The Newsboy with Known Demand and Uncertain Replenishment: Applications to Quality Control and Container Fill

John S. Rose
University of Richmond

Follow this and additional works at: https://scholarship.richmond.edu/robins-white-papers

Part of the Business Commons

Recommended Citation
THE NEWSBOY WITH KNOWN DEMAND AND UNCERTAIN REPLENISHMENT:

APPLICATIONS TO QUALITY CONTROL AND CONTAINER FILL

John S. Rose

1990-1
THE NEWSBOY WITH KNOWN DEMAND AND UNCERTAIN REPLENISHMENT:
APPLICATIONS TO QUALITY CONTROL AND CONTAINER FILL

JOHN S. ROSE
E. C. Robins School of Business
University of Richmond
Richmond, VA 23173
U.S.A.
(804) 289-8579

Abstract

Consider a newsboy problem with known demand and random replenishment. The distribution of the replenishment quantity is determined by a location-scale parameter, $(\mu, \sigma)$, which can be controlled. The usual expected cost function includes a separate term associated with the choice of $\sigma$. The asymptotic behavior of the optimal parameter values, as functions of the holding and shortage costs, is examined.

KEYWORDS  Inventory; Uncertainty; Sensitivity; Newsboy; Container Fill
1. Inventory Models with Uncertain Fill

Inventory models with uncertain fill have been investigated by several researchers; among them are Silver [11], Shih [10], Moinzadeh and Lee [6], Posner and Berg [8], and Henig and Gerchak [4]. Explicit integration of quality control with inventory control has been considered by Peters et. al. [7] and Lee and Rosenblatt [5].

Investigating a container fill problem in which the fill is normally distributed, Golhar and Pollock [2] assume that the mean is controllable but that the variance is not. However, they can affect the quality of the outgoing containers by setting an upper control limit - fuller containers are scrapped and refilled. Recently, Golhar and Pollock [3] computed the cost saving expected from a reduction in the variance of the fill distribution. Gerchak and Parlar [1] consider an EOQ model in which the variance of the fill distribution is \( \sigma^2Q \), where \( Q \) denotes the order quantity. The proportionality parameter, \( \sigma^2 \), can be controlled, and they minimize the long-run average cost with respect to \( Q \) and \( \sigma^2 \).

2. Problem Statement

Consider a newsboy problem characterized by the following elements: \( A = \) known (deterministic) demand; \( X = \) fill (replenishment quantity), a random variable with absolutely continuous distribution \( F \); \( c = \) material cost; \( p = \) shortage cost, with \( p > c \); \( h = \) holding cost; no initial inventory; and no fixed cost. The expected cost for the period is

\[
c\mu + p\int_{-D_F}^{A}(A-x)F(dx) + h\int_{A}^{B_F}(x-A)F(dx),
\]

where \( \mu = E(X) \) and the interval \( (-D_F, B_F) = \text{supp}(F) \), the support of \( F \); \( D_F \leq \infty, B_F \leq \infty \).

The formulation given above is a nearly complete reversal of the usual newsboy problem; see, for example, Ross [9, pp. 169-171]. Indeed, (1) is also the expected
cost for a newsboy with demand distribution \( F \), reorder quantity \( A \), holding cost \( p \), and penalty cost \( h+c \). The optimal order, \( A^* \), is known:

\[
F(A^*) = \frac{h}{p+h}.
\]  

(2)

The demand, \( A \), is treated like any other parameter, and our objective henceforth is to control \( F \), the distribution of the fill, \( X \). We shall restrict our choices to a family of absolutely continuous distributions, \( \{F_{\mu,\sigma}\} \), determined by a location-scale parameter, \( (\mu,\sigma) \), \( \sigma > 0 \). Letting \( f_{\mu,\sigma} \) denote the density of \( F_{\mu,\sigma} \), then \( F_{\mu,\sigma}(x) = F((x-\mu)/\sigma) \) and \( f_{\mu,\sigma}(x) = f((x-\mu)/\sigma)/\sigma \) for some distribution \( F \) with density \( f \).

