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AN INVARIANT SUBSPACE PROBLEM FOR $p = 1$ BERGMAN SPACES ON SLIT DOMAINS

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Abstract. In this paper, we characterize the z-invariant subspaces that lie between the Bergman spaces $A^1(G)$ and $A^1(G\backslash K)$, where G is a bounded region in the complex plane and K is a compact subset of a simple arc of class C^1.

1. Introduction

For a bounded region $U \subset \mathbb{C}$, we define the *Bergman space* $A^1(U)$ as the space of analytic functions f on U with $\int_U |f| dA < \infty$ (Here dA is Lebesgue measure on $\mathbb{C}$) and the operator S on $A^1(U)$ by $(Sf)(z) = zf(z)$. Characterizing the subspaces $\mathcal{M}$ of $A^1(U)$ for which $S\mathcal{M} \subset \mathcal{M}$ is a difficult and unsolved problem which has received considerable attention over the past 40 years. In this paper, we give a complete characterization of the S-invariant subspaces $\mathcal{M}$ with

$$A^1(G) \subset \mathcal{M} \subset A^1(G\backslash K). \tag{1.1}$$

Here G is a bounded region in $\mathbb{C}$ and K is a compact subset of a simple arc of class C^1. This paper will be a continuation of an L^p version of this problem [5] to the largest of the Bergman spaces $p = 1$. Different techniques will be used here since the papers mentioned above use duality and the reflexivity of L^p, a luxury not afforded us in the non-reflexive setting of L^1. Our main theorem is:

Theorem 1.1. *For $\mathcal{M}$ of type (1.1) and S-invariant, there is a closed $F \subset K$ with $\mathcal{M} = A^1(G\backslash F)$.*

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2. Preliminaries

Before proceeding, we point out that some of the techniques used here are somewhat standard and fall under the general name of 'Havin's lemma'. We refer the reader to [3] and [8], Ch. 4, section 2, for further details. For the sake of completeness, we will outline these results here.

For a bounded region U in the complex plane $\mathbb{C}$, we identify the dual of $L^1(U) = L^1(U, dA)$ with $L^\infty(U) = L^\infty(U, dA)$ by the bi-linear pairing

$$< f, g >= \int_U fg dA, \quad f \in L^1(U),\ g \in L^\infty(U).$$

For a linear manifold X in $L^1(U)$ we let $X^\perp$ be the annihilator of X and note that $X^\perp$ is weak-star closed in $L^\infty(U)$. For a linear manifold Y in $L^\infty(U)$, we let ${}^\perp Y$ be the pre-annihilator of Y and note that ${}^\perp Y$ is norm closed in $L^1(U)$. We also note that by the Hahn-Banach theorem ${}^\perp(X^\perp)$ is the norm closure of X in $L^1(U)$ and $({}^\perp Y)^\perp$ is the weak-star closure of Y in $L^\infty(U)$.

Lemma 2.1. *$A^1(U)^\perp$ is the weak-star closure of $\overline{\partial} C_0^\infty(U)$, where $\overline{\partial} = \frac{1}{2}(\partial/\partial x + i\partial/\partial y)$.*

Proof. By Weyl's lemma [2], p. 172, ${}^\perp(\overline{\partial} C_0^\infty(U)) = A^1(U)$, hence

$$\left({}^\perp\, (\overline{\partial} C_0^\infty(U))\,\right)^\perp = A^1(U)^\perp. \tag{2.1}$$

By Hahn-Banach, the left-hand side of (2.1) is the weak-star closure of $\overline{\partial} C_0^\infty(U)$. □

Remark: We will show, in Proposition 3.1, that in fact $\overline{\partial} C_0^\infty(U)$ is weak-star sequentially dense in $A^1(U)^\perp$, a technicality that will be important later in the paper.

We now relate $A^1(U)^\perp$ with a certain type of Sobolev space on U via $\overline{\partial}$. Let $\mathcal{W} = \mathcal{W}(\mathbb{C})$ be the Banach space of $f \in L^\infty = L^\infty(\mathbb{C}, dA)$ such that $\overline{\partial} f$ (in the sense of distributions) belongs to L^∞. We norm $\mathcal{W}$ by

$$\|f\|_{\mathcal{W}} = \|f\|_\infty + \|\overline{\partial} f\|_\infty.$$

Remark: We pause here for a moment to mention that $\mathcal{W}$ contains, but is not equal to $W^{1,\infty}(\mathbb{C})$, the Sobolev space of $f \in L^\infty$ whose first partial derivatives (in the sense of distributions) also belong to L^∞. In fact, if $f \in \mathcal{W}$, then the first partial derivatives belong to BMO but are not always bounded (see [9] and [10], p. 164, and [4]).

