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The matrix algebra associated with finding minimum variance portfolio weights, mapping the efficient frontier, and determining the tangency portfolio weights is greatly simplified in Excel by applying Cramer's Rule. Only a scant knowledge of linear algebra is necessary for producing a very intuitive presentation for a multi-asset portfolio. The technique is very easily replicated for an assignment or for providing a classroom resource.

INTRODUCTION

An optimal portfolio can be constructed such that the portfolio's variance is minimized or overall reward per unit of risk is maximized given a vector of assets' historical return, risk, and return correlation data. The asset weights within these portfolios can be found using a Lagrange multiplier method, with one of the conditions being that asset weights in the portfolio sum to 1 (i.e., 100%). This optimization process can be intuitively and efficiently presented in Excel by making use of Cramer's rule and the =MDETERM function that returns the matrix determinant of an array.

In this paper, we first provide a brief overview of Cramer's rule, and highlight how linear algebra, presented effectively, greatly simplifies the optimization problem. Next, we provide an application of Cramer's rule by determining the minimum variance portfolio wieghts, efficient frontier weights, and the tangency portfolio weights given three risky assets and a risk-free security.

A REVIEW OF CRAMER'S RULE

Cramer's Rule can be found in most basic linear algebra texts (e.g. Strang, 2020, Simmons, 1987 provides a very good numerical application). Basically, given an equal number of non-redundant equations for a set of variables, solutions for the variables can be found using matrix determinants. For example, suppose there are three unknown variables X_1, X_2, and X_3 and three equations:

$$(a_1 \times X_1) + (b_1 \times X_2) + (c_1 \times X_3) = d_1 \tag{1}$$

$$(a_2 \times X_1) + (b_2 \times X_2) + (c_2 \times X_3) = d_2 \tag{2}$$

$$(a_3 \times X_1) + (b_3 \times X_2) + (c_3 \times X_3) = d_3 \tag{3}$$

Convert the equations into matrices (ABC, X, and D):

$$\begin{array}{ccc} ABC & X & D \\ \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \times & \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = & \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \end{array} \quad (4)$$

Create matrices DBC, ADC, and ABD, in which the column matrix D is substituted for the first (column of a_i), second (column of b_i), and third column (column of c_i) respectively.

$$\begin{array}{ccc} DBC & ADC & ABD \\ \begin{bmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{bmatrix} & \begin{bmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{bmatrix} & \begin{bmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{bmatrix} \end{array} \quad (5)$$

By using determinants ("det") for matrices ABC, DBC, ADC, and ABD, solutions for X_1, X_2, and X_3 can be found based on Cramer's Rule:

$X_1 = \det(DBC) \div \det(ABC)$ (6)

$X_2 = \det(ADC) \div \det(ABC)$ (7)

$X_3 = \det(ABD) \div \det(ABC)$ (8)

A determinant does have a geometric/spatial interpretation, however, that will not be important for our purposes. Further, calculation of the determinant can be found in the earlier references for linear algebra and in the appendix. Calculating the determinant will not be necessary for our purposes because Excel has a function, =MDETERM( ), that will perform the calculation. What is important is the structure/method of finding the solution for the unknown variables based on Cramer's Rule and how to structure the portfolio math to apply Cramer's Rule.

Arnold (2002) based on Roll (1977) demonstrates the optimization criteria for finding minimum variance portfolio weights, tangency portfolio weights (however, Bitten-Jones' method , 1999, will be used here), and for mapping the efficient frontier. Arnold and Nixon (2021) develop methods in Excel to perform these calculations using matrix inversion techniques. By applying Cramer's Rule, the method illustrated in this paper avoids using matrix inversion.

In the next three successive sections, Cramer's Rule will be applied to solve for minimum variance portfolio weights, mapping the efficient frontier, and finding the tangency portfolio weights. The fourth section concludes the paper.

MINIMUM VARIANCE PORTFOLIO WEIGHTS:

In Table 1, information is provided for three risky securities (A, B, and C) and a risk-free security.

Table 1: Information for Three Risky Securities (A, B, and C) and a Risk-free Security

Security:	Mean:	Standard Deviation:	Variance:
A	5.00%	25.00%	0.0625
B	6.00%	34.00%	0.1156
C	7.00%	48.00%	0.2304
Risk-free	1.00%	0.00%	0.0000
Correlations (CORR) and Covariances (COV):			
CORR (A,B):	0.600	COV (A,B):	0.0510
CORR (A,C):	0.400	COV (A,C):	0.0480
CORR (B,C):	0.500	COV (B,C):	0.0816

Based on Arnold (2002), the weights for the minimum variance portfolio can be found using the following equation with a Lagrange condition for the portfolio weights summing to 1 (i.e., 100%).

