2002

Generalized Analytic Continuation

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University
LECTURE
Series

Volume 25

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American Mathematical Society
Providence, Rhode Island

Preface

In these notes, we shall present a body of mathematical work devoted to generalizing the classical Weierstrassian notion of analytic continuation. These works by Poincaré, Borel, Beurling, Tumarkin, Gončar, and others have been undertaken by various methods and for a variety of reasons, but for brevity, we will refer to them as studies in 'generalized analytic continuation' (GAC). To better explain the motivation behind our assembling this little book, we employ an analogy with the, now classical, subject of divergent series.

In the 18th century, there was little agreement on how to attach an appropriate number s to represent the 'sum' $a_0 + a_1 + a_2 + \cdots$ of complex numbers a_i. From G. H. Hardy's *Divergent Series* [**68**, p. 5]:

> It is plain that the first step towards such an interpretation must be some definition, or definitions, of the 'sum' of an infinite series,. . . . This remark is trivial now: it does not occur to a modern mathematician that a collection of mathematical symbols should have a 'meaning' until one has been assigned to it by definition. It was not a triviality even to the greatest mathematicians of the eighteenth century. They had not the habit of definition: it was not natural to them to say, in so many words, 'by X we *mean* Y'. There are reservations to be made... but it is broadly true to say that mathematicians before Cauchy asked not 'How shall we *define* $1 - 1 + 1 - 1 + \cdots$?' but 'What *is* $1 - 1 + 1 - 1 + \cdots$?', and that this habit of mind led them into unnecessary perplexities and controversies which were often really verbal.

Some order was established in this regard when Cauchy first defined, in a way that is now standard, the notion of a 'limit' of a sequence, allowing him to define the sum $a_0 + a_1 + a_2 + \cdots$ to be the number s if

$$\lim_{n \to \infty} (a_0 + a_1 + \cdots + a_n) = s.$$

This definition banished series such as $1 - 1 + 1 - 1 + \cdots$ to the realm of 'divergent' series.

However, various investigations pointed towards the usefulness of certain divergent series. From James Pierpont:

> It is indeed a strange vicissitude of our science that those series which early in the century were supposed to be banished once and for all from rigorous mathematics should, at its close, be knocking at the door for readmission.

For example, the initial treatment (by Fourier) of Fourier series operated extensively with divergent series. As another example, important series expansions arising in celestial mechanics - and also in connection with the Euler-Maclaurin summation formula - turned out to diverge, yet gave useful results if truncated suitably, a phenomenon later explained by Poincaré and others in the theory of asymptotic series. Still further, Euler, Borel, and Mittag-Leffler had devised procedures which 'summed' certain Taylor expansions in regions outside their circle of convergence to the value of the analytically continued function [**75**, p. 74]. Even further, after the disappointing discovery, by du Bois-Reymond, that the Fourier series of a continuous function could diverge at some points, order was restored in a remarkable way by L. Fejér's discovery that none the less, the arithmetic means of those partial sums do converge uniformly to the function. Lebesgue proceeded further and showed that the same arithmetic means formed from the partial sums of the Fourier series of a merely integrable function converge almost everywhere to the function. These are only some examples. A more detailed discussion can be found in [**68**].

As various methods of 'summing' divergent series proved their worth, and also became systematized by Toeplitz and others, a new 'modern' point of view emerged: there was no need for polemics as to what was the 'correct' sum of a series like $1 - 1 + 1 - 1 + \cdots$ Indeed, various reasonable looking summation procedures lead to different answers. Instead, what was truly needed was a classification and a comparison of the various summation procedures of Abel, Cesàro, Borel, and Euler, and an even greater emphasis on their applications.

Among the new challenges in the field of divergent series was the task of proving that the various new types of summation were compatible with Cauchy's definition of the sum in the case where the series was convergent in the first place. These results are generally referred to as 'Abelian' theorems since the first was a result of Abel asserting that if $a_0 + a_1 + a_2 + \cdots$ converges to s then

$$\lim_{x\to 1^-} \sum_{n=0}^{\infty} a_n x^n = s,$$

with the above formula being the 'Abel sum' of the a_i's. A similar type of result holds for Cesàro convergence, that is

$$\lim_{n\to\infty} \frac{s_0 + s_1 + \cdots + s_n}{n+1} = s,$$

where $s_n = a_0 + a_1 + \cdots + a_n$, and the above formula being the 'Cesáro sum' of the a_i's. Another important task was to explore the converse question, and more generally the compatibility of different types of summability - what are referred to as 'Tauberian' theorems [**68**, p. 149]. For example, a result of Tauber says that if a sequence $\{a_n\}$ is Abel summable to s, as above, and $a_n = o(1/n)$, then $a_0 + a_1 + \cdots = s$.

