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RANGE SPACES OF CO-ANALYTIC TOEPLITZ OPERATORS

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ABSTRACT. We discuss the range spaces of Toeplitz operators with co-analytic symbols where we focus on the boundary behavior of the functions in these spaces as well as a natural orthogonal decomposition of this range.

1. INTRODUCTION

In this paper we examine the range of co-analytic Toeplitz operators on the classical Hardy space H^2 of the open unit disk $\mathbb{D}$. In particular, we explore both the boundary behavior of functions in the range as well as a natural orthogonal decomposition of the range in a suitable Hilbert space structure.

To explain our results, let T_φ be the Toeplitz operator on H^2 with symbol $\varphi \in L^\infty$ and define its range space

$$\mathscr{M}(\varphi) := T_\varphi H^2.$$

This range space is endowed with the inner product $\langle \cdot, \cdot \rangle_\varphi$ defined by

$$\langle T_\varphi f, T_\varphi g \rangle_\varphi := \langle f, g \rangle_{H^2}, \quad f, g \in H^2 \ominus \operatorname{Ker} T_\varphi,$$

where $\langle \cdot, \cdot \rangle_{H^2}$ is the inner product in H^2. We remind the reader of some standard facts in the next section.

When $a \in H^\infty$, the bounded analytic functions on $\mathbb{D}$, and is outer, the co-analytic Toeplitz operator $T_{\overline{a}}$ is injective with dense range $\mathscr{M}(\overline{a})$ in H^2 (Proposition 2.3). In this case, the corresponding inner product $\langle \cdot, \cdot \rangle_{\overline{a}}$ on $\mathscr{M}(\overline{a})$ becomes

$$\langle T_{\overline{a}} f, T_{\overline{a}} g \rangle_{\overline{a}} = \langle f, g \rangle_{H^2}, \quad f, g \in H^2. \tag{1.1}$$

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Many properties of Toeplitz operators have been well investigated (see e.g. [3, 25, 26]). The less studied range spaces make important connections with the de Branges–Rovnyak spaces [13, 30], and the paper [23] characterizes the common range of the co-analytic Toeplitz operators. In this paper we begin a more focussed discussion of $\mathscr{M}(\overline{a})$ and its various properties.

Our first goal is to study the boundary behavior of functions in $\mathscr{M}(\overline{a})$. Functions, along with their derivatives, in the so-called sub-Hardy Hilbert spaces can have more regularity at particular ζ_0 on the unit circle $\mathbb{T}$ than generic functions in H^2. Broadly speaking, these type of results say that if certain conditions are satisfied, then *every* function in a given sub-Hardy Hilbert space has a non-tangential limit at a particular $\zeta_0 \in \mathbb{T}$.

As a specific example of these kind of results, suppose that I is an inner function factored (canonically) as $I = Bs_\mu$, where the first factor B is a Blaschke product with zeros $\{a_n\}_{n \geqslant 1} \subset \mathbb{D}$ while the second factor s_μ is a singular inner function with corresponding positive measure μ on $\mathbb{T}$ with $\mu \perp d\theta$ [7, 14]. One can define the well-studied model space

$$K_I := H^2 \ominus IH^2 = (IH^2)^\perp \tag{1.2}$$

[24, 25, 26]. A theorem of Ahern and Clark [1] says that if $\zeta_0 \in \mathbb{T}$ and $N \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, then every $f \in K_I$, along with the derivatives $f', \ldots, f^{(N)}$, has a finite non-tangential limit at ζ_0 if and only if

$$\sum_{n \geqslant 1} \frac{1 - |a_n|}{|\zeta_0 - a_n|^{2N+2}} + \int_{\mathbb{T}} \frac{d\mu(\xi)}{|\zeta_0 - \xi|^{2N+2}} < \infty. \tag{1.3}$$

This work was extended by Fricain and Mashreghi [11, 12] to the closely related de Branges-Rovnyak spaces $\mathscr{H}(b)$ (defined below), where b is in the closed unit ball H^∞_1 of H^∞, and factored (canonically) as $b = Bs_\mu b_0$, where Bs_μ is the inner factor of b and b_0 its outer factor. Here the necessary and sufficient condition that every $f \in \mathscr{H}(b)$, along with $f', \ldots, f^{(N)}$, has a finite non-tangential limit at ζ_0 becomes

$$\sum_{n \geqslant 1} \frac{1 - |a_n|}{|\zeta_0 - a_n|^{2N+2}} + \int_{\mathbb{T}} \frac{d\mu(\xi)}{|\zeta_0 - \xi|^{2N+2}} + \int_0^{2\pi} \frac{|\log |b(e^{i\theta})||}{|\zeta_0 - e^{i\theta}|^{2N+2}} d\theta \tag{1.4}$$

is finite. See [30, 2] for some related results.

The technique originally used by Ahern and Clark, and extended by others, to discover conditions like (1.3) was to control the norm of the reproducing kernels as one approached the boundary point $\zeta_0 \in \mathbb{T}$. We will explore this Ahern-Clark technique in a broader setting to not only capture the boundary behavior of functions in the range spaces $\mathscr{M}(\overline{a})$, the primary focus of this paper, but also the de Branges-Rovnyak spaces $\mathscr{H}(b)$, and even the harmonically weighted Dirichlet spaces $\mathscr{D}(\mu)$.

To describe the boundary behavior in $\mathscr{M}(\overline{a})$, we first observe that we can always assume that a is an outer function (Proposition 3.6). Furthermore, in Theorem 4.5 and Corollary 4.11, we will show that if $\zeta_0 \in \mathbb{T}$ and $N \in \mathbb{N}_0$, then every $f \in \mathscr{M}(\overline{a})$, along with $f', f'', \ldots, f^{(N)}$, has a finite non-tangential limit at ζ_0 if and only if

$$\int_0^{2\pi} \frac{|a(e^{i\theta})|^2}{|e^{i\theta} - \zeta_0|^{2N+2}} d\theta < \infty. \tag{1.5}$$

Obviously, the convergence of the integral in (1.5) depends on the strength of the zero of a at ζ_0. We will use this observation to show (Proposition 4.17) that there is no point $\zeta_0 \in \mathbb{T}$ for which every function in $\mathscr{M}(\overline{a})$ has an analytic continuation to an open neighborhood of ζ_0. This is in contrast to the model spaces K_I where, under certain circumstances, every function in K_I has an analytic continuation across a portion of $\mathbb{T}$ [6]. We point out that our boundary behavior results for $\mathscr{M}(\overline{a})$ make connections to similar types of results for $T_{\overline{a}}K_I$ [16].

To discuss the internal Hilbert space structure of $\mathscr{M}(\overline{a})$, we first observe (Proposition 3.8) that $\mathscr{M}(a) \subset \mathscr{M}(\overline{a})$ with contractive inclusion. The space $\mathscr{M}(a)$ has an obvious description as

$$aH^2 = \{af : f \in H^2\},$$

and we are interested in how $\mathscr{M}(a)$ completes to $\mathscr{M}(\overline{a})$ when $\mathscr{M}(a)$ is complemented in $\mathscr{M}(\overline{a})$. This happens when $\mathscr{M}(a)$ is closed in the topology of $\mathscr{M}(\overline{a})$, which takes place when the Toeplitz operator $T_{\overline{a}/a}$ is surjective (Proposition 5.9) [17]. In this case we have an orthogonal decomposition

$$\mathscr{M}(\overline{a}) = \mathscr{M}(a) \oplus_{\overline{a}} K$$

for some closed subspace K of $\mathscr{M}(\overline{a})$. Here $\oplus_{\overline{a}}$ denotes the orthogonal sum in the inner product $\langle \cdot, \cdot \rangle_{\overline{a}}$. To identify the summand K, we will show that

$$\mathscr{M}(\overline{a}) = \mathscr{M}(a) \oplus_{\overline{a}} T_{\overline{a}} \operatorname{Ker} T_{\overline{a}/a}$$

and then proceed to use the well developed theory of the kernels of Toeplitz operators from [17, 18, 19, 20, 21, 29] to identify, in certain cases, $T_{\overline{a}} \operatorname{Ker} T_{\overline{a}/a}$. Our previous results on the boundary behavior naturally come into play here. Indeed, when (1.5) is satisfied, then point evaluation kernels as well as their derivatives up to order N are elements of K (see Proposition 5.15) and, in certain situations, span the complementary space K (Corollary 5.16).

In particular, but not all, cases, the decomposition takes the form $\mathscr{M}(\overline{a}) = \mathscr{M}(a) \oplus_{\overline{a}} K_I$, where K_I is a model space corresponding to an inner function I associated with a.

Finally, we will use our techniques to generalize the results from [10, 22] to decompose the de Branges Rovnyak spaces $\mathscr{H}(b)$ for certain b (Theorem 5.17).

2. Some reminders

Let H^2 denote the classical Hardy space of the unit disk $\mathbb{D}$ [7, 14] endowed with the standard L^2 inner product

$$\langle f, g\rangle_{H^2} := \int_{\mathbb{T}} f\overline{g}dm,$$

where m is normalized Lebesgue measure on $\mathbb{T}$.

Recall that H^2 is a reproducing kernel Hilbert space with reproducing (Cauchy) kernel

$$k_\lambda(z) := \frac{1}{1-\overline{\lambda}z}, \quad \lambda, z \in \mathbb{D}, \tag{2.1}$$

meaning that

$$f(\lambda) = \langle f, k_\lambda\rangle_{H^2}, \quad f \in H^2, \lambda \in \mathbb{D}.$$

Let $P_+ : L^2 \to H^2$ the usual (orthogonal) *Riesz projection* given by the formula

$$(P_+ f)(\lambda) = \langle f, k_\lambda\rangle_{L^2}, \quad f \in L^2, \lambda \in \mathbb{D}.$$

If $n \in \mathbb{N}_0$, $\lambda \in \mathbb{D}$, and

$$k_{\lambda,n}(z) := \frac{n!z^n}{(1-\overline{\lambda}z)^{n+1}},$$

then $k_{\lambda,n}$ is the reproducing kernel for the n-th derivative at λ in that

$$f^{(n)}(\lambda) = \langle f, k_{\lambda,n}\rangle_{H^2}, \quad f \in H^2. \tag{2.2}$$

For a symbol $\varphi \in L^\infty$, the space of essentially bounded Lebesgue measurable functions on $\mathbb{T}$, define the *Toeplitz operator* T_φ on H^2 by

$$T_\varphi f := P_+(\varphi f), \quad f \in H^2.$$

When $\varphi \in H^\infty$, T_φ is called an *analytic Toeplitz operator* (sometimes called a Laurent operator), and is given by the simple formula $T_\varphi f = \varphi f$, while $T_\varphi^* = T_{\overline{\varphi}}$ is called a *co-analytic Toeplitz operator*.

