g_{m+1}.

Theorem 4.3. *Let $\alpha \in (0,1)$, $\mu = 1/4$ and Σ be a (URT)–hypersurface in $\mathbb{R}^{m+1}$ with a tubular neighborhood of radius a.*

(a) *Then for any $\rho_0 \in V_\mu := \{\rho \in bc^{1+\alpha}(\Sigma) : \|\rho\|_\infty < \mathsf{a}\}$, (4.12) has a unique solution*

$$\rho(\cdot, \rho_0) \in C_{1-\mu}(J, bc^{4+\alpha}(\Sigma)) \cap C_{1-\mu}^1(J, bc^\alpha(\Sigma))$$

on a maximal interval $J = [0, T) = [0, T(\rho_0))$, with the additional property that $\rho(\cdot, \rho_0) \in C(J, bc^{1+\alpha}(\Sigma))$.

(b)

$$\mathcal{M} := \bigcup_{t \in (0,T)} (\{t\} \times \Gamma(t))$$

is a C^∞–hypersurface in $\mathbb{R}^{m+2}$. In particular, each manifold $\Gamma(t)$ is C^∞ for $t \in (0, T)$. If, in addition, Σ is C^ω–uniformly regular, then $\mathcal{M}$ is a C^ω–hypersurface in $\mathbb{R}^{m+2}$.

(c) *The map $[(t, \rho_0) \mapsto \rho(t, \rho_0)]$ defines a semiflow on V_μ which is analytic for $t > 0$ and Lipschitz continuous for $t \geq 0$.*

Proof. We have already proved part (a) above. Part (b) follows directly from the argument in [33, Sections 3 and 5]. For part (c), we first note that Lipschitz continuity of the semiflow follows from [22, Corollary 2.3]. Regarding additional regularity of the semiflow; for any $\tau > 0$, we note that

$$\rho(\tau, \rho_0) \in V_\gamma = bc^{3+\alpha}(\Sigma) \cap [\|\rho\|_\infty < \mathsf{a}],$$

and so the result holds in V_γ because of [10, Theorem 6.1] and the mapping properties of $A(\cdot)$ and $F(\cdot)$. Regularity of the semiflow in V_μ then follows by embedding. $\square$

5. The Willmore flow

In this section, we take Σ to be a (URT)–hypersurface in $\mathbb{R}^3$. For the Willmore flow, we seek a family of hypersurfaces $\{\Gamma(t) : t \geq 0\}$ satisfying the evolution equation

$$\begin{cases} V(t) = -\Delta_{\Gamma(t)} H_{\Gamma(t)} - 2H_{\Gamma(t)}\big(H_{\Gamma(t)}^2 - K_{\Gamma(t)}\big), & t > 0, \\ \Gamma(0) = \Gamma_0, \end{cases} \tag{5.1}$$

where the term $K_{\Gamma(t)}$ denotes Gaussian curvature of $\Gamma(t)$.

Working in the same setting as Section 4 above, we consider (5.1) acting on surfaces $\Gamma(t) = \Gamma_{\rho(t)}$ defined over Σ via height functions $\rho(t) : \Sigma \to \mathbb{R}$. Assuming that Σ has a tubular neighborhood U_a of radius $\mathsf{a} > 0$, we recall that $E_\mu = bc^{1+\alpha}(\Sigma)$ and $E_\gamma = bc^{3+\alpha}(\Sigma)$ are interpolation spaces between $E_0 := bc^\alpha(\Sigma)$ and $E_1 := bc^{4+\alpha}(\Sigma)$, and we consider initial functions from $V_\mu := \{\rho \in E_\mu : \|\rho\|_\infty < \mathsf{a}\}$.

Treating (5.1) as a lower–order perturbation of (4.1), we again define

$$A(\rho) : E_1 \to E_0 : \quad [u \mapsto \frac{1}{m}\mathsf{C}(g^*(\rho) \otimes g^*(\rho), \nabla^4 u)]$$

for all $\rho \in V_\mu$, and we introduce the mapping $Q : V_\mu \cap E_\gamma \to E_0$ defined as

$$Q(\rho) := A(\rho)\rho - \frac{1}{\beta(\rho)}\Psi_\rho^*\Big(\Delta_{g_3|\Gamma_\rho} H_{\Gamma_\rho} + 2H_{\Gamma_\rho}\big(H_{\Gamma_\rho}^2 - K_{\Gamma_\rho}\big)\Big)$$

$$= A(\rho)\rho - \frac{1}{\beta(\rho)}\big(\Delta_\rho H_\rho + 2H_\rho(H_\rho^2 - K_\rho)\big).$$

