Cosmological Inflation in *N*-Dimensional Gaussian Random Fields with Algorithmic Data Compression

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Abstract

The leading modern theories of cosmological inflation are increasingly multi-dimensional. The "**inflaton field**" $\boldsymbol{\phi}$ that has been postulated to drive accelerating expansion in the very early universe has a corresponding **potential function** V, the details of which, such as the number of dimensions and shape, have yet to be specified. We consider a natural hypothesis that V ought to be maximally random. We realize this idea by **defining the V as a Gaussian random field in some number N of dimensions**. We repeatedly simulate of the evolution of $\boldsymbol{\phi}$ given a set of conditions on the "landscape" of V. We simulate a "path" stepwise through $\boldsymbol{\phi}$-space while simultaneously computing V and its derivatives along the path via a constrained Gaussian random process, incorporating the information from prior steps. When N is large, this method significantly reduces computational load as compared to methods which generate the potential landscape all at once. Even so, computation of the **covariance matrix $\boldsymbol{\Gamma}$** of constraints on V can quickly become intractable. Inspired by this problem, we present **data compression algorithms to prioritize the necessary information already simulated**, then keep an arbitrarily large portion. Information such as the evolution of the scale factor and tensor and scalar perturbations can be extracted from any particular path, then statistical information about these quantities can be gathered from repeated trials. In these ways, **we present a versatile multi-variable program** for exploration into how accurately this emergent model can fit to observation.

Problem

Creating an inflationary model means defining a potential function V of many variables $\phi^{(0)}, \phi^{(1)}, \dots, \phi^{(N-1)} \equiv \boldsymbol{\phi}$. The function is required to have some **key properties**:

- *Minimum condition.* The function must have a local minimum at $V = 0$.
- *Slow roll condition.* The function must have gradual and gradually changing slopes near the minimum.
- *Energy condition.* V represents an energy density in the early universe, so its values must be large enough to make it dominant at that time. This ensures that V governed large-scale changes in the size of the universe.

There are many examples of inflationary potentials, but we are interested in creating V from a **random process**. We do this point-by-point. The first potential value $V(\boldsymbol{\phi}_0)$ is a completely random value, drawn from a Normal distribution with mean zero and variance V_0^2 , or $V(\boldsymbol{\phi}_0) \sim \mathcal{N}(0, V_0^2)$. The second value $V(\boldsymbol{\phi}_1)$ is constrained by the first value depending on the difference between $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_0$. In general, $V(\boldsymbol{\phi}_t) \sim \mathcal{N}(\mu, \Gamma_C)$, where

$$\mu = \boldsymbol{\Gamma}_{NO}\boldsymbol{\Gamma}_{OO}^{-1}\boldsymbol{v}_O \qquad \Gamma_C = \Gamma_{NN} - \boldsymbol{\Gamma}_{NO}\boldsymbol{\Gamma}_{OO}^{-1}\boldsymbol{\Gamma}_{ON}$$

$$\boldsymbol{\Gamma} = \begin{bmatrix} \boldsymbol{\Gamma}_{OO} & \boldsymbol{\Gamma}_{ON} \\ \boldsymbol{\Gamma}_{NO} & \Gamma_{NN} \end{bmatrix} \qquad \Gamma^{(ij)} = \langle V(\boldsymbol{\phi}_i)V(\boldsymbol{\phi}_j)\rangle = V_0^2 \cdot \mathrm{Exp}\left(-\frac{|\boldsymbol{\phi}_i - \boldsymbol{\phi}_j|^2}{2s^2}\right)$$

$\mathbf{v}_O$ is the list of all previous potential values and $\boldsymbol{\Gamma}$ is the covariance matrix of constraints on V. Subscripts O and N abbreviate "old" and "new", respectively. This process of randomly generating V is used when solving the Klein-Gordon differential equations for the inflaton field evolution of a model universe:

$$\frac{d^2\phi^{(\alpha)}}{dN_e^2} + (3-\epsilon)\frac{d\phi^{(\alpha)}}{dN_e} + \frac{1}{H^2}\frac{\partial V}{\partial \phi^{(\alpha)}} = 0 \qquad \boldsymbol{\phi}(0) = \boldsymbol{\phi}_0, \quad \frac{d\boldsymbol{\phi}}{dN_e}(0) = -\frac{\nabla_\phi V(\boldsymbol{\phi}_0)}{V(\boldsymbol{\phi}_0)}$$