We gather up the following useful facts about Toeplitz operators. See [13, 24, 25] for more details.

Proposition 2.3. *Let $\varphi, \psi \in L^\infty$.*

(1) If $\varphi \in H^\infty$, then $T_{\overline{\varphi}} k_\lambda = \overline{\varphi(\lambda)} k_\lambda$ for every $\lambda \in \mathbb{D}$.

(2) If $\varphi \in H^\infty$ and outer, then the Toeplitz operators $T_\varphi, T_{\overline{\varphi}}$, and $T_{\varphi/\overline{\varphi}}$ are injective.

(3) If at least one of φ, ψ belongs to H^∞, then $T_{\overline{\psi}} T_\varphi = T_{\overline{\psi}\varphi}$.

(4) If $\varphi \in H^\infty$ and I is the inner factor of φ, then

$$\operatorname{Ker} T_{\overline{\varphi}} = K_I.$$

(5) If $\varphi \in H^\infty$ and I is inner, then $T_{\overline{\varphi}} K_I \subset K_I$.

The kernel $\operatorname{Ker} T_\varphi$ of a Toeplitz operator has been well studied and will play an important role in our orthogonal decomposition. Let us recall a few results in this area. A closed linear subspace M of H^2 is said to be *nearly invariant* if

$$f \in M, \quad f(0) = 0 \implies \frac{f}{z} \in M.$$

We will only consider the non-trivial nearly invariant subspaces of H^2: $\{0\} \subsetneq M \subsetneq H^2$.

Theorem 2.4 (Hitt [21], Sarason [29])**.** *Let M be a non-trivial nearly invariant subspace of H^2. If γ is the unique solution to the extremal problem*

$$\sup\{\Re g(0) : g \in M, \|g\|_{H^2} \leqslant 1\},$$

then there is an inner function I with $I(0) = 0$ such that

$$M = \gamma K_I.$$

Furthermore, γ is an isometric multiplier from K_I onto γK_I and can be written as

$$\gamma = \frac{\alpha}{1 - \beta_0 I},$$

where $\alpha, \beta_0 \in H^\infty$ and $|\alpha|^2 + |\beta_0|^2 = 1$ a.e. on $\mathbb{T}$.

Conversely, every space of the form $M = \gamma K_I$, with

$$\gamma = \frac{\alpha}{1 - I\beta_0},$$

$\alpha, \beta_0 \in H^\infty$, $|\alpha|^2 + |\beta_0|^2 = 1$ a.e. on $\mathbb{T}$, and I inner with $I(0) = 0$, is nearly invariant with associated extremal function γ.

The parameters γ and $\beta = I\beta_0$ are related by the following formula from [29]:

$$\frac{1+\beta(z)}{1-\beta(z)} = \int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z} |\gamma(\zeta)|^2 dm(\zeta), \quad z \in \mathbb{D}. \tag{2.5}$$

Clearly, when $\varphi \in L^\infty$, then $\operatorname{Ker} T_\varphi$ is nearly invariant. Hayashi identified those nearly invariant subspaces which are kernels of Toeplitz operators. With the notation from Theorem 2.4, set

$$\gamma_0 := \frac{\alpha}{1-\beta_0}.$$

Theorem 2.6 (Hayashi [20])**.** *A non trivial nearly invariant subspace M is the kernel of a Toeplitz operator if and only if γ_0^2 is rigid in H^1.*

The H^1 function γ_0^2 is said to be *rigid* if the only H^1 functions having the same argument as γ_0^2 almost everywhere on $\mathbb{T}$ are $\{c\gamma_0^2 : c > 0\}$. One can show that if g and $1/g$ both belong to H^1 then g is rigid. The converse is not always true.

Observe that the extremal function for the kernel of a Toeplitz operator is necessarily outer (one can always divide out the inner factor). In particular, for this situation, α is always outer.

If γ is the extremal function for $\operatorname{Ker} T_\varphi$, with associated inner function I, then

$$\operatorname{Ker} T_\varphi = \gamma K_I = \operatorname{Ker} T_{\overline{I\gamma}/\gamma}. \tag{2.7}$$

Note that when γ_0^2 is rigid, then $T_{\overline{\gamma_0}/\gamma_0}$ is injective [30, Theorem X-2]. In this paper we will also need the stronger property, namely the invertibility of $T_{\overline{\gamma_0}/\gamma_0}$. This is characterized in [17] by the well-known (A_2)-condition.

Theorem 2.8. *With the notation above, suppose that $\operatorname{Ker} T_\varphi \neq \{0\}$. Then the Toeplitz operator T_φ is surjective if and only if $|\gamma_0|^2$ is an (A_2) weight, meaning*

$$\sup_J \left(\frac{1}{J}\int_J |\gamma_0|^2 dm\right)\left(\frac{1}{J}\int_J |\gamma_0|^{-2} dm\right) < \infty, \tag{2.9}$$

where the supremum above is taken over all arcs $J \subset \mathbb{T}$.

3. Range spaces

For a bounded linear operator $A : H^2 \to H^2$, define the *range space*

$$\mathscr{M}(A) := AH^2$$

and endow it with the *range norm*

$$\|Af\|_{\mathscr{M}(A)} := \|f\|_{H^2}, \quad f \in H^2 \ominus \operatorname{Ker} A. \tag{3.1}$$

The induced inner product

$$\langle Af, Ag\rangle_{\mathscr{M}(A)} := \langle f, g\rangle_{\mathscr{M}(A)}, \quad f, g \in H^2 \ominus \operatorname{Ker} A$$

makes $\mathscr{M}(A)$ a Hilbert space and makes A a partial isometry with initial space $H^2 \ominus \operatorname{Ker} A$ and final space AH^2. In fact, using the identity $(\operatorname{Ker} A)^{\perp} = (\operatorname{Rng} A^*)^{-}$, we see that

$$\langle f, AA^*g\rangle_{\mathscr{M}(A)} = \langle f, g\rangle_{H^2}, \quad f \in \mathscr{M}(A), \, g \in H^2. \tag{3.2}$$

These range spaces $\mathscr{M}(A)$, as well as their complementary spaces, were formally introduced by Sarason [30] though they appeared earlier in the context of square summable power series in the work of de Branges and Rovnyak [4, 5]. We will discuss this connection in a moment.

Since $\mathscr{M}(A)$ is boundedly contained in H^2, meaning that the inclusion operator is bounded, we see that for fixed $n \in \mathbb{N}_0$ and $\lambda \in \mathbb{D}$, the linear functional $f \mapsto f^{(n)}(\lambda)$ is continuous on $\mathscr{M}(A)$. By the Riesz representation theorem, this functional is given by a reproducing kernel $k_{\lambda,n}^{\mathscr{M}(A)} \in \mathscr{M}(A)$, that is to say,

$$f^{(n)}(\lambda) = \langle f, k_{\lambda,n}^{\mathscr{M}(A)}\rangle_{\mathscr{M}(A)}, \qquad f \in \mathscr{M}(A).$$

Proposition 3.3. *For fixed $\lambda \in \mathbb{D}$ and $n \in \mathbb{N}_0$, we have*

$$k_{\lambda,n}^{\mathscr{M}(A)} = AA^* k_{\lambda,n}.$$

Proof. For any $f \in \mathscr{M}(A)$, use (3.2) along with (2.2) to get

$$\langle f, AA^* k_{\lambda,n}\rangle_{\mathscr{M}(A)} = \langle f, k_{\lambda,n}\rangle_{H^2} = f^{(n)}(\lambda). \quad \square$$

When A is a co-analytic Toeplitz operator $T_{\overline{a}}$ ($a \in H^\infty$), we obtain a special form for the reproducing kernel.

Corollary 3.4. *For each $\lambda \in \mathbb{D}$ and $n \in \mathbb{N}_0$ we have*

$$k_{\lambda,n}^{\mathscr{M}(T_{\overline{a}})} = T_{\overline{a}} a k_{\lambda,n} = T_{|a|^2} k_{\lambda,n}.$$

Proof. Observe that $T_{\overline{a}}^* = T_a$ and apply Proposition 3.3 and Proposition 2.3(2). $\square$

Remark 3.5. Since the range space of a co-analytic Toeplitz operator is the primary focus on this paper, we will use the less cumbersome notation

$$\mathscr{M}(a) := \mathscr{M}(T_a), \qquad \mathscr{M}(\overline{a}) := \mathscr{M}(T_{\overline{a}}),$$
$$\langle \cdot, \cdot \rangle_{\overline{a}} := \langle \cdot, \cdot \rangle_{\mathscr{M}(T_{\overline{a}})},$$
$$k^{\overline{a}}_{\lambda,n} := k^{\mathscr{M}(T_{\overline{a}})}_{\lambda,n}, \qquad k^{\overline{a}}_{\lambda} := k^{\overline{a}}_{\lambda,0}.$$

Let us mention a few more structural details concerning $\mathscr{M}(\overline{a})$. For any $a \in H^\infty$, let a_0 be the outer factor of a.

Proposition 3.6. [13, Corollary 16.8] *$\mathscr{M}(\overline{a}) = \mathscr{M}(\overline{a_0})$ as Hilbert spaces.*

Remark 3.7. Thus, when discussing $\mathscr{M}(\overline{a})$ spaces, we can always assume that $a = a_0$ is outer.

Proposition 3.8. [13, 30] *For $a \in H^\infty$ we have $\mathscr{M}(a) \subset \mathscr{M}(\overline{a})$ and the inclusion is contractive.*

The previous proposition can be seen from from the simple identity $T_a = T_{\overline{a}} T_{a/\overline{a}}$ which we will use later.

