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Perturbed Obstacle Problems in Lipschitz Domains: Linear Stability and Nondegeneracy in Measure

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PERTURBED OBSTACLE PROBLEMS IN LIPSCHITZ DOMAINS: LINEAR STABILITY AND NONDEGENERACY IN MEASURE

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ABSTRACT. We consider the classical obstacle problem on bounded, connected Lipschitz domains $D \subset \mathbb{R}^n$. We derive quantitative bounds on the changes to contact sets under general perturbations to both the right-hand side and the boundary data for obstacle problems. In particular, we show that the Lebesgue measure of the symmetric difference between two contact sets is linearly comparable to the L^1-norm of perturbations in the data.

1. Introduction. Given functions $g_1, g_2 : D \to [\lambda, \mu]$ and $\psi_1, \psi_2 : \partial D \to [0, \infty)$, with sufficient regularity and $0 < \lambda \le \mu$, we denote by $OP(Lap = g_i, Bdry = \psi_i)$ the nonnegative functions $u_i \in W^{1,2}(D)$ satisfying the semilinear pdes

$$(1.1) \qquad \begin{cases} \Delta u_i = \chi_{\{u_i > 0\}} g_i & \text{in } D, \\ u_i = \psi_i & \text{on } \partial D, \end{cases} \qquad i = 1, 2.$$

We mention that the obstacle problem can also be formulated in terms of variational inequalities and functional optimization, though the equivalence of these settings is well-known; see **[3, 7]**, for instance. The existence and uniqueness of solutions to (1.1) is shown in the same references, via standard methods in functional analysis.

Under minimal assumptions on the data (g_i, ψ_i) and the content of contact sets $\Lambda(u_i) := \{x \in D : u_i(x) = 0\}$, we prove that the Lebesgue measure of the symmetric difference $\Lambda(u_1) \,\Delta\, \Lambda(u_2)$ is linearly comparable to the L^1-norms of the perturbations to data over appropriate sets. We denote by $\Omega(u_i) := D \setminus \Lambda(u_i)$ the noncontact set associated with solution u_i. Our main result is stated in the following theorem:

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Theorem 1.1. *Let $D \subset \mathbb{R}^n$ be a bounded, connected Lipschitz domain and let*

$$(g_i, \psi_i) \in L^\infty(D) \times C(\partial D), \quad i = 1, 2,$$

with $0 < \lambda \le g_i \le \mu$ and $\psi_i \ge 0$. Consider the following obstacle problem solutions:

$$\begin{aligned} u_i &= OP(Lap = g_i, Bdry = \psi_i), \\ \bar{v} &= OP(Lap = \min(g_1, g_2), Bdry = \max(\psi_1, \psi_2)). \end{aligned} \tag{1.2}$$

Assume there exist $\bar{y} \in D$ and $\delta > 0$ such that $B_\delta(\bar{y}) \subset \Lambda(\bar{v}) := \{\bar{v} = 0\}$. Further, for $\eta > 0$, define the set

$$D_{-\eta} := D \setminus \mathcal{N}_\eta(\partial D) = D \setminus \{x \in \mathbb{R}^n : \operatorname{dist}(x, \partial D) < \eta\}.$$

(a) (linear stability) *For $\eta > 0$, there exist positive constants C_1 and C_2 such that*

$$\begin{aligned} &|(\Lambda(u_1) \,\Delta\, \Lambda(u_2)) \cap D_{-\eta}| \\ &\qquad \le C_1 \|\psi_1 - \psi_2\|_{L^1(\partial D)} + C_2 \|g_1 - g_2\|_{L^1(\Omega(\bar{v}))}. \end{aligned} \tag{1.3}$$

The constants C_1 and C_2 depend upon n, D, η, δ, and λ.

(b) (linear nondegeneracy) *If $\psi_1 \ge \psi_2$ on ∂D and $g_1 \le g_2$ in D, then for $\eta > 0$ there exist positive constants C_3 and C_4 such that*

$$|\Lambda(u_1) \,\Delta\, \Lambda(u_2)| \ge C_3 \|\psi_1 - \psi_2\|_{L^1(\partial D)} + C_4 \|g_1 - g_2\|_{L^1(\Omega(u_1) \cap D_{-\eta})}. \tag{1.4}$$

The constants C_3 and C_4 both depend upon n, D, δ, and μ, while C_4 additionally depends upon η.

