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ABSTRACT

The arrival times of what we shall call offers constitute a Poisson process. The value of any offer is a nonnegative random variable with known distribution. At an arrival epoch, we may select, reject, or hold the offer. Rejected offers may not be recalled, but an offer on hold is available for future consideration. However, cost accrues during the holding period. We seek a holding and selection strategy that maximizes the expected value of the offer selected less holding costs. The discrete time version of the problem is also considered.

1. INTRODUCTION AND SUMMARY

What we shall call "offers" arrive randomly over time and the value of an offer is itself a random quantity. After some time, t, there are no more arrivals. A rejected offer is irrevocably lost and we perforce wait for the next arrival, if there is one. As a hedge against the event that no better offer occur in the future, we may purchase an option to hold an offer, thereby keeping it available for our future consideration. Holding cost increases proportionately with time and is nondecreasing with an offer's value. At some future time, a held offer will be accepted or rejected in favor of a better one. We seek a decision policy which maximizes expected (net) return, which is the value of the offer accepted, if any, minus holding costs.

Scenarios which conform somewhat to our model could include both the acquisition and the disposition of assets: a prospector negotiating an option to purchase a parcel of land, or its mineral rights, thereby keeping it off the market while the search continues; or an enterprise aspiring to sell a property and offering an incentive, such as free rent, to a would-be purchaser for the privilege of later accepting the offer.

Our primary contribution is the notion of holding an offer to keep it available for possible future acceptance. Furthermore, it is crucial that the decision to hold be taken immediately after an offer's arrival - a rejected offer cannot be recalled. Were the holding action unavailable, we would have a pure stopping problem (PSP).

Apparently Karlin [1962] was among the first to investigate such a PSP. In his model, the arrival process could be a renewal process, but he later assumes it to be Poisson; and the values of the offers are i.i.d. Allowing for the possibility that $t = \infty$, Elfving [1967] considers a discounted version of the PSP. The theory and a couple examples are nicely presented by Chow et. al. [1971, pp. 113-118]. Problems in which multiple offers may be accepted have been considered by, among others, Sakaguchi [1976] and Stadje [1987], both of whom assume a Poisson arrival process.

A related but very specialized problem is the well-known secretary problem: the number of offers is known; at any arrival time, what is observed is not the offer's value but rather its relative rank among those offers already observed; and, finally, the objective is to maximize the probability of selecting the best offer, i.e., the so-called best-choice criterion. Yang [1974] suggested that we might attempt to recall a previously rejected offer, whose availability is uncertain and may be decreasing stochastically with time. A more general formulation of problems involving uncertain recall is provided by Petruccelli [1984]. Samuels [1985] suggested that a rejected offer could be recalled by paying a cost proportional to the time elapsed since its arrival. The idea of purchasing a call option in order to hold an arrival was introduced by Rose [1984]. Ferenstein and Enns [1988] employ this concept when the offers' values are observable i.i.d. random variables. The secretary problem with random (Poisson) arrivals, but with no mechanism for recall, was analyzed by Cowan and Zabczyk [1978]. Their work has been generalized by Bruss [1987], who supposes that the arrival process is nonhomogeneous Poisson. He also considers the inference problem associated with unknown intensity parameter.

In the present paper, the arrival process is Poisson with known intensity λ. The amounts of the offers are independent random variables with common distribution F. Assume that F is absolutely continuous and $F(0) = 0$. Holding an offer incurs cost at the rate $c > 0$.

The next section formalizes the model and introduces some notation. The structure of the optimal policy is obtained in Section 3; it is intuitively appealing. In Section 4, we obtain a surprisingly elegant expression for the optimal return expected from holding an offer. This expression is obtained by solving a renewal equation for its derivative. To evaluate the optimal return expected from passing an offer, we must compute the (lower) critical curve, the boundary between the optimal holding and rejecting support sets. We obtain a nicely succinct but nonlinear differential equation describing this boundary. As an approximation to the optimal policy, we propose the one-stage look ahead (OLA) policy.

The solution to the discrete-time model is presented in Section 5. We perceive a close relationship between it and the earlier results. Section 6 compares our optimal procedure to the optimal stopping rule for the equivalent PSP. Finally, in Section 7, we allow the holding cost rate to increase with the value of the offer held.