To connect the results of this paper with those of [10, 22], let us briefly recall some facts about the de Branges-Rovnyak spaces [13, 30]. For $b \in H^\infty_1 = \{f \in H^\infty : \|f\|_\infty \leqslant 1\}$, the closed unit ball in H^∞, define

$$A := (I - T_b T_{\overline{b}})^{1/2}.$$

The *de Branges-Rovnyak space $\mathscr{H}(b)$* is defined to be

$$\mathscr{H}(b) := \mathscr{M}(A), \tag{3.9}$$

endowed with the range norm from (3.1).

Remark 3.10. In a similar vein to Remark 3.5, we set

$$\langle \cdot, \cdot \rangle_b := \langle \cdot, \cdot \rangle_{\mathscr{M}(A)}, \qquad k^b_{\lambda,n} := k^{\mathscr{M}(A)}_{\lambda,n}, \qquad k^b_\lambda := k^b_{\lambda,0},$$

when $A = (I - T_b T_{\overline{b}})^{1/2}$ and $n \in \mathbb{N}_0$.

When $\|b\|_\infty < 1$ it turns out that $\mathscr{H}(b) = H^2$ with an equivalent norm. When $b = I$ is an inner function, then $\mathscr{H}(I) = K_I$ is one of the model spaces from (1.2) endowed with the H^2 norm.

Suppose $a \in H^\infty_1$ is outer and satisfies $\log(1 - |a|) \in L^1 = L^1(\mathbb{T}, m)$. This log integrability condition is equivalent to the fact that a is a *non-extreme* point of H^∞_1. Let b be the outer function, unique if we require the additional condition that $b(0) > 0$, which satisfies

$$|a|^2 + |b|^2 = 1 \quad \text{a.e. on } \mathbb{T}.$$

We call b, necessarily in H^∞_1, the *Pythagorean mate* for a. If $\mathscr{H}(b)$ is the associated de Branges-Rovnyak space from (3.9), it is known [30, p. 24] that

$$\mathscr{M}(a) \subset \mathscr{M}(\overline{a}) \subset \mathscr{H}(b),$$

though neither $\mathscr{M}(a)$ nor $\mathscr{M}(\overline{a})$ is necessarily closed in $\mathscr{H}(b)$. Still, $\mathscr{M}(\overline{a})$ is always dense in $\mathscr{H}(b)$. Furthermore, when (a, b) is a *corona pair*, that is to say,

$$\inf\{|a(z)| + |b(z)| : z \in \mathbb{D}\} > 0, \tag{3.11}$$

then $\mathscr{H}(b) = \mathscr{M}(\overline{a})$ [13, Theorem 28.7] or [30]. The equality $\mathscr{M}(\overline{a}) = \mathscr{H}(b)$ is a set equality but the norms, though equivalent by the closed graph theorem, need not be equal.

4. Boundary behavior in sub-Hardy Hilbert spaces

While the focus of this paper is the boundary behavior of functions in $\mathscr{M}(\overline{a})$, or more generally the range spaces $\mathscr{M}(A)$, our discussion of boundary behavior can be broadened to a class of "admissible" reproducing kernel Hilbert spaces of analytic functions on $\mathbb{D}$.

To get started, let $\mathscr{H}$ be a Hilbert space of analytic functions on $\mathbb{D}$ with norm $\|\cdot\|_{\mathscr{H}}$ such that for each $\lambda \in \mathbb{D}$, the evaluation functional $f \mapsto f(\lambda)$ is continuous on $\mathscr{H}$. By the Riesz representation theorem, there is a $k^{\mathscr{H}}_\lambda \in \mathscr{H}$ such that

$$f(\lambda) = \langle f, k^{\mathscr{H}}_\lambda \rangle_{\mathscr{H}}.$$

This function $k^{\mathscr{H}}_\lambda(z)$, called the *reproducing kernel* for $\mathscr{H}$, is an analytic function of z and a co-analytic function of λ. The space $\mathscr{H}$ with such a kernel function is called a *reproducing kernel Hilbert space* [27].

For each $n \in \mathbb{N}_0$ it follows that the linear functional $f \mapsto f^{(n)}(\lambda)$ is also continuous on $\mathscr{H}$ and thus given by a reproducing kernel $k^{\mathscr{H}}_{\lambda,j} \in \mathscr{H}$:

$$f^{(j)}(\lambda) = \langle f, k^{\mathscr{H}}_{\lambda,j} \rangle_{\mathscr{H}}, \quad f \in \mathscr{H}, \lambda \in \mathbb{D}.$$

A brief argument from [13, p. 911] will show that

$$k^{\mathscr{H}}_{\lambda,j} = \frac{\partial^j}{\partial \overline{\lambda}^j} k^{\mathscr{H}}_\lambda. \tag{4.1}$$

When $j = 0$ we set $k^{\mathscr{H}}_\lambda := k^{\mathscr{H}}_{\lambda,0}$.

Define the following linear transformations T and B on $\mathscr{O}(\mathbb{D})$ (the vector space of analytic functions on $\mathbb{D}$) by

$$(Tf)(z) = zf(z), \quad (Bf)(z) = \frac{f(z) - f(0)}{z}.$$

Observe that $S := T|_{H^2}$ is the well-known unilateral shift operator on H^2 and $S^* = B|_{H^2}$ is the equally well-known backward shift. Observe further that, in terms of Toeplitz operators on H^2, we have $S = T_z$ and $S^* = T_{\overline{z}}$.

Definition 4.2. A reproducing kernel Hilbert space $\mathscr{H}$ of analytic functions on $\mathbb{D}$ satisfying the two conditions

(1) $B\mathscr{H} \subset \mathscr{H}$ and $\|B\|_{\mathscr{H}\to\mathscr{H}} \leqslant 1$,

(2) $\sigma_p(X_{\mathscr{H}}^*) \subset \mathbb{D}$, where $X_{\mathscr{H}} := B|_{\mathscr{H}}$,

will be called *admissible*. In the above, $\sigma_p(X_{\mathscr{H}}^*)$ is the point spectrum of the operator $X_{\mathscr{H}}^*$.

We will discuss some examples, such as $\mathscr{M}(\overline{a})$, $\mathscr{H}(b)$, and $\mathscr{D}(\mu)$ towards the end of this section.

The following result, valid beyond the setting of admissible spaces (see [13, p. 912] for an alternate proof given in terms of $\mathscr{H}(b)$ spaces), gives us a useful formula for the reproducing kernels $k_{\lambda,j}^{\mathscr{H}}$.

Lemma 4.3. *Let $\mathscr{H}$ be a reproducing kernel Hilbert space of analytic functions on $\mathbb{D}$ such that $B\mathscr{H} \subset \mathscr{H}$ and $\|B\| \leqslant 1$. Then for each $j \in \mathbb{N}_0$ and $\lambda \in \mathbb{D}$ we have*

$$k_{\lambda,j}^{\mathscr{H}} = j!(I - \overline{\lambda}X_{\mathscr{H}}^*)^{-(j+1)} {X_{\mathscr{H}}^*}^j k_0^{\mathscr{H}}. \tag{4.4}$$

Proof. We first establish (4.4) when $j = 0$. Since B is a contraction, the operator $(I - \overline{\lambda}X_{\mathscr{H}}^*)$ is invertible when $\lambda \in \mathbb{D}$ and the formula in (4.4), for $j = 0$, is equivalent to the identity $(I - \overline{\lambda}X_{\mathscr{H}}^*)k_\lambda^{\mathscr{H}} = k_0^{\mathscr{H}}$. Observe how this identity holds if and only if for every $f \in \mathscr{H}$,

$$\langle f, (I - \overline{\lambda}X_{\mathscr{H}}^*)k_\lambda^{\mathscr{H}} \rangle_{\mathscr{H}} = \langle f, k_0^{\mathscr{H}} \rangle_{\mathscr{H}} = f(0).$$

To prove this last identity, observe that

$$\begin{aligned}\langle f, (I - \overline{\lambda} X_{\mathscr{H}}^*) k_\lambda^{\mathscr{H}} \rangle_{\mathscr{H}} &= \langle f, k_\lambda^{\mathscr{H}} \rangle_{\mathscr{H}} - \lambda \langle f, X_{\mathscr{H}}^* k_\lambda^{\mathscr{H}} \rangle_{\mathscr{H}} \\ &= f(\lambda) - \lambda \langle X_{\mathscr{H}} f, k_\lambda^{\mathscr{H}} \rangle_{\mathscr{H}} \\ &= f(\lambda) - \lambda \frac{f(\lambda) - f(0)}{\lambda} \\ &= f(0).\end{aligned}$$

This proves (4.4) when $j = 0$.

The formula for $k_{\lambda,j}^{\mathscr{H}}$ now follows from (4.1) by differentiating the identity

$$k_\lambda^{\mathscr{H}} = (I - \overline{\lambda} X_{\mathscr{H}}^*)^{-1} k_0^{\mathscr{H}}$$

j times with respect to the variable $\overline{\lambda}$. $\square$

We are now ready to state the main result of this section. For fixed $\zeta_0 \in \mathbb{T}$ and $\alpha > 1$ let

$$\Gamma_\alpha(\zeta_0) := \big\{ z \in \mathbb{D} : |z - \zeta_0| < \alpha(1 - |z|) \big\}$$

be a standard Stolz domain anchored at ζ_0. We say that an $f \in \mathscr{O}(\mathbb{D})$ has a finite *non-tangential limit* L at ζ_0 if $f(z) \to L$ whenever $z \to \zeta_0$ within any Stolz domain $\Gamma_\alpha(\zeta_0)$. When $\alpha = 1$, $\Gamma_1(\zeta_0)$ degenerates to the radius connecting 0 and ζ_0 and the limit within $\Gamma_1(\zeta_0)$ becomes a radial limit. The non-tangential limit L is denoted by $L = f(\zeta_0)$.

The following result was inspired by an operator theory result of Ahern and Clark [1] where they discussed non-tangential limits of functions in the classical model spaces K_I.