We thus arrive at the following expression for (5.1) in our current setting:

$$\begin{cases} \partial_t\rho + A(\rho)\rho = Q(\rho) & \text{in} \quad (0,\infty) \times \Sigma, \\ \qquad\quad \rho(0) = \rho_0 & \text{in} \quad \Sigma. \end{cases} \tag{5.2}$$

By Proposition 4.2, we know that $A \in C^\omega(V_\mu, \mathcal{M}_\mu(E_1, E_0))$ so we focus on showing regularity and structural properties for $Q(\rho)$.

By (4.11) and the definition of $Q(\rho)$, we note that

$$Q(\rho) = F(\rho) - \frac{2}{\beta(\rho)}H_\rho^3 + \frac{2}{\beta(\rho)}H_\rho K_\rho$$

and it follows that the local expression for $Q(\rho)$ is of the form

$$Q(\rho) = \sum_{|\eta|=3,|\tau|\leq 2} c_{\eta,\tau}(\rho,\partial\rho)\,\partial^\tau\rho\,\partial^\eta\rho + \sum_{|\eta|,|\sigma|,|\tau|\leq 2} d_{\eta,\sigma,\tau}(\rho,\partial\rho)\,\partial^\eta\rho\,\partial^\sigma\rho\,\partial^\tau\rho.$$

To confirm this local expression for $Q(\rho)$, we first note that all third–order derivatives of ρ appear in $F(\rho)$, while the terms $\frac{2}{\beta(\rho)}H_\rho^3$ and $\frac{2}{\beta(\rho)}H_\rho K_\rho$ depend only on up to second–order derivatives. With the structure for $F(\rho)$ already established in (4.15), it suffices to confirm that $Q(\rho)$ only contributes additional terms of the form

$$\sum_{|\eta|,|\sigma|,|\tau|\leq 2} d_{\eta,\sigma,\tau}(\rho,\partial\rho)\,\partial^\eta\rho\,\partial^\sigma\rho\,\partial^\tau\rho.$$

Local expressions for $\beta(\rho)$ and H_ρ are given in (4.4) and (4.8), respectively. Since $\beta(\rho)$ depends on at most first–order derivatives of ρ and H_ρ depends linearly on second–order derivatives, we see that at most cubic powers of $\partial^2\rho$ appear in $\frac{2}{\beta(\rho)}H_\rho^3$.

Regarding the term $(2/\beta(\rho))H_\rho K_\rho$, we first express Gaussian curvature

$$K_\rho = \det[g^{ki}(\rho)l_{ij}(\rho)],$$

as derived in [32, Section 2]. Here $l_{ij}(\rho)$ are the components of the pull–back of the second fundamental form of Γ_ρ. It follows from (4.8) that

$$
\begin{aligned}
l_{ij}(\rho) = \beta(\rho)\Big\{ &\partial_i\partial_j\rho + (l_j^k\partial_i\rho - \Gamma_{ij}^k)\partial_k\rho \\
&+ \big[r_k^l(\rho)l_i^k\partial_j\rho + r_k^l(\rho)(\partial_j l_i^k + \Gamma_{jh}^k l_i^h - \Gamma_{ij}^h l_h^k)\rho + r_k^l(\rho)l_j^h l_h^k\rho\partial_i\rho \big] \Big\}\partial_l\rho \\
&+ \beta(\rho)(l_{ij} - l_{ik}l_j^k\rho).
\end{aligned}
$$

Observing that each $l_{ij}(\rho)$ is linear with respect to $\partial^2\rho$, it follows that $\partial^2\rho$ appears at most quadratically in $\det[l_{ij}(\rho)]$, since it is a 2×2 matrix. Therefore, we conclude that K_ρ contains at most quadratic factors of $\partial^2\rho$ and thus, multiplying with the second–order quasilinear term H_ρ, we conclude that the term $(2/\beta(\rho))H_\rho K_\rho$ contains at most cubic powers of $\partial^2\rho$.