Remark 1. (a) Use of the term "nondegeneracy" in Theorem 1.1(b) differs from most literature related to the obstacle problem. Typically, one refers to the nondegenerate quadratic growth enjoyed by solutions to the obstacle problem in noncontact regions, while here we refer to nondegenerate changes to contact (or likewise noncontact) regions induced by data perturbations.

(b) Extracting explicit dependence of the constants C_1, C_2, C_3, C_4 on the domain D does not appear to be feasible in the generality considered here. The constants are determined (among other things) by the location of the specified interior point $\bar{y}$ and L^∞ bounds for Green's functions and Poisson kernels for D, over subsets of D that are bounded away from ∂D or $\bar{y}$.

(c) To complement $\bar{v}$, we also introduce the function

$$\underline{v} := OP(Lap = \max(g_1, g_2), Bdry = \min(\psi_1, \psi_2))$$

which is used in the proof of Theorem 1.1. By [**1,** Theorem 2.7(d)], one immediately concludes $\underline{v} \le u_i \le \bar{v}$, $i = 1, 2$.

(d) In the case of monotone perturbations, as considered in Theorem 1.1(b), one may wish to combine bounds (1.3) and (1.4) to derive a *linear comparability* result for $\Lambda(u_1)\,\Delta\,\Lambda(u_2) = \Lambda(u_2)\setminus\Lambda(u_1)$. With $\bar{v} = u_1$ in this case, note that the terms in these two inequalities differ only by specific appearances of intersections with the restricted domain $D_{-\eta}$. Without further assumptions on either the structure of the domain D, or the nature of the data g_i and ψ_i, these intersections with $D_{-\eta}$ are necessary to produce positive lower bounds on the Green's function (which is zero on ∂D). The following corollary provides one set of assumptions under which linear comparability holds.

Corollary 1.2. *Under the assumptions of Theorem* 1.1(b)*, suppose there exists* $\eta > 0$ *such that* $\Lambda(u_2) \subset D_{-\eta}$ *and* $g_2 - g_1$ *is supported in* $D_{-\eta}$*. Then* $|\Lambda(u_1)\ \Delta\ \Lambda(u_2)|$ *is bounded above and below by constant multiples of the sum*

$$\|\psi_1 - \psi_2\|_{L^1(\partial D)} + \|g_1 - g_2\|_{L^1(\Omega(u_1))},$$

with multiplicative constants depending on n*,* D*,* η*,* δ*,* λ*, and* μ*.*

Comparing our results with the literature, a form of measure stability is proved in [**2**], with square root dependence on changes to the data, while many more stability results appear in [**7**], including stability with respect to perturbations to the operator itself, which we do not treat here. On the other hand, all of the quantitative bounds established in [**7**] also involve the square root of data perturbations (along with many convergence results without giving a rate). The closest result to our current linear stability (Theorem 1.1(a)) can be found in [**1,** Theorem 4.1], where the first author worked in the specific setting of $D = B_1$, the unit ball in $\mathbb{R}^n$. We note that the result in [**1**] measures the full set $\Lambda(u_1)\ \Delta\ \Lambda(u_2)$, while the current work measures only the portion of this symmetric difference that is away from the boundary ∂D by some distance $\eta > 0$. However we are working in a more general setting here and consider both perturbations to the right-hand side and boundary data for obstacle problems.

Regarding linear nondegeneracy, our result (Theorem 1.1(b)) appears to be new in the literature. One can find a form of linear nondegeneracy bounds in [**1,** Theorem 5.7], where it is established that the Hausdorff distance between free boundaries is linearly comparable to perturbations of the Laplacian data, in the special case when free boundaries are assumed to be regular. We note that the current work differs from [**1,** Theorem 5.7] as we do not assume any regularity on the free boundaries, we permit perturbations to the Laplacian that are supported on proper subsets of the domain B (whereas the argument in [**1**] requires the difference $g_2 - g_1$ to be uniformly bounded below by some positive constant), and we allow perturbations to both the right-hand side and boundary data.

