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The ABCs of Modified Bond Duration and WXYZs of Bond Convexity

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By breaking the mathematical derivation of Macaulay Duration, Modified Duration, and Bond Convexity into smaller easily calculated component parts, a more manageable means of calculation for these bond measures emerges for the student. Further, an Excel spreadsheet or an algorithm within a programming language can also be implemented using these smaller component calculations. The Excel template provided can be made into an assignment or used as a resource for the student.

INTRODUCTION:

Macaulay Duration (Macaulay, 1938) and Modified Duration are the standard measures of an option-free bond's sensitivity to a change in its yield-to-maturity. They are related to the first derivative of the bond price relative to the yield-to-maturity (k). More precisely, Macaulay Duration (D) is:

$$D = (1+k) \times \left[\frac{\partial Bond\ price}{\partial k}\right] \div Bond\ price \qquad (1)$$

Dividing Macaulay Duration by (1 + k) produces the Modified Duration (D_{MOD}):

$$D \div (1+k) = D_{MOD} = \left[\frac{\partial Bond\ price}{\partial k}\right] \div Bond\ price \qquad (2)$$

The percentage change in bond price (%ΔBond price) is equal to the negative of the change in the yield-to-maturity (Δk) multiplied by the Modified Duration:

$$\%\Delta Bond\ price = -\Delta k \times D_{MOD} \qquad (3)$$

Macauley Duration is commonly used in hedging strategies. An individual investor who purchases a bond with a Macauley Duration equal to his/her desired holding period/investment horizon is immunized against changes in market interest rates, because price fluctuation risk and reinvestment rate risk will offset if the bond is sold at that horizon and the investor will (approximately) earn a total annualized return that equals the bond's original yield to maturity. Similarly, assuming parallel shifts in the yield curve, a financial institution with both assets and liabilities is largely immunized against changes in market interest rates if the market value weighted average Macauley Durations of its assets and liabilities are equal. In contrast, Modified Duration is more typically used to measure sensitivity to market interest rates. It is only accurate for small changes in a bond's yield to maturity because, given the non-linear nature of bond pricing, convexity (and higher order effects) also become important when yield changes are large.

The computation of Macaulay Duration for a coupon bond paying “M” coupons annually with a maturity of “N” years can be performed by evaluating the (M × N + 1) individual cash flows or by using a “short-cut formula” with a ratio (see Fabozzi and Fabozzi, 2022). Define “H” as the ratio of the discounted bond coupons over the bond price:

$$H = \left[\frac{coupon}{k/M}\left(1 - \frac{1}{(1+k/M)^{M\times N}}\right)\right] \Big/ \left[\frac{coupon}{k/M}\left(1 - \frac{1}{(1+k/M)^{M\times N}}\right) + \frac{par}{(1+k/M)^{M\times N}}\right]$$

$$H = \left[\frac{coupon}{k/M}\left(1 - \frac{1}{(1+k/M)^{M\times N}}\right)\right] \Big/ [Bond\ price] \tag{4}$$

where, k = annual yield to maturity at purchase, coupon = (par × k_C) ÷ M and k_C is the annual coupon rate for the bond

Macaulay Duration is:

$$D = \left[\frac{(1+k/M)}{k/M}\right] \times H + \left[\frac{(k/M)-(k_C/M)}{k/M}\right] \times (N \times M) \times (1-H) \tag{5}$$

To convert “D” to “D_{MOD},” divide D by (1 + k / M). To convert either duration measure to an annual value, divide the measure by “M.”

In this treatment, the bond price formula is converted to a form introduced by Arnold and Earl (2014) and Arnold (2015).

$$Bond\ price = par\ \times \left\{1 + \frac{\left({k_C}/{M} - {k}/{M}\right)}{k/M}\left[1 - \frac{1}{(1+k/M)^{M\times N}}\right]\right\}$$

$$Bond\ price = par\ \times \left\{1 + \left(\frac{1}{M}\right)(k_C - k) \times PVA(k/M, M\times N)\right\}$$

$$where{:}\ PVA(k/M, M\times N) = \frac{1}{k/M}\left[1 - \frac{1}{(1+k/M)^{M\times N}}\right] \tag{6}$$

Based on equation (6), derivatives relative to the yield-to-maturity (k / M) are computed to find a "short-cut" calculation for Modified Duration (D_{MOD}) using "ABC" components and for bond convexity (CVX) using "WXYZ" components. These calculations are much faster than traditional individual cash flow calculations and can be readily performed in Excel or by using a calculator. The instructor can supply Excel templates for the calculations or have the students generate an Excel template as an assignment.