Though there was initially little unity in the subject of divergent series, since the motivations as well as the methods were diverse, unifying studies began to appear in the work of Toeplitz and others. Finally, a comprehensive synthesis of the whole field, including historical background, was done by G. H. Hardy in his masterful work *Divergent Series* [**68**].

In much the same way as Euler, Abel, Borel, Cesàro and others attempted to extend the notion of convergence of a series, there were those who attempted to extend the notion of analytic continuation of a function f. In brief, 'generalized analytic continuation' (GAC), as the name suggests, studies ways in which the component functions $f|\Omega_j$ - where f is a meromorphic function on a disconnected open set Ω in the complex plane $\mathbb{C}$ and $\{\Omega_j\}$ are the connected components of Ω - may possibly be related to each other in certain cases where the Weierstrassian notion of analytic continuation says there is a 'natural boundary'. We say that $\partial\Omega_j$, the boundary of Ω_j, is a 'natural boundary' for the component function $f|\Omega_j$ if $f|\Omega_j$ does not have an analytic continuation across any point of $\partial\Omega_j$. As an example of what is meant here, consider the function f defined by the series

$$f(z) = \sum_{n=1}^{\infty} \frac{c_n}{z - e^{i\theta_n}},$$

where $\{e^{i\theta_n}\}$ is a sequence of points everywhere dense in the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and $\{c_n\}$ is an absolutely summable sequence of non-zero complex numbers. This function is analytic on $\mathbb{C}_\infty \backslash \mathbb{T}$ and in 1883, Poincaré [**106**] was able to prove that $\mathbb{T}$ is a natural boundary for both $f|\mathbb{D}$ and $f|\mathbb{D}_e$, where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the unit disk and $\mathbb{D}_e = \{z \in \mathbb{C}_\infty : 1 < |z| \leq \infty\}$ is the extended exterior disk.

Certainly, if Ω_j and Ω_k are adjacent components (they share a common boundary arc) and $f|\Omega_j$ is an analytic continuation of $f|\Omega_k$ across a portion of that common boundary arc, then $f|\Omega_j$ and $f|\Omega_k$ are related, and, by the uniqueness of analytic continuation, each uniquely determines the other. However, there are other types of 'continuations', or ways of relating $f|\Omega_j$ and $f|\Omega_k$, beyond analytic continuation. From É. Borel's work [**28**, p. 100], where such ideas began to be studied:

> ...we wished only to show how one could introduce into the calculations analytic expressions whose values, in different regions of their domain of convergence, are mutually linked in a simple way. It seems, on the basis of that, that one could envision extending Weierstrass' definition of *analytic function* and regarding in certain cases as being [parts of] *the same function*, analytic functions having separate domains of existence. But for that it is necessary to impose restrictions on the analytic expressions one considers, and because he did not wish to impose such restrictions Weierstrass answered in the negative [this] question:
>
> "Therefore the thought was not to be ignored, as to whether in the case where an arithmetic expression $F(x)$ represents different monogenic functions in different portions of its domain of validity, there is an essential connection, with the consequence that the properties of the one should determine the properties of the other. Were this the case, it would follow that the concept 'monogenic function' must be widened."- (Weierstrass, *Mathematische Werke*, vol. 2, p. 212)
>
> It is not possible for us to give to this Chapter a decisive conclusion; for, in our opinion, the question addressed here is not entirely resolved and calls for further research. We would be content if we have convinced our readers that neither the fundamental works of Weierstrass, nor the later ones of Mittag-Leffler, Appell, Poincaré, Runge, Painlevé entirely answer the question

> as to the relations between the notions of analytic function, and analytic expression. One can even say without exaggeration, that the classification of analytic expressions which are incapable of representing zero [on some domain] without doing so everywhere, is yet to be brought to completion.