As a final note on literature related to perturbed obstacle problems, the reader should refer to [**8**] for precise formulas for normal velocity and acceleration of free boundaries under sufficiently regular variations to Laplacian and boundary data. The authors of [**8**] work in a global setting (i.e., $D = \mathbb{R}^n$) with compactly supported perturbations to Laplacian data and constant "boundary" data (at $|x| \to \infty$). Finally, we note that regularity of free boundaries is assumed in [**8**], as one may expect to make sense of pointwise normal velocity.

Outlining the current work, in Section 2 we introduce notation and state necessary lemmas from elliptic theory and potential theory. Then, in Section 3, we prove Theorem 1.1 by splitting into cases where either boundary data or Laplacian data are fixed.

2. Setting, notation, and preliminary bounds. We assume the set $D \subset \mathbb{R}^n$ is a bounded, connected Lipschitz domain. In this section, we collect preliminary lemmas we will use in the proof of Theorem 1.1.

2.1. The inhomogeneous Dirichlet problem in D. Considering the situation in (1.1) when boundary data is fixed (i.e., assuming $\psi_1 = \psi_2$), the difference $w = u_1 - u_2$ will satisfy an inhomogeneous Dirichlet problem of the form

$$\begin{cases} \Delta w = f & \text{in } D, \\ w = 0 & \text{on } \partial D. \end{cases} \tag{2.1}$$

The precise expression of the function f is not important at the moment (though it may be instructive for the reader to identify values of f on subsets of D depending upon the contact sets $\Lambda(u_1)$, $\Lambda(u_2)$, and regions of overlap between these), rather we note that tools for controlling solutions to (2.1) with rough data f will thus help control differences between u_1 and u_2. We direct the reader to **[5]** for a detailed treatment of inhomogeneous Dirichlet problems in Lipschitz domains, though many of the statements below come from **[9]**.

We first note that (2.1) is solvable for general domains A and data f:

Theorem 2.1. [**9,** Theorem 1.2.1] *Let A be a bounded domain in $\mathbb{R}^n$. Given any $f \in W^{-1}(A)$ (the dual space to $W_0^{1,2}(A)$), there exists a unique solution $u = Tf \in W_0^{1,2}(A)$ to* (2.1), *in the sense that*

$$\int_A \nabla u \, \nabla v = \int_A fv \quad \textit{for all } v \in W_0^{1,2}(A).$$

There exists a *Dirichlet Green's function* for any bounded A:

Theorem 2.2. [**9,** Theorem 1.2.2] *Let A be a bounded domain in $\mathbb{R}^n$ and let $T : W^{-1}(A) \to W_0^{1,2}(A)$ be the operator defined in Theorem 2.1. There exists a kernel function $G(x,y)$ in $A \times A$ satisfying the following:*

(a) $G(x,y) \in C^\infty(A \times A \setminus \{(x,x) : x \in A\})$.
(b) $(1 - \eta_y(x))G(x,y) \in W_0^{1,2}(A)$ *where $\eta_y(x) \in C_0^\infty(A)$ is any cutoff function satisfying $\eta \geq 0$ and $\eta = 1$ in $B_\varepsilon(y)$, $\varepsilon > 0$.*
(c) $G(x,y) = G(y,x)$ *for every $y \neq x$.*
(d) $G(x, \cdot\,) \in L^1(A)$ *and*

$$Tf(x) = \int_A G(x,y) f(y) dy \quad \textit{for all } f \in C_0^\infty(A).$$

Considering the low regularity expected for f in (2.1) in the context of functions $w = u_1 - u_2$, we extend the representation found in Theorem 2.2(d) to more general functions f:

Lemma 2.3. *Let D be a bounded, connected, Lipschitz domain in $\mathbb{R}^n$, let G be the Dirichlet Green's function on D, and consider*

$$f \in L^q(D) \quad \textit{with } q > n.$$

Then the solution $u = Tf$ to (2.1) *satisfies the representation*

$$u(x) = \int_D G(x,y) f(y) dy \quad \textit{for all } x \in D.$$

Proof. Fix $x \in D$. Since the Green's function $G(x, \cdot)$ belongs to $W_0^{1,p}(D)$ for all $p \in [1, n/(n-1))$ (see [**6,** Theorem 1.2.8]), the map given by

$$If := \int_D G(x,y) f(y) dy$$

is a bounded continuous linear functional on $L^q(D)$ for all $q > n/2$ by Hölder's inequality. By Calderon-Zygmund theory (see [**4,** Chapter 9]), it follows that the solution map $T : f \mapsto u$, taking $f \in L^q(D)$ with $n/2 < q < \infty$ to the solution $u := Tf \in W_0^{2,q}(D)$ of

$$\begin{cases} \Delta u = f & \text{in } D, \\ u = 0 & \text{on } \partial D, \end{cases} \tag{2.2}$$

is a bounded linear map.

Further, since $W_0^{2,q}(D) \subset C_0^{1,\alpha}(\overline{D})$ when $q > n$, it follows that the map $\tilde{T} : f \mapsto u(x)$ (composition of T and pointwise evaluation at $x \in D$) is also continuous. Thus, we know by Theorem 2.2(d) that the maps $\tilde{T}$ and I agree whenever $f \in C_0^\infty(D)$. Since $C_0^\infty(D)$ is dense in $L^q(D)$ for all $n < q < \infty$, we know that $I(f)$ and $\tilde{T}(f)$ must agree for all $f \in L^q(D)$, when $q > n$. $\square$

For any parameter $\eta > 0$, we note that the restricted domain

$$D_{-\eta} := D \setminus \mathcal{N}_\eta(\partial D) = \{x \in D : \operatorname{dist}(x, \partial D) \geq \eta\}$$

is a compact subset of D. Thus, the following uniform bounds on the Green's function follow from regularity and positivity of G (away from the pole and away from the boundary ∂D).

Proposition 2.4. *Fix $\delta > 0$ such that $D_{-\delta} \neq \varnothing$ and consider Green's function $G(\bar{y}, \cdot)$ with pole at $\bar{y} \in D_{-\delta}$.*

(a) *For $\eta > 0$, there is a constant $\underline{G} = \underline{G}(n, D, \delta, \eta) > 0$ such that*

$$-G(x, \bar{y}) \geq \underline{G} \quad \textit{for } x \in D_{-\eta}.$$

(b) *There is a constant $\overline{G} = \overline{G}(n, D, \delta) > 0$ such that*

$$-G(x, \bar{y}) \leq \overline{G} \quad \textit{for } x \in D \setminus B_\delta(\bar{y}).$$

2.2. The homogeneous Dirichlet problem in D. Turning to the situation in (1.1) when Laplacian data is fixed (i.e., assuming $g_1 = g_2$), the difference $w = u_1 - u_2$ can be written as the sum of a solution to the inhomogeneous Dirichlet problem (2.1) and a harmonic function Θ satisfying a homogeneous equation of the form

$$\begin{cases} \Delta\Theta = 0 & \text{in } D, \\ \Theta = \phi & \text{on } \partial D. \end{cases} \tag{2.3}$$

To bound the function Θ and access the boundary data $\phi = \psi_1 - \psi_2$, we utilize harmonic measures and properties of Poisson kernels in Lipschitz domains. The sensitive dependence of solutions to boundary value problems and the regularity of the boundaries themselves has been an area of deep inquiry with contributions from many mathematicians. Although many great references can be included in this context, we refer the reader to [**6**] for a detailed development of the content necessary for our setting.