Depending on the instructor's agenda, the student can be exposed to the derivations of D_{MOD} and CVX provided in this article to see mathematically why these values measure sensitivity to the yield-to-maturity. Alternatively, the goal may be to have students perform calculations using D, D_{MOD}, and CVX with the formulas/templates provided as a means for producing values for D, D_{MOD}, and CVX.

The next section provides the mathematical derivations for D_{MOD} and D using "ABC" components. The second section provides the mathematical derivation for CVX using "WXYZ" components. The third section provides the Excel template to compute D, D_{MOD}, and CVX. The fourth section concludes the article.

THE "ABC" COMPONENTS FOR MODIFIED AND MACAULAY DURATION:

Based on equation (2) and using the partial derivative of Equation (6) relative to (k / M), the modified duration (D_{MOD}) for a bond can be computed as:

$$D_{MOD} = \frac{par \times \left[(k_C/k) \times PVA(k/M, M \times N) + \ (1 - k_C/k) \times \left(\frac{M \times N}{(1 + k/M)^{M \times N + 1}}\right)\right]}{par \times \left[1 + \left(\frac{1}{M}\right)(k_C - k) \times PVA(k/M, M \times N)\right]}$$

$$D_{MOD} = \frac{(k_C/k) \times PVA(k/M, M \times N) + \ (1 - k_C/k) \times \left(\frac{M \times N}{(1 + k/M)^{M \times N + 1}}\right)}{1 + \left(\frac{1}{M}\right)(k_C - k) \times PVA(k/M, M \times N)}$$

$$\text{where: } PVA(k/M, M \times N) = \frac{1}{k/_M}\left[1 - \frac{1}{(1+k/M)^{M \times N}}\right] \qquad (7)$$

Equation (7) looks very daunting, however, breaking the equation into component parts simplifies the calculation. Define A, B, and C:

$$A = \left[\frac{k_C}{k}\right] \times PVA(k/M\,, M \times N) \qquad (8)$$

$$B = \left[1 - \frac{k_C}{k}\right] \times \left[\frac{M \times N}{(1+k/M)^{M \times N+1}}\right] \qquad (9)$$

$$C = 1 + \left(\frac{1}{M}\right)(k_C - k) \times PVA(k/M, M \times N) \qquad (10)$$

Modified Duration and Macaulay Duration become:

$$D_{MOD} = \frac{A+B}{C} \qquad (11)$$

$$D = D_{MOD} \times (1 + k/M) = \left[\frac{A+B}{C}\right] \times (1 + k/M) \qquad (12)$$

To annualize either measure, divide the calculation by "M." See Fabozzi and Fabozzi (2022) for an expanded discussion of bond duration.[1]

A numerical example illustrating our approach follows. A semiannual pay bond has a 6% (annual) coupon rate, a 4% (annual) yield to maturity and 20 years left until maturity. The bond's Modified Duration is calculated as follows:

PVA(4%/2, 2×20) = $\frac{1}{.04/2}\left[1 - \frac{1}{(1+.04/2)^{2\times 20}}\right] \approx 27.35548$

[1] While the focus of our paper is the calculation of duration on coupon anniversary dates, the ABC approach for duration can be easily modified to reflect a situation where the bond is purchased between coupon payment dates. Let N now equal the number of years to maturity remaining as of the beginning of the current coupon period (need not be an integer value), t = number of days between the last coupon payment and the settlement date and T = total number of days in the current coupon period. The Modified Duration using our approach is then calculated as $D_{MOD} = \frac{A+B}{C} - \left(1 + {}^{k}/_{M}\right)\frac{t}{T}$.

A = $\left[\frac{.06}{.04}\right] \times 27.35548 \approx 41.03322$

B = $\left[1 - \frac{.06}{.04}\right] \times \left[\frac{2\times20}{(1+.04/2)^{2\times20+1}}\right] \approx -8.88020$

C = $1 + \left(\frac{1}{2}\right)(.06 - .04) \times 27.35548 \approx 1.27355$

$D_{MOD} \approx \frac{41.03322 - 8.88020}{1.273555} \approx$ 25.24667 semiannual periods.

The Modified Duration in annual terms is then approximately 25.24667 / 2 = 12.62334.