Theorem 4.5. *Let $\mathscr{H}$ be an admissible space, $\zeta_0 \in \mathbb{T}$, and $N \in \mathbb{N}_0$. Then the following are equivalent:*

(i) For every $f \in \mathscr{H}$, the functions $f, f', f'', \ldots, f^{(N)}$ have finite non-tangential limits at ζ_0.

(ii) For each fixed $\alpha > 1$, we have

$$\sup\{\|k_{\lambda,N}^{\mathscr{H}}\|_{\mathscr{H}} : \lambda \in \Gamma_\alpha(\zeta_0)\} < \infty.$$

(iii) ${X_{\mathscr{H}}^*}^N k_0^{\mathscr{H}} \in \operatorname{Rng}(I - \overline{\zeta}_0 X_{\mathscr{H}}^*)^{N+1}$.

Moreover, if any one of the above equivalent conditions hold then

$$(I - \overline{\zeta_0} X_{\mathscr{H}}^*)^{N+1} k_{\zeta_0,N}^{\mathscr{H}} = N! {X_{\mathscr{H}}^*}^N k_0^{\mathscr{H}}, \tag{4.6}$$

where $k^{\mathscr{H}}_{\zeta_0,N} \in \mathscr{H}$ *and satisfies*

$$f^{(N)}(\zeta_0) = \langle f, k^{\mathscr{H}}_{\zeta_0,N}\rangle_{\mathscr{H}}, \qquad f \in \mathscr{H}.$$

The proof of this requires the following technical lemma from [13, Cor. 21.22] (see also [11]) which generalizes an operator theory result of Ahern and Clark [1].

Lemma 4.7. *Let T be a contraction on a Hilbert space $\mathscr{H}$, $\zeta \in \mathbb{T}$, and $\{\lambda_n\}_{n\geqslant 1} \subset \mathbb{D}$ with the following properties:*

(1) $(I - \zeta T)$ is injective;

(2) λ_n tends to ζ non-tangentially as $n \to \infty$.

Let $x \in \mathscr{H}$ and $p \in \mathbb{N}$. Then the sequence

$$\left\{(I - \lambda_n T)^{-p} x\right\}_{n\geqslant 1}$$

is uniformly bounded if and only if $x \in \operatorname{Rng}(I - \zeta T)^p$, in which case,

$$(I - \lambda_n T)^{-p} x \to (I - \zeta T)^{-p} x$$

weakly in $\mathscr{H}$.

Proof of Theorem 4.5. $(i) \implies (ii)$: Since the norms of the reproducing kernels $k^{\mathscr{H}}_{\lambda,N}$ are the norms of the evaluation functionals $f \mapsto f^{(N)}(\lambda)$, we can apply the uniform boundedness principle to see, for fixed $\alpha > 1$, that if the N-th derivative of every function in $\mathscr{H}$ has a finite limit as $\lambda \to \zeta_0$ with $\lambda \in \Gamma_\alpha(\zeta_0)$, then the norms of the kernels $k^{\mathscr{H}}_{\lambda,N}$ are uniformly bounded for $\lambda \in \Gamma_\alpha(\zeta_0)$.

$(ii) \implies (iii)$: By Lemma 4.3, the vectors

$$(I - \overline{z_n} X^*_{\mathscr{H}})^{-(N+1)} {X^*_{\mathscr{H}}}^N k_0^{\mathscr{H}}$$

are uniformly bounded for any sequence $\{z_n\}_{n\geqslant 1} \subset \Gamma_\alpha(\zeta_0)$ tending to ζ_0. By our assumption $\sigma_p(X^*_{\mathscr{H}}) \subset \mathbb{D}$ (Definition 4.2) we see that the operator $I - \overline{\zeta_0} X^*_{\mathscr{H}}$ is injective. Now apply Lemma 4.7 to conclude that ${X^*_{\mathscr{H}}}^N k_0^{\mathscr{H}} \in \operatorname{Rng}(I - \overline{\zeta}_0 X^*_{\mathscr{H}})^{N+1}$.

$(iii) \implies (i)$: Again using Lemma 4.7, we see that

$$(I - \overline{z_n} X^*_{\mathscr{H}})^{-(N+1)} {X^*_{\mathscr{H}}}^N k_0^{\mathscr{H}} \to (I - \overline{\zeta_0} X^*_{\mathscr{H}})^{-(N+1)} {X^*_{\mathscr{H}}}^N k_0^{\mathscr{H}}$$

weakly for any sequence $\{z_n\}_{n\geqslant 1} \subset \Gamma_\alpha(\zeta_0)$ tending to ζ_0. However, Lemma 4.3 says that the left hand side of the identity above is precisely $\frac{1}{N!} k^{\mathscr{H}}_{z_n,N}$. Hence, for any $f \in \mathscr{H}$, the N-th derivative $f^{(N)}(z_n)$ has a finite limit as z_n tends to ζ_0 within $\Gamma_\alpha(\zeta_0)$.

To see that the lower order derivatives $f, f', f'', \ldots, f^{(N-1)}$ have finite non-tangential limits at ζ_0, use an argument from the proof of Theorem 21.26 in [13].

Finally, the equivalent conditions of the theorem show that the linear functional $f \mapsto f^{(N)}(\zeta_0)$ is continuous on $\mathscr{H}$ and thus, by the Riesz representation theorem, it is induced by a kernel $k^{\mathscr{H}}_{\zeta_0,N} \in \mathscr{H}$ satisfying

$$(I - \overline{\zeta_0} X^*_{\mathscr{H}})^{-(N+1)} {X^*_{\mathscr{H}}}^N k^{\mathscr{H}}_0 = \frac{1}{N!} k^{\mathscr{H}}_{\zeta_0,N}.$$

This proves (4.6). □

This next result helps us to produce a large class of admissible reproducing kernel Hilbert spaces.

Lemma 4.8. *Let $\mathscr{H}$ be a B-invariant reproducing kernel Hilbert space of analytic functions on $\mathbb{D}$ such that the analytic polynomials are dense in $\mathscr{H}$. Then $\sigma_p(X^*_{\mathscr{H}}) = \emptyset$. In particular, if $X_{\mathscr{H}} = B|_{\mathscr{H}}$ acts as a contraction on $\mathscr{H}$, then $\mathscr{H}$ is an admissible space.*

Proof. Suppose $\lambda \in \mathbb{C}$ and $f \in \mathscr{H} \setminus \{0\}$ with $X^*_{\mathscr{H}} f = \lambda f$. On one hand, $\langle X^*_{\mathscr{H}} f, z^n \rangle_{\mathscr{H}} = \lambda \langle f, z^n\rangle_{\mathscr{H}}$, while on the other hand,

$$\langle X^*_{\mathscr{H}} f, z^n \rangle_{\mathscr{H}} = \langle f, X_{\mathscr{H}} z^n \rangle_{\mathscr{H}} = \langle f, z^{n-1}\rangle_{\mathscr{H}}, \quad n \geqslant 1.$$

Combining these two facts yields

$$\lambda \langle f, z^n\rangle_{\mathscr{H}} = \langle f, z^{n-1}\rangle_{\mathscr{H}} \quad n \geqslant 1. \tag{4.9}$$

If $\lambda = 0$, the previous identity shows that $\langle f, z^k\rangle_{\mathscr{H}} = 0$ for all $k \geqslant 0$. By the density of the polynomials in $\mathscr{H}$ we see that $f = 0$ – a contradiction.

If $\lambda \neq 0$ then

$$\lambda \langle f, 1\rangle_{\mathscr{H}} = \langle X^*_{\mathscr{H}} f, 1\rangle_{\mathscr{H}} = \langle f, X_{\mathscr{H}} 1\rangle_{\mathscr{H}} = 0$$

and thus $\langle f, 1\rangle_{\mathscr{H}} = 0$. Use this last identity and repeatedly apply (4.9) to see that $\langle f, z^k\rangle_{\mathscr{H}} = 0$ for all $k \geqslant 0$. Again, by our assumption that the polynomials are dense in $\mathscr{H}$, we see that $f = 0$. □

Remark 4.10. If $\mathscr{H}$ contains all of the Cauchy kernels k_w, $w \in \mathbb{D}$ (see (2.1)), then we can use the fact that $X_{\mathscr{H}} k_w = \overline{w} k_w$ to replace the identity in (4.9) with $\lambda \langle f, k_w\rangle_{\mathscr{H}} = w \langle f, k_w\rangle_{\mathscr{H}}$. Thus the hypothesis "the polynomials are dense in $\mathscr{H}$" in Lemma 4.8 can be replaced with "the linear span of Cauchy kernels are dense in $\mathscr{H}$". We would like to thank Omar El Fallah for some fruitful discussions concerning an earlier version of this result.

Here are three applications of Theorem 4.5.

$\mathscr{M}(\overline{a})$-spaces. For $a \in H^\infty$ we want to show that $\mathscr{M}(\overline{a})$ is admissible. By Proposition 3.6 we can assume that a is outer. To verify that $\mathscr{M}(\overline{a})$ is admissible, we will check the hypothesis of Lemma 4.8. It is clear that $\mathscr{M}(\overline{a})$ is B-invariant (use the identity $T_{\overline{z}}T_{\overline{a}} = T_{\overline{a}}T_{\overline{z}}$ from Proposition 2.3(3)).

To show that $B = T_{\overline{z}}$ is contractive on $\mathscr{M}(\overline{a})$, notice that for any $g \in H^2$ we have

$$\|BT_{\overline{a}}g\|_{\overline{a}} = \|T_{\overline{z}}T_{\overline{a}}g\|_{\overline{a}} = \|T_{\overline{a}}T_{\overline{z}}g\|_{\overline{a}} = \|T_{\overline{z}}g\|_{H^2} \leqslant \|g\|_{H^2} = \|T_{\overline{a}}g\|_{\overline{a}}.$$

Thus $\|B\|_{\mathscr{M}(\overline{a}) \to \mathscr{M}(\overline{a})} \leqslant 1$.