For $g \in L^\infty$ with compact support, define the *Cauchy transform* Tg by

$$(Tg)(w) = -\frac{1}{\pi}\int \frac{g(z)}{z-w}\, dA(z) \tag{2.2}$$

and note that Tg is continuous on $\mathbb{C}$ ([11], p. 40), analytic off of the support of g, $(Tg)(\infty) = 0$, and $\phi = T(\overline{\partial}\phi)$ for all $\phi \in C_0^\infty$ ([2], p. 170).

Lemma 2.2. *Every $f \in \mathcal{W}$ has a continuous representative.*

Proof. Let $f \in \mathcal{W}$ and $\phi \in C_0^\infty$. Note that $\phi f \in \mathcal{W}$ and, by distribution theory [2], p. 174 - 175, $\phi f = T(\overline{\partial}(\phi f))$ a.e. (dA). Since $T(\overline{\partial}(\phi f))$ is continuous, we can conclude that f has a continuous representative. □

Assuming now, and for the rest of the paper that all functions in $\mathcal{W}$ are continuous, we let $\mathcal{W}_0(U)$ be the subspace of functions in $\mathcal{W}$ which vanish off of U. ($\mathcal{W}_0(U)$ is not the same as the closure of $C_0^\infty(U)$ in the $\mathcal{W}$ norm.) For $f \in \mathcal{W}_0(U)$, one sees from Lemma 2.2 that $f = T(\overline{\partial} f)$, and thus for $w \in U$

$$|f(w)| \le \frac{1}{\pi}\int_U \frac{|\overline{\partial} f(z)|}{|z-w|}\, dA(z) \le C_U \|\overline{\partial} f\|_\infty. \tag{2.3}$$

Here C_U is a positive constant depending only on the region U. Hence an equivalent norm on $\mathcal{W}_0(U)$ is

$$\|f\|_{\mathcal{W}_0} = \|\overline{\partial} f\|_\infty.$$

If $f \in \mathcal{W}_0(U)$ and $w \notin U$, then

$$0 = f(w) = -\frac{1}{\pi}\int_U \frac{\overline{\partial} f(z)}{z-w}\, dA(z).$$

But since $A^1(U)$ is the closed linear span of $\{(z-w)^{-1} : w \notin U\}$ [1], then $\overline{\partial} f \in A^1(U)^\perp$ and moreover $\overline{\partial} : \mathcal{W}_0(U) \to A^1(U)^\perp$ is an isometry.

Proposition 2.3. *$\overline{\partial} : \mathcal{W}_0(U) \to A^1(U)^\perp$ is invertible.*

Proof. Since $\overline{\partial}$ is an isometry, it suffices to show that $\overline{\partial}$ is onto. To this end, let $g \in A^1(U)^\perp$. By distribution theory [2], p. 174, $\overline{\partial}(Tg) = g \in L^\infty$, so $Tg \in \mathcal{W}$. Since $g \in A^1(U)^\perp$, then by (2.2), $(Tg)(w) = 0$ for all $w \notin U$. Hence $Tg \in \mathcal{W}_0(U)$. □

Proposition 2.4. *$\mathcal{W}_0(U)$ can be equivalently re-normed to make it a Banach algebra.*

Proof. If $f = T(\overline{\partial} f)$ and $g = T(\overline{\partial} g)$ both belong to $\mathcal{W}_0(U)$, then one has ([2], p. 178, Lemma 3.11) $fg = T(f\overline{\partial} g + g\overline{\partial} f)$, hence

$$\overline{\partial}(fg) = f\overline{\partial} g + g\overline{\partial} f, \tag{2.4}$$

from which we obtain $\|\overline{\partial}(fg)\|_\infty \le \|f\overline{\partial} g\|_\infty + \|g\overline{\partial} f\|_\infty$. Using (2.3) will yield $\|\overline{\partial}(fg)\|_\infty \le 2C_U\|\overline{\partial} f\|_\infty\|\overline{\partial} g\|_\infty$. We conclude from this that $\mathcal{W}_0(U)$ can be equivalently re-normed to make it a Banach algebra. □

3. Invariant subspaces

Define the operator R on $A^1(U)^\perp$ by $(Rg)(z) = zg(z)$ and M on the Sobolev space $\mathcal{W}_0(U)$ by $(Mh)(z) = zh(z)$ and notice that R and M are well defined and continuous. For $f \in \mathcal{W}_0(U)$, observe that

$$\overline{\partial}(zf) = z\overline{\partial} f,$$

and thus $\overline{\partial} M = R\overline{\partial}$.