2. THE CONTINUOUS MODEL

Suppose that an offer in the amount x (hereafter called an x-offer) arrives with time t (or an (x,t)-offer) remaining. There are three actions: pass (reject), hold, and stop (accept). Obviously, the return from stopping is x. If the offer is passed, we merely wait for the next one, if there is one, and decide anew. If there isn't another offer, then the return is zero. We shall denote by $v(t)$ the maximum expected payoff that can be achieved by waiting for the next offer. Suppose we decide to hold the offer. We incur cost at a rate $c > 0$. If this were the last offer, then our return would be $x - ct$. If not, suppose the next offer arrives at time $t - s$ and has value y. Then, we have incurred cost cs and we are confronted with the same decision, pass or hold or stop, for a $(\max\{x,y\},t-s)$-offer. Note that holding ensures a value at least x at the next decision point. Denote by $w(x,t)$ the maximum expected net payoff obtained by holding an (x,t)-offer. Thus, the maximal return expected from the arrival of an x-offer at time t is

$$V(x,t) = \max\{x,v(t),w(x,t)\},$$

assuming that no previous offer were currently being held.

We shall employ the following notation:

$$V(t) = \int_0^\infty V(x,t)F(dx),$$

$$W(x,t) = F(x)V(x,t) + \int_x^\infty V(y,t)F(dy),$$

$$G(x) = 1 - F(x), \quad R(x) = \int_x^\infty yF(dy), \text{ and}$$

$$T(x) = \int_x^\infty (y-x)F(dy) = R(x) - xG(x).$$

The symbol "∞" as the upper limit of integration is used to denote the supremum of the support of F.

By definition,

$$v(t) = \lambda\int_0^t V(t-s)e^{-\lambda s}ds. \tag{1}$$

Differentiating (1) results in

$$v'(t) = \lambda[V(t)-v(t)], \quad v(0) = 0. \tag{2}$$

From (2), it is apparent that v is strictly increasing. Also by definition,

$$w(x,t) = xe^{-\lambda t} - (c/\lambda)[1-e^{-\lambda t}] + \lambda\int_0^t W(x,t-s)e^{-\lambda s}ds. \tag{3}$$

The first term in (3) represents our accepting the held x-offer if none other appears. The second term is the expected holding cost until the next arrival or $t = 0$. Differentiate (3) with respect to each variable to obtain

$$w_2(x,t) = \lambda[W(x,t)-w(x,t)] - c, \quad w(x,0) = x; \tag{4}$$

$$w_1(x,t) = e^{-\lambda t} + \lambda F(x)\int_0^t V_1(x,t-s)e^{-\lambda s}ds. \tag{5}$$

Finally, plugging $x = 0$ into (3) and noting that $W(0,\cdot) = V(\cdot)$, we obtain

$$v(t) = w(0,t) + (c/\lambda)\ [1-e^{-\lambda t}]. \tag{6}$$

3. THE FORM OF THE SOLUTION

The optimal policy may be characterized by the support set of each action. Let $S_t = \{x:V(x,t) = x\}$, $P_t = \{x:V(x,t) = v(t)\} - S_t$, and $H_t = \{x:V(x,t) = w(x,t)\} - (P_t \cup S_t)$. Note our tie-breaking convention that favors stop over pass over hold. We also write $S = \{S_t\}_{t\geq 0}$ and similarly for P and H. We refer to S or S_t, P or P_t, and H or H_t as the stop, pass, or hold sets respectively. The nature of the optimal policy is given by Lemmas 3, 4, and 5, but we first need some technical results.

LEMMA 1 For all t, $V(x,t)(w(x,t))$ is nondecreasing (strictly increasing) on $x \geq 0$.

PROOF By definition of V, it suffices to prove the assertion about w. Let $x > y$ and consider the difference, D, between holding an (x,t)-offer rather than a (y,t)-offer. If there is no arrival in $(0,t]$, then $D = x-y > 0$. Suppose the next offer occurs at time $t-s$. If it's a z-offer with $z \geq x$, then $D = 0$. If $z < x$, then $y_1 = \max(z,y) < x$. In this case, we shall employ a suboptimal strategy for deciding about x: do whatever is optimal for a $(y_1,t-s)$-offer. If $y_1 \in P_{t-s}$, then $D = 0$, and if $y_1 \in S_{t-s}$, then $D = x-y_1 > 0$. If $y_1 \in H_{t-s}$, then go to the next arrival (or time zero) and reapply the same suboptimal strategy if necessary. We are assured that $D \geq 0$, and $P(D > 0) > 0$, so $w(x,t)-w(y,t) \geq ED > 0$. ∎

LEMMA 2 On $t > 0$, $w_1(x,t) < 1$.