$$L = (W_A)^2 \times \text{Variance (A)} + (W_B)^2 \times \text{Variance (B)} + (W_C)^2 \times \text{Variance (C)}^2$$
$$+ 2 \times W_A \times W_B \times \text{Covariance (A, B)} + 2 \times W_A \times W_C \times \text{Covariance (A, C)}$$
$$+ 2 \times W_B \times W_C \times \text{Covariance (B, C)} + \lambda\ [\ W_A + W_B + W_C - 1] \qquad (9)$$

After taking the partial derivatives relative to each weight (W_A, W_B, and W_C) and relative to the Lagrange multiplier (λ), the following matrices are generated based on setting each partial derivative equation to zero[1]:

$$\underbrace{\begin{bmatrix} V(A) & C(A,B) & C(A,C) & 1 \\ C(A,B) & V(B) & C(B,C) & 1 \\ C(A,C) & C(B,C) & V(C) & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}}_{\text{VCOVL-ABC}} \times \underbrace{\begin{bmatrix} W_A \\ W_B \\ W_C \\ \lambda \end{bmatrix}}_{\text{WL}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\text{Z}} \qquad (10)$$

To implement Cramer's Rule, substitute column matrix Z into the first column of VCOVL-ABC to create the square matrix VCOVL-ZBC:

VCOVL-ZBC

$$\begin{bmatrix} 0 & C(A,B) & C(A,C) & 1 \\ 0 & V(B) & C(B,C) & 1 \\ 0 & C(B,C) & V(C) & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \qquad (11)$$

Substitute column matrix Z into the second column of VCOVL-ABC to create the square matrix VCOVL-AZC:

VCOVL-AZC

$$\begin{bmatrix} V(A) & 0 & C(A,C) & 1 \\ C(A,B) & 0 & C(B,C) & 1 \\ C(A,C) & 0 & V(C) & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \qquad (12)$$

Substitute column matrix Z into the third column of VCOVL-ABC to create the square matrix VCOVL-ABZ:

[1] Set to zero to find the minimum variance portfolio weights. Note that V(X) is the variance of X and C(X,Y) is the covariance between X and Y)

VCOVL-ABZ

$$\begin{bmatrix} V(A) & C(A,B) & 0 & 1 \\ C(A,B) & V(B) & 0 & 1 \\ C(A,C) & C(B,C) & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \tag{13}$$

Apply the determinants of the matrices to find the minimum variance portfolio weights:

$$W_A = \det(\text{VCOVL-ZBC}) \div \det(\text{VCOVL-ABC}) \tag{14}$$

$$W_B = \det(\text{VCOVL-AZC}) \div \det(\text{VCOVL-ABC}) \tag{15}$$

$$W_C = \det(\text{VCOVL-ABZ}) \div \det(\text{VCOVL-ABC}) \tag{16}$$

In Table 2, the associated Excel sheet provides the minimum variance portfolio weight calculations associated with the securities in Table 1 and applying equations (10) through (16).

Table 2: Excel Solution for Minimum Variance Portfolio Weights

	A	B	C	D	E	F	G	H	I
1		Mean:	STDEV:	VAR:					
2	Sec. A:	5.00%	25.00%	0.0625		CORR(A,C):	0.600	C(A,B):	0.0510
3	Sec. B:	6.00%	34.00%	0.1156		CORR(A,B):	0.400	C(A,C):	0.0480
4	Sec. C:	7.00%	48.00%	0.2304		CORR(B,C):	0.500	C(B,C):	0.0816
5	Risk-free:	1.00%	0.00%	0.0000					
6									
7	VCOVL-ABC								
8		A	B	C	L		Z		
9		0.0625	0.0510	0.0480	1		0		
10		0.0510	0.1156	0.0816	1		0		
11		0.0480	0.0816	0.2304	1		0		
12		1	1	1	0		1		
13									
14	VCOVL-ZBC								
15		Z	B	C	L		Weight A:		83.05%
16		0	0.0510	0.0480	1		Weight B:		12.44%
17		0	0.1156	0.0816	1		Weight C:		4.52%
18		0	0.0816	0.2304	1				
19		1	1	1	0				
20									
21	VCOVL-AZC								
22		A	Z	C	L				
23		0.0625	0	0.0480	1				

24		0.0510	0	0.0816	1				
25		0.0480	0	0.2304	1				
26		1	1	1	0				
27									
28	VCOVL-ABZ								
29		A	B	Z	L				
30		0.0625	0.0510	0	1				
31		0.0510	0.1156	0	1				
32		0.0480	0.0816	0	1				
33		1	1	1	0				

STDEV: standard deviation
VAR: variance
CORR (X,Y): Correlation (X,Y)
C(X,Y): Covariance (X,Y)