If $\mathcal{M}$ is invariant with

$$A^1(G) \subset \mathcal{M} \subset A^1(G\backslash K), \tag{3.1}$$

we can take annihilators to get

$$A^1(G\backslash K)^\perp \subset \mathcal{M}^\perp \subset A^1(G)^\perp \tag{3.2}$$

with $R\mathcal{M}^\perp \subset \mathcal{M}^\perp$. Taking $T = \overline{\partial}^{-1}$ (Proposition 2.3) of both sides of (3.2) will yield

$$\mathcal{W}_0(G\backslash K) \subset T\mathcal{M}^\perp \subset \mathcal{W}_0(G)$$

and using $\overline{\partial} M = R\overline{\partial}$, we get that $T\mathcal{M}^\perp$ is z-invariant. We will eventually show that $T\mathcal{M}^\perp$ is an ideal of the Banach algebra $\mathcal{W}_0(G)$ and that $T\mathcal{M}^\perp = \mathcal{W}_0(G\backslash Z_{\mathcal{M}})$, and hence

$$\mathcal{M} = A^1(G\backslash Z_{\mathcal{M}}),$$

where

$$Z_{\mathcal{M}} = \{z \in K : (Tg)(z) = 0 \ \ \forall g \in \mathcal{M}^\perp\}, \tag{3.3}$$

but first we need a few preliminary lemmas.

In Lemma 2.1, we saw that $\overline{\partial} C_0^\infty(G)$ is weak-star dense in $A^1(G)^\perp$. This next result gives us slightly more.

Proposition 3.1. *$\overline{\partial} C_0^\infty(G)$ is weak-star sequentially dense in $A^1(G)^\perp$.*

The proof of Proposition 3.1 will depend on the following lemma which uses a certain "mollifier" of Ahlfors [1].

Lemma 3.2. *Let $h \in \mathcal{W}_0(G)$. Then there is a sequence $h_n \in \mathcal{W}_0(G)$ with supp$(h_n) \subset G$ and $\overline{\partial} h_n \to \overline{\partial} h$ weak-star.*

Proof. Since $h \equiv 0$ off of G, then one can show [11], p. 40, that for all z and w in $\mathbb{C}$

$$|h(z) - h(w)| \leq C|z - w||\log|z - w||.$$

Thus if $d(z)$ equals the minimum of e^{-2} and dist$(z, \partial G)$, then

$$|h(z)| \leq C d(z)|\log d(z)|. \tag{3.4}$$

We now construct the "Ahlfors mollifier" w_n as follows [1]: Let $j(t)$ be an infinitely differentiable function on $\mathbb{R}$ with $0 \leq j \leq 1$, $j(t) = 0$ for all $t \leq 1$, and $j(t) = 1$ for all $t > 2$. For $n \in \mathbb{N}$ and $z \in G$ let

$$w_n(z) = j\left(\frac{n}{\log\log d(z)}\right) \tag{3.5}$$

and notice that $w_n \equiv 0$ near ∂G. Thus define w_n on $\mathbb{C}$ be defining $w_n \equiv 0$ off G.

Since $d(z)$ is Lipschitz continuous with constant 1 and $j'(t) = 0$ outside $1 < t < 2$, one can check [1] that

$$|\overline{\partial} w_n(z)| \leq \frac{C}{n} \frac{1}{d(z)|\log d(z)|} \quad \forall z \in G. \tag{3.6}$$

Hence $w_n \in \mathcal{W}_0(G)$ and so, by Proposition 2.4, $h_n \equiv w_n h$ also belongs to $\mathcal{W}_0(G)$ with supp$(h_n) \subset G$.