PROOF Use an argument very similar to that of Lemma 1. After holding at (y,t), apply a suboptimal policy which will stop whenever stop is the optimal action for the sample path starting at (x,t). Then, $D \leq x-y$ and $P(D=0) > 0$, so $w(x,t)-w(y,t) < x-y$. ∎

It follows almost immediately that the stop and pass sets are value-connected, with the stop set lying above the pass set. More precisely, we state without proof

LEMMA 3 (a) If $y \in P_t$, then $x \in P_t$ for $x < y$.

(b) If $y \in S_t$, then $x \in S_t$ for $x > y$.

The hold set, if it's not null, must lie between the pass and stop sets. We shall refer to the locus of points separating the hold and stop (pass) sets as the upper (lower) boundary, b (a), i.e., if $H_t \neq \phi$, then $b(t)$ ($a(t)$) is the unique solution of $w(x,t) = x$ ($w(x,t) = v(t)$). Note that the uniqueness follows from the first two lemmas. Note too that $a(0) = 0$.

It turns out that the upper boundary is a straight line, $b(t) = y_o$. At the value y_o, the expected rate of incremental return, $\lambda T(x)$, and the cost rate, c, are in balance. See Chow et. al. [1971, p. 118, example (a)] for an identical result for a PSP with infinite horizon and discounted returns. The final lemma shows that the hold set is also time-connected. Obviously, it follows that $a(\cdot)$ is nondecreasing. Let's denote the mean offer amount by $\mu = EX$, where X is a random variable with distribution F.

LEMMA 4 If $c \geq \lambda\mu$, then $H = \phi$. If $H_t \neq \phi$, then $b(t) = y_o$, the unique solution of $\lambda T(x) = c$.

PROOF By definition, $w(b(t),t) = b(t)$, so

$$b'(t) = w_2(b(t),t) + b'(t)w_1(b(t),t). \tag{7}$$

Now, $V(b(t)) = b(t)$ and $W(b(t),t) = E\max\{X,b(t)\}$, which we substitute into (4) to get $w_2(b(t),t) = \lambda T(b(t)) - c$. Thus, $b(t) = y_o$ is a solution to (7). Plugging this solution into (3) gives us $w(y_o,t) = y_o$, which establishes the second part. If $c \geq \lambda\mu$, the preceding argument gives $y_0 \leq 0$, so $H_t = \phi$. ∎

LEMMA 5 If $x \in H_t$, then $x \in H_s$, $s < t$.

PROOF By Lemma 4, $x \notin S_s$. Let $r = \max\{s < t : v(s) = w(x,s)\}$. If we agree that $\max \phi = 0$, then r is well-defined and $r \geq 0$. By continuity we have $w(x,s) > v(s)$ on $s > r$. From (6), $x > 0$, so $w(x,0) = x > v(0) = 0$. Therefore, if the lemma were false, we must have $r > 0$. With $x \notin S_r$ and $w(x,r) = v(r)$, it follows that $W(x,r) \leq V(r)$. From (4) and (2), we obtain $w_2(x,r) \leq v'(r) - c < v'(r)$. It must follow that $v(s) > w(x,s)$ in an interval with $s > r$. This contradiction implies that $r = 0$. ∎

4. THE SOLUTION

If $x \in H_t$, then Lemma 5 allows w to replace V in (5) which, except for the factor $F(x)$, would be a renewal equation for $w_1(x,\cdot)$. Multiply both sides of (5) by $\exp(\lambda G(x)t)$ to get

$$u(t) = e^{-\lambda F(x)t} + \int_0^t u(t-s)\lambda F(x)e^{-\lambda F(x)s}ds,$$

a proper renewal equation for $u(t) = \exp(\lambda G(x)t)\cdot w_1(x,t)$. Its solution is $u(t) = 1$ or

$$w_1(x,t) = e^{-\lambda G(x)t}. \tag{8}$$

Integrating (8) yields

$$w(x,t) = y_o - \int_x^{y_o} e^{-\lambda G(y)t}dy, \quad x \in H_t. \tag{9}$$

Now, substitute (9) into (3) to verify w.

On the lower boundary, $w(a(t),t) = v(t)$, $H_t \neq \phi$. Given $a(t)$, use (9) to compute

$$v(t) = y_o - \int_{a(t)}^{y_o} e^{-\lambda G(y)t}dy. \tag{10}$$

It remains to compute a.