With confirmation that $Q(\rho)$ satisfies the same structural condition (2.5) as $F(\rho)$ in Section 4, we employ the same argument outlined in (4.16)–(4.21) to conclude that (A, Q) satisfies conditions **(H1)–(H2)**. The following well–posedness result for (5.1) then follows from Theorem 2.4.

Theorem 5.1. *Let $\alpha \in (0, 1)$, $\mu = 1/4$ and Σ be a (URT)–hypersurface in $\mathbb{R}^3$ with tubular neighborhood of radius* a.

(a) *Then for any $\rho_0 \in V_\mu := \{\rho \in bc^{1+\alpha}(\Sigma) : \|\rho\|_\infty < $ a$\}$, (5.2) has a unique solution*

$$\rho(\cdot, \rho_0) \in C_{1-\mu}(J, bc^{4+\alpha}(\Sigma)) \cap C_{1-\mu}^1(J, bc^\alpha(\Sigma))$$

on a maximal interval $J = [0, T) = [0, T(\rho_0))$, with the additional property that $\rho(\cdot, \rho_0) \in C(J, bc^{1+\alpha}(\Sigma))$.

(b)

$$\mathcal{M} := \bigcup_{t\in(0,T)} (\{t\} \times \Gamma(t))$$

is a C^∞–hypersurface in $\mathbb{R}^4$. In particular, each manifold $\Gamma(t)$ is C^∞ for $t \in (0, T)$. If, in addition, Σ is C^ω–uniformly regular, then $\mathcal{M}$ is a C^ω–hypersurface in $\mathbb{R}^{m+2}$.

(c) *The map $[(t, \rho_0) \mapsto \rho(t, \rho_0)]$ defines a semiflow on V_μ which is analytic for $t > 0$ and Lipschitz continuous for $t \geq 0$.*

Proof. Part (b) follows from [32] and [33, Section 3]. Part (c) follows exactly as in the proof of Theorem 4.3(c) above. $\qquad\square$

5.1. **Stability of spheres.** In the case Σ is a Euclidean sphere in $\mathbb{R}^3$, we apply the generalized principle of linearized stability (c.f. [22, Section 3]) to prove the following result regarding stability of spheres under the Willmore flow, with control on only first–order derivatives of perturbations.

Theorem 5.2. *Fix $\alpha \in (0,1)$, $\mu = 1/4$, and $\bar{\mu} \in (0,1)$, and let Σ be a Euclidean sphere in $\mathbb{R}^3$ with radius $r > 0$. There exists a constant $\delta \in (0,r)$ such that, given any admissible perturbation Γ_{ρ_0} for*

$$\rho_0 \in V_{\mu,\delta} := \{\rho \in bc^{1+\alpha}(\Sigma) : \|\rho\|_\infty < r \text{ and } \|\rho_0\|_{1+\alpha,\infty} < \delta\},$$

the solution $\rho(\cdot, \rho_0)$ of (5.2) exists globally in time and converges to some $\bar{\rho} \in \mathcal{M}_{sph}$ at an exponential rate, in the topology of $E_{\bar{\mu}}$. Here, $\mathcal{M}_{sph}$ denotes the family of functions $\rho \in C^\infty(\Sigma, \mathbb{R})$ for which Γ_ρ is a sphere that is close to Σ in $\mathbb{R}^3$.

Proof. It is shown in the proof of [35, Theorem 1.2] that $\rho_* = 0$ is normally stable under (5.2). The result then follows from [22, Theorem 3.2]. $\square$

Corollary 5.3. *There exist non–convex hypersurfaces Γ_0 such that the solution $\rho(\cdot, \rho_0)$ to (5.2) with $\Gamma(\rho_0) = \Gamma_0$, exists globally in time and converges exponentially fast to a sphere.*

We note here that Theorem 5.2 also holds true for the surface diffusion flow, as was shown in [22, Section 4.5].

Appendix A.

Suppose Σ is a (URT)-hypersurface with tubular neighborhood of radius a. Given $\rho \in C(\Sigma)$ with $\|\rho\|_\infty < \mathsf{a}$, let $\Gamma_\rho := \Psi_\rho(\Sigma)$, where $\Psi_\rho(\mathsf{p}) = \mathsf{p} + \rho(\mathsf{p})\nu_\Sigma(\mathsf{p})$ for $\mathsf{p} \in \Sigma$.
Then Γ_ρ enjoys the following properties.