Cell I15: =MDETERM(B16:E19) / MDETERM(B9:E12)
Cell I16: =MDETERM(B23:E26) / MDETERM(B9:E12)
Cell I17: =MDETERM(B30:E33) / MDETERM(B9:E12)

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As one can see in Table 2, the Excel programing is minimal and the minimum variance portfolio weights for Security A, Security B, and Security C are: 83.05%, 12.44%, and 4.52% respectively. [2]

EFFICIENT FRONTIER PORTFOLIO WEIGHTS

Based on Arnold (2002), the weights for a portfolio on the efficient frontier that generate a return of "K" can be found using the following equation with two Lagrange conditions: portfolio weights summing to 1 (i.e., 100%) and a portfolio mean set to a return of "K".

$$\begin{aligned} L = {} & (W_A)^2 \times \text{Variance (A)} + (W_B)^2 \times \text{Variance (B)} + (W_C)^2 \times \text{Variance (C)}^2 \\ & + 2 \times W_A \times W_B \times \text{Covariance (A, B)} + 2 \times W_A \times W_C \times \text{Covariance (A, C)} \\ & + 2 \times W_B \times W_C \times \text{Covariance (B, C)} + \lambda\, [\, W_A + W_B + W_C - 1] \\ & + \delta\, [\, K - (W_A \times \text{Mean(A)}) - (W_B \times \text{Mean (B)}) - (W_C \times \text{Mean (C)})] \qquad (17) \end{aligned}$$

[2] For those readers interested in a refresher regarding the calculation of determinants, the calculation for the determinant of VCOVL-ABC and VCOVL-ZBC are included in the Appendix.

After taking the partial derivatives relative to each weight (W_A, W_B, and W_C) and relative to the Lagrange multipliers (λ and δ), the following matrices are generated based on setting each partial derivative equation to zero:[3]

$$\underbrace{\begin{bmatrix} V(A) & C(A,B) & C(A,C) & 1 & M(A) \\ C(A,B) & V(B) & C(B,C) & 1 & M(B) \\ C(A,C) & C(B,C) & V(C) & 1 & M(C) \\ 1 & 1 & 1 & 0 & 0 \\ M(A) & M(B) & M(C) & 0 & 0 \end{bmatrix}}_{\text{VCOVLG-ABC}} \times \underbrace{\begin{bmatrix} W_A \\ W_B \\ W_C \\ \lambda \\ \delta \end{bmatrix}}_{\text{WLG}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ K \end{bmatrix}}_{\text{K}} \quad (18)$$

To implement Cramer's Rule, substitute column matrix K into the first column of VCOVLG-ABC to create the square matrix VCOVLG-KBC:

VCOVLG-KBC

$$\begin{bmatrix} 0 & C(A,B) & C(A,C) & 1 & M(A) \\ 0 & V(B) & C(B,C) & 1 & M(B) \\ 0 & C(B,C) & V(C) & 1 & M(C) \\ 1 & 1 & 1 & 0 & 0 \\ K & M(B) & M(C) & 0 & 0 \end{bmatrix} \quad (19)$$

Substitute column matrix K into the second column of VCOVLG-ABC to create the square matrix VCOVLG-AKC:

VCOVLG-AKC

$$\begin{bmatrix} V(A) & 0 & C(A,C) & 1 & M(A) \\ C(A,B) & 0 & C(B,C) & 1 & M(B) \\ C(A,C) & 0 & V(C) & 1 & M(C) \\ 1 & 1 & 1 & 0 & 0 \\ M(A) & K & M(C) & 0 & 0 \end{bmatrix} \quad (19)$$

[3] Here, partial derivatives are set to zero to find the weights for the efficient frontier portfolios. Note that V(X) is the variance of X, C(X,Y) is the covariance between X and Y, and M(X) is the mean of X.

Substitute column matrix K into the third column of VCOVLG-ABC to create the square matrix VCOVLG-ABK:

VCOVLG-ABK

$$\begin{bmatrix} V(A) & C(A,B) & 0 & 1 & M(A) \\ C(A,B) & V(B) & 0 & 1 & M(B) \\ C(A,C) & C(B,C) & 0 & 1 & M(C) \\ 1 & 1 & 1 & 0 & 0 \\ M(A) & M(B) & K & 0 & 0 \end{bmatrix} \quad (20)$$

Apply the determinants of the matrices to find the minimum variance portfolio weights:

$W_A = \det(\text{VCOVLG-KBC}) \div \det(\text{VCOVLG-ABC})$ (21)

$W_B = \det(\text{VCOVLG-AKC}) \div \det(\text{VCOVLG-ABC})$ (22)

$W_C = \det(\text{VCOVLG-ABK}) \div \det(\text{VCOVLG-ABC})$ (23)

In Table 3, the associated Excel sheet provides the weight calculations for a portfolio on the efficient frontier with a mean of 5.70% (i.e. K = 5.70%) based on the securities in Table 1 and applying equations (18) through (23).