We first note that (2.3) is solvable for Lipschitz D and continuous ϕ:

Theorem 2.5. [**9,** Theorems 1.3.1, 1.3.2(3) and equation (1.3.6)] *Let D be a bounded Lipschitz domain. Given any $\phi \in C(\partial D)$, there exists a $\Theta \in C(\overline{D})$ satisfying* (2.3). *Moreover, for every $y \in D$ there exists a function $K(y, \cdot\,) \in C^\alpha(\partial D)$, for some $0 < \alpha < 1$, so that Θ satisfies the expression*

$$\Theta(y) = \int_{\partial D} \phi(x) K(y, x) d\sigma(x).$$

The function $K(y, \cdot\,)$ is the Poisson kernel on D, which can be defined in general as the Radon-Nikodym derivative of harmonic measure ω^y with respect to surface measure σ on ∂D. Other expressions for $K(y, \cdot\,)$ can also be found in [**6,** Corollaries 1.3.18 and 1.3.19], for instance. Moreover, by [**6,** Theorem 1.3.17] and the definition of kernel function, we conclude that $K(y, x) > 0$ whenever $y \notin \partial D$. Thus, by compactness of $D_{-\delta}$ and continuity of $K(y, \cdot\,)$ on ∂D, we derive the following bounds on K:

Proposition 2.6. *Fix $\delta > 0$ such that $D_{-\delta} \neq \varnothing$. Then there exist positive constants $\overline{K} = \overline{K}(n, D, \delta)$ and $\underline{K} = \underline{K}(n, D, \delta)$ such that*

$$\underline{K} \le K(y, \cdot\,) \le \overline{K} \quad \textit{for all } y \in D_{-\delta}. \tag{2.4}$$

3. Measure-theoretic changes to contact sets. We now proceed with the proof of Theorem 1.1. As a general overview, we first isolate cases where either the Laplacian or the boundary data are fixed. We prove results in each of these cases first, then we conclude the proof of our main result by applying standard ordering principles on solutions to the obstacle problem.

Lemma 3.1 (linear control with perturbed boundary data)**.** *Take u_i and $\bar{v}$ as in Theorem 1.1 and assume that $g = g_1 = g_2$.*

(a) (linear stability) *Suppose $\bar{y} \in D \cap \Lambda(\bar{v})$ with $dist(\bar{y}, \partial D) \geq \delta > 0$, and choose $\eta > 0$. Then*

$$|(\Lambda(u_1) \,\Delta\, \Lambda(u_2)) \cap D_{-\eta}| \leq \left(\frac{\overline{K}(n, D, \delta)}{\lambda\, \underline{G}(n, D, \eta, \delta)} \right) \|\psi_1 - \psi_2\|_{L^1(\partial D)}.$$

(b) (linear nondegeneracy) *Suppose $\psi_1 \geq \psi_2$ on ∂D and $B_\delta(\bar{y}) \subset \Lambda(u_1)$ for some $\delta > 0$. Then*

$$|\Lambda(u_1) \,\Delta\, \Lambda(u_2)| \geq \left(\frac{\underline{K}(n, D, \delta)}{\mu\, \overline{G}(n, D, \delta)} \right) \|\psi_1 - \psi_2\|_{L^1(\partial D)}.$$

Proof. (a) To prove linear stability, we define $\underline{v} := OP(Lap = g, Bdry = \min(\psi_1, \psi_2))$ and note that $\underline{v} \leq u_i \leq \bar{v}$ holds in D, $i = 1, 2$. Therefore, we have

$$\Lambda(u_1) \,\Delta\, \Lambda(u_2) \subset \Lambda(\underline{v}) \,\Delta\, \Lambda(\bar{v}) = \Lambda(\underline{v}) \setminus \Lambda(\bar{v}) =: \mathcal{L},$$

and it suffices to prove the desired bound for $\mathcal{L} \cap D_{-\eta}$.

Define the auxiliary function Θ solving

$$\begin{cases} \Delta\Theta = 0 & \text{in } D, \\ \Theta = |\psi_1 - \psi_2| & \text{on } \partial D, \end{cases} \tag{3.1}$$

and define $h := \bar{v} - \underline{v} - \Theta$. Note that h verifies $h(\bar{y}) = -\Theta(\bar{y})$ and

$$\begin{cases} \Delta h = \chi_{\mathcal{L}}\, g & \text{in } D, \\ h = 0 & \text{on } \partial D. \end{cases} \tag{3.2}$$