Proposition A.1. *Let $k \in \mathbb{N} \cup \{\infty\}$.*
  (a) *Suppose $\rho \in BC^{k+1}(\Sigma)$ and $\|\rho\|_\infty < \mathsf{a}$. Then Γ_ρ is C^k–uniformly regular.*
  (b) *There exists $\varepsilon_1 > 0$ such that, for any $\rho \in BC^2(\Sigma)$ with $\|\rho\|_\infty \leq \varepsilon_1$, the hypersurface Γ_ρ has a tubular neighborhood of radius a_1 for some positive number $\mathsf{a}_1 = \mathsf{a}_1(\varepsilon_1, \rho)$.*

Proof. (a) We can construct an atlas $\mathfrak{A}_\rho = \{\mathsf{O}_{\kappa,\rho}, \varphi_{\kappa,\rho}) : \kappa \in \mathfrak{K}\}$ for Γ_ρ as follows. Define

$$\mathsf{O}_{\kappa,\rho} := \Psi_\rho \circ \psi_\kappa(\mathbb{B}^m), \quad \varphi_{\kappa,\rho} := \varphi_\kappa \circ \Psi_\rho^{-1}, \quad \psi_{\kappa,\rho} = \varphi_{\kappa,\rho}^{-1}.$$

Then $\mathfrak{A}_\rho$ inherits properties (R1)–(R3) from $\mathfrak{A}$. Next we note that

$$\psi_{\kappa,\rho}^*(g_{m+1}|_{\Gamma_\rho}) = \psi_\kappa^* g(\rho),$$

and that by (4.5), $g(\rho)$ involves first order derivatives of ρ. Hence, $\psi_\kappa^* g(\rho)$ is C^k for $\rho \in C^{k+1}$. It follows readily from (4.5) that

$$|(I - \rho L_\Sigma)\xi|^2(\mathsf{p}) \leq g_{ij}(\rho)(\mathsf{p})(\xi,\xi) \leq (1 + |a(\rho)|^2)|(I - \rho L_\Sigma)\xi|^2(\mathsf{p})$$

for $\mathsf{p} \in \Sigma$ and $\xi = \xi^i \tau_i(\mathsf{p}) \in T_\mathsf{p}\Sigma$. Properties (R4)-(R5) now follow from

$$\min\{(1 - (\rho\kappa_r)(\mathsf{p}))^2\}|\xi|^2 \leq |(I - \rho L_\Sigma)\xi|^2(\mathsf{p}) \leq \max\{(1 - (\rho\kappa_r)(\mathsf{p}))^2\}|\xi|^2$$

and Remark 3.1(d).

(b) Let $r_0 \in (0,1)$ be the constant related to the uniformly shrinkable property.

Claim 1: Let $\tilde{r}_0 := \frac{1+r_0}{2}$. There exists a uniform constant r_1 such that for any $\kappa \in \mathfrak{K}$ and $\mathsf{p} \in \psi_\kappa(\tilde{r}_0\mathbb{B}^m)$, $\psi_\kappa(\mathbb{B}^m(x_\mathsf{p}, r_1))$ is a graph $f_{\kappa,\mathsf{p}}$ over $T_\mathsf{p}\Sigma$ with $x_\mathsf{p} := \varphi_\kappa(\mathsf{p})$ satisfying

$$\|f_{\kappa,\mathsf{p}}\|_{2,\infty} \leq c_0 \tag{A.1}$$

for some $c_0 > 0$ independent of κ and p.

Proof of Claim 1. Let $\kappa \in \mathfrak{K}$ and $\mathsf{p} \in \psi_\kappa(\tilde{r}_0 \mathbb{B}^m)$ be given. Then there exists x_p in $\tilde{r}_0 \mathbb{B}^m$ such that $\mathsf{p} = \psi_k(x_\mathsf{p})$. Let

$$\mathcal{P}_\mathsf{p} := I - \nu_\Sigma(\mathsf{p}) \otimes \nu_\Sigma(\mathsf{p})$$

be the orthogonal projection of $\mathbb{R}^{m+1}$ onto $T_\mathsf{p}\Sigma$. In the following we will identify $T_\mathsf{p}\Sigma$ with $\mathbb{R}^m$. It follows from the boundedness of $\|L_\Sigma\|_\infty$ that there is a universal constant b_0 such that $\mathcal{P}_\mathsf{p} : \psi_\kappa(\mathbb{B}(x_\mathsf{p}, b_0)) \to T_\mathsf{p}(\Sigma) \doteq \mathbb{R}^m$ is injective. Let

$$F_{\kappa,\mathsf{p}}(x) := (\mathcal{P}_\mathsf{p} \circ \psi_\kappa)(x), \quad x \in \mathbb{B}(x_\mathsf{p}, b_0).$$