To finish, using Lemma 4.8 and Remark 4.10, we need to show that the Cauchy kernels k_λ belong to $\mathscr{M}(\overline{a})$ and have dense linear span. From Proposition 2.3(1) we have $k_\lambda = T_{\overline{a}}(k_\lambda/\overline{a(\lambda)}) \in \mathscr{M}(\overline{a})$. Furthermore, since $T_{\overline{a}}$ is a partial isometry from H^2 *onto* $\mathscr{M}(\overline{a})$, it maps a dense subset of H^2 onto a dense subset of $\mathscr{M}(\overline{a})$. Thus the density of the linear span of k_λ, $\lambda \in \mathbb{D}$, in $\mathscr{M}(\overline{a})$ follows from the well-known density of this span in H^2. We remark that one can also obtain the admissibility of $\mathscr{M}(\overline{a})$ by showing the density of the polynomials in $\mathscr{M}(\overline{a})$ [13, p. 745].

Using Theorem 4.5, we obtain the following explicit characterization of the boundary behavior for $\mathscr{M}(\overline{a})$.

Corollary 4.11. *Let $a \in H^\infty$, $\zeta_0 \in \mathbb{T}$, and $N \in \mathbb{N}_0$. Then for every $f \in \mathscr{M}(\overline{a})$, the functions $f, f', f'', \ldots, f^{(N)}$ have finite non-tangential limits at ζ_0 if and only if*

$$\int_0^{2\pi} \frac{|a(e^{it})|^2}{|e^{it} - \zeta_0|^{2N+2}} dt < \infty. \tag{4.12}$$

We will write $\zeta_0 \in (AC)_{\overline{a},N}$ if the condition (4.12) *holds. In this case, we have*

$$k^{\overline{a}}_{\zeta_0,\ell} = T_{\overline{a}}(ak_{\zeta_0,\ell}), \qquad 0 \leqslant \ell \leqslant N,$$

where

$$ak_{\zeta_0,\ell} = \ell! \frac{z^\ell a}{(1-\overline{\zeta_0}z)^{\ell+1}}.$$

Moreover, for each $\alpha > 1$ we have

$$\lim_{\substack{\lambda \to \zeta_0 \\ \lambda \in \Gamma_\alpha(\zeta_0)}} \|k^{\overline{a}}_{\lambda,\ell} - k^{\overline{a}}_{\zeta_0,\ell}\|_{\overline{a}} = 0.$$

Proof. Corollary 3.4 gives us
$$\|k^{\overline{a}}_{\lambda,N}\|^2_{\overline{a}} = (N!)^2 \int_0^{2\pi} \frac{|a(e^{it})|^2}{|e^{it}-\lambda|^{2N+2}} \frac{dt}{2\pi}.$$
If λ approaches ζ_0 from within a fixed Stolz domain $\Gamma_\alpha(\zeta_0)$, then
$$\frac{1}{|e^{it}-\lambda|} \leqslant \frac{\alpha+1}{|e^{it}-\zeta_0|}, \quad t \in [0, 2\pi],$$
and so
$$\frac{|a(e^{it})|^2}{|e^{it}-\lambda|^{2N+2}} \leqslant (\alpha+1)^{2N+2} \frac{|a(e^{it})|^2}{|e^{it}-\zeta_0|^{2N+2}}. \tag{4.13}$$
If
$$\int_0^{2\pi} \frac{|a(e^{it})|^2}{|e^{it}-\zeta_0|^{2N+2}} \frac{dt}{2\pi} < \infty$$
we see that
$$\sup\{\|k^{\overline{a}}_{\lambda,N}\|_{\overline{a}} : \lambda \in \Gamma_\alpha(\zeta_0)\} < \infty. \tag{4.14}$$
Now apply Theorem 4.5.

Conversely, if for every $f \in \mathscr{M}(\overline{a})$, the functions $f, f', f'', \dots, f^{(N)}$ have non-tangential limits at ζ_0, then Theorem 4.5 implies that for each fixed $\alpha > 1$, the condition (4.14) is satisfied. Thus
$$\sup_{\lambda \in \Gamma_\alpha(\zeta_0)} \int_0^{2\pi} \frac{|a(e^{it})|^2}{|e^{it}-\lambda|^{2N+2}} \frac{dt}{2\pi} < \infty.$$
By Fatou's Lemma
$$\int_0^{2\pi} \frac{|a(e^{it})|^2}{|e^{it}-\zeta_0|^{2N+2}} \frac{dt}{2\pi} \leqslant \liminf_{\substack{\lambda \to \zeta_0 \\ \lambda \in \Gamma_\alpha(\zeta_0)}} \int_0^{2\pi} \frac{|a(e^{it})|^2}{|e^{it}-\lambda|^{2N+2}} \frac{dt}{2\pi} < \infty.$$

Now let $\zeta_0 \in (AC)_{\overline{a},N}$. Then, for any $f = T_{\overline{a}} g \in \mathscr{M}(\bar{a})$ and $0 \leqslant \ell \leqslant N$, we have
$$\langle f, T_{\overline{a}}(ak_{\zeta_0,\ell})\rangle_{\overline{a}} = \langle g, ak_{\zeta_0,\ell}\rangle_{H^2}.$$
Note that $ak_{\lambda,\ell} \to ak_{\zeta_0,\ell}$ in H^2 as $\lambda \in \zeta_0$ from within $\Gamma_\alpha(\zeta_0)$. Indeed this is true pointwise and, by using the inequality in (4.13) and the dominated convergence theorem, we also have
$$\|ak_{\lambda,\ell}\|_{H^2} \to \|ak_{\zeta_0,\ell}\|_{H^2}$$
as $\lambda \to \zeta_0$ from within $\Gamma_\alpha(\zeta_0)$. By a standard Hilbert space argument we have
$$\|ak_{\lambda,\ell} - ak_{\zeta_0,\ell}\|_{H^2} \to 0. \tag{4.15}$$

The above analysis says that

$$\begin{aligned}\langle f, T_{\overline{a}}(ak_{\zeta_0,\ell})\rangle_{\overline{a}} &= \lim_{\substack{\lambda\to\zeta_0\\ \lambda\in\Gamma_\alpha(\zeta_0)}} \langle g, ak_{\lambda,\ell}\rangle_{H^2}\\ &= \lim_{\substack{\lambda\to\zeta_0\\ \lambda\in\Gamma_\alpha(\zeta_0)}} \langle f, T_{\overline{a}}ak_{\lambda,\ell}\rangle_{\overline{a}}.\end{aligned}$$

By Corollary 3.4, $T_{\overline{a}}(ak_{\lambda,\ell}) = k^{\overline{a}}_{\lambda,\ell}$, whence

$$\begin{aligned}\langle f, T_{\overline{a}}(ak_{\zeta_0,\ell})\rangle_{\overline{a}} &= \lim_{\substack{\lambda\to\zeta_0\\ \lambda\in\Gamma_\alpha(\zeta_0)}} \langle f, k^{\overline{a}}_{\lambda,\ell}\rangle_{\overline{a}}\\ &= \lim_{\substack{\lambda\to\zeta_0\\ \lambda\in\Gamma_\alpha(\zeta_0)}} f^{(\ell)}(\lambda)\\ &= f^{(\ell)}(\zeta_0)\\ &= \langle f, k^{\overline{a}}_{\zeta_0,\ell}\rangle_{\overline{a}},\end{aligned}$$

which proves that $k^{\overline{a}}_{\zeta_0,\ell} = T_{\overline{a}}(ak_{\zeta_0,\ell})$. Finally, from (4.15)

$$\|k^{\overline{a}}_{\lambda,\ell} - k^{\overline{a}}_{\zeta_0,\ell}\|_{\overline{a}} = \|ak_{\lambda,\ell} - ak_{\zeta_0,\ell}\|_{H^2} \to 0, \quad \lambda\to\zeta_0, \lambda\in\Gamma_\alpha(\zeta_0). \quad \square$$

Remark 4.16.

(1) In a general admissible space $\mathscr{H}$ we see that if

$$\sup\{\|k^{\mathscr{H}}_{\lambda,N}\|_{\mathscr{H}} : \lambda\in\Gamma_\alpha(\zeta_0)\} < \infty$$

for each $\alpha > 1$, then

$$k^{\mathscr{H}}_{\lambda,N} \to k^{\mathscr{H}}_{\zeta_0,N}$$

weakly in $\mathscr{H}$ as $\lambda\to\zeta_0$ non-tangentially. However, it is not immediately clear if we also have norm convergence of the kernels. Corollary 4.11 shows this is true when $\mathscr{H} = \mathscr{M}(\overline{a})$. See also [13] where this was shown to be true when $\mathscr{H}$ is one of the de Branges-Rovnyak spaces $\mathscr{H}(b)$.

(2) The condition (4.12) yields an estimate on the rate of decrease of the outer function a, along with its derivatives, at the distinguished point ζ_0. Indeed, using the facts that $(\zeta_0 - z)^{N+1}$ is an outer function, along with the condition (4.12), and Smirnov's theorem [7] (if the boundary function of an outer function belongs to L^2 then the function belongs to H^2), the function

$$h(z) := \frac{a(z)}{(z-\zeta_0)^{N+1}}$$

belongs to H^2. Recall the following standard estimates for the derivatives of $h \in H^2$:

$$|h^{(\ell)}(r\zeta)| = o((1-r)^{-\ell-\frac{1}{2}}), \quad r \to 1^-.$$

Thus Leibniz' formula yields

$$\begin{aligned} a^{(k)}(r\zeta_0) &= \sum_{\ell=0}^{k} \binom{k}{\ell} h^{(\ell)}(r\zeta_0) \frac{d^{k-\ell}}{dz^{k-\ell}} (z-\zeta_0)^{N+1}\Big|_{z=r\zeta_0} \\ &= o((1-r)^{N+\frac{1}{2}-\ell}), \quad r \to 1^-. \end{aligned}$$

In particular, we see that the functions $a, a', \ldots, a^{(N)}$ have radial (and even non-tangential) limits $a^{(\ell)}(\zeta_0)$ which vanish for each $0 \leqslant \ell \leqslant N$.