Differentiate (10) and then integrate by parts, recalling that $w(y_o, t) = y_o$ and $w(a(t),t) = v(t)$, and we obtain $v'(t) = \lambda y_o G(y_o) - \lambda v(t)G(a(t)) + a'(t)\exp(-\lambda G(a(t))t) + \lambda\int_{a(t)}^{y_o} w(y,t)F(dy)$. From (2), $v'(t) = \lambda v(t)F(a(t)) + \lambda\int_{a(t)}^{y_o} w(y,t)F(dy) + \lambda R(y_o) - \lambda v(t)$. Equating these two expressions for v' yields $a'(t)\exp(-\lambda G(a(t))t) = \lambda R(y_o) - \lambda y_o G(y_o) = \lambda T(y_o) = c$, so

$$a'(t) = ce^{\lambda G(a(t))t} \quad (11)$$

We are unable to solve (11). Note that $a'(0) = c$ and $a'(t) > c, t > 0$.

If t is large enough, there is no incentive to hold. Define $t_o = \inf\{t: H_t \neq \phi\}$. At the point (y_o, t_o), we have $v(t_o) = a(t_o) = y_o = w(y_o, t_o)$ -- we are indifferent among all three actions. For the moment, let's define a new time origin at t_o and let $u(t) = v(t_o + t)$, $t > 0$. For this PSP, modify the argument in Chow et. al. [1971, pp.115-117], to show that u is the unique solution of

$$u'(t) = \lambda T(u(t)), \; u(0) = y_o. \quad (12)$$

The first offer (x,t) with $x \geq u(t)$ should be accepted.

Return now to $t < t_o$. In principle we should be able to compute $v(t)$ by solving (11) and using (10). At first glance, the relation (6) looks promising, but $w(0,t)$ is just as difficult to compute as $v(t)$ itself. Out of frustration, we might use (9) as a lower bound for $w(0,t)$, even though $x = 0 \notin H_t$. Another approach is to solve for an attractive suboptimal policy. Hence, we shall obtain the so-called one-stage look ahead (OLA) policy, which is itself optimal for a large class of stopping problems. As we shall see, these two approaches are equivalent.

In the present context, the OLA policy is defined as follows. We say that an offer is "captured" if it is held or accepted. If an offer is held, then any subsequent better offer must also be captured. Now, suppose that an x-offer arrives at time t. Obviously, if $x \geq y_o$, we stop, so assume that $x < y_o$. Let $\tilde{w}(x,t)$ denote the expected return from holding, and let $g(t)$ be the expected return from rejecting the current offer and capturing the next arrival, whatever its value, provided there is another. Then, the OLA policy prescribes hold if $\tilde{w}(x,t) > g(t)$ and pass otherwise.

According to Lemma 5, the OLA policy and the optimal policy behave identically once they have already held an offer. Hence, $\tilde{w}(x,t) = w(x,t)$ if

$x \in H_t$. Note too that the derivation of (9) is still applicable if the condition $x \in H_t$ is dropped and we substitute the functions $\tilde{w}$ for w. Thus, $\tilde{w}(x,t)$ equals the RHS of (9) for <u>all</u> $x < y_o$.

We can now write

$$g(t) = \int_0^t \lambda e^{-\lambda s} ds \{\int_0^{y_o} \tilde{w}(x,t-s)F(dx) + R(y_0)\}.$$

We omit the details but outline the order of the calculations. Substituting from (9) for $\tilde{w}$, we need to integrate w.r.t. y, say. Reverse the order of integration between y and x, obtaining an integral w.r.t. y alone. Now, reverse the order between y and s to obtain again an integral w.r.t. y alone and which is exactly the integral in (9) with $x = 0$. It turns out that

$$g(t) = \tilde{w}(0,t) + (c/\lambda)\ [1-e^{-\lambda t}].$$

Note the similarity to (6). Thus, to use the lower boundary of the OLA policy as a substitute for a(t) is equivalent to substituting $\tilde{w}$ for w. The resulting lower bounds for a(t) and v(t) might be fairly good if t isn't too large.

5. THE DISCRETE MODEL

Assume now that the number, n, of offers is known and that they occur at regular intervals, $t = 1, \ldots, n$. (We shall now be counting forward with the discrete time parameter, t.) The only other notational changes are that c is the single period holding cost and that implicitly $\lambda = 1$ -- one arrival per period w.p.1. We can employ discrete analogues of our earlier (continuous) methods to attack this problem. That n is known and the opportunity to invoke inductive arguments make the work easy. Consequently, we omit the analysis altogether.