We now show that $\overline{\partial} h_n \to \overline{\partial} h$ weak-star. If $f \in L^1(G)$, then by (2.4)

$$|\int_G f(\overline{\partial} h_n - \overline{\partial} h) dA| \leq |\int_G f \overline{\partial} h (w_n - 1) dA| + \int_G |f||h||\overline{\partial} w_n| dA. \tag{3.7}$$

By (3.4) and (3.6), we get

$$\int_G |f||h||\overline{\partial} w_n| dA \leq \frac{C}{n} \int_G |f| dA$$

which goes to zero as $n \to \infty$. The first integral in (3.7) goes to zero since $w_n \to 1$ pointwise and $w_n \leq 1$. Thus $\overline{\partial} h_n \to \overline{\partial} h$ weak-star. ☐

Proof of Proposition 3.1

Let $h_n = w_n h$ be as in Lemma 3.2. For $n, k \in \mathbb{N}$ let φ_k be a mollifier [2], p. 171, and define

$$h_{n,k}(w) = \int \varphi_k(w - z) h_n(z) dA(z).$$

Notice that $h_{n,k} \in C_0^\infty(G)$ (since h_n has compact support in G) and $h_{n,k} \to h_n$ uniformly as $k \to \infty$. By a change of variables and Fubini's theorem, one checks that

$$\overline{\partial} h_{n,k}(w) = -\int \varphi_k(z) \overline{\partial} h_n(w - z) dA(z).$$

Since $\overline{\partial} h_n \to \overline{\partial} h$ weak-star, then

$$\sup_n \|\overline{\partial} h_n\|_\infty = M < \infty$$

and so $|\overline{\partial} h_{n,k}(w)| \leq M \int \varphi_k(z) dA(z) = M$. Hence

$$\sup_{n,k} \|\overline{\partial} h_{n,k}\|_\infty \leq M < \infty.$$

Choose $k(n)$ so that $\|h_{n,k(n)} - h_n\|_\infty \le 1/n$ and let $H_n = h_{n,k(n)}$. We shall conclude by showing that $\overline{\partial} H_n \to \overline{\partial} h$ weak-star. Let $f \in L^1(G)$ and choose a sequence $\phi_j \in C_0^\infty(G)$ with $\phi_j \to f$ in L^1. Then

$$\begin{aligned} \left| \int f(\overline{\partial} H_n - \overline{\partial} h) dA \right| &\le \left| \int (f-\phi_j)(\overline{\partial} H_n - \overline{\partial} h) dA \right| + \left| \int \phi_j (\overline{\partial} H_n - \overline{\partial} h_n) dA \right| \\ &+ \left| \int \phi_j (\overline{\partial} h_n - \overline{\partial} h) dA \right| \\ &\le M\|f-\phi_j\|_{L^1} + \int |\overline{\partial}\phi_j| \frac{1}{n} dA + \left| \int \phi_j (\overline{\partial} h_n - \overline{\partial} h) dA \right|. \end{aligned}$$

For $\varepsilon > 0$ given, choose j' such that

$$\|f - \phi_{j'}\|_{L^1} \le \varepsilon.$$

Hence

$$\left| \int f(\overline{\partial} H_n - \overline{\partial} h) dA \right| \le M\varepsilon + \frac{1}{n}\int |\overline{\partial}\phi_{j'}| dA + \left| \int \phi_{j'} (\overline{\partial} h_n - \overline{\partial} h) dA \right|.$$

Now use Lemma 3.2 and let $n \to \infty$ to get the desired conclusion. Λ

Lemma 3.3. *If $f, g \in \mathcal{W}_0(G)$ and ϕ_n in $C_0^\infty(G)$ with $\overline{\partial}\phi_n \to \overline{\partial} g$ weak-star, then $\overline{\partial}(f\phi_n) \to \overline{\partial}(fg)$ weak-star.*

Proof. Let $h \in L^1(G)$. By (2.4) we have

$$\int_G \overline{\partial}(f\phi_n - fg) h dA = \int_G hf\overline{\partial}(\phi_n - g) dA + \int_G h(\phi_n - g)\overline{\partial} f dA. \tag{3.8}$$

The first integral on the right-hand side of (3.8) goes to zero since $hf \in L^1(G)$ and $\overline{\partial}\phi_n \to \overline{\partial} g$ weak-star. For the second integral, we first notice that by (2.3) $\|\phi_n\|_\infty \le C\|\overline{\partial}\phi_n\|_\infty$ and that $\|\overline{\partial}\phi_n\|_\infty$ is uniformly bounded in n (since $\overline{\partial}\phi_n$ is weak-star convergent). Since $(z-w)^{-1} \in L^1(G)$ for all $w \in G$, and $\overline{\partial}\phi_n \to \overline{\partial} g$ weak-star, then

$$\phi_n(w) = -\pi^{-1} \int \overline{\partial}\phi_n(z)(z-w)^{-1} dA \to -\pi \int \overline{\partial} g(z)(z-w)^{-1} dA = g(w) \quad \forall w \in G.$$

Now apply the dominated convergence theorem to get the second integral on the right-hand side of (3.8) goes to zero as $n \to \infty$. $\square$

This next lemma can be found in [5] and uses very strongly that K lies on γ a simple compact C^1 arc.