Then we obtain for the Fréchet derivative of $F_{\kappa,\mathsf{p}}$

$$DF_{\kappa,\mathsf{p}}(x_\mathsf{p}) = \mathcal{P}_\mathsf{p} D\psi_\kappa(x_\mathsf{p}) = D\psi_\kappa(x_\mathsf{p}), \tag{A.2}$$

as $D\psi_\kappa(x_\mathsf{p})\xi \in T_\mathsf{p}\Sigma$ for all $\xi \in \mathbb{R}^m$. We infer from (R5) that

$$(1/\gamma_1)^2 |\xi|^2 \le |D\psi_\kappa(x)\xi|^2 = (\psi_\kappa^* g)(x)(\xi, \xi) \le \gamma_1^2 |\xi|^2, \quad x \in \mathbb{B}^m, \ \xi \in \mathbb{R}^m, \tag{A.3}$$

for some uniform constant $\gamma_1 \ge 1$. It follows from (A.2) and (A.3) that the spectrum of $DF_{\kappa,\mathsf{p}}(x)$ lies outside the ball $\mathbb{B}_\mathbb{C}(0, 1/\gamma_1)$ for any $x \in \mathbb{B}^m$. Indeed, suppose $\mu v = DF_{\kappa,p}(x)v$ for some $\mu \in \mathbb{C}$ and $v = \xi + i\eta \in \mathbb{C}^m$ with $|v| = 1$. Then

$$|\mu|^2 = |DF_{\kappa,p}(x)v|^2 = |D\psi_\kappa(x)\xi|^2 + |D\psi_\kappa(x)\eta|^2 \ge (1/\gamma_1)^2.$$

Lemma 4.1 in [7] implies that $DF_{\kappa,\mathsf{p}}(x)$ is invertible with

$$|[DF_{\kappa,\mathsf{p}}(x)]^{-1}| \le \gamma_2, \tag{A.4}$$

where the constant γ_2 is independent of $x \in \mathbb{B}^m$ and κ, p. By the inverse function theorem, there exists a uniform constant r_1 which is independent of κ and $\mathsf{p} \in \psi_\kappa(\tilde{r}_0 \mathbb{B}^m)$ such that

$$F_{\kappa,\mathsf{p}} : \mathbb{B}^m(x_\mathsf{p}, r_1) \to \mathcal{P}_\mathsf{p} \psi_\kappa(\mathbb{B}^m(x_\mathsf{p}, r_1))$$

is a diffeomorphism. Next we note that

$$\partial_j F_{\kappa,\mathsf{p}}^{-1}(y) = [DF_{\kappa,\mathsf{p}}(F_{\kappa,\mathsf{p}}^{-1}(y))]^{-1} e_j$$

$$\partial_i \partial_j F_{\kappa,\mathsf{p}}^{-1}(y) = -[DF_{\kappa,\mathsf{p}}(F_{\kappa,\mathsf{p}}^{-1}(y))]^{-1} \, \partial_i [DF_{\kappa,\mathsf{p}}(F_{\kappa,\mathsf{p}}^{-1}(y))] \, [DF_{\kappa,\mathsf{p}}(F_{\kappa,\mathsf{p}}^{-1}(y))]^{-1} e_j$$

Recall that $\partial_i \partial_j \psi_\kappa = \Gamma_{ij}^k \partial_k \psi_\kappa + l_{ij} \nu_\Sigma$. In view of [3, Formula (3.19)], (A.3) and the boundedness of $\|L_\Sigma\|_\infty$, we conclude that

$$\|\psi_\kappa\|_{2,\infty} \le \gamma_3, \qquad \text{for all } \kappa \in \mathfrak{K}. \tag{A.5}$$