Table 3: Excel Solution for Efficient Frontier Portfolio Weights for a Specific Portfolio Mean

	A	B	C	D	E	F	G	H	I
1		Mean:	STDEV:	VAR:					
2	Sec. A:	5.00%	25.00%	0.0625		CORR(A,C):	0.600	C(A,B):	0.0510
3	Sec. B:	6.00%	34.00%	0.1156		CORR(A,B):	0.400	C(A,C):	0.0480
4	Sec. C:	7.00%	48.00%	0.2304		CORR(B,C):	0.500	C(B,C):	0.0816
5	Risk-free:	1.00%	0.00%	0.0000					
6									
7	VCOVLG-ABC								
8		A	B	C	L	G		K	
9		0.0625	0.0510	0.0480	1	5.00%		0	
10		0.0510	0.1156	0.0816	1	6.00%		0	
11		0.0480	0.0816	0.2304	1	7.00%		0	
12		1	1	1	0	0		1	
13		5.00%	6.00%	7.00%	0	0		5.70%	
14									
15	VCOVLG-KBC								
16		A	B	C	L	G		Weight A:	50.71%
17		0	0.0510	0.0480	1	5.00%		Weight B:	28.57%
18		0	0.1156	0.0816	1	6.00%		Weight C:	20.71%
19		0	0.0816	0.2304	1	7.00%			
20		1	1	1	0	0			
21		5.70%	6.00%	7.00%	0	0			
22									

23	VCOVLG-AKC								
24		A	B	C	L	G			
25		0.0625	0	0.0480	1	5.00%			
26		0.0510	0	0.0816	1	6.00%			
27		0.0480	0	0.2304	1	7.00%			
28		1	1	1	0	0			
29		5.00%	5.70%	7.00%	0	0			
30									
31	VCOVLG-ABK								
32		A	B	C	L	G			
33		0.0625	0.0510	0	1	5.00%			
34		0.0510	0.1156	0	1	6.00%			
35		0.0480	0.0816	0	1	7.00%			
36		1	1	1	0	0			
37		5.00%	6.00%	5.70%	0	0			

STDEV: standard deviation
VAR: variance
CORR (X,Y): Correlation (X,Y)
C(X,Y): Covariance (X,Y)

Cell I15: =MDETERM(B17:F21) / MDETERM(B9:F13)
Cell I16: =MDETERM(B25:F29) / MDETERM(B9:F13)
Cell I17: =MDETERM(B33:F37) / MDETERM(B9:F13)

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Changing “K” to various values will map out the efficient frontier, see Table 4.

Table 4: The Efficient Frontier Based on Securities from Table 1

Portfolio Mean:	Portfolio Standard Deviation:	Portfolio Sharpe Ratio:	Weight A:	Weight B:	Weight C:
6.00%	29.21%	17.119%	30.73%	38.55%	30.73%
5.90%	28.17%	17.393%	37.39%	25.22%	27.39%
5.80%	27.25%	17.616%	44.05%	31.90%	24.05%
5.70%	26.44%	17.774%	50.71%	28.57%	20.71%
5.60%	***25.77%***	***17.851%***	***57.38%***	***25.25%***	***17.38%***
5.50%	25.24%	17.830%	64.04%	21.92%	14.04%
5.40%	24.86%	17.699%	70.70%	18.60%	10.70%
5.30%	24.64%	17.452%	77.36%	15.27%	7.36%
5.20%	**24.58%**	**17.086%**	**84.03%**	**11.95%**	**4.03%**
5.10%	24.69%	16.608%	90.69%	8.62%	0.69%
5.00%	24.95%	16.029%	97.35%	5.30%	-2.65%