Without loss of generality, we assume that the mean of \( F \) is zero:

\[
\int_{-D}^{B} xf(x)dx = 0,
\]  

(3)

where \( \text{supp}(F) = (-D, B) \). In what follows, (3) is invoked frequently and shall not explicitly be alluded to. It follows that, if \( X - F_{\mu,\sigma} \), then \( \text{EX} = \mu \), \( \text{VarX} = \sigma^2 \), and \( \text{supp}(F_{\mu,\sigma}) = (-\mu-\sigma D, \mu+\sigma B) \). Common examples for the types of applications we envision are the normal, uniform, and triangular distributions. We shall frequently use another parameter, \( \theta = (A-\mu)/\sigma \), the number of "deviation" units below demand at which we set the expected fill.

Ideally, we should select the degenerate distribution, \( F_{\mu,0} \), thereby exactly satisfying demand with no deviation. Unfortunately, \( \sigma = 0 \) may be prohibitively expensive to achieve, so we perforce settle for less "quality." Quality, in this context, refers to the precision of \( F \). Denote the cost associated with the choice of \( \sigma \) by \( v(\sigma) \), which we add to (1) to obtain the expected cost function,

\[
L(\mu,\sigma) = c\mu + p(A-\mu)F(\theta) + h(\mu-A)[1-F(\theta)] - p\int_{-D}^{\theta} yf(y)dy + h\int_{\theta}^{B} yf(y)dy + v(\sigma).
\]

(4)

We want to choose \( \mu \geq 0 \) and \( \sigma > 0 \) to minimize \( L \).
Assume that the quality cost function, \( v(\sigma) \), \( \sigma \geq 0 \), is strictly convex, decreasing, and nonnegative. Indeed, we allow the possibility that \( v(0) = \infty \). Finally, we shall impose an upper bound, \( S \), on \( \sigma \), even if the support of \( F_{\mu,\sigma} \) is infinite \( (D = \infty \) or \( B = \infty) \) - greater variability cannot be tolerated. Assume without loss of generality that \( v(S) = 0 \).

3. Examples

\textbf{CONTAINER FILL} A company sells its product in containers of \( A \) pounds, the so-called label weight. The container filling process is subject to some degree of randomness, \( \sigma \). More or less independently of \( \sigma \), the fill process is calibrated to provide a mean weight of \( \mu \) pounds of product per container. If \( \mu \) is set nearly equal to \( A \), then some containers will be underfilled, subjecting the company to penalty. Such penalty could include fines from a regulatory agency, the cost of handling customer returns, and lost goodwill. To compensate, the fill process may be recalibrated to yield higher mean fill. This action results in giving away free product, at unit cost \( c \).

Another possibility is to keep \( \mu \) near \( A \) but improve the quality of the fill process by reducing \( \sigma \). Obviously, average outgoing quality could be improved simply through the application of more rigorous sampling rules at the end of the production line, but changes in management practices and/or technological innovations might be more cost effective. In any event, to improve quality is also expensive. Note that we have made no allusion to holding cost; it is a meaningless concept for this application. It should be mentioned that this application was the motivational stimulus for the present paper.

\textbf{QUALITY CONTROL} A product's specifications call for a value, \( A \), on some criterion, or dimension. For any unit of product, let \( X \) denote its value on this criterion, and suppose that costs incurred from producing an item with an excess or a deficit relative to \( A \) are locally approximately linear. If the process
variability can be controlled, through sampling if necessary, with cost function, \( v \), then (4) is applicable, possibly with \( c = 0 \).

Suppose differently that product quality is an attribute, "good" or "defective," and that our goal is to ship batches of A good items. Let \( N \) denote the overall batch size, including defects; let \( \sigma \) denote the ratio of defective items; and let \( X \) denote the number of good items per batch. Then, \( X \) is binomially distributed with parameters \( N \) and \( 1-\sigma \). The costs associated with exceeding or falling short of \( A \) have obvious interpretations, and (4) is applicable using \( N \) in lieu of \( \mu \).