In these equations, N_e is the "number of *e*-folds", a measure of both time and the relative size of the universe and ϵ and H are functions of the other quantities. If we know the values of V in the relevant region, we can solve for $\boldsymbol{\phi}$ from its initial position to its final position, ideally in the minimum of V. The trajectory (evolution) of any $\boldsymbol{\phi}$ depends on its underlying potential function, which is randomized to some extent but controlled by parameters s, V_0, and N. Today, there exist are some **observable quantities** A_s, n_s, A_t, n_t, r that are theoretically linked back to the trajectory, and thus to the choice of potential. Testing the plausibility that the Gaussian random potential represents *our* physical universe involves simulating many potentials, solving for many trajectories, and extracting observable quantities from each trajectory for statistical analysis.

Program

Restricting $N = 2$ for visual purposes, the Gaussian random potentials end up looking something like this:

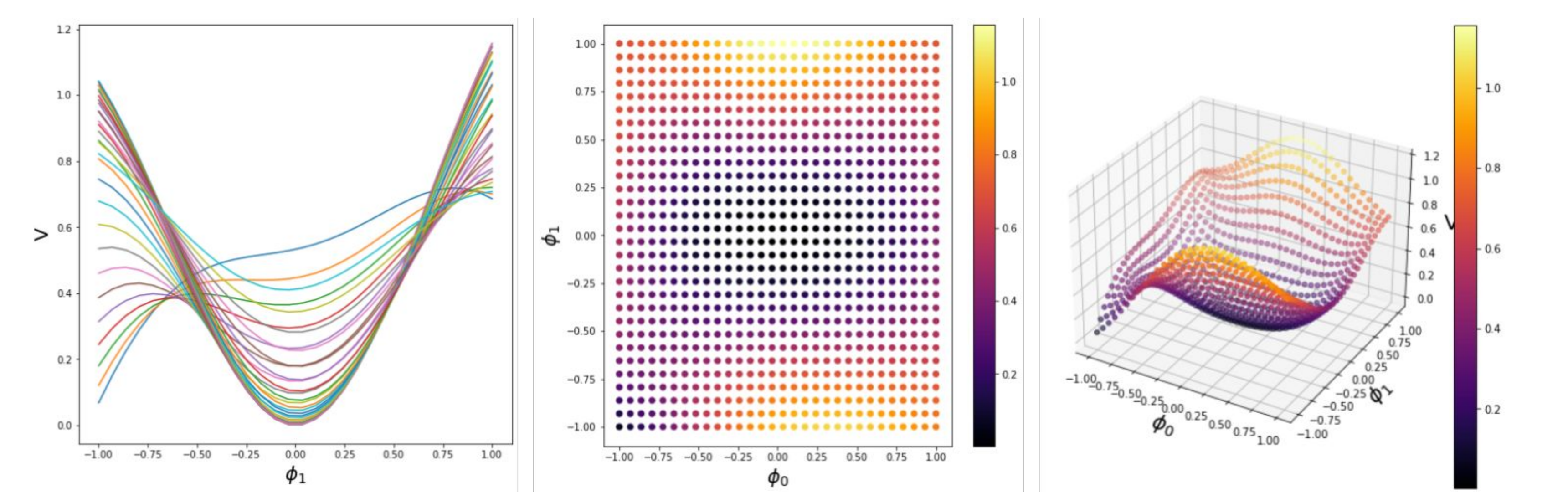


FIGURE 1: *A **Gaussian random potential** V simulated at discrete points in its domain, N = 2 inflaton field components.*

To use this function, or one like it, to solve for the inflaton field evolution in a model universe, imagine releasing a ball at some point around the minimum. The Klein-Gordon equations tell us that the ball will accelerate downhill (down the slopes of V) subjected to air resistance. This is wonderfully analogous to the trajectory of the inflaton field. We built a multi-component program to solve for the trajectory of $\boldsymbol{\phi}$ under a random potential, then extract from it observable quantities. Figure 2 describes the process, from input to output, that we designed in Python to do the tasks.