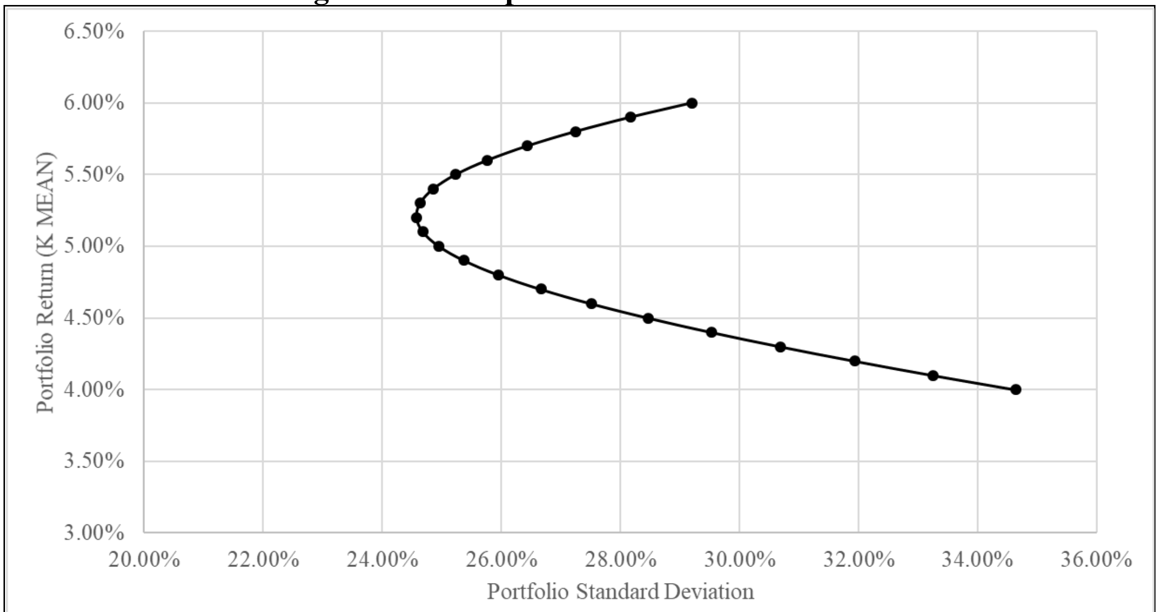
Sharpe Ratio = (portfolio return – risk-free rate) ÷ Standard deviation of portfolio

The tangency portfolio is located on the efficient frontier where the Sharpe ratio is maximized (in bold and italic in the table). The actual tangency portfolio is where the portfolio mean is 5.57%. The exact weights for the tangency portfolio will be found in the next section.

The “approximate” minimum variance portfolio is indicated with bold. Notice how the standard deviation increases with portfolio returns above and below it. The exact minimum variance portfolio has a mean return of 5.21% when applying the portfolio weights from Table 2.

Again, the Excel programming is minimal and if programmed appropriately, changing cell H13 within the spreadsheet (i.e. the value for "K") will produce the different portfolio combinations that produce the efficient frontier. Additionally, a simple scatterplot connected with a smooth curve can graphically depict the portfolios presented in Table 4. The y-axis presents the return figures from the "Portfolio Mean" column of Table 4 while the x-axis presents the values in the "Portfolio Standard Deviation" column of Table 4. Producing and interpreting this figure might be left to the student as an exercise. Here, the minimum variance portfolio is clear: the portfolio where the curve reaches furthest to the left is the portfolio with the lowest risk. Note the return (y) and risk (x) values of this portfolio are about equal to the "approximate" minimum variance portfolio in Table 4.

Figure 1: A Graph of the Efficient Frontier



To insert the graph in Excel: Insert > Charts > Scatter with Smooth Line and Markers

TANGENCY PORTFOLIO WEIGHTS

Based on Arnold (2002), a risk-free security can be introduced as a fourth security in the previous analysis. The risk-free security will have a variance of zero and its covariance with any risky security is also zero. However, Bitten-Jones (1999) finds the weights of the tangency portfolio through a regression routine that Arnold and Nixon (2021) equate to as having a Lagrange condition of the portfolio risk premium being equal to 1 when minimizing the portfolio variance.[4]

$$\begin{aligned} L = {} & (W_A)^2 \times \text{Variance (A)} + (W_B)^2 \times \text{Variance (B)} + (W_C)^2 \times \text{Variance (C)}^2 \\ & + 2 \times W_A \times W_B \times \text{Covariance (A, B)} + 2 \times W_A \times W_C \times \text{Covariance (A, C)} \\ & + 2 \times W_B \times W_C \times \text{Covariance (B, C)} \\ & + \lambda\,[\,1 - W_A(\text{Mean (A)} - R_F) - W_B(\text{Mean (B)} - R_F) - W_C(\text{Mean (C)} - R_F)] \end{aligned} \qquad (24)$$

After taking the partial derivatives relative to each weight (W_A, W_B, and W_C) and relative to the Lagrange multiplier (λ), the following matrices are generated based on setting each partial derivative equation to zero[5]:

$$\underset{\text{VCOVRP-ABC}}{\begin{bmatrix} V(A) & C(A,B) & C(A,C) & RP(A) \\ C(A,B) & V(B) & C(B,C) & RP(B) \\ C(A,C) & C(B,C) & V(C) & RP(C) \\ RP(A) & RP(B) & RP(C) & 0 \end{bmatrix}} \times \underset{\text{WRP}}{\begin{bmatrix} W_A \\ W_B \\ W_C \\ \lambda \end{bmatrix}} = \underset{\text{T}}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}} \qquad (25)$$

To implement Cramer's Rule, substitute column matrix T into the first column of VCOVRP-ABC to create the square matrix VCOVRP-TBC:

---

[4] The risk premium is the mean return less the risk-free rate.