Since $\bar{y} \in D_{-\delta}$ and Θ solves (3.1), we apply Theorem 2.5 and Proposition 2.6 to conclude the existence of $\overline{K} > 0$ such that

$$\overline{K}\, \|\psi_1 - \psi_2\|_{L^1(\partial D)} \geq \int_{\partial D} |\psi_1 - \psi_2|\, K(\bar{y}, \cdot\,)\, d\sigma = \Theta(\bar{y}). \tag{3.3}$$

By Proposition 2.4(a) there exists $\underline{G} > 0$ such that $-G(\bar{y}, x) \geq \underline{G}$ for all $x \in D_{-\eta}$. Further, by $g \in L^\infty(D)$ and $\mathcal{L}$ measurable, we know that $\chi_{\mathcal{L}} g \in L^q(D)$ for any $q > n$, so combining (3.2) and Lemma 2.3, we compute

$$\begin{aligned}\Theta(\bar{y}) &= -h(\bar{y}) = -\int_{\mathcal{L}} g(x)\, G(\bar{y}, x)\, dx \\ &\geq \lambda \int_{\mathcal{L}} -G(\bar{y}, x)\, dx \geq \lambda \int_{\mathcal{L} \cap D_{-\eta}} -G(\bar{y}, x)\, dx \;\geq\; \lambda \underline{G}\, |\mathcal{L} \cap D_{-\eta}|\,.\end{aligned}$$

Together with (3.3), this completes the proof of (a).

(b) To prove linear nondegeneracy, we use the same tools as in the proof of (a), noting that $\psi_1 \geq \psi_2$ implies $\bar{v} = u_1$, $\underline{v} = u_2$, and $\mathcal{L} = \Lambda(u_1)\, \Delta\, \Lambda(u_2) = \Lambda(u_2) \setminus \Lambda(u_1)$ in this case. Also note that $h = u_1 - u_2 - \Theta$ satisfies (3.2).

By assumption that $B_\delta(\bar{y}) \subset \Lambda(u_1)$, we have $\Lambda(u_1)\, \Delta\, \Lambda(u_2) \subset D \setminus B_\delta(\bar{y})$ and so it follows from Proposition 2.4(b) that there exists $\overline{G} > 0$ such that $-G(\bar{y}, x) \leq \overline{G}$ for all $x \in \Lambda(u_1)\, \Delta\, \Lambda(u_2)$. Therefore, employing Theorem 2.5, Proposition 2.6, and Lemma 2.3, we compute

$$\begin{aligned}\underline{K}\, \|\psi_1 - \psi_2\|_{L^1(\partial D)} &\leq \int_{\partial D} |\psi_1 - \psi_2|\, K(\bar{y}, \cdot\,)\, d\sigma \\ &= \Theta(\bar{y}) = -h(\bar{y}) = -\int_{\mathcal{L}} g(x)\, G(\bar{y}, x)\, dx \\ &\leq \mu \int_{\mathcal{L}} -G(\bar{y}, x)\, dx \;\leq\; \mu\, \overline{G}\, |\mathcal{L}|,\end{aligned}$$

which completes the proof of (b). $\square$

Lemma 3.2 (linear control with perturbed right-hand side)**.** *Take u_i and $\bar{v}$ as in Theorem 1.1 and assume that $\psi = \psi_1 = \psi_2$. Assume $\delta, \eta > 0$ are fixed and $B_\delta(\bar{y}) \subset \Lambda(\bar{v})$ for some $\bar{y} \in D$.*

- (linear stability) *We have*

$$|(\Lambda(u_1)\, \Delta\, \Lambda(u_2)) \cap D_{-\eta}| \leq \left(\frac{\overline{G}(n, D, \delta)}{\lambda\, \underline{G}(n, D, \eta, \delta)} \right) \|g_1 - g_2\|_{L^1(\Omega(\bar{v}))}.$$

- (linear nondegeneracy) *If $g_1 \leq g_2$ holds in D then*

$$|\Lambda(u_1)\, \Delta\, \Lambda(u_2)| \geq \left(\frac{\underline{G}(n, D, \eta, \delta)}{\mu\, \overline{G}(n, D, \delta)} \right) \|g_1 - g_2\|_{L^1(\Omega(u_1) \cap D_{-\eta})}.$$