The form of the solution is intact. In particular, if $H_t \neq \phi$, then $b(t) = y_o$, where $T(y_o) = c$. Also, if $x \in H_t$,

$$w(x,t) = y_o - \int_x^{y_o} F(y)^{n-t} dy. \qquad (13)$$

Note the close resemblance between (9) and (13). The integrand in (9) is the probability of no more offers greater than y, and that is exactly the integrand of (13). For $t < n-1$ and $H_t \neq \phi$, the lower boundary satisfies

$$\int_{a(t+1)}^{a(t)} F(x)^{n-t} dx = c. \qquad (14)$$

For some $a(t+1) < x_o < a(t)$, we get $a(t) - a(t+1) = cF(x_o)^{-(n-t)}$, or the difference equals cost times (approximately) the inverse of the probability that no more offers will exceed the boundary. The prosaic description of (11) is identical, except for the derivative in lieu of the difference.

At $t = n - 1$, only one offer remains. Obviously, $v(n-1) = \mu$ and the lower boundary satisfies $v(n-1) = w(x,n-1)$ or

$$\mu = y_o - \int_{a(n-1)}^{y_o} F(x)dx.$$

Because n is known, the lower boundary doesn't drop to zero.

6. THE PURE STOPPING PROBLEM

We employ the tilde symbol, $\tilde{}$, over notation from Sections 2 through 4 to represent the corresponding element from the associated PSP, i.e., no holding action available. It is known that

$$\tilde{v}'(t) = \lambda T(\tilde{v}(t)),\ \tilde{v}(0) = 0,$$

as we indicated in (12) with different initial condition. Differentiating again shows that $\tilde{v}$ is strictly concave. Unless $H = \phi$, $\tilde{v}(t) < v(t)$ on $t > 0$.

We want to compare $\tilde{v}$ and a. Note first that $\tilde{v}'(0) = \lambda T(0) = \lambda\mu > c = a'(0)$, from (11). Differentiating (11), we find that a is strictly convex in an open neighborhood of the origin. (Indeed, if we assume that F is an increasing failure rate distribution, then it can be demonstrated that a possesses a single point of inflection.) It follows that $a(t) < \tilde{v}(t)$ on $t < t_1$, for some $t_1 > 0$. By the definition of t_o, $a(t_o) = y_o = v(t_o)$, so there is $t_2 < t_o$ such that $a(t) > \tilde{v}(t)$, $t_2 < t < t_o$.

Suppose that $a(t) < x < \tilde{v}(t)$ ($\tilde{v}(t) < x < a(t)$) for $0 < t < t_1$ ($t_2 < t < t_o$). Then, according to the pure stopping rule, an (x,t)-offer should be rejected (accepted), but it should be held (rejected) by our policy. Obviously, if $\tilde{v}(t) < x < v(t)$ for $t > t_o$, the pure stopping rule accepts while we reject. Thus, when t is large enough, we can afford to be more demanding than the pure stopping procedure. (Originally, we had anticipated that, if $H_t \neq \phi$, then $a(t) \leq \tilde{v}(t)$; or, equivalently, an offer accepted by the pure stop rule would be captured by our rule.) For small t, we can afford to spend (a maximum of ct) as a hedge against the prospect of no future arrivals, even though the pure stop rule would reject the offer.

7. GENERAL HOLDING COST

We allow the holding cost rate to depend on the value of the offer held. Equations (3) and (4) remain intact if c is replaced by $c(x)$, but (5) becomes

$$w_1(x,t) = e^{-\lambda t} - (c'(x)/\lambda)[1-e^{-\lambda t}] + F(x)\int_0^t \lambda e^{-\lambda s} V_1(x,t-s)ds. \quad (15)$$

We have assumed that $c \in C^1[0,\infty)$ with $c'(x) \geq 0$. Lemma 2 is proved as before and Lemma 3(b) follows immediately. However, without Lemma 3(a), we cannot assert that H lie between P and S, so Lemma 4 needs modification. Let y_o solve $\lambda T(x) = c(x)$ and redefine t_o as the solution of $v(t) = y_o$.

LEMMA 4′ If $t < t_o$, then $S_t = [y_o,\infty)$ and there exists $y_t < y_o$ such that $(y_t, y_o) \subset H_t$. If $t > t_o$, then $S_t = [v(t),\infty)$ and $\overline{H}_t \cap S_t = \phi$.