[5] The partial derivates are set to zero to find the tangency portfolio weights. Note that V(X) is the variance of X, C(X,Y) is the covariance between X and Y, and RP(X) = Mean (X) less risk-free rate.

VCOVRP-TBC

$$\begin{bmatrix} 0 & C(A,B) & C(A,C) & RP(A) \\ 0 & V(B) & C(B,C) & RP(B) \\ 0 & C(B,C) & V(C) & RP(C) \\ 1 & RP(B) & RP(C) & 0 \end{bmatrix} \tag{26}$$

Substitute column matrix T into the second column of VCOVTP-ABC to create the square matrix VCOVRP-ATC:

VCOVRP-ATC

$$\begin{bmatrix} V(A) & 0 & C(A,C) & RP(A) \\ C(A,B) & 0 & C(B,C) & RP(B) \\ C(A,C) & 0 & V(C) & RP(C) \\ RP(A) & 1 & RP(C) & 0 \end{bmatrix} \tag{27}$$

Substitute column matrix T into the third column of VCOVRP-ABC to create the square matrix VCOVRP-ABT:

VCOVRP-ABT

$$\begin{bmatrix} V(A) & C(A,B) & 0 & RP(A) \\ C(A,B) & V(B) & 0 & RP(B) \\ C(A,C) & C(B,C) & 0 & RP(C) \\ RP(A) & RP(B) & 1 & 0 \end{bmatrix} \tag{28}$$

Apply the determinants of the matrices to find the portfolio weights:

$$W_A = \det(\text{VCOVRP-TBC}) \div \det(\text{VCOVRP-ABC}) \tag{29}$$

$$W_B = \det(\text{VCOVRP-ATC}) \div \det(\text{VCOVRP-ABC}) \tag{30}$$

$$W_C = \det(\text{VCOVRP-ABT}) \div \det(\text{VCOVRP-ABC}) \tag{31}$$

These portfolio weights are set to include the weight of the risk-free security. To adjust to the tangency portfolio weights, one has to re-apportion the above portfolio weights in the following manner:

$$W_{A\text{-}TAN} = W_A \div (W_A + W_B + W_C) \quad (32)$$

$$W_{B\text{-}TAN} = W_B \div (W_A + W_B + W_C) \quad (33)$$

$$W_{C\text{-}TAN} = W_C \div (W_A + W_B + W_C) \quad (34)$$

In Table 5, the associated Excel sheet provides the tangency portfolio weight calculations associated with the securities in Table 1 and applying equations (25) through (34).

Table 5: Excel Solution for Tangency Portfolio Weights

	A	B	C	D	E	F	G	H	I
1		Mean:	STDEV:	VAR:					
2	Sec. A:	5.00%	25.00%	0.0625		CORR(A,C):	0.600	C(A,B):	0.0510
3	Sec. B:	6.00%	34.00%	0.1156		CORR(A,B):	0.400	C(A,C):	0.0480
4	Sec. C:	7.00%	48.00%	0.2304		CORR(B,C):	0.500	C(B,C):	0.0816
5	Risk-free:	1.00%	0.00%	0.0000					
6									
7	VCOVRP-ABC								
8		A	B	C	RP		T		
9		0.0625	0.0510	0.0480	4.00%		0		
10		0.0510	0.1156	0.0816	5.00%		0		
11		0.0480	0.0816	0.2304	6.00%		0		
12		4.00%	5.00%	6.00%	0		1		
13									
14	VCOVL-TBC								
15		T	B	C	RP			Weight A:	1299.45%
16		0	0.0510	0.0480	4.00%			Weight B:	530.56%
17		0	0.1156	0.0816	5.00%			Weight C:	358.23%
18		0	0.0816	0.2304	6.00%				
19		1	5.00%	6.00%	0			Weight A-TAN:	59.38%
20								Weight B-TAN:	24.25%
21	VCOVL-ATC							Weight C-TAN:	16.37%
22		A	T	C	RP				
23		0.0625	0	0.0480	4.00%				
24		0.0510	0	0.0816	5.00%				
25		0.0480	0	0.2304	6.00%				
26		4.00%	1	6.00%	0				
27									
28	VCOVL-ABT								
29		A	B	T	RP				
30		0.0625	0.0510	0	4.00%				
31		0.0510	0.1156	0	5.00%				
32		0.0480	0.0816	0	6.00%				
33		4.00%	5.00%	1	0				

STDEV: standard deviation
VAR: variance
CORR (X,Y): Correlation (X,Y)
C(X,Y): Covariance (X,Y)