The Macauley Duration is 25.24667 × (1 + .04/2) = 25.75160 semiannual periods, and 25.75160 / 2 = 12.87580 in annual terms.

Before leaving this section, it should be noted that another popular measure is Dollar Duration, which is the Modified Duration multiplied by the bond price. Following Equation (2), it can be demonstrated that Dollar Duration is the partial derivative of the bond price relative to the yield to maturity:

$$Dollar\ Duration = D_{MOD} \times Bond\ price$$

$$= \left\{\left[\frac{\partial Bond\ price}{\partial k}\right] \div Bond\ price\right\} \times Bond\ price = \left[\frac{\partial Bond\ price}{\partial k}\right] \tag{13}$$

Using the ABC components and Equation (7), the Dollar Duration is:

$$Dollar\ Duration = par \times (A + B) \tag{14}$$

Continuing with our earlier numerical example and assuming the bond has $1,000 par value, Dollar Duration ≈ $1,000 × (41.0333 – 8.8802) ≈ $32,153.

THE "WXYZ" COMPONENTS FOR CONVEXITY:

Convexity (CVX) is the second derivative of the bond price relative to the yield to maturity divided by the bond price. Taking the second derivative of equation (6) relative to (k / M) and dividing by the bond price generates CVX:

$$CVX = \frac{2 \times M \times (k_C/k^2) \times PVA(k/M, M \times N)}{1 + \left(\frac{1}{M}\right)(k_C - k) \times PVA(k/M, M \times N)}$$

$$- \frac{2 \times M \times (k_C/k^2) \times \left(\frac{M \times N}{(1 + k/M)^{M \times N + 1}}\right) + (1 - k_C/k) \times \left(\frac{(M \times N) \times (M \times N + 1)}{(1 + k)^{M \times N + 2}}\right)}{1 + \left(\frac{1}{M}\right)(k_C - k) \times PVA(k/M, M \times N)}$$

$$+ \frac{(1 - k_C/k) \times \left(\frac{(M \times N) \times (M \times N + 1)}{(1 + k)^{M \times N + 2}}\right)}{1 + \left(\frac{1}{M}\right)(k_C - k) \times PVA(k/M, M \times N)}$$

$$\text{where: } PVA(k/M, M \times N) = \frac{1}{k/_M}\left[1 - \frac{1}{(1+k/M)^{M \times N}}\right] \tag{15}$$

Again, Equation (15) looks very daunting, however, breaking the equation into component parts simplifies the calculation. Define W, X, Y, and Z:

$$W = 2 \times M \times \left[\frac{k_C}{k^2}\right] \times PVA(k/M\,, M \times N) \tag{16}$$

$$X = 2 \times M \times (k_C/k^2) \times \left(\frac{M \times N}{(1+k/M)^{M \times N+1}}\right) \tag{17}$$

$$Y = \left[1 - \frac{k_C}{k}\right] \times \left[\frac{(M \times N) \times (M \times N+1)}{(1+k/M)^{M \times N+2}}\right] \tag{18}$$

$$Z = 1 + \left(\frac{1}{M}\right)(k_C - k) \times PVA(k/M, M \times N) \tag{19}$$

$$CVX = \frac{W - X + Y}{Z} \tag{20}$$

It should be noted that Equations (10) and (19), are actually the same calculation, i.e. C = Z. Further, to annualize CVX, divide the calculation by "M^2." Again, see Fabozzi and Fabozzi (2022) for an expanded discussion of bond convexity.

Applying the WXYZ approach to our earlier numerical example (semiannual pay bond, 6% annual coupon rate, 4% annual yield to maturity, 20 years to maturity) yields the following:

PVA(4%/2, 2×20) ≈ 27.35548 (calculated previously)

W = $2 \times 2 \times \left[\frac{.06}{.04^2}\right] \times 27.35548 \approx 4{,}103.322$

$$X = 2 \times 2 \times \left[\frac{.06}{.04^2}\right] \times \left[\frac{2\times 20}{(1+.04/2)^{2\times 20+1}}\right] \approx 2{,}664.061$$

$$Y = \left[1 - \frac{.06}{.04}\right] \times \left[\frac{(2\times 20)\times(2\times 20+1)}{(1+.04/2)^{2\times 20+2}}\right] \approx -356.949$$

Z = C = 1.27355 (calculated previously)

$$CVX = \frac{4{,}103.322 - 2{,}664.061 - 356.949}{1.27355} \approx 849.839 \text{ semiannual periods}$$

The annual convexity $\approx \frac{849.839}{2^2} \approx 212.460$

EXCEL TEMPLATE FOR BOND DURATION AND CONVEXITY MEASURES:

The Excel template for the calculation of D, D_{MOD}, and CVX is displayed in Exhibit 1. Except for slight rounding differences, the template reproduces the values provided in the numerical example used earlier in the paper; as is well-known, the convexity calculations are particularly sensitive to rounding.