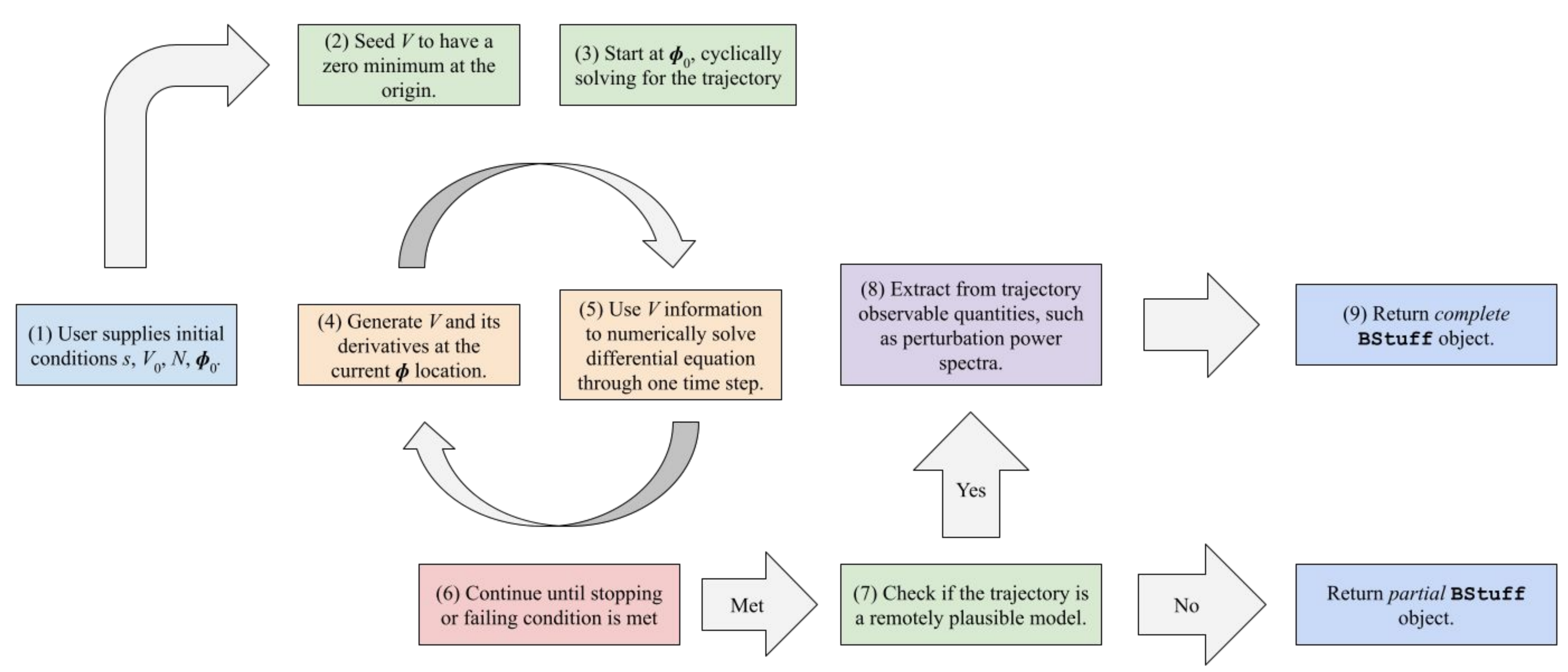


FIGURE 2: *To test our inflationary model against observation, we coded a **versatile program** which executes a set process given initial conditions on the potential and the differential equations. The potential is seeded to satisfy the Minimum condition, then the differential equations are solved numerically, simulating the requisite potential information at each step. Once all relevant data is collected, a stopping condition is met and attention turns to extracting quantities which are comparable to modern observations.*