Corollary 4.11 yields the following interesting observation which shows a sharp difference between $\mathscr{M}(\overline{a})$ spaces and the model, or more generally, de Branges-Rovnyak spaces $\mathscr{H}(b)$. More precisely, when $\log(1 - |b|) \notin L^1$, it is sometimes the case that every function in $\mathscr{H}(b)$ can be analytically continued to an open neighborhood of a point $\zeta_0 \in \mathbb{T}$. For example, if b is an inner function and $\zeta_0 \in \mathbb{T}$ with

$$\liminf_{\lambda \to \zeta_0} |b(\lambda)| > 0,$$

then every $f \in \mathscr{H}(b)$ (which turns out to be a model space K_b) can be analytically continued to some open neighborhood Ω_{ζ_0} of ζ_0 (see [6, Cor. 3.1.8] for details). This phenomenon never happens in $\mathscr{M}(\overline{a})$.

Proposition 4.17. *There is no point $\zeta_0 \in \mathbb{T}$ such that every $f \in \mathscr{M}(\overline{a})$ can be analytically continued to some open neighborhood of ζ_0.*

Proof. Suppose there exists such a $\zeta_0 \in \mathbb{T}$ where every function in $\mathscr{M}(\overline{a})$ has an analytic continuation to an open neighborhood Ω_{ζ_0} of ζ_0. Then the function $a \in \mathscr{M}(a) \subset \mathscr{M}(\overline{a})$ would have an analytic continuation to Ω_{ζ_0} and thus could be expanded in a power series around ζ_0. If every function in $\mathscr{M}(\overline{a})$ had an analytic continuation to Ω_{ζ_0}, then every function in $\mathscr{M}(\overline{a})$, and its derivatives of all orders, would have finite non-tangential limits at ζ_0. In particular, the condition (4.12) would hold for every $N \in \mathbb{N}$ at ζ_0. By Remark 4.16, this would imply that all of the Taylor coefficients of a at ζ_0 would vanish, implying $a \equiv 0$ on $\mathbb{D}$. $\square$

$\mathscr{H}(b)$-spaces. We have seen that $\mathscr{H}(b)$ spaces are special cases of $\mathscr{M}(A)$-spaces. It turns out that they are admissible. Indeed, they are B-invariant reproducing kernel Hilbert spaces contained in H^2 with $\|B\|_{\mathscr{H}(b)\to\mathscr{H}(b)} \leqslant 1$ [13, Theorem 18.13]. Furthermore, $\sigma_p(X^*_{\mathscr{H}}) \subset \mathbb{D}$ [13, Theorem 18.26]. Thus Theorem 4.5 applies, allowing us to reproduce some of the results of [12]. In particular, the condition that every $f \in \mathscr{H}(b)$, along with $f', \ldots, f^{(N)}$, has a non-tangential limit at ζ_0 is equivalent to the condition that the norm of the reproducing kernels for $\mathscr{H}(b)$ are uniformly bounded in every Stolz domain anchored at ζ_0. The difficult part of [12] is to prove that the boundedness of the kernels is equivalent to the condition in (1.4).

Remark 4.18. As already mentioned in Section 3, if $a \in H^\infty_1$ is such that $\log(1-|a|) \in L^1$ and b is its (outer) Pythagorean mate, then we have $\mathscr{M}(\overline{a}) \subset \mathscr{H}(b)$. If $N \in \mathbb{N}_0$ and $\zeta_0 \in \mathbb{T}$ are such that for every $f \in \mathscr{H}(b)$, the functions $f, f', \ldots, f^{(N)}$ admit a finite non-tangential limit at ζ_0, then this is also true for every function $f \in \mathscr{M}(\overline{a})$. What is more surprising here is that the converse is true. This is a byproduct of Corollary 4.11 and [11, Theorem 3.2]. Indeed, since $|b|^2 = 1 - |a|^2$ almost everywhere on $\mathbb{T}$, we see (remembering b is outer) that condition (1.4) implies

$$\int_{\mathbb{T}} \frac{|\log(1-|a(\zeta)|^2)|}{|\zeta-\zeta_0|^{2N+2}}\, dm(\zeta) < \infty,$$

which is equivalent to

$$\int_0^{2\pi} \frac{|a(e^{it})|^2}{|e^{it}-\zeta_0|^{2N+2}} dt < \infty.$$

Thus the conditions in (1.4) and (4.12) are equivalent which shows that the existence of boundary derivatives for functions in $\mathscr{H}(b)$ and $\mathscr{M}(\overline{a})$ (in the case when b is outer) are equivalent.

$\mathscr{D}(\mu)$-spaces. For a positive finite Borel measure μ on $\mathbb{T}$ let

$$\varphi_\mu(z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|\xi - z|^2} d\mu(\xi), \quad z \in \mathbb{D},$$

be the Poisson integral of μ. The *harmonically weighted Dirichlet space* $\mathscr{D}(\mu)$ [9, 28] is the set of all $f \in \mathscr{O}(\mathbb{D})$ for which

$$\int_{\mathbb{D}} |f'|^2 \varphi_\mu dA < \infty,$$

where $dA = dxdy/\pi$ is normalized planar measure on $\mathbb{D}$. Notice that when μ is Lebesgue measure on $\mathbb{T}$, then $\varphi_\mu \equiv 1$ and $\mathscr{D}(\mu)$ becomes the classical Dirichlet space [9]. One can show that $\mathscr{D}(\mu) \subset H^2$ [28, Lemma 3.1] and the norm $\|\cdot\|_{\mathscr{D}(\mu)}$ satisfying

$$\|f\|^2_{\mathscr{D}(\mu)} := \|f\|^2_{H^2} + \int_{\mathbb{D}} |f'|^2 \varphi_\mu dA$$

makes $\mathscr{D}(\mu)$ into a reproducing kernel Hilbert space of analytic functions on $\mathbb{D}$. It is known that both the polynomials and the linear span of the Cauchy kernels form a dense subset of $\mathscr{D}(\mu)$ [28, Corollary 3.8].

The backward shift B is a well-defined contraction on $\mathscr{D}(\mu)$. Indeed, we have

$$\|zf\|_{\mathscr{D}(\mu)} \geqslant \|f\|_{\mathscr{D}(\mu)}, \quad f \in \mathscr{D}(\mu),$$

and the constant function 1 is orthogonal to $z\mathscr{D}(\mu)$ [28, Theorem 3.6]. Thus

$$\|f\|^2_{\mathscr{D}(\mu)} = \|f(0) + zBf\|^2_{\mathscr{D}(\mu)} = |f(0)|^2 + \|zBf\|^2_{\mathscr{D}(\mu)} \geqslant \|Bf\|^2_{\mathscr{D}(\mu)}.$$

We thank Stefan Richter for showing us this elegant argument. From Lemma 4.8 we see that $\mathscr{D}(\mu)$ is an admissible space.

Using a kernel function estimate from [8], one can show that if

$$\mu = \sum_{1 \leqslant j \leqslant n} c_j \delta_{\zeta_j}, \quad c_j > 0, \zeta_j \in \mathbb{T},$$

then each of the kernels

$$k^{\mathscr{D}(\mu)}_{r\zeta_j}, \quad 1 \leqslant j \leqslant n,$$

remains norm bounded as $r \to 1^-$. Thus the radial limits of every function from $\mathscr{D}(\mu)$ exist at each of the ζ_j. Other radial limit results along these lines can be stated in terms of an associated capacity for $\mathscr{D}(\mu)$ [8, 15].

5. An orthogonal decomposition

The goal of this last section is to determine, whenever it exists, the orthogonal complement of $\mathscr{M}(a)$ in $\mathscr{M}(\overline{a})$. We begin our discussion with a few interesting and representative examples.

Example 5.1. If I is inner, then $a := 1 + I$ is outer. Moreover, one can quickly verify that

$$\frac{a}{\overline{a}} = I \quad \text{a.e. on } \mathbb{T}.$$

Since IH^2 is a closed subspace of H^2 (multiplication by an inner function is an isometry on H^2), we see that $T_{a/\overline{a}} = T_I$ has closed range. Hence,

$$H^2 = T_{a/\overline{a}} H^2 \oplus_{H^2} (H^2 \ominus_{H^2} T_{a/\overline{a}} H^2) = T_{a/\overline{a}} H^2 \oplus_{H^2} \operatorname{Ker} T_{\overline{a}/a}. \tag{5.2}$$

Since a is outer, then $T_{\overline{a}}$ is injective (Proposition 2.3(2)) and so by (1.1), $T_{\overline{a}}$ is an isometry from H^2 onto $\mathscr{M}(\overline{a})$. Applying $T_{\overline{a}}$ to both sides of (5.2) and using the earlier mentioned operator identity

$$T_{\overline{a}} T_{a/\overline{a}} = T_a$$

(Proposition 2.3(3)), we obtain

$$\mathscr{M}(\overline{a}) = \mathscr{M}(a) \oplus_{\overline{a}} T_{\overline{a}} \operatorname{Ker} T_{\overline{a}/a}.$$

Now bring in the identity $T_{\overline{a}/a} = T_{\overline{I}}$ and the facts that $\operatorname{Ker} T_{\overline{I}} = K_I$ (Proposition 2.3(4)) and $T_{\overline{a}} K_I = K_I$ (to see this last fact, observe that $T_{\overline{a}} K_I \subset K_I$ – Proposition 2.3(2) – and if $f \in K_I$ then $T_{\overline{a}} f = f + T_{\overline{I}} f = f$ and so $T_{\overline{a}} K_I = K_I$), to finally obtain the orthogonal decomposition

$$\mathscr{M}(\overline{a}) = \mathscr{M}(a) \oplus_{\overline{a}} K_I.$$

Example 5.3. For the outer function

$$a := \prod_{1 \leqslant j \leqslant n} (z - \zeta_j)^{m_j}, \quad \zeta_j \in \mathbb{T}, m_j \in \mathbb{N},$$

one can verify that

$$\frac{a}{\overline{a}} = cI \quad \text{on } \mathbb{T},$$

where

$$I(z) = z^N, \quad |c| = 1, \quad N = \sum_{1 \leqslant j \leqslant n} m_j.$$

The same analysis as in the previous example shows that

$$\mathscr{M}(\overline{a}) = \mathscr{M}(a) \oplus_{\overline{a}} T_{\overline{a}} K_I.$$

Now observe that $K_I = \mathcal{P}_{N-1}$ (the analytic polynomials of degree at most $N-1$) and $T_{\overline{a}} \mathcal{P}_{N-1} = \mathcal{P}_{N-1}$. Indeed $T_{\overline{a}} \mathcal{P}_{N-1} \subset \mathcal{P}_{N-1}$ and equality follows since $\mathcal{P}_{N-1}$ is finite dimensional and $T_{\overline{a}}$ is injective. Thus we get

$$\mathscr{M}(\overline{a}) = \mathscr{M}(a) \oplus_{\overline{a}} \mathcal{P}_{N-1}.$$

Example 5.4. Suppose I is inner, $m \in \mathbb{N}$, and

$$a := (1 - I)^m.$$

Again, as we have seen in the previous two examples,

$$\frac{a}{\overline{a}} = cI^m \quad \text{a.e. on } \mathbb{T},$$

for some suitable unimodular constant c, and so

$$\mathscr{M}(\overline{a}) = \mathscr{M}(a) \oplus_{\overline{a}} T_{\overline{a}} K_{I^m}.$$

Here things are a bit more tricky since it is not as clear as it was before that $T_{\overline{a}} K_{I^m} = K_{I^m}$. However, by applying the following technical lemma, this is indeed the case.

