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Portfolio Optimization Tools in Excel

The calculus and matrix algebra associated with finding the optimal portfolio weights for a set of securities is tedious. However, Excel tools make the computations simple, with minimal programming needed to arrive at optimal portfolio weights for securities in a portfolio. We provide this Excel template and techniques for acquiring optimal weights, which is useful for personal and institutional investors alike.

Takeaways:

- The calculus and matrix algebra associated with optimizing portfolio weights is tedious, but not necessary for finding optimal portfolio weights
- Based on the results of the associated mathematics for optimizing portfolio weights, a very accessible Excel template can be produced to perform the optimization for an individual investor. Excel's =MDETERM function makes the programming very easy.
- Although demonstrated using three funds and a risk-free rate, the Excel template can be extended to allow for many more funds and/or securities.

INTRODUCTION

It is not uncommon for individual investors to hold multiple securities or multiple funds even if they do not consider themselves active investors. However, like a fund manager, investors may wish to construct a Markowitz (1952) efficient portfolio of securities designed to maximize return relative to risk. If one envisions the securities within the portfolio as the only securities available for investment, they can create an efficient frontier in which a tangency portfolio maximizes the Sharpe ratio for the portfolio (Sharpe, 1966). By maximizing the Sharpe ratio for the portfolio, the investor has the maximum excess portfolio return (i.e., portfolio return less the risk-free rate (R_F)) relative to the risk (i.e., standard deviation) achievable, given the securities in the portfolio. The weights of the assets in such a portfolio that achieves the highest reward per unit of risk are known as optimal weights.

Further, should the investor prefer less risk, the investor can offset the amount invested in the portfolio with additional investment in the risk-free security to lower the overall risk while still achieving the maximized Sharpe ratio. Consequently, finding the optimal weights of the assets in the tangency portfolio is important and should be re-evaluated periodically.

Finding the tangency portfolio requires taking partial derivatives of the portfolio variance with Lagrange conditions (see Arnold, 2002, and Arnold and Nixon, 2022) and solving for the optimal weights of the securities in the portfolio. Applying Cramer's Rule (see Strang, 2020 or Simmons, 1987) avoids inverting matrices and is easily implemented in Excel using the =MDETERM function (see Arnold, Farizo, and Nixon, 2024). In this treatment, we briefly allude to the necessary math; however, we emphasize how an Excel template can easily generate the optimal portfolio weights for multiple securities. Further, the Excel programming is minimal, and its result can provide a significant benefit to the investor, whether they are an institution or an individual.

Section 1 presents the math behind the optimization and may be optional reading for practitioners and academics looking to implement the optimization procedure. Section 2 presents the relevant Excel framework for determining the optimized portfolio weights.

SECTION 1: TANGENCY PORTFOLIO WEIGHTS FOR A PORTFOLIO OF THREE FUNDS (THE MATH):

Table 1 presents information on three funds (A, B, and C) and a risk-free security, T-bills (R_F).

Table 1: Fund Information and a Risk-free Security

Security:	Mean:	Standard Deviation:	Variance:
Fund A	4.50%	22.00%	0.0484
Fund B	5.90%	36.00%	0.1296
Fund C	6.30%	42.00%	0.1764
Risk-free	2.50%	0.00%	0.0000
Correlations (CORR) and Covariances (COV):			
CORR (A,B):	0.550	COV (A,B):	0.043560
CORR (A,C):	0.720	COV (A,C):	0.066528
CORR (B,C):	0.650	COV (B,C):	0.098280
The correlation between the risk-free security and any risky security/fund is zero			
COV(X, Y) = CORR(X, Y) × Standard Deviation (X) × Standard Deviation (Y)			

Britten-Jones (1999) computes the weights of the tangency portfolio through a regression routine, however, this can also be accomplished using a Lagrange condition of excess portfolio return being equal to “1” (or 100%, this value is set for convenience) while minimizing the portfolio variance.