4. Cost Minimization

Let's find the critical points, \( (\mu^*, \sigma^*) \), of \( L \). Note that \( L_1(\mu, \sigma) = c + h - (p+h)F(\theta) \) and \( L_2(\mu, \sigma) = -(p+h)\int_{-D}^{\theta} yf(y)dy + v'(\sigma) \). Setting \( L_1 = 0 \) gives

\[
F((A-\mu^*)/\sigma^*) = F(\theta^*) = (c+h)/(p+h).
\]

(5)

Setting \( L_2 = 0 \) yields

\[
-v'(\sigma^*) = -(p+h)\int_{-D}^{\theta^*} yf(y)dy.
\]

(6)

For any \( \sigma^*>0 \), there is a unique solution to (5), although it's possible that \( \mu^*<0 \). If there exist a solution to (6), then it too is unique. To examine the second-order conditions, let \( \alpha = L_{11}(\mu^*, \sigma^*) \), \( \gamma = L_{22}(\mu^*, \sigma^*) \), and \( \beta = L_{12}(\mu^*, \sigma^*) \). Then, \( \gamma = (p+h)\sigma^2f(\theta^*)/\sigma^* + v''(\sigma^*) \) and \( \alpha - \beta^2 = (p+h)f(\theta^*)v''(\sigma^*)/\sigma^* \), both of which are positive due to the strict convexity of \( v \). Hence, the unique critical point satisfying (5) and (6) minimizes \( L \). Using the notation \( \mu^*, \sigma^* \) (and \( \theta^* \)) to denote the optimal solution, we just set them equal to the critical points, \( \mu^*, \sigma^* \) (and \( \theta^* \)). Plugging (5) and (6) into (4) gives

\[
L(\mu^*, \sigma^*) = cA - \sigma^*v'(\sigma^*) + v(\sigma^*),
\]

(7)
which, surprisingly, does not explicitly depend on $\mu^\circ$.

If the solution to (5) is negative, set $\mu^* = 0$ — the holding cost is so great relative to $p$ that we want the mean fill to be as far below $A$ as possible. (Because the notion of negative fill has little meaning in a bonafide single-period inventory problem, the constraint $\mu^* \geq 0$ may be easily replaced by $\mu^* \geq U$, say, where $U$ is chosen, in conjunction with $S$, such that $F_{U,S}(0) = F(-U/S)$ is appropriately small.) Let $R(h,p,c)$ denote the right hand side of (6). If there is no solution to (6), then either

$$-v'(S) \geq R(h,p,c), \quad \text{or}$$

$$-v'(0) \leq R(h,p,c). \quad (8)$$

In the former case, $L_2(\mu,\sigma) < 0$, so let $\sigma^* = S$, with $\mu^* = \mu^*$, if $\mu^* > 0$ in (5) with $\sigma^* = S$, and $\mu^* = 0$ otherwise. If (9) holds, then $L_2(\mu,\sigma) \geq 0$, so let $\sigma^* = 0$ and $\mu^* = A$. Then, $(\mu^*, \sigma^*)$ minimizes $L$. Essentially, what (8) ((9)) says is that quality is dear (cheap) relative to the expected deviation costs, so we allow maximum variability (eliminate variability altogether) in the fill.

Note the local behavior of $\mu^*$:

$$\mu^* > (\leq) A \text{ only if } [1-F(0)]h < (>) pF(0)-c. \quad (10)$$

The inequality indicates which error is more serious. Also, note the implications of a linear quality cost function. If $v(\sigma) = a(S-\sigma)$, for some $a > 0$, then either (8) or (9) holds and we should select either the most variable fill or fill exactly at $A$. The remainder of the paper is devoted to the asymptotic behavior of the optimal solution.