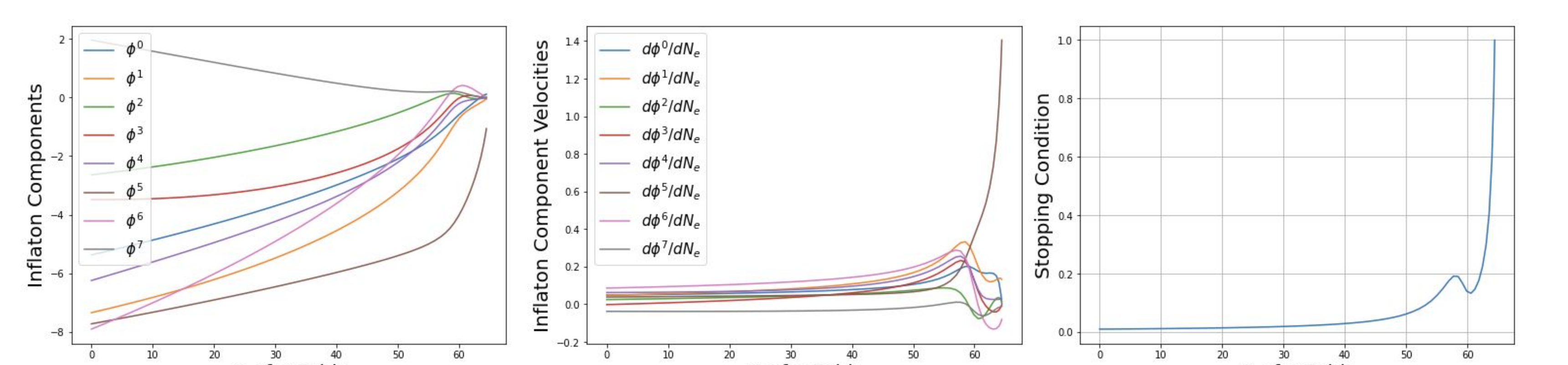


FIGURE 3: *A **solution** $\boldsymbol{\phi}$ to the Klein-Gordon equations of motion evolving under a Gaussian random potential with N = 8. Shown are its vector components (left), component velocities (center), and monitored stopping condition over time. The stopping condition is a specific function of the velocities, and when it approaches 1 from below, the program knows to stop solving differential equations. Notice how each vector component approaches zero, indicating a convergence to the origin and preliminary success.*

Predictions

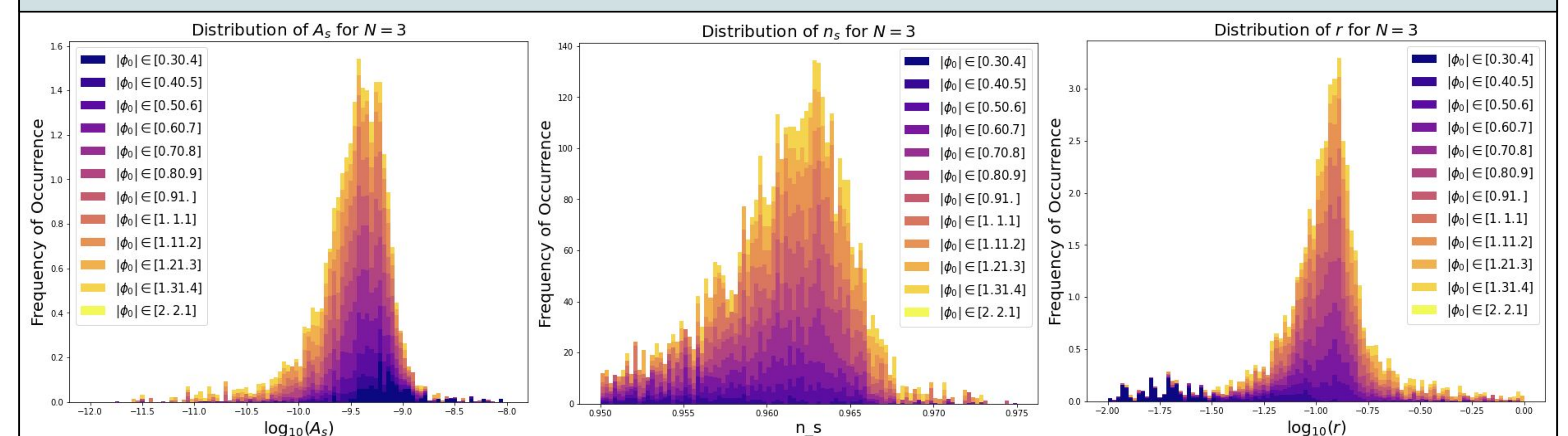
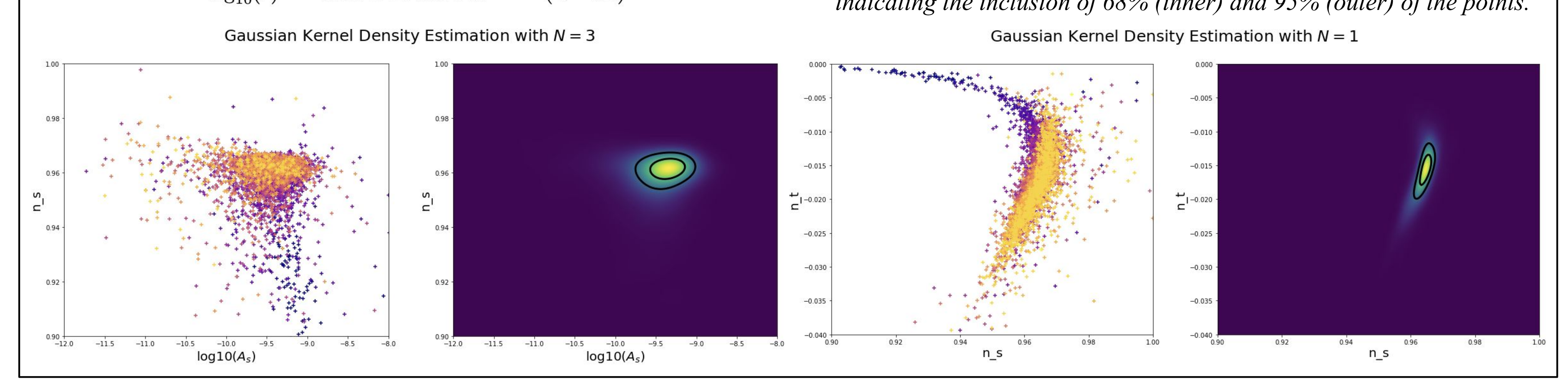