Lemma 3.4. *Let $\psi \in C^1(\mathbb{C})$ and $\varepsilon > 0$ be given. Then there is a polynomial $p(z)$ and a $\Psi \in C^1(\mathbb{C})$ with*

(i) $\Psi = \psi$ *on* γ
(ii) $\|p - \Psi\|_\infty < \varepsilon$
(iii) $\|\overline{\partial}(p - \Psi)\|_\infty < \varepsilon$.

As a consequence of this lemma, we have that $T\mathcal{M}^\perp$ is not only z-invariant but invariant under multiplication by any $C_0^\infty(G)$ function.

Corollary 3.5. *If $\psi \in C_0^\infty(G)$, then $\psi(T\mathcal{M}^\perp) \subset T\mathcal{M}^\perp$.*

Proof. Let $\psi \in C_0^\infty(G)$ and $\varepsilon > 0$ be given and Ψ and p be as in Lemma 3.4. If $f \in T\mathcal{M}^\perp$, then $\Psi f \in \mathcal{W}_0(G)$ and $\Psi f - \psi f = 0$ on K, so $\Psi f - \psi f \in \mathcal{W}_0(G \backslash K) \subset T\mathcal{M}^\perp$. Hence

$$\operatorname{dist}(\psi f, T\mathcal{M}^\perp) = \operatorname{dist}(\Psi f, T\mathcal{M}^\perp) \le \|\overline{\partial}(pf - \Psi f)\|_\infty \le C\varepsilon\|\overline{\partial} f\|_\infty \qquad \square$$

This immediately yields the following:

Proposition 3.6. *$T\mathcal{M}^{\perp}$ is an ideal of $\mathcal{W}_0(G)$.*

Proof. Let $f \in T\mathcal{M}^{\perp}$ and $g \in \mathcal{W}_0(G)$ and notice that $fg \in \mathcal{W}_0(G)$. Employing the weak-star sequential density of $\overline{\partial} C_0^{\infty}(G)$ in $A^1(G)^{\perp}$, Proposition 3.1, we can find a sequence $\phi_n \in C_0^{\infty}(G)$ with $\overline{\partial}\phi_n \to \overline{\partial} g$ weak-star. By Corollary 3.5, $\phi_n f \in T\mathcal{M}^{\perp}$ and by Lemma 3.3, $\overline{\partial}(\phi_n f) \to \overline{\partial}(fg)$ weak-star. Since $\mathcal{M}^{\perp}$ is weak-star closed, then $\overline{\partial}(fg) \in \mathcal{M}^{\perp}$, hence $fg \in T\mathcal{M}^{\perp}$. □

Proof of Theorem 1.1

Let $Z_{\mathcal{M}}$ be as in (3.3). Since $\mathcal{W}_0(G \backslash K) \subset T\mathcal{M}^{\perp}$, then $T\mathcal{M}^{\perp} \subset \mathcal{W}_0(G \backslash Z_{\mathcal{M}})$ and so $\mathcal{M}^{\perp} \subset A^1(G \backslash Z_{\mathcal{M}})^{\perp}$. To prove $A^1(G \backslash Z_{\mathcal{M}})^{\perp} \subset \mathcal{M}^{\perp}$, we apply Lemma 3.2 to see that it suffices to show that $\overline{\partial}\phi \in \mathcal{M}^{\perp}$ for all $\phi \in \mathcal{W}_0(G \backslash Z_{\mathcal{M}})$ with support in $G \backslash Z_{\mathcal{M}}$. For this, we use an argument of Sarason [7], p. 41, Lemma 1, along with the fact that $T\mathcal{M}^{\perp}$ is an ideal to find a $g \in T\mathcal{M}^{\perp}$ with $g \equiv 1$ on the support of ϕ. Thus, since $T\mathcal{M}^{\perp}$ is an ideal, $\phi = g\phi \in T\mathcal{M}^{\perp}$ and hence $\overline{\partial}\phi \in \mathcal{M}^{\perp}$. Λ

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