PROOF If we ignore the passing action and pretend the decision is between only stop and hold, then the proof of Lemma 4 shows that $x(t) = y_o$ is the critical curve. Let $t < t_o$, so $v(t) < y_o$. We know $w(y_o,t) = y_o$, so continuity implies that $(y,y_o) \subset H_t$ for some $y < y_o$. If $t > t_o$, then $v(t) > y_o$, Lemma 2 ensures that $w(v(t),t) < v(t)$, and the lemma is proved. ∎

As before, then, $b(t) = y_o$ is the boundary between H and S. Finally, the proof of Lemma 5 holds, assuming $c(x) > 0$. However, if $c(x) = 0$ for some $x > 0$, then $w(x,s) > v(s)$, so $r = 0$ at the outset. Thus, the time-connectedness of H is maintained.

If $x \in H_t$, then Lemma 5 again gives $V_1(x,\cdot) = w_1(x,\cdot)$ on $(0,t]$ and (15) would also be a renewal equation were the factor $F(x)$ absent. Using the method of Section 4, we get the solution

$$w_1(x,t) = e^{-\lambda G(x)t} - (c'(x)/\lambda G(x))[1 - e^{-\lambda G(x)t}]. \quad (16)$$

Lemma 3(a) is not generally true. Instead, we offer the following.

LEMMA 6 A necessary and sufficient condition for $y \in P_t$ to imply $x \in P_t$, $x < y$, is that $w_1(x,t) \geq 0$ for all $x \in \overline{H}_t$.

PROOF The proof relies heavily on the continuity of the return functions. Lemmas 4′ and 5 imply that $H_t \cap [y_o,\infty) = \phi$. For any $0 < x < y_o$, there exists $t(x) > 0$ such that $w(x,t) > (\leq)\ v(t)$ on $t < (\geq)\ t(x)$. Clearly a N.A.S.C. for P_t to be value-connected for all t is that $t(x)$ be nondecreasing. Suppose $x \in H_t$ and $w_1(x,t) < 0$. From (16), $w_1(x,s) < 0$ for all $s > t$, and in particular $w_1(x,t(x)) < 0$. It follows that $v(t(x)) > w(y,t(x))$ for $y \in (x,x+\varepsilon)$, for some $\varepsilon > 0$. In turn, given $y \in (x, x + \varepsilon)$, there exists $\delta > 0$ such that $v(t) > w(y,t)$ for $t \in (t(x)-\delta, t(x))$. Hence $t(y) < t(x)$ and necessity is established. Sufficiency is trivial. ∎

The condition $w_1(x,t) \geq 0$ is equivalent to

$$c'(x) \leq \lambda G(x)e^{-\lambda G(x)t}/(1 - e^{-\lambda G(x)t}), \quad x \in H_t. \tag{17}$$

The RHS of (17) decreases in t, so it is sufficient that it hold for $t = t_o$. We don't know t_o, but we could use $\tilde{t}_o$, the solution to $\tilde{v}(t) = y_o$ from the PSP, for $\tilde{t}_o > t_o$. More easily yet, we can bound $\tilde{t}_o$ as follows. By concavity, $\tilde{v}'(t) > \tilde{v}'(\tilde{t}_o) = \lambda T(\tilde{v}(\tilde{t}_o)) = \lambda T(y_o) = c(y_o)$, so let $\hat{t} > \tilde{t}_o$ be given by $c(y_o)\hat{t}_o = y_o$. An easy sufficient condition, then, is to use $t = y_o/c(y_o)$ in (17).

Assume that (17) holds. Then, H lies between S and P and the function t introduced in the proof of Lemma 6 is the inverse of the lower boundary function, a; i.e., $a(t(x)) = x$. As before, differentiate $w(a(t),t) = v(t)$ to get

$$a'(t)w_1(a(t),t) = c(a(t)),$$

which reduces to (11) when $c(x) \equiv c$.

As a simple example, consider the discrete problem with $n = 2$ and $F(x) = x$, $0 \leq x \leq 1$. The corresponding discrete analogue of (17), with $t = 1$, is $c'(x) \leq x$. This condition is also easy to verify directly, but we omit it. Obviously, $v(1) = 1/2$ and $w(x,1) = 1/2 + x^2/2 - c(x)$, so $x \in H_1$ iff $c(x) < x^2/2$. Certainly $c(x)$ can be contrived to oscillate about $x^2/2$, even up to countably many times, and H_t will then be the union of disjoint intervals.

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