5. Asymptotic Behavior of Optimal Solution

1) AS FUNCTION OF $h$, WITH $B = \infty$ 

Because of the constraint, $\mu^* \geq 0$, the behavior of $\mu^*$ and $L(\mu^*, \sigma^*)$ as $h$ increases requires a fair amount of work, especially
if there is no upper bound on the support of $F$. By considering this case first, we should have an easier time with the subsequent analysis.

First, IGNORE THE CONSTRAINT THAT $\mu^* \geq 0$ and consider the extreme point, $(\mu^*, \sigma^*)$, as a function of $h$. To mitigate against rising holding costs, it seems reasonable to let $\sigma^* \to 0$, unless perhaps the quality cost becomes prohibitive, in which case we might let $\mu^*$ decrease, which in turn would yield larger shortage costs. It turns out that indeed $\sigma^* \to 0$, but the limiting behavior of $\mu^*$ depends on the behavior of $v$ near the origin. Consequently, we shall suppose $v$ varies regularly at the origin -- definitions and the necessary results are presented in the Appendix.

From (5), we have $\lim_{h \to \infty} \theta^* = B = \infty$ and $\partial \theta^*/\partial h = (p-c)/f(\theta^*)(p+h)^2 > 0$. Then, $R(h,p,c) = -\theta^*[1-F(\theta^*)] + \int_{\theta^*}^{\infty} yf(y)dy > 0$, so $R(\cdot,p,c)$ is increasing. Apply l'Hospital's rule to obtain

$$\lim_{h \to \infty} R(h,p,c) = (p-c)B = \infty. \quad (11)$$

Suppose $-v'(0) < \infty$, which, of course, implies that $v(0) < \infty$. Then, (9) holds for large $h$, so $\sigma^* = 0$ and $\mu^* = A$, a reasonable result. Quality is cheap compared to the holding cost, so $X = A w.p.l$; and the total expected cost is constant with respect to $h$.

Hereafter, assume that $-v'(0) = \infty$, which does not necessarily imply that $v(0) = \infty$. Then, (6) is soluble and $\lim_{h \to \infty} -v'(\sigma^*) = \infty$, from (11). Also, from (6),

we have $-v''(\sigma^*)\partial \sigma^*/\partial h = R_1(h,p,c) > 0$. From the convexity of $v$, $\sigma^*$ is decreasing, and we have established

$$\sigma^* \downarrow 0 \text{ as } h \uparrow \infty. \quad (12)$$

Suppose now that $v(0) < \infty$. Then, $v$ is slowly varying at the origin, and we easily obtain some intuitively appealing results. From (5) and (6),
\[-v'(\sigma^o) = R(h, p, c) > (p+h)\theta^o[1-F(\theta^o)] = (p-c)\theta^o, \text{ or } -\sigma^ov'(\sigma^o) > (p-c)(A-\mu^o).\]

From (A.1) and (12), \(-\sigma^ov'(\sigma^o) \to 0\). It follows from (10) that

\[\lim_{h \to \infty} \mu^o = A.\]  \hspace{1cm} (13)

From (7) we have

\[\lim_{h \to \infty} L(\mu^o, \sigma^o) = cA + v(0).\]  \hspace{1cm} (14)

Obviously, \(\mu^o = \mu^0\) and \(\sigma^o = \sigma^0\) for large \(h\), and the analysis is complete.

Hereafter, we shall also assume that \(v(0) = \infty\). Although \(v(\sigma^o) \to \infty\), we find that \(v(\sigma^o)\) is small relative to \(h\), i.e., \(v(\sigma^o) = O(h)\). Indeed,

\[\lim_{h \to \infty} v'(\sigma^o) = \lim_{h \to \infty} v'(\sigma^o)/\delta h = \lim_{h \to \infty} [-v'(\sigma^o)/v''(\sigma^o)]R_1(h, p, c).\]