It follows from (A.4) and (A.5) that $\|F_{\kappa,\mathsf{p}}^{-1}\|_{2,\infty} \le c$ for some c independent of κ, p. Define $\Phi_{\kappa,\mathsf{p}} : \mathcal{P}_\mathsf{p} \psi_\kappa(\mathbb{B}^m(x_\mathsf{p}, r_1)) \to \Sigma$ by $\Phi_{\kappa,\mathsf{p}} := \psi_\kappa \circ F_{\kappa,\mathsf{p}}^{-1}$. Note that

$$\Phi_{\kappa,\mathsf{p}}(y) = \mathcal{P}_\mathsf{p} \circ \Phi_{\kappa,\mathsf{p}}(y) + (I - \mathcal{P}_\mathsf{p}) \circ \Phi_{\kappa,\mathsf{p}}(y)$$

$$= y + (\nu_\Sigma(\mathsf{p}) | \Phi_{\kappa,\mathsf{p}}(y)) \nu_\Sigma(\mathsf{p})) =: y + f_{\kappa,\mathsf{p}}(y) \nu_\Sigma(p).$$

We can now conclude that (A.1) holds.   ∎

In the following, we assume that

$$r_1 < \min \left\{ \frac{1 - r_0}{2}, \frac{1}{2\gamma_1 \gamma_3} \right\}. \tag{A.6}$$

By Claim 1, we can find $L \in \mathbb{N}$ such that, in every O_κ, there exist $x_{\kappa,i} \in r_0 \bar{\mathbb{B}}^m$ with $i = 1, \cdots, L$ such that

$$\bigcup_{i=1}^L \psi_\kappa(\mathbb{B}^m(x_{\kappa,i}, r_1/4)) \text{ covers } \psi_\kappa(r_0 \bar{\mathbb{B}}^m).$$

Taking new local patches $\psi_\kappa(\mathbb{B}^m(x_{\kappa,i}, r_1/2))$, after relabelling, translation and scaling, we obtain a new atlas satisfying (R1)–(R5), still denoted by $\mathfrak{A} = \{(\mathsf{O}_\kappa, \varphi_\kappa) : \kappa \in \mathfrak{K}\}$. Note that for this new atlas, O_κ is the graph of a function $f_{\kappa,\mathsf{p}}$ over $T_\mathsf{p}\Sigma$ for any $\kappa \in \mathfrak{K}$ and $\mathsf{p} \in \mathsf{O}_\kappa$. Moreover, (A.1) still holds true. In addition, we can take uniformly shrinkable constant $r_0 = 1/2$. Note also that, by (R5), we can assume that r_1 is chosen so small that

$$|\mathsf{p} - \mathsf{q}| < \mathsf{a}/8, \quad \mathsf{p}, \mathsf{q} \in \mathsf{O}_\kappa. \tag{A.7}$$

Let $\operatorname{dist}(\cdot, \cdot)$ denote the Euclidean distance between two compact subsets in $\mathbb{R}^{m+1}$.

Claim 2: There exists $c_1 > 0$ such that $\operatorname{dist}(\mathsf{p}, \partial X(\mathsf{O}_\kappa, [-\mathsf{a}/2, \mathsf{a}/2])) > c_1$ for all $\kappa \in \mathfrak{K}$ and $\mathsf{p} \in \psi_\kappa(\frac{1}{2}\mathbb{B}^m)$.

Proof of Claim 2. We set

$$D_\kappa := X(\mathsf{O}_\kappa \times [-\mathsf{a}/2, \mathsf{a}/2]),$$

and

$$S_{1,\kappa} := X(\mathsf{O}_\kappa \times \{-\mathsf{a}/2\}) \cup X(\mathsf{O}_\kappa \times \{\mathsf{a}/2\}), \quad S_{2,\kappa} := \partial D_\kappa \setminus S_{1,\kappa}.$$

We now show that $\operatorname{dist}(\mathsf{p}, \partial D_\kappa)$ is uniformly positive.

Case 1: $|\mathsf{p} - q| = \operatorname{dist}(\mathsf{p}, \partial D_\kappa)$ for some $q \in S_{1,\kappa}$

Since $q \in S_{1,\kappa}$, we can find some $\mathsf{q} \in \mathsf{O}_\kappa$ such that, without loss of generality, we have $X(\mathsf{q}, \mathsf{a}/2) = q$. This implies

$$|\mathsf{p} - q| \geq \mathsf{a}/2 - |\mathsf{p} - \mathsf{q}| \geq \mathsf{a}/4.$$

Case 2: $|\mathsf{p} - q| = \operatorname{dist}(\mathsf{p}, \partial D_\kappa)$ for some $q \in S_{2,\kappa}$.