For an example of a 'continuation' beyond analytic continuation, we return to the above Poincaré example

$$f(z) = \sum_{n=1}^{\infty} \frac{c_n}{z - e^{i\theta_n}}.$$

Even though the component functions $f|\mathbb{D}$ and $f|\mathbb{D}_e$ are not analytic continuations of each other across any point of $\mathbb{T}$, they can, in light of more modern complex analysis techniques, be regarded as 'continuations' of each other. Indeed, $f|\mathbb{D}$ and $f|\mathbb{D}_e$ are both in the H^p $(0 < p < 1)$ spaces of their respective domains and moreover,

$$\lim_{r\to 1^-} f(re^{i\theta}) = \lim_{r\to 1^+} f(re^{i\theta})$$

for almost every θ. Borel [**27, 28**] began to generalize Poincaré's example to functions of the type

$$f(z) = \sum_{n=1}^{\infty} \frac{c_n}{z - z_n},$$

where $\{z_n\}$ is a sequence of points which need not *lie* on $\mathbb{T}$ but merely *accumulate* on all of $\mathbb{T}$, and further explored the relationship between the component functions $f|\mathbb{D}$ and $f|\mathbb{D}_e$. This investigation of 'coherence', often in various other settings, was continued in the 1920's and 1930's by Walsh and more recently in the 1960's and 1970's by one of the current authors as well as Tumarkin and Gončar.

An important insight emerged from these pioneering works: if a sequence of rational functions converges uniformly on compact subsets of a disconnected open set - as the partial sums of the above series do - and the approximants are restricted by either of two very different types of requirements, then the limit functions on the components of the open set exhibit a kind of mutual 'coherence'. In particular, they mutually determine one another. These two types of requirements are, on the one hand, a sufficiently rapid convergence as related to the degree of the rational function (a line pursued by Borel, Walsh, and Gončar), and on the other, a geometric restriction of the locations of the poles, without regard to the speed of convergence (a line pursued by Walsh and Tumarkin). One of the current authors and his co-workers later revealed a surprising connection between Tumarkin's results and a problem in operator theory, namely the classification of the cyclic vectors for the backward shift operator on the Hardy space.

The results we have assembled in this little book possess a mathematical richness and potential to constitute an interesting field in much the same way as divergent series. But we are not there yet. To again pursue the analogy with divergent series, so far we have a variety of proposed schemes for continuing (classes of) noncontinuable analytic functions and 'Abelian' theorems, guaranteeing they will produce the correct result when applied to functions that already possess ordinary analytic continuations. But the analog of the Tauberian theorems is almost wholly lacking. For example, it is not known, at least to us, whether pseudocontinuation and Gončar continuation, as defined in this book, applied to the same function can yield incompatible results. There is also another interesting analogy with divergent series. Just as the Tauberian theorems showed that some divergent series, for example, $1/2 + 1/3 + 1/4 + \cdots$, are, roughly speaking, 'universally divergent' (they cannot be summed by any of the natural methods), some Taylor series, such as the ones which have an isolated winding point on their circle of convergence, are 'universally non-continuable' and do not allow a GAC by any method satisfying a few natural conditions. This limitation of the scope of GAC is a sign that we are dealing with a *bona fide* concept here since it cannot be stretched to make 'everything' continuable.

It is our hope that the present book, which juxtaposes results in a way that has not been done before, may help pave the way towards the study of GAC in a unifying context. We feel our pursuit is worthwhile since for one, much of the mathematics involved here is quite beautiful but unfortunately widely scattered throughout the literature, often in sources difficult to obtain. Secondly, it involves a central notion in function theory, analytic continuation, which historically has given rise to heated polemics. In the same spirit as the modern theory of summability methods for divergent series, there is nothing controversial here, only technical questions of how different continuation schemes relate to one another and to analytic continuation. Thirdly, many attributes of analytic continuation can be studied afresh in the context of one or another notion of GAC, such as the overconvergence theorems of Ostrowski or noncontinuability of Taylor series with various types of gaps. Fourth, GAC appears in one form or another in the study of the backward shift operator on many Banach spaces of analytic functions on the unit disk. We have already mentioned this with regard to the Hardy space. Recently however, several authors have employed GAC in their investigations of the backward shift on other function spaces such as the Bergman and Dirichlet spaces. In fact, a new type of 'continuation' to be introduced in these notes leads to further progress in this context. Finally, GAC appears, almost surprisingly, in the study of electrical networks (the Darlington synthesis problem) as well as in questions related to linear differential equations of infinite order.

Though the subject of GAC is not ready for a *Divergent Series* like treatise, we feel this humble offering begins to organize the results in a coherent way and presents the future author of such a treatise with some open questions that need to be answered before such a comprehensive work can be written.

We wish to give our warmest thanks to Dima Khavinson, Genrikh Tumarkin, and Lawrence Zalcman who read earlier drafts of this manuscript and provided valuable suggestions and corrections. We also thank Evgeny Abakumov for introducing us to Aleksandrov's papers on gap series.

We will be keeping any updates/corrections/additions at

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We welcome your comments.

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