Lemma 5.5. *Let $a \in H^\infty$ be outer and I inner. Then the following are equivalent:*

(i) $T_{\overline{a}} K_I = K_I$;

(ii) There exists a $\psi \in H^\infty$ such that $a\psi - 1 \in IH^\infty$;

(iii) There exists a constant $\delta > 0$ such that $|a| + |I| \geqslant \delta$ on $\mathbb{D}$.

Proof. $(i) \iff (ii)$: As we have already seen, since a is outer, $T_{\overline{a}}$ is injective on H^2 and hence on K_I. In order to have $T_{\overline{a}} K_I = K_I$, the operator $T_{\overline{a}}$ must be invertible on K_I. This is equivalent to saying that the compression of the analytic Toeplitz operator T_a to K_I (a truncated Toeplitz operator), i.e.,

$$a(S_I) := P_I T_a|_{K_I}$$

(where P_I is the orthogonal projection of L^2 onto K_I, $S_I := P_I T_z|_{K_I}$ is the compression of the shift T_z, and $a(S_I)$ is defined via the functional calculus) is invertible on K_I. If $a(S_I)$ is invertible then its inverse commutes with S_I [25, p. 231]. By the commutant lifting theorem, there is a $\psi \in H^\infty$ such that

$$(a(S_I))^{-1} = \psi(S_I)$$

and thus for every $f \in K_I$, $P_I(a\psi f) = f$, or equivalently, $(a\psi - 1)f \in IH^2$. This translates to the condition $a\psi - 1 \in IH^\infty$ (pick for instance $f = 1 - \overline{I(0)}I$ which is outer with bounded recicprocal). Clearly, when $a\psi - 1 \in IH^\infty$, we can reverse the argument.

The equivalence $(2) \iff (3)$ is an application of the corona theorem [14]. $\square$

Example 5.6. Let

$$a := \prod_{1 \leqslant j \leqslant n} (\zeta_j - I_j)^{m_j},$$

where I_j are inner functions, $\zeta_j \in \mathbb{T}$, and $m_j \in \mathbb{N}$.

As with the previous two examples,

$$\frac{a}{\overline{a}} = cI \quad \text{a.e. on } \mathbb{T},$$

where
$$I = \prod_{1 \leqslant j \leqslant n} I_j^{m_j}, \quad |c| = 1.$$
Hence
$$\mathscr{M}(\overline{a}) = \mathscr{M}(a) \oplus_{\overline{a}} T_{\overline{a}}(K_I).$$
Here things become more complicated than in our previous examples since, as we will see shortly, $T_{\overline{a}} K_I$ can be a proper subspace of K_I that is difficult to identify. Note however that since a is outer then one can easily prove that $T_{\overline{a}} K_I$ is dense in K_I (in the H^2 norm).

For example, if
$$a := (1 - I_1)(1 - I_2), \quad I = I_1 I_2,$$
then (a, I) is not always a corona pair and so, by Lemma 5.5, $T_{\overline{a}} K_I$ is a proper subspace of K_I.

More precisely, let
$$\lambda_n = 1 - 4^{-n^2}, \quad \Lambda_1 = (\lambda_n)_{n \geqslant 1},$$
$$\mu_n = 1 - 4^{-n^2 - n}, \quad \Lambda_2 = (\mu_n)_{n \geqslant 1},$$
I_1 and I_2 the Blaschke products with these zeros. In order to show that
$$\inf\{|a(z)| + |I(z)| : z \in \mathbb{D}\} = 0,$$
it is enough to show that $I_1(\mu_{n_k}) \to 1$ when $k \to \infty$ for some suitable sub-sequence (μ_{n_k}). Clearly $I_1(\mu_n)$ is a real number. Since the zeros of I_1 are simple, I_1 changes sign on $[0, 1)$ at each λ_n. We can thus assume that for alternating μ_n, we have $I_1(\mu_n) > 0$. Note these μ_n by μ_n^+. Finally, since the sequence is interpolating with increasing pseudohyperbolic distances between successive points, we necessarily have $I_1(\mu_n^+) \to 1$. Hence
$$a(\mu_n^+) = (1 - I_1(\mu_n^+))(1 - I_2(\mu_n^+)) \to 0, \quad n \to \infty,$$
and $I(\mu_n^+) = 0$, which proves the claim.

In general, we see from the discussion in our first example that if $a \in H^\infty$ is outer and $\mathscr{M}(a)$ is a closed subspace of $\mathscr{M}(\overline{a})$ (and this is not always the case), then, as we will explain why in a moment,
$$\mathscr{M}(\overline{a}) = \mathscr{M}(a) \oplus_{\overline{a}} T_{\overline{a}} \operatorname{Ker} T_{\overline{a}/a}.$$
So the issues we need to discuss further are:

(1) When is $\mathscr{M}(a)$ a closed subspace of $\mathscr{M}(\overline{a})$?

(2) Identify $\operatorname{Ker} T_{\overline{a}/a}$.

(3) Identify $T_{\overline{a}} \operatorname{Ker} T_{\overline{a}/a}$.

In order to avoid trivialities, we point out the following:.

Proposition 5.7. *Let $a \in H^\infty$ and outer.*

(1) If $T_{\overline{a}}$ is surjective, then $\mathscr{M}(a) = \mathscr{M}(\overline{a}) = H^2$.

(2) $\mathscr{M}(a) = \mathscr{M}(\overline{a})$ if and only if $T_{a/\overline{a}}$ is surjective.

Proof. (1): From Proposition 2.3(2) we know that $T_{\overline{a}}$ is injective. Thus if $T_{\overline{a}}$ were surjective it would also be invertible (as would T_a). Hence $\mathscr{M}(a) = \mathscr{M}(\overline{a}) = H^2$.

(2): Note that

$$\mathscr{M}(a) = T_{\overline{a}} T_{a/\overline{a}} H^2, \tag{5.8}$$

and since $T_{\overline{a}}$ is injective, we get that

$$\mathscr{M}(a) = \mathscr{M}(\overline{a}) \iff T_{a/\overline{a}} H^2 = H^2. \quad \square$$

From now on we will assume that $T_{a/\overline{a}}$ is not surjective. This next result helps us determine when $\mathscr{M}(a)$ is closed in $\mathscr{M}(\overline{a})$.

Proposition 5.9. *For $a \in H^\infty$ and outer, the following are equivalent:*

(i) $\mathscr{M}(a)$ is a closed subspace of $\mathscr{M}(\overline{a})$.

(ii) $T_{a/\overline{a}} H^2$ is a closed subspace of H^2.

(iii) $T_{a/\overline{a}}$ is left invertible.

(iv) $T_{\overline{a}/a}$ is surjective.

Proof. Using (5.8) and the fact that $T_{\overline{a}}$ is an isometry from H^2 onto $\mathscr{M}(\overline{a})$, we see that $\mathscr{M}(a)$ is a closed subspace of $\mathscr{M}(\overline{a})$ if and only if $T_{a/\overline{a}} H^2$ is a closed subspace of H^2. This proves $(i) \iff (ii)$. For the remaining implications, use the fact that $T_{a/\overline{a}}$ is injective (Proposition 2.3(2)) along with the general fact that for a bounded linear operator A on a Hilbert space, the conditions A is left invertible; A is injective with closed range; A^* is surjective – are equivalent. $\square$

When $\mathscr{M}(a)$ is a closed subspace of $\mathscr{M}(\overline{a})$ then $T_{a/\overline{a}}$ has closed range and so, by using the analysis from Example 5.1,

$$\mathscr{M}(\overline{a}) = \mathscr{M}(a) \oplus_{\overline{a}} T_{\overline{a}} \operatorname{Ker} T_{\overline{a}/a}.$$

This brings us to some of the subtleties of $\operatorname{Ker} T_{\overline{a}/a}$ discussed earlier. Note that $\operatorname{Ker} T_{\overline{a}/a} \neq \{0\}$ since $T_{a/\overline{a}}$ is not surjective but left invertible. Recall from Theorem 2.4 and the discussion thereafter that

$$\operatorname{Ker} T_{\overline{a}/a} = \gamma K_I,$$

where

$$\gamma = \frac{\alpha}{1 - \beta_0 I}$$

and $\alpha \in H_1^\infty$ and outer, β_0 is a Pythagorean mate, and I is an inner function with $I(0) = 0$. As a consequence of Proposition 5.9 and Theorem 2.8, we see that $T_{a/\overline{a}}$ has closed range if and only if $|\gamma_0|^2$ is an (A_2) weight, where

$$\gamma_0 = \frac{\alpha}{1 - \beta_0}.$$

Thus $\mathscr{M}(a)$ is a closed non-trivial subspace of $\mathscr{M}(\overline{a})$ if and only if $|\gamma_0|^2$ is an (A_2) weight. We summarize this discussion with the following:

Theorem 5.10. *Let $a \in H^\infty$ be outer. Then*

(1) $\mathscr{M}(a)$ is a closed subspace of $\mathscr{M}(\overline{a})$ if and only if $|\gamma_0|^2$ is an (A_2) weight.