We first observe that, for any $\mathsf{p} \in \psi_\kappa(\frac{1}{2}(\mathbb{B}^m))$, we have by (A.3), (A.5) and Taylor expansion that
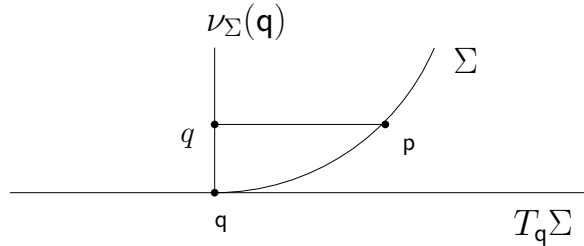
$$|\mathsf{p} - \mathsf{q}| = |\psi_\kappa(x_\mathsf{p}) - \psi_\kappa(x_\mathsf{q})| \geq \frac{r_1}{2}\left[\frac{1}{2\gamma_1} - \frac{r_1}{2}\gamma_3\right],$$

where $\mathsf{p} = \psi_\kappa(x_\mathsf{p})$ with x_p in $\frac{1}{2}\mathbb{B}^m$ and $\mathsf{q} = \psi_\kappa(x_\mathsf{q})$ with $x_\mathsf{q} \in \partial\mathbb{B}^m$. The assumption (A.6) now implies

$$\operatorname{dist}(\mathsf{p}, \partial\mathsf{O}_\kappa) \geq \eta_0 = r_1/(8\gamma_1). \tag{A.8}$$

Moreover, $S_{2,\kappa} = X(\partial\mathsf{O}_\kappa \times (-\mathsf{a}/2, \mathsf{a}/2))$ implies that there exist $\mathsf{q} \in \partial\mathsf{O}_\kappa$ and $s \in (-\mathsf{a}/2, \mathsf{a}/2)$ such that $q = X(\mathsf{q}, s)$.

Because of (A.7), we can realize p as a point on the graph of $f_{\kappa,\mathsf{q}}$, cf. the following figure.

By (A.8), we observe that $|\mathsf{p} - \mathsf{q}| \geq \eta_0$. Let $d = |\mathsf{p} - q|$. Using (A.1), we have

$$d^2 + d^2 c_0^2 \geq \eta_0^2,$$

which implies

$$|\mathsf{p} - q| \geq \eta_1$$

for some uniform constant $\eta_1 > 0$. Thus we can take $c_1 = \min\{\mathsf{a}/4, \eta_1\}$ independent of p and κ. ∎

Claim 3: Let $\delta \in (0, \mathsf{a})$ and $M > 0$ be fixed. Then there exists a constant $r_2 > 0$ such that for any $\rho \in BC^2(\Sigma)$ with $\|\rho\|_\infty \leq \delta$, $\|\rho\|_{2,\infty} \leq M$, and any $\mathsf{p} \in \psi_\kappa(\frac{3}{4}\mathbb{B}^m)$, $\Psi_\rho(\psi_\kappa(x_\mathsf{p}, r_2))$ is the graph of a C^2-function $h_{\kappa, \mathsf{p}}$ over $T_{\Psi_\rho(\mathsf{p})}\Gamma_\rho$ satisfying

$$\|h_{\kappa, \mathsf{p}}\|_{2,\infty} \leq c_2, \tag{A.9}$$

for some $c_2 > 0$ independent of κ, p and ρ.

Proof of Claim 3. The proof is basically the same as that of Claim 1, as Γ_ρ is a C^2-hypersurface and C^1-uniformly regular by part (a) of the proposition; and all we need for the proof of Claim 1 is this property. ∎

We assume

$$\varepsilon_1 < \min\{\mathsf{a}/8, c_1/2\}, \tag{A.10}$$

where c_1 is the constant in Claim 2. By our choice of ε_1, following the construction below Claim 1, we can further modify the atlas $\mathfrak{A}$, still with uniform shrinkable parameter $r_0 = 1/2$, such that for every $\rho \in BC^2(\Sigma)$ with $\|\rho\|_\infty \leq \varepsilon_1$, $\Psi_\rho(\mathsf{O}_\kappa)$ is a graph over $T_{\Psi_\rho(\mathsf{p})}\Gamma_\rho$, for any $\kappa \in \mathfrak{K}$ and $\mathsf{p} \in \mathsf{O}_\kappa$.