$L = (W_A)^2 \times \text{Variance (A)} + (W_B)^2 \times \text{Variance (B)} + (W_C)^2 \times \text{Variance (C)}^2$

$+ 2 \times W_A \times W_B \times \text{Covariance (A, B)} + 2 \times W_A \times W_C \times \text{Covariance (A, C)}$

$+ 2 \times W_B \times W_C \times \text{Covariance (B, C)}$

$+ \lambda\,[\,1 - W_A(\text{Mean (A)} - R_F) - W_B(\text{Mean (B)} - R_F) - W_C(\text{Mean (C)} - R_F)] \quad (1)$

After taking the partial derivatives relative to each weight (W_A, W_B, and W_C) and relative to the Lagrange multiplier (λ), the following matrices are generated based on setting each partial derivative equation to zero [note: V(X) is the variance of X, C(X,Y) is the covariance between X and Y, and XR(X) is the excess return of X, which equals (Mean (X) – R_F)]:

$$\underbrace{\begin{bmatrix} V(A) & C(A,B) & C(A,C) & XR(A) \\ C(A,B) & V(B) & C(B,C) & XR(B) \\ C(A,C) & C(B,C) & V(C) & XR(C) \\ XR(A) & XR(B) & XR(C) & 0 \end{bmatrix}}_{\text{VCOVXR-ABC}} \times \underbrace{\begin{bmatrix} W_A \\ W_B \\ W_C \\ \lambda \end{bmatrix}}_{W} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{T} \quad (2)$$

VCOVXR-ABC is the variance-covariance matrix for Funds A, B, and C, with borders containing the excess returns. W is a column matrix of portfolio weights and the Lagrange multiplier (λ). T is a column matrix that is the result of setting the partial derivatives to zero and simplifying the equations (this results in "1" being at the bottom of the column matrix).

To solve for the portfolio weights, implement Cramer's Rule:

- Substitute column matrix T into the first column of VCOVXR-ABC to create the square matrix VCOVXR-TBC:

VCOVXR-TBC

$$\begin{bmatrix} 0 & C(A,B) & C(A,C) & XR(A) \\ 0 & V(B) & C(B,C) & XR(B) \\ 0 & C(B,C) & V(C) & XR(C) \\ 1 & XR(B) & XR(C) & 0 \end{bmatrix} \quad (3)$$

- Substitute column matrix T into the second column of VCOVXR-ABC to create the square matrix VCOVXR-ATC:

VCOVXR-ATC

$$\begin{bmatrix} V(A) & 0 & C(A,C) & XR(A) \\ C(A,B) & 0 & C(B,C) & XR(B) \\ C(A,C) & 0 & V(C) & XR(C) \\ XR(A) & 1 & XR(C) & 0 \end{bmatrix} \quad (4)$$

- Substitute column matrix T into the third column of VCOVXR-ABC to create the square matrix VCOVXR-ABT:

VCOVXR-ABT

$$\begin{bmatrix} V(A) & C(A,B) & 0 & XR(A) \\ C(A,B) & V(B) & 0 & XR(B) \\ C(A,C) & C(B,C) & 0 & XR(C) \\ XR(A) & XR(B) & 1 & 0 \end{bmatrix} \quad (5)$$

- Use calculations based on the determinants of the matrices to find the portfolio weights (i.e., Cramer's Rule, "det" means the matrix determinant):

W_A = det(VCOVXR-TBC) ÷ det(VCOVXR-ABC) = 1772.78% (6)

W_B = det(VCOVXR-ATC) ÷ det(VCOVXR-ABC) = 1363.41% (7)

W_C = det(VCOVXR-ABT) ÷ det(VCOVXR-ABC) = 478.65% (8)

These portfolio weights are set to include the weight of the risk-free security as part of the portfolio. To transition to the tangency portfolio weights, the above weights need to be reapportioned in the following manner:

$$W_{A\text{-}TAN} = W_A \div (W_A + W_B + W_C) = 49.04\% \quad (9)$$

$$W_{B\text{-}TAN} = W_B \div (W_A + W_B + W_C) = 37.72\% \quad (10)$$

$$W_{C\text{-}TAN} = W_C \div (W_A + W_B + W_C) = 13.24\% \quad (11)$$

A matrix determinant has a numerical/spatial interpretation that is not particularly important in this setting. The determinant is usually calculated when inverting a matrix, which is a calculation that Cramer's Rule allows us to avoid.