(2) If γ and I are the associated functions as above, then

$$\mathscr{M}(\overline{a}) = \mathscr{M}(a) \oplus_{\overline{a}} T_{\overline{a}}(\gamma K_I). \tag{5.11}$$

Although Theorem 5.10 might appear implicit, it actually yields a recipe to construct further, more subtle, decompositions. For example, choose an outer $\alpha \in H_1^\infty$ such that its Pythagorean mate β_0 satisfies the property that $|\gamma_0|^2$ is an (A_2) weight. We will see a specific example of this in a moment. As mentioned earlier, the (A_2) condition implies that γ_0^2 is a rigid function. Let I be any inner function with $I(0) = 0$ and $\gamma = \alpha/(1 - I\beta_0)$. From (2.7) we have $\gamma K_I = \operatorname{Ker} T_{\overline{I\gamma}/\gamma}$. Set

$$a = (1 + I)\gamma.$$

Then

$$\frac{\overline{a}}{a} = \frac{\overline{I\gamma}}{\gamma} \quad \text{a.e. on } \mathbb{T}$$

and so

$$\operatorname{Ker} T_{\overline{a}/a} = \gamma K_I,$$

whence

$$\mathscr{M}(\overline{a}) = \mathscr{M}(a) \oplus_{\overline{a}} T_{\overline{a}}(\gamma K_I). \tag{5.12}$$

Here is an example which uses this recipe.

Example 5.13. Let $\varepsilon \in (0, \frac{1}{2})$ and define the outer function $\alpha \in H_1^\infty$ by

$$\alpha(z) = \left(\frac{1-z}{2}\right)^{\varepsilon}.$$

With β_0 the outer Pythagorean mate for α, an estimate from [18, p. 359-360] yields

$$|1 - \beta_0(\zeta)| \asymp |1-\zeta|^{2\varepsilon}, \quad \zeta \in \mathbb{T}.$$

The function $\gamma_0 = \alpha/(1-\beta_0)$ satisfies

$$|\gamma_0(\zeta)| \asymp |1-\zeta|^{-\varepsilon}, \quad \zeta \in \mathbb{T}.$$

A routine estimate will show that the condition (2.9) holds and so $|\gamma_0|^2$ is an (A_2) weight. For any inner I with $I(0) = 0$, define $\gamma = \alpha/(1 - I\beta_0)$ and $a = \gamma(1 + I)$ and follow the above argument to obtain the decomposition in (5.12).

It is also possible to start from $\gamma_0(z) = (1-z)^{\varepsilon}$. Then β_0 can be expressed using the integral representation (2.5) and $\alpha = \gamma_0(1-\beta_0)$.

We now produce a formula for the orthogonal projection P from $\mathscr{M}(\overline{a})$ onto $T_{\overline{a}}(\gamma K_I)$.

Theorem 5.14. *In the above notation, let P_I denote the orthogonal projection of H^2 onto K_I. Then $P = T_{\overline{a}}\gamma P_I \overline{\gamma} T_{1/\overline{a}}$ is the orthogonal projection from $\mathscr{M}(\overline{a})$ onto $T_{\overline{a}}(\gamma K_I)$.*

Proof. From Theorem 2.4 we know that γ is an isometric multiplier of K_I. The operator $P_0 := \gamma P_I \overline{\gamma}$ is the orthogonal projection from H^2 onto γK_I. Indeed, it is clear that its range is γK_I. From Theorem 2.4 we deduce that $P_0(\gamma f) = \gamma f$, when $f \in K_I$. Finally it is straight forward to see that $P_0 f = 0$ whenever $f \perp \gamma K_I$. Since $T_{\overline{a}}$ is a an isometric isomorphism from H^2 onto $\mathscr{M}(\overline{a})$, we can define its inverse, which is just $T_{1/\overline{a}}$. The result now follows by composition. □

To help us better understand some of the contents of $T_{\overline{a}}(\gamma K_I)$ we have the following:

Proposition 5.15. *If $\zeta_0 \in (AC)_{\overline{a},N}$, then*

$$k^{\overline{a}}_{\zeta_0,\ell} \in T_{\overline{a}}(\gamma K_I), \qquad 0 \leqslant \ell \leqslant N.$$

Proof. Notice that

$$\zeta_0 \in (AC)_{\overline{a},\ell} \implies \zeta_0 \in (AC)_{\overline{a},\ell'}, \quad 0 \leqslant \ell' \leqslant \ell,$$

and so it suffices to prove the result when $\ell = N$. By Theorem 5.10, we can do this by proving

$$k^{\overline{a}}_{\zeta_0,N} \perp_{\overline{a}} aH^2.$$

To prove this last fact, set $f = ah$, where $h \in H^2$. By Leibniz's formula,

$$\langle f, k^{\overline{a}}_{r\zeta_0,N}\rangle_{\overline{a}} = f^{(N)}(r\zeta_0) = \sum_{0\leqslant k\leqslant N} \binom{N}{k} a^{(k)}(r\zeta_0)h^{(N-k)}(r\zeta_0).$$

But according to Remark 4.16 we have

$$|a^{(k)}(r\zeta_0)h^{(N-k)}(r\zeta_0)| = o((1-r)^{N+\frac{1}{2}-k}(1-r)^{k-N-\frac{1}{2}}) = o(1).$$

Thus

$$\lim_{r\to 1}\langle f, k^{\overline{a}}_{r\zeta_0,N}\rangle_{\overline{a}} = 0,$$

and, using Corollary 4.11, yields

$$\langle f, k^{\overline{a}}_{\zeta_0,N}\rangle_{\overline{a}} = 0.$$

This proves the result. □

Using Proposition 5.15, we can revisit Example 5.3 and give an alternate description of the orthogonal complement of $\mathscr{M}(a)$ in $\mathscr{M}(\overline{a})$ when

$$a = \prod_{1\leqslant j\leqslant n} (z-\zeta_j)^{m_j}.$$

Indeed, since a is a polynomial, it is clear that $\zeta_j \in (AC)_{\overline{a},m_j-1}$, and so

$$k^{\overline{a}}_{\zeta_j,\ell} \in T_{\overline{a}}(\gamma K_I) = \mathcal{P}_{N-1}, \quad 1 \leqslant j \leqslant n, 0 \leqslant \ell \leqslant m_j - 1.$$

Since the functions

$$\{k^{\overline{a}}_{\zeta_j,\ell} : j = 1, \cdots, n, \ell = 0, \cdots, m_i - 1\}$$

are linearly independent, we obtain

$$\mathcal{P}_{N-1} = \bigvee\{k^{\overline{a}}_{\zeta_j,\ell} : j = 1, \cdots, n, \ell = 0, \cdots, m_i = 1\}.$$

Corollary 5.16. *If $a = \prod_{j=1}^{n}(z-\zeta_j)^{m_j}$, then*

$$\mathscr{M}(\overline{a}) = \mathscr{M}(a) \oplus_{\overline{a}} \bigvee\{k^{\overline{a}}_{\zeta_j,\ell} : j = 1, \cdots, n, \ell = 0, \cdots, m_i = 1\}.$$

The techniques above also give the following which generalizes a result from [10, 22].

Theorem 5.17. *Let I be any inner function vanishing at 0, set $a = (1-I)/2$ and $b = (1+I)/2$. Then*

$$\mathscr{H}(b) = \mathscr{M}(a) \oplus_b K_I.$$

Proof. From [30] we know that (a, b) forms a corona pair (see (3.11)), whence $\mathscr{H}(b) = \mathscr{M}(\overline{a})$. It follows from our first example of this section that we can decompose $\mathscr{H}(b)$ as the direct sum of $\mathscr{M}(a)$ and K_I with respect to $\langle \cdot, \cdot \rangle_{\overline{a}}$. It remains to prove that $\mathscr{M}(a)$ and K_I are orthogonal in the inner product of $\mathscr{H}(b)$. In other words, we need to check that given any function $f \in H^2$ and any function $g \in K_I$, we have

$$\langle af, g \rangle_b = 0.$$

Using again that $\operatorname{Ker} T_{\overline{I}} = K_I$ so that $g = 2T_{\overline{a}}g$, and using a well-known formula for the inner product in $\mathscr{H}(b)$ [30], we have

$$\langle af, g/2 \rangle_b = \langle af, T_{\overline{a}}g \rangle_{H^2} + \langle T_{\overline{b}}(af), T_{\overline{b}}T_{\overline{a}}g \rangle_{\mathscr{H}(\bar{b})}.$$

Note that

$$T_{\overline{b}}(af) = T_{\overline{a}}T_{a/\overline{a}}(\overline{b}f) \quad \text{and} \quad T_{\overline{b}}T_{\overline{a}}g = T_{\overline{a}}T_{\overline{b}}g.$$

Since $\mathscr{H}(\overline{b})$ and $\mathscr{M}(\overline{a})$ coincide as Hilbert spaces, we deduce that

$$\langle af, g/2 \rangle_b = \langle af, T_{\overline{a}}g \rangle_{H^2} + \langle T_{a/\overline{a}}(\overline{b}f), T_{\overline{b}}g \rangle_{H^2}.$$

Note that $T_{\overline{a}}g = \frac{1}{2}g = T_{\overline{b}}g$ and $a + b = 1$. Hence

$$\begin{aligned}
\langle af, g \rangle_b &= \langle af, g \rangle_{H^2} + \langle T_{a/\overline{a}}(\overline{b}f), g \rangle_{H^2} \\
&= \langle af, g \rangle_{H^2} + \langle af, \frac{b}{a}g \rangle_{H^2} \\
&= \langle af, g + \frac{b}{a}g \rangle_{H^2} \\
&= \langle af, \frac{1}{a}g \rangle_{H^2} \\
&= \langle T_{a/\overline{a}}f, g \rangle_{H^2}.
\end{aligned}$$

Recall that $T_{a/\overline{a}}H^2$ is a closed subspace with

$$T_{a/\overline{a}}H^2 = (\operatorname{Ker} T_{\overline{a}/a})^{\perp} = K_I^{\perp} = IH^2$$

(see also Example 5.1) which proves the claim. □

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