Referring to the paragraph leading to (11), we see that

\[\lim_{h \to \infty} R_1(h, p, c) = \lim_{h \to \infty} \int_{x \to \infty}^{\infty} yf(y)dy - x(1-F(x)) = 0,\]

so it remains to bound \(-v'(\sigma^o)/v''(\sigma^o)\). If \(-\rho\) is the exponent of \(v\), then

\[\lim_{t \to 0} v(xt)/v(t) = x^{-\rho} = \lim_{t \to 0} v(xt)/v(t)\]

\[= \lim_{t \to 0} xv'(xt)/v'(t),\]

so \(v'\) is regular with exponent \(-(1+\rho)\). Write \(-v'(x) = x^{-(1+\rho)}M(x)\), where \(M\) is slow. Then,

\[-v''(x)/v'(x) = x^{-(1+\rho)}[1+\rho - xM'(x)/M(x)] \to \infty\]

as \(x \to 0\). Obviously, the inverse tends to zero, and we have established that

\[\lim_{h \to \infty} v'(\sigma^o)/h = 0.\]  \hspace{1cm} (15)

Next, it would be interesting to know what proportion of the expected cost is due to the improvement in quality. Indeed, from (7), (12), and (A.2),

\[\lim_{h \to \infty} L(\mu^o, \sigma^o)/v(\sigma^o) = 1 + \rho,\]  \hspace{1cm} (16)

where \(-\rho\) is the exponent of \(v\). If \(v\) is slow, then \(\rho = 0\) and, in the limit, \(v\) accounts for 100% of the total cost. If \(\rho > 0\), then there is a fraction, \(\rho/(1+\rho)\), of the cost.
due to the deviation from demand, \( A \). We shall interpret this fraction after giving some consideration to the behavior of \( \mu^0 \).

To this end, assume that

\[
\lim_{x \to \infty} \frac{f(x)}{[1-F(x)]} > 0. \tag{17}
\]

If \( F \) is nondecreasing failure rate (NDFR), at least on some right hand half-line, then (17) will be satisfied. Next, let

\[
K = \lim_{x \to 0} -x v'(x), \tag{18}
\]

where \( 0 \leq K \leq \infty \). The existence of the limit follows from the convexity of \( v \). Examples are given in the Appendix. Applying (17) gives

\[
\lim_{x \to \infty} x[1-F(x)]/\int_{x}^{\infty} yf(y)dy = \lim_{x \to \infty} [1-[1-F(x)]/xf(x)] = 1.
\]

Hence, \( 1 = \lim_{h \to \infty} \theta^*[1-F(\theta^*)]/\int_{\theta^*}^{\infty} yf(y)dy = \lim_{h \to \infty} (p-c)(A-\mu^0)/[-\sigma^*v'(\sigma^*)], \)

so \( (p-c)(A-\mu^0) \to K \) or

\[
\lim_{h \to \infty} \mu^0 = \frac{A - K}{(p-c)}. \tag{19}
\]

For the moment, think of \( K \) as a measure of how steeply \( v \) grows at the origin. Then, in the limit, \( \mu^0 \) is decreasing with \( K \). It's a compromising result. We want \( \sigma \) to vanish, in order to control holding costs. But, if \( \sigma \to 0 \) too fast, then \( v \to \infty \) rapidly, so we let \( \mu \) drift below demand, \( A \), thereby saving somewhat on quality while still containing the excess. Such "saving" is not free, however; a (net) shortage cost, \( (p-c)(A-\mu) \), is incurred. Indeed, we can now explain \( \rho > 0 \) in (16).

From (A.2), \( \rho = \lim_{x \to 0} -x v'(x)/v(x) \), so \( K = \infty \) and \( \mu^0 \to -\infty \). From the argument immediately preceding (19), we have

\[
\lim_{h \to \infty} (p-c)(A-\mu^0)/v(\sigma^*) = \rho. \tag{20}
\]
Asymptotically, the (net) shortage cost grows at a multiple \( \rho \) of the quality cost.