Cell I15: =MDETERM(B16:E19) / MDETERM(B9:E12)
Cell I16: =MDETERM(B23:E26) / MDETERM(B9:E12)
Cell I17: =MDETERM(B30:E33) / MDETERM(B9:E12)

Cell I19: = I15 / SUM(I15:I17)
Cell I20: = I16 / SUM(I15:I17)

Cell I21: = I17 / SUM(I15:I17)

This file can be downloaded from: https://scholarship.richmond.edu/finance-faculty-publications/XX/

The Excel programming is still very minimal even with the inclusion of an additional step. When compared to the portfolio weights in Table 4 for the maximum Sharpe ratio portfolio, one can see the subtle difference between the table and the actual tangency portfolio. The tangency portfolio has an actual portfolio mean of 5.57% with a standard deviation of 25.59% and a Sharpe ratio of 17.855% compared to the earlier approximate portfolio mean of 5.60% with a standard deviation of 25.77% and a Sharpe ratio of 17.851%.

CONCLUSION

Cramer's Rule with the =MDETERM( ) function greatly reduces the necessary calculations for portfolio optimization and can easily be extended to more than three risky securities (possibly as an assignment). When compared to Arnold (2002) and Arnold and Nixon (2021), much of the math regarding optimization disappears which may not be advantageous for a more advanced course. However, for a course that features the use of the results from an optimization rather than actually performing the optimization, these methods become very useful and practical.

The methods can be demonstrated during a live or virtual class very easily and then potentially supplied as a resource. Actual data can be downloaded from internet resources or Bloomberg and readily applied to the spreadsheet templates.

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APPENDIX: CALCULATING A DETERMINANT

There are two common approaches for calculating a determinant: using cofactors in a recursive manner and the "basket weaving" technique, also known as Sarrus' Rule. The latter is more illustrative, but can only be used for 2 × 2 and 3 × 3 square matrices.

For a 2 × 2 matrix, the determinant is a fairly simple calculation:

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \tag{A1}$$

The determinant is $(a_1 \times b_2) - (b_1 \times a_2)$ (A2)

For a 3 × 3 square matrix the "basket weaving" techniques starts by repeating the first two columns after the third column of the matrix.

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{matrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{matrix} \tag{A3}$$

Define "right diagonals," RD1 = $a_1 \times b_2 \times c_3$ (in bold), RD2 = $b_1 \times c_2 \times a_3$ (in italic), and RD3 = $c_1 \times a_2 \times b_3$ (in bold-italic) in equation (A4).

$$\begin{array}{cc} \begin{matrix} \textbf{RD1:} & \textbf{RD2:} & \textbf{RD3:} \end{matrix} & \\ \begin{bmatrix} \mathbf{a_1} & \mathit{b_1} & \boldsymbol{c_1} \\ a_2 & \mathbf{b_2} & \mathit{c_2} \\ a_3 & b_3 & \mathbf{c_3} \end{bmatrix} & \begin{matrix} a_1 & b_1 \\ \boldsymbol{a_2} & b_2 \\ \mathit{a_3} & \boldsymbol{b_3} \end{matrix} \end{array} \tag{A4}$$

Define "left diagonals," LD1 = $c_1 \times b_2 \times a_3$ (in bold), LD2 = $a_1 \times c_2 \times b_3$ (in italic), and LD3 = $b_1 \times a_2 \times c_3$ (in bold-italic) in equation (A5).

$$\begin{array}{cc} \begin{matrix} & & \textbf{LD1:} \end{matrix} & \begin{matrix} \textbf{LD2:} & \textbf{LD3:} \end{matrix} \\ \begin{bmatrix} a_1 & b_1 & \mathbf{c_1} \\ a_2 & \mathbf{b_2} & \mathit{c_2} \\ \mathbf{a_3} & \mathit{b_3} & \boldsymbol{c_3} \end{bmatrix} & \begin{matrix} \mathit{a_1} & \boldsymbol{b_1} \\ \boldsymbol{a_2} & b_2 \\ a_3 & b_3 \end{matrix} \end{array} \tag{A5}$$

The determinant is [RD1 + RD2 + RD3] – [LD1 + LD2 +LD3] (A6)

Because the basket weaving technique is limited to 2 × 2 and 3 × 3 matrices only, the determinant of a 4 × 4 matrix requires the use of cofactors. A cofactor is a smaller matrix within a larger matrix and can be used recursively to reduce a large matrix into many 3 × 3 or 2 × 2 matrices depending on the complexity of the 4 × 4 matrix. Consider a 4 × 4 matrix and choose a particular value inside the matrix, say a_1.