Indeed, there is value to understanding the math behind the portfolio construction. However, applying the implications of the math is perhaps more important to the practitioner. In this case: invest 49.04% of the portfolio in Fund A, 37.72% of the portfolio in Fund B, and 13.24% of the portfolio in Fund C. The mean return on the portfolio is 5.27% with a standard deviation of 26.03% and a Sharpe ratio of 0.106.

If the investor desires to reduce the risk by 50%, invest 50% in the risk-free security and 50% in the relative optimal weights of the three funds: 24.52% (= 50% × 49.04%) in Fund A, 18.86% (= 50% × 37.72%) in Fund B, and 6.62% (= 50% × 13.24%) in Fund C. The mean return on this portfolio is 3.885% (= 50% × 2.50% + 50% × 5.27%) and the standard deviation is 13.015% (= 50% × 26.03%). The Sharpe ratio for this less risky portfolio will still be 0.106 and an optimized portfolio.

SECTION 2: TANGENCY PORTFOLIO WEIGHTS FOR A PORTFOLIO OF THREE FUNDS (EXCEL TEMPLATE):

In Table 2, we present the Excel sheet that provides the tangency portfolio weight calculations associated with the securities in Table 1. The calculations in the previous section are applied using the =MDETERM function to compute matrix determinants. The =MDETERM function is used in cells I15, I16, and I17.

Table 2: Excel Template for Tangency Portfolio Weights

	A	B	C	D	E	F	G	H	I
1		Mean:	STDEV:	VAR:					
2	FUND A:	4.50%	22.00%	0.0484		CORR(A,C):	0.550	C(A,B):	0.043560
3	FUND B:	5.90%	36.00%	0.1296		CORR(A,B):	0.720	C(A,C):	0.066528
4	FUND C:	6.30%	42.00%	0.1764		CORR(B,C):	0.650	C(B,C):	0.098280
5	Risk-free:	2.50%	0.00%	0.0000					
6									
7	VCOVXR-ABC								
8		A	B	C	XR		T		
9		0.0484	0.043560	0.066528	2.00%		0		
10		0.0435650	0.1296	0.098280	3.40%		0		
11		0.066528	0.098280	0.1764	3.80%		0		
12		2.00%	3.40%	3.80%	0		1		
13									
14	VCOVXR-TBC								
15		T	B	C	XR			Weight A:	1772.78%
16		0	0.043560	0.066528	2.00%			Weight B:	1363.41%
17		0	0.1296	0.098280	3.40%			Weight C:	478.65%
18		0	0.098280	0.1764	3.80%				
19		1	3.40%	3.80%	0			Weight A-TAN:	49.04%
20								Weight B-TAN:	37.72%
21	VCOVXR-ATC							Weight C-TAN:	13.24%
22		A	T	C	XR				
23		0.0484	0	0.066528	2.00%			MEAN-TAN:	5.27%
24		0.0435650	0	0.098280	3.40%			VAR-TAN:	0.067741
25		0.066528	0	0.1764	3.80%			STDEV-TAN:	26.03%
26		2.00%	1	3.80%	0			Sharpe ratio:	0.10629
27									
28	VCOVXR-ABT								
29		A	B	T	XR				
30		0.0484	0.043560	0	2.00%				
31		0.0435650	0.1296	0	3.40%				
32		0.066528	0.098280	0	3.80%				
33		2.00%	3.40%	1	0				

STDEV: standard deviation
VAR: variance
CORR (X,Y): Correlation (X,Y)
C(X,Y): Covariance (X,Y)
MEAN-TAN : mean of tangency portfolio
VAR-TAN: variance of the tangency portfolio
STDEV-TAN: standard deviation of the tangency portfolio
Sharpe Ratio: the Shapre Ratio of the tangency portfolio

Cell I15: =MDETERM(B16:E19) / MDETERM(B9:E12)
Cell I16: =MDETERM(B23:E26) / MDETERM(B9:E12)
Cell I17: =MDETERM(B30:E33) / MDETERM(B9:E12)

Cell I19: = I15 / SUM(I15:I17)
Cell I20: = I16 / SUM(I15:I17)
Cell I21: = I17 / SUM(I15:I17)
Cell I23: = SUMPRODUCT(I19:I21, B2:B4)
Cell I24: = MMULT(MMULT(TRANSPOSE(I19:I21),B9:D11),I19:I21))…hit CRTL-SHIFT-ENTER for entering this formula