Obviously, (15) and (11) imply that \((A-\mu^o)/h\) tends to zero, but a much stronger convergence exists. Noting that \(-v'(\sigma^o)/h \to 0\) and slightly rearranging the limits leading to (19) give

\[
\lim_{h \to \infty} \theta^o/h = \lim_{h \to \infty} (A-\mu^o)/h\sigma^o = 0. \tag{21}
\]

Even as \(\sigma^o \downarrow 0\), the expected number of deviation units of demand shortfall is quite small compared to \(h\).

Now, REIMPOSE THE CONSTRAINT, \(\mu^* \geq 0\). If the limit in (19) is positive, or if zero is approached through a nonnegative subsequence, then there is nothing more to do. In particular, the problem has been solved in (13) and (14) if \(v(0) < \infty\), so assume \(v(0) = \infty\). If \(\mu^o < 0\), recall that we set \(\mu^* = 0\) and let \(\sigma^*\) solve

\[
-v'(\sigma^*) = -(p+h) \int_{-D}^{A/\sigma^*} yf(y)dy.
\]

Then, \(\sigma^* < \sigma^o\) and \(\theta^* = A/\sigma^* < \theta^o\). We want to see how \(L(0, \sigma^*)\) compares with \(L(\mu^o, \sigma^o)\), in the limit, for \(K < \infty\).

From (4), we have

\[
\lim_{h \to \infty} [L(0, \sigma^*) - v(\sigma^*)] = pA + \lim_{h \to \infty} h[1-F(A/\sigma^*)] + K.
\]

With a little work, we obtain

\[
\lim_{h \to \infty} h[1-F(A/\sigma^*)] = K/A, \text{ so }
\]

\[
\lim_{h \to \infty} [L(0, \sigma^*) - v(\sigma^*)] = pA. \tag{22}
\]

Thus, with \(\mu^* = 0\), \(\sigma^*\) is set small enough that the holding cost tends to zero, as it did in the unconstrained case, and (22) is merely the penalty cost. The difference in quality costs in the two cases is

\[
v(\sigma^*) - v(\sigma^o) < -(\theta^o - \sigma^o)v'(\sigma^*) < -\sigma^ov'(\sigma^*)(-\mu^o/A), \text{ so } \lim_{h \to \infty} [v(\sigma^*) - v(\sigma^o)] \leq -K + K^2/(p-c)A, \text{ from (19). Combining K with (7) gives a nice bound,}
\]
\[ \lim_{h \to \infty} [L(0, \sigma^*) - L(\mu^*, \sigma^*)] \leq [K - (p-c)A]^2 / (p-c)A. \]

This bound is tight -- just try \( K = 0 \). We have been unable to make a meaningful comparison of the two cases when \( K = \infty \).

2) **AS FUNCTION OF h, WITH B < \infty.** As in the previous subsection, it is relatively straightforward to show that \( \sigma^* \) is nonincreasing in \( h \) and that \( \theta^* \) is nondecreasing, with \( \lim_{h \to \infty} \theta^* = B \). Let \( \hat{\theta} = \lim_{h \to \infty} \sigma^* \). By monotonicity, we must have

\[ \hat{\mu} = \lim_{h \to \infty} \mu^* = A - B \hat{\theta}. \]

Intuitively, we could have started at this point, viz., \( B_{\hat{\mu}, \hat{\theta}} - (A-(B+D)\hat{\theta}, A) \). Unless \( \hat{\theta} = 0 \), which is expensive, the only way to keep the holding costs from exploding is, in the limit, to require the support of the fill distribution to lie below demand. Then, \( \hat{\theta} \) will be chosen to balance quality cost with shortage cost.