Proof. (a) For linear stability, we again define

$$\underline{v} := OP(Lap = \max(g_1, g_2), Bdry = \psi),$$

so that $\underline{v} \le u_i \le \bar{v}$ again holds; thus it suffices to prove the result for

$$\mathcal{L} := \Lambda(\underline{v}) \setminus \Lambda(\bar{v})\Lambda(\underline{v}) \,\Delta\, \Lambda(\bar{v}) \supset \Lambda(u_1) \,\Delta\, \Lambda(u_2).$$

We define the auxiliary function Φ solving

$$\begin{cases} \Delta\Phi = \chi_{\Omega(\bar{v})} |g_1 - g_2| & \text{in } D, \\ \Phi = 0 & \text{on } \partial D, \end{cases}$$

and define $h := \bar{v} - \underline{v} + \Phi$. It follows that $h(\bar{y}) = \Phi(\bar{y})$ and

$$\begin{cases} \Delta h = \chi_{\mathcal{L}} \max(g_1, g_2) & \text{in } D, \\ h = 0 & \text{on } \partial D. \end{cases} \tag{3.4}$$

Note that we have $\chi_{\mathcal{L}} \max(g_1, g_2) \in L^q(D)$ for any $q > n$, and the assumption on $\bar{y}$ ensures $\Omega(\bar{v}) \subset D \setminus B_\delta(\bar{y})$. Thus, we apply Proposition 2.4, Lemma 2.3, and $g_i \ge \lambda$ in D to compute

$$\begin{aligned} \overline{G}\, \|g_1 - g_2\|_{L^1(\Omega(\bar{v}))} &\ge -\int_{\Omega(\bar{v})} |g_1(x) - g_2(x)|\, G(\bar{y}, x)\, dx \\ &= -\Phi(\bar{y}) = -h(\bar{y}) \\ &= -\int_{\mathcal{L}} \max(g_1(x), g_2(x))\, G(\bar{y}, x)\, dx \\ &\ge \lambda \int_{\mathcal{L}} -G(\bar{y}, x)\, dx \\ &\ge \lambda \int_{\mathcal{L} \cap D_{-\eta}} -G(\bar{y}, x)\, dx \;\ge\; \lambda\, \underline{G}\, |\mathcal{L} \cap D_{-\eta}|\,, \end{aligned}$$

which completes the proof of (a).

(b) For linear nondegeneracy, we again use the tools introduced in the proof of (a). With $g_1 \le g_2$, we have $\bar{v} = u_1$, $\underline{v} = u_2$, and $\mathcal{L} = \Lambda(u_1) \,\Delta\, \Lambda(u_2) = \Lambda(u_2) \setminus \Lambda(u_2)$ in this case. Note that $h = u_1 - u_2 + \Phi$ satisfies (3.4) where $\max(g_1, g_2) = g_2$ in this case. Thus, applying

Proposition 2.4, Lemma 2.3 and $g_i \le \mu$ in D, we have

$$
\begin{aligned}
\underline{G}\,\|g_1 - g_2\|_{L^1(\Omega(\bar v)\cap D_{-\eta})} &\le -\int_{\Omega(\bar v)\cap D_{-\eta}} (g_2(x) - g_1(x))\, G(x,\bar y)\, dx \\
&\le -\int_{\Omega(\bar v)} (g_2(x) - g_1(x))\, G(x,\bar y)\, dx \\
&= -\Phi(\bar y) = -h(\bar y) \\
&= -\int_{\mathcal{L}} g_2(x)\, G(x,\bar y)\, dx \\
&\le \mu \int_{\mathcal{L}} -G(x,\bar y)\, dx \ \le\ \mu\, \overline{G}\, |\mathcal{L}|,
\end{aligned}
$$

which completes the proof of (b). □

3.1. Proof of Theorem 1.1 and Corollary 1.2. We conclude the note with a quick comment on bringing together the results from the preceding lemmata to prove Theorem 1.1 and Corollary 1.2.