Claim 3 and Example 3.2(c) imply that $\Psi_\rho(\mathsf{O}_\kappa)$ has a tubular neighborhood of radius a_1, where a_1 is independent of κ, p. In order to prove that Γ_ρ has a tubular neighborhood of radius a_1, it suffices to show that

$$X_\rho : \Sigma \times (-\mathsf{a}_1, \mathsf{a}_1) : (\mathsf{p}, s) \mapsto \Psi_\rho(\mathsf{p}) + s\nu_{\Gamma_\rho}(\Psi_\rho(\mathsf{p}))$$

is injective.

Claim 4: For sufficiently small $\mathsf{a}_1 > 0$ and any $\rho \in BC^2(\Sigma)$ with $\|\rho\|_\infty \leq \varepsilon_1$ satisfying (A.10) and $\mathsf{p} \in \psi_\kappa(\frac{1}{2}\mathbb{B}^m)$, it holds that $\mathbb{B}_{\mathbb{R}^{m+1}}(\Psi_\rho(\mathsf{p}), 2\mathsf{a}_1)$ is contained in $X(\mathsf{O}_\kappa, [-\mathsf{a}/2, \mathsf{a}/2])$.

Proof of Claim 4. Define D_κ, $S_{1,\kappa}$ and $S_{2,\kappa}$ as in Claim 2. Given any $\mathsf{p} \in \psi_\kappa(\frac{1}{2}\mathbb{B}^m)$, there exists some $q \in \partial D_\kappa$ such that

$$|\Psi_\rho(\mathsf{p}) - q| = \text{dist}(\Psi_\rho(\mathsf{p}), \partial D_\kappa).$$

If $q \in S_{1,\kappa}$, then there exists $\mathsf{q} \in \mathsf{O}_\kappa$ so that, without loss of generality, $X(\mathsf{q}, \mathsf{a}/2) = q$. By (A.7) and (A.10), we infer that

$$|\Psi_\rho(\mathsf{p}) - q| \geq \mathsf{a}/2 - |\Psi_\rho(\mathsf{p}) - \mathsf{p}| - |\mathsf{p} - \mathsf{q}| \geq \mathsf{a}/4.$$

If $q \in S_{2,\kappa}$, by Claim 2 and (A.10)

$$|\Psi_\rho(\mathsf{p}) - q| \geq |\mathsf{p} - q| - |\rho(\mathsf{p})| \geq c_1/2.$$

Therefore, it suffices to take $\mathsf{a}_1 \leq \min\{\mathsf{a}/8, c_1/4\} = c_1/4$. ∎

If $X_\rho(\mathsf{p}, s) = X_\rho(\mathsf{q}, t)$ for some $\mathsf{p}, \mathsf{q} \in \Sigma$ and $s, t \in (-\mathsf{a}_1, \mathsf{a}_1)$, we may assume that $\mathsf{p} \in \psi_\kappa(\frac{1}{2}\mathbb{B}^m)$ for some $\kappa \in \mathfrak{K}$. It follows that

$$|\Psi_\rho(\mathsf{p}) - \Psi_\rho(\mathsf{q})| = |s\nu_\Sigma(\mathsf{p}) - t\nu_\Sigma(\mathsf{q})| < 2\mathsf{a}_1.$$

We conclude from Claim 4 that $\Psi_\rho(\mathsf{q}) \in X(\mathsf{O}_\kappa, [-\mathsf{a}/2, \mathsf{a}/2])$ and thus $\mathsf{q} \in \mathsf{O}_\kappa$ as well. However, in this case, we already know that $\Psi_\rho(\mathsf{O}_\kappa)$ has an a_1–tubular neighborhood, which implies that $\mathsf{p} = \mathsf{q}$ and $s = t$. $\qquad\square$

Remark A.2. *In Proposition A.1(b) it would be desirable to be able to replace the smallness condition $\|\rho\|_\infty \le \varepsilon_1$ by the more natural condition $\|\rho\|_\infty < \mathsf{a}$.*

In the special case that Σ is compact, this property holds by Remark 3.1(b), as Γ_ρ is a compact (closed) C^2-hypersurface.

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