It is apparent from the preceding that \( \lim_{h \to \infty} L(\mu^*, \sigma^*) = L(A-B\hat{\theta}, \hat{\theta}) \). Let \( \lambda = L(\mu, \sigma) = L(A-B\sigma, \sigma) \). From (4), \( \lambda(\sigma) = cA + (p-c)B\sigma + v(\sigma) \), which we want to minimize, subject to the constraints \( A-B\sigma \geq 0 \) and \( \sigma \leq S \). Setting \( \lambda'(\sigma) = 0 \) yields

\[ -v'(\sigma) = (p-c)B, \]  

implied also by (6) and (11).

Suppose first there is no solution to (23) in the interval \([0, \min\{A/B, S\}]\). If \(-v'(0) < (p-c)B\), then setting \( \hat{\theta} = 0 \) minimizes \( \lambda \), with \( \lambda(0) = L(A, 0) - cA + v(0) \) which, of course, is finite. "Perfect" quality is (asymptotically) cheapest, with no penalty costs. If \( A/B < S \), then set \( \hat{\theta} = A/B \). The minimum limiting cost is then

\[ \lambda(A/B) = L(0, A/B) = pA + v(A/B). \]

The maximum shortage is cheapest, with no material cost. Obviously, if \( S < A/B \), then \( \hat{\theta} = S \) is the minimizer, and \( \lambda(S) = c(A-BS) + pBS \). Zero quality cost is cheapest, with some combination of material and shortage costs.

Otherwise, there is a unique solution, \( \hat{\theta} \), to (23), with \( 0 \leq \hat{\theta} \leq \min\{A/B, S\} \). The
minimum cost is

\[ \lambda(\hat{\sigma}) = L(A-B\hat{\sigma},\hat{\sigma}) = cA - \hat{\sigma}v'(\hat{\sigma}) + v(\hat{\sigma}), \]

which is identical to (7). Obviously, it's finite.

3) **AS FUNCTION OF p**  

The results are more or less symmetric to those in h without the nonnegativity constraint on \( \mu^* \), and the analysis is nearly identical. Consequently, for the sake of completeness, we list only the salient points, without embellishment.

For \( D = \infty \), (11) becomes \( \lim_{p \to \infty} R(h,p,c) = (c+h)D = \infty \) and (12) - (16) and (21) hold as is with h replaced by p. The assumption (17) now concerns the left tail of the distribution, viz.

\[ \lim_{x \to -\infty} \frac{f(x)}{F(x)} > 0, \quad (17)' \]

which will be satisfied if, for example, F is NDFR and symmetric. Then,

\[ \lim_{p \to \infty} \mu^* = A + K/(c+h), \quad (19)' \]

\[ \lim_{p \to \infty} (c+h)(A-\mu^*)/v(\sigma^*) = \rho, \quad (20)' \]

which illustrates how the (gross) holding cost behaves.

For \( D << \infty \), \( \sigma_{\mu^*} \), say; \( \sigma_{\mu^*}D \), and \( \mu^* \to A+D\sigma \). Thus, \( D_{\mu^*,\sigma^*}(A,A+(D+B)\sigma) \). The limiting quality, \( \sigma \), is given by the solution to

\[ -v'(\sigma) = (c+h)D, \quad (23)' \]

if it exist in \([0,S]\), or by \( \sigma = 0 \) or \( S \), otherwise.

4) **AS FUNCTION OF A**  

Differentiating (5) and (6) gives immediately that \( \partial \sigma^*/\partial A = 0 \) and \( \partial \mu^*/\partial A = 1 \), both of which are intuitively logical results.

We may write
\[ \mu^* = A - \delta \quad (24) \]

on \( A \geq \delta \), a constant w.r.t. \( A \). Indeed, let \( \delta = A - \mu^* \) which, from (5), means \( \delta = \sigma^* F^{-1} \left( \frac{c+h}{p+h} \right) \). According to (10), \( \delta \leq 0 \) if \( [1-F(0)]h \leq pF(0) - c \), so \( \mu^* = \mu^* \).

Otherwise, \( \delta > 0 \) and \( \mu^* = A - \delta \) on \( A \geq \delta \), which proves (24). Obviously, \( \mu^* = 0 \) if \( A < \delta \).