Proof of Theorem 1.1. (a) Regarding linear stability, we recall $\bar v$ as defined in the statement of the theorem and further define

$$
\begin{aligned}
\underline{v} &= OP(Lap = \max(g_1, g_2), Bdry = \min(\psi_1, \psi_2)) \quad \text{and} \\
w &= OP(Lap = \max(g_1, g_2), Bdry = \max(\psi_1, \psi_2)).
\end{aligned}
$$

Notice that we can apply Lemma 3.1(a) to the set difference $\Lambda(w) \,\Delta\, \Lambda(\underline{v})$, while Lemma 3.2(a) applies to $\Lambda(\bar v) \,\Delta\, \Lambda(w)$. The proof of part (a) of Theorem 1.1 thus follows from these lemmata and the simple bound

$$
\begin{aligned}
&|(\Lambda(u_1) \,\Delta\, \Lambda(u_2)) \cap D_{-\eta}| \\
&\qquad \le |(\Lambda(\bar v) \,\Delta\, \Lambda(\underline{v})) \cap D_{-\eta}| \\
&\qquad \le |(\Lambda(\bar v) \,\Delta\, \Lambda(w)) \cap D_{-\eta}| + |(\Lambda(w) \,\Delta\, \Lambda(\underline{v})) \cap D_{-\eta}|.
\end{aligned}
$$

(b) Proving nondegeneracy follows in a similar manner, where here the function w satisfies

$$
w = OP(Lap = g_2, Bdry = \psi_1),
$$

due to the monotonicity assumptions on ψ_i and g_i. The result now follows by applying Lemma 3.1(b) to $\Lambda(u_1) \,\Delta\, \Lambda(w)$, applying Lemma 3.2(b) to $\Lambda(w) \,\Delta\, \Lambda(u_2)$, and noting that these sets form a disjoint decomposition of $\Lambda(u_1) \,\Delta\, \Lambda(u_2) = \Lambda(u_2) \setminus \Lambda(u_1)$ in this case. □

Proof of Corollary 1.2. By assumption $\Lambda(u_2) \subset D_{-\eta}$ and $\mathcal{L} = \Lambda(u_2) \setminus \Lambda(u_1)$, thus Theorem 1.1(a) produces the desired linear upper bound. Meanwhile, assuming that $g_1 - g_2$ is supported in $D_{-\eta}$, we have $\|g_1 - g_2\|_{L^1(\Omega(u_1)\cap D_{-\eta})} = \|g_1 - g_2\|_{L^1(\Omega(u_1))}$ and the desired linear lower bound then follows from Theorem 1.1(b). $\square$

REFERENCES

1. I. Blank, *Sharp results for the regularity and stability of the free boundary in the obstacle problem*, Indiana Univ. Math. J. **50** (2001), no. 3, 1077–1112.

2. L.A. Caffarelli, *A remark on the Hausdorff measure of a free boundary, and the convergence of coincidence sets*, Boll. Un. Mat. Ital. A (5) **18** (1981), no. 1, 109–113.

3. A. Friedman, *Variational principles and free-boundary problems*, R.E. Krieger, Malabar, FL (1988).

4. D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed., Springer (1983).

5. D. Jerison and C. Kenig, *The inhomogeneous Dirichlet problem in Lipschitz domains*, J. Funct. Anal. **130** (1995), 161–219.

6. C. Kenig, *Harmonic analysis techniques for second order elliptic boundary value problems*, Amer. Math. Soc. (1994).

7. J.-F. Rodrigues, *Obstacle problems in mathematical physics*, North-Holland Math. Studies **134**, North-Holland, Amsterdam (1987).

8. S. Serfaty and J. Serra, *Quantitative stability of the free boundary in the obstacle problem*, Anal. PDE **11** (2018), no. 7, 1803–1839.

9. M. Shaw, *Boundary value problems on Lipschitz domains in $\mathbb{R}^n$ or $\mathbb{C}^n$*, pp. 375–404 in *Geometric analysis of PDE and several complex variables*, Contemp. Math. **368**, Amer. Math. Soc. (2005).

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