Cell I25: = SQRT(I24)
Cell I26: = (I23 – B5) / I25

This file can be downloaded from: https://scholarship.richmond.edu/finance-faculty-publications/XX/

Implementing the necessary computations within the Excel template is fairly straightforward to do. The first five rows require information about the funds and the risk-free rate. The "boxed" cells represent the matrices in equations (2) through (5), however, within each matrix, one is referencing information from the first five rows or typing in "ones" and "zeroes." Actual calculations occur within cells I15, I16, I17, I19, I10, I21, I23, I24, I25, and I26. The most complicated calculation is for the variance of the tangency portfolio, which requires matrix operations. If the investor's goal is to only find the optimal portfolio weights for the three funds, cells I23 through I26 are not necessary, and the matrix operations leave the spreadsheet.

An investor may wish to expand the results to a portfolio of four or more securities. In this case, the math is implemented in the same way; however, the matrices become larger. For example, the spreadsheet for four funds (A, B, C, and D) adjusts in this manner:

Table 3: Excel Template for Four Funds

	A	B	C	D	E	F	G	H	I
9	VCOVXR-ABCD								
10		A	B	C	D	XR		T	
11		V(A)	C(B,A)	C(C,A)	C(D,A)	XR(A)		0	
12		C(A,B)	V(B)	C(C,B)	C(D,B)	XR(B)		0	
13		C(A,C)	C(B,C)	V(C)	C(D,C)	XR(C)		0	
14		C(A,D)	C(B,D)	C(C,D)	V(D)	XR(D)		0	
15		XR(A)	XR(B)	XR(C)	XR(D)	0		1	
16									

V(X): variance of X
C(X,Y) = C(Y,X): covariance between X and Y
XR(X): excess return on X = (Mean(X) – R_F)

This file can be downloaded from: https://scholarship.richmond.edu/finance-faculty-publications/XX/

The VCOVXR-ABCD matrix has one more row and one more column relative to the VCOVXR-ABC matrix to accommodate the fourth fund. The "T" matrix has one more row and has the same structure of containing all zeroes and a one at the bottom. The variations from the VCOVXR-ABCD matrix will also be structured as five rows and five columns. The associated calculations become:

W_A = det(VCOVXR-TBCD) ÷ det(VCOVXR-ABCD) (12)

W_B = det(VCOVXR-ATCD) ÷ det(VCOVXR-ABCD) (13)

W_C = det(VCOVXR-ABTD) ÷ det(VCOVXR-ABCD) (14)

W_D = det(VCOVXR-ABCT) ÷ det(VCOVXR-ABCD) (15)

$W_{A\text{-}TAN} = W_A \div (W_A + W_B + W_C + W_D)$ (16)

$W_{B\text{-}TAN} = W_B \div (W_A + W_B + W_C + W_D)$ (17)

$W_{C\text{-}TAN} = W_C \div (W_A + W_B + W_C + W_D)$ (18)

$W_{D\text{-}TAN} = W_C \div (W_A + W_B + W_C + W_D)$ (19)

Five funds (include Fund E) will require the VCOVXR-ABCDE matrix to have six columns and six rows to accommodate C(E,A), C(E,B), C(E,C), C(E,D), V(E), and XR(E). The process continues with enlarging the matrices and adding two more calculations: one calculation with determinants, and one calculation to adjust to a tangency portfolio weight.

Consequently, each new fund increases the VCOVXR matrix by an additional row and column due to the new information being added by the new fund in terms of variance, covariances, and excess return. The portfolio can become rather large, however, the individual portfolio weight is still a matrix determinant divided by another matrix determinant (e.g. equation (12)) with a re-adjustment to tangency portfolio weights (e.g. equation (16)).

The three-fund and four-fund example spreadsheet is available at:

https://scholarship.richmond.edu/finance-faculty-publications/XX/

CONCLUSION:

Determining mean-variance optimized portfolio weights intended to maximize reward and minimize risk requires calculus and matrix algebra. However, by applying Cramer's Rule with the =MDETERM function in Excel, investors can obtain these results fairly easily. This is very beneficial for the individual investor and by periodically updating the information in the spreadsheet, the individual investor can also update portfolio weights as needed.

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