FIGURE 4 (Above): *Inflationary models, like our Gaussian random potential, yield predictions about how the universe should appear today. Important predictable quantities are **power spectra of tensor and scalar perturbations**, which relate to gravitational waves in the Cosmic Microwave Background (CMB) and density of galaxies. The spectra are summarized by their amplitudes and indices A_S, n_S, A_T, n_T, and r. Recent data from a 2018 Planck mission constrained three of these values to some extent (right), offering powerful plausibility tests for our model. Shown are histograms compiling predictions from 11,000 potentials randomized with parameters $s = 30$, $V_0 = 5 \cdot 10^{-9}$, $N = 3$, weighted and colored by initial condition $|\boldsymbol{\phi}_0|$.*

$$\log_{10}(A_S) = -9.4727 \pm 0.0909 \qquad (-8.680)$$
$$n_S = 0.96360 \pm 0.00205 \qquad (0.9626)$$
$$\log_{10}(A_T) = -10.404 \pm 0.118$$
$$n_T = -0.01600 \pm 0.00218$$
$$\log_{10}(r) = -0.90242 \pm 0.07248 \qquad (< -1.0)$$

Fixing V parameters s and V_0 while varying N and initial condition $|\boldsymbol{\phi}_0|$, we compiled predictions from thousands of trajectories using the UR supercomputer. Results show promise for the Gaussian random model, predicting mean values for A_s, n_s, A_t, n_t, r that closely match observations of these cosmological parameters from 2018 Planck missions. Below are predictions and observations (in parentheses) compared side-by-side.

FIGURE 5 (below): *Another way to visualize the predictions our model gives for perturbation spectra is with a **Gaussian KDE**. Put one quantity on the horizontal axis and another on the vertical axis, then scatter-plot the values extracted from thousands of solutions. The KDE fits a multivariate distribution to the data with two closed black curves indicating the inclusion of 68% (inner) and 95% (outer) of the points.*



Future

Since our model shows promise, there are several ways in which we can further its verification as plausible:

- *Explore parameter space.* Varying s, V_0, N, and $|\boldsymbol{\phi}_0|$ and cataloguing many simulations will reveal the fuller range of the our model.
- *Translate to other languages.* Other programming languages, like C and FORTRAN, can quicken the execution of the program detailed in Figure 2.
- *Incorporate other observable quantities.* Some models have been tested against other quantities, such as the non-Gaussianity of the CMB, additional tests which could "make or break" the model

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