EXHIBIT 1: EXCEL TEMPLATE FOR MACAULAY DURATION, MODIFIED DURATION, AND CONVEXITY

	A	B	C	D	E
1	**BOND INFO:**				
2					
3	COUPON RATE:	6.00%	APR		
4	PAR:	$1,000.00			
5	YIELD-TO-MATURITY	4.00%	APR		
6	MATURITY (N):	20	years		
7	COUPONS per year (M):	2			
8					
9	BOND PRICE:	$1,273.55			
10	PVA:	27.355479			
11					
12	A:	41.03321886			
13	B:	-8.880204219			
14	C:	1.2735548			
15					
16	W:	4103.3218861			
17	X:	2664.0612658			
18	Y:	-356.9493853			
19	Z:	1.273555			
20					
21	**MOD-DURATION:**	25.2467		12.6233	annual
22	**MAC-DURATION:**	25.7516		12.8758	annual
23					
24	**CONVEXITY (CVX):**	849.8348		212.4587	annual

Cell formulas are provided in the appendix

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The Excel programming is not very extensive, which makes building the spreadsheet a very workable assignment or it can be simply made available to the student as a resource. The Fabozzi and Fabozzi (2022) ratio calculation for Macaulay Duration can be used as a means of checking the programming for D and D_{MOD}.

Further, approximations for D, D_{MOD}, and CVX also exist that utilize three bond prices based on a small change in the yield-to-maturity (Δk): the bond price with a slightly lower yield-to-maturity ($k - \Delta k$), the actual bond price based on "k," and the bond price with a slightly higher yield-to-maturity ($k + \Delta k$). These approximations are available in Fabozzi and Fabozzi (2022) and other fixed income security texts. An interesting exercise is determining how precise Δk needs to be for an appropriate level of accuracy with the correct values available in the spreadsheet.

CONCLUSION:

Macaulay Duration (Macaulay, 1938) and Modified Duration are the standard measures of an option-free bond's sensitivity to a change in its yield-to-maturity. Macaulay Duration is most commonly used in hedging strategies, while Modified Duration is more typically used to measure sensitivity to changes in market interest rates. The "ABC" and "WXYZ" component formulas for Macauley Duration (D), Modified Duration (D_{MOD}), and Convexity (CVX) are based on the mathematical derivations of the bond measures and are in a context that can be readily calculated in the undergraduate classroom by hand with a calculator or within an Excel spreadsheet. Developing the Excel spreadsheet can be an assignment or given as a resource to the student. If desired, the structure of the calculations are very amenable for developing a function or algorithm within a programming language, such as, Python.

Within Excel, Data Tables can be used to demonstrate how each bond sensitivity measure changes relative to the coupon rate, coupon frequency, yield to maturity, and length of maturity (i.e. computational inputs). Further, Dollar Duration can easily be added to the spreadsheet calculations if desired.

APPENDIX: Excel Formulas

Cell B9: =PV(B5/B7, B6*B7, -B3*B4/B7, -B4)

Cell B10: =PV(B5/B7, B6*B7, -1)

Cell B12: =B10*(B3/B5)

Cell B13: =(1 – B3/B5)*(B6*B7) / ((1 + B5/B7)^(B6*B7+1))

Cell B14: = 1 + ((B3 – B5)/B7)*B10

Cell B16: = 2*B7*(B3 / (B5^2))*B10

Cell B17: = 2*B7*(B3 / (B5^2))*B7*B6/((1 + B5/B7)^(B7*B6 + 1))

Cell B18: = (1 – B3/B5)*(B6*B7)*(B6*B7 + 1) /((1 + B5/B7)^(B7*B6 + 2))

Cell B19: = 1 + ((B3 – B5)/B7)*B10

Cell B21: =(B12 + B13) / B14

Cell B22: = B21*(1 +B5/B7)

Cell B24: = (B16 – B17 + B18) / B19

Cell D21: = B21 / B7

Cell D22: = B22 / B7

Cell D24: = B24 / (B7^2)

A copy of this file is available at:

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