Differentiating (6) again, with \( \theta^* \) replaced by \( A/\sigma^* \), shows that \( \partial \sigma^*/\partial A > 0 \) on \( A < \delta \).

5) AS FUNCTION OF C

As \( c \uparrow p \), then \( \theta^* \uparrow \omega \) by (5) and \( R(h, p, c) \downarrow 0 \). Then, from (6), either \( \sigma^* = S \) for all large \( c \) or \( \sigma^* \rightarrow S \) as \( c \uparrow p \). In either event, \( \mu^* \rightarrow -D \). What's happening is that, by letting \( \mu^* \rightarrow -D \), the shortage cost is balanced by the savings in material, we can spend nothing on quality, and still the holding cost becomes nil, too. Invoking the nonnegativity constraint, we shall have \( \mu^* = 0 \) for all large \( c \) - we just don't want to purchase anything.

6. Summary

Our concern has been the minimization of an expected cost function, \( L(\mu, \sigma) \), which could arise in certain situations where both the mean and the precision of the fill distribution can be controlled. The minimizing parameter, \( (\mu^*, \sigma^*) \), is derived by conventional technique in Section 4. It is important to ascertain how the solution behaves as a function of the per unit holding and shortage costs, \( h \) and \( p \), respectively. In particular, as either \( h \) or \( p \) gets large, we show in Section 5 how to adjust \( \mu \) and \( \sigma \) accordingly. The amount to be spent on quality, in order to balance the increasing holding or shortage cost, depends in a rather complicated way on the behavior of the quality cost function, \( v(\sigma) \), near the origin. The asymptotic behavior of the minimal cost, \( L(\mu^*, \sigma^*) \), is also analyzed.
A nonnegative function, \( v \), defined on a positive neighborhood of the origin is said to vary slowly (at the origin) if

\[
\lim_{x \to 0} \frac{v(\xi x)}{v(x)} = 1
\]

for all \( \xi > 0 \). It is said to vary regularly (at the origin) with exponent \( -\rho \) if \( v(x) = x^{-\rho} L(x) \) and \( L \) is slowly varying. We need a couple results about the quality cost function, which we assume to be regular.

First, suppose that \( v \) is slow. Then,

\[
\lim_{x \to 0} -xv'(x)/v(x) = 0. \tag{A.1}
\]

To verify (A.1), note that \(-v'(x) < [v(x-\delta)-v(x)]/\delta\), for any \( \delta < x \), by monotonicity and convexity of \( v \). So, write \( \delta = tx \), \( t < 1 \). Because \( v \) is slowly varying, we then have

\[
\lim_{x \to 0} -xv'(x)/v(x) \leq \lim_{x \to 0} \left[ v((1-t)x)-v(x) \right]/tv(x) = 1/t-1/t = 0.
\]

Next, let \( v \) be regular with exponent \( -\rho \), i.e., \( v(x) = x^{-\rho}L(x) \). Then,

\[
\lim_{x \to 0} -xv'(x)/v(x) = \rho, \tag{A.2}
\]

and \( \rho \geq 0 \). The nonnegativity of \( \rho \) is obvious. Compute \(-xv'(x)/v(x) = \rho - xL'(x)/L(x)\). Then, (A.2) follows from (A.1) and the slowness of \( L \).

Examples: (1) Let \( v(x) = \log \log Se/x \). Then, \( v \) is slow and \(-xv'(x) \to 0\).

(2) If \( v(0) < \infty \), then \( v \) is slow and \(-xv'(x) \to 0\).

(3) Let \( v(x) = K \log S/X \). Then, \( v \) is slow and \(-xv'(x) = K\).

(4) Let \( v(x) = k(S-x)/Sx \). Then, \( v \) is regular with exponent \(-1\).

(5) Let \( v(x) = \exp(x^{-1}-S^{-1}) \). Then, \( v \) is regular with \( \rho = \infty \).
REFERENCES