$$\begin{bmatrix} \mathbf{a_1} & b_1 & c_1 & d_1 \\ a_2 & \mathbf{b_2} & \mathbf{c_2} & \mathbf{d_2} \\ a_3 & \mathbf{b_3} & \mathbf{c_3} & \mathbf{d_3} \\ a_4 & \mathbf{b_4} & \mathbf{c_4} & \mathbf{d_4} \end{bmatrix} \qquad \text{(A7)}$$

The cofactor for a_1 is the 3 × 3 matrix in bold. The 3 × 3 matrix that is the cofactor consists of columns and rows that do not match the column or row of the value selected within the 4 × 4 matrix (i.e. a_1 in this case). In a similar manner, cofactors of 3 × 3 matrices can be found for a_2, a_3, and a_4:

$$\begin{bmatrix} a_1 & \mathbf{b_1} & \mathbf{c_1} & \mathbf{d_1} \\ \mathbf{a_2} & b_2 & c_2 & d_2 \\ a_3 & \mathbf{b_3} & \mathbf{c_3} & \mathbf{d_3} \\ a_4 & \mathbf{b_4} & \mathbf{c_4} & \mathbf{d_4} \end{bmatrix} \qquad \text{(A8)}$$

$$\begin{bmatrix} a_1 & \mathbf{b_1} & \mathbf{c_1} & \mathbf{d_1} \\ a_2 & \mathbf{b_2} & \mathbf{c_2} & \mathbf{d_2} \\ \mathbf{a_3} & b_3 & c_3 & d_3 \\ a_4 & \mathbf{b_4} & \mathbf{c_4} & \mathbf{d_4} \end{bmatrix} \qquad \text{(A9)}$$

$$\begin{bmatrix} a_1 & \mathbf{b_1} & \mathbf{c_1} & \mathbf{d_1} \\ a_2 & \mathbf{b_2} & \mathbf{c_2} & \mathbf{d_2} \\ a_3 & \mathbf{b_3} & \mathbf{c_3} & \mathbf{d_3} \\ \mathbf{a_4} & b_4 & c_4 & d_4 \end{bmatrix} \qquad \text{(A10)}$$

The determinant for a 4 × 4 matrix becomes:

a_1 × determinant of the cofactor of a_1

$-\ a_2 \times$ determinant of the cofactor of a_2
$+\ a_3 \times$ determinant of the cofactor of a_3
$-\ a_4 \times$ determinant of the cofactor of a_4 (A11)

The above techniques can be applied the VCOVL-ABC and VCOVL-ZBC matrices in Table 2. Following equation (A11), the determinant for VCOVL-ABC is:

$$\begin{bmatrix} 0.0625 & 0.0510 & 0.0480 & 1 \\ 0.0510 & 0.1156 & 0.0816 & 1 \\ 0.0480 & 0.0816 & 0.2304 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \tag{A12}$$

$0.0625 \times$ determinant of the cofactor of a_1
$-\ 0.0510 \times$ determinant of the cofactor of a_2
$+\ 0.0480 \times$ determinant of the cofactor of a_3
$-\ 1 \times$ determinant of the cofactor of a_4 (A13)

Using the basket weaving technique, the determinants of the 3 × 3 cofactor matrices can be calculated and the determinant of the 4 × 4 matrix can them be computed.

$$0.0625 \times (-0.182800) - 0.0510 \times (-0.151800) + 0.0480 \times (0.031000) - 1 \times (0.010755)$$
$$= -0.012950 \tag{A14}$$

Again, following equation (A11), the determinant for VCOVL-ZBC is:

$$\begin{bmatrix} 0 & 0.0510 & 0.0480 & 1 \\ 0 & 0.1156 & 0.0816 & 1 \\ 0 & 0.0816 & 0.2304 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \tag{A15}$$

$0 \times$ determinant of the cofactor of a_1
$-\ 0 \times$ determinant of the cofactor of a_2
$+\ 0 \times$ determinant of the cofactor of a_3
$-\ 1 \times$ determinant of the cofactor of $a_4 = -\ 1 \times$ determinant of the cofactor of a_4 (A16)

Because only one cofactor remains, it is easy to demonstrate the calculations for the determinant of the cofactor using the basket weaving technique.

$RD1 = 0.0510 \times 0.0816 \times 1 = 0.004162$
$RD2 = 0.0480 \times 1 \times 0.0816 = 0.003917$
$RD3 = 1 \times 0.1156 \times 0.2304 = 0.026634$
$LD1 = 1 \times 0.0816 \times 0.0816 = 0.006659$
$LD2 = 0.0510 \times 1 \times 0.2304 = 0.011750$
$LD3 = 0.0480 \times 0.1156 \times 1 = 0.005549$

Determinant of the cofactor:

$$(0.004162 + 0.003917 + 0.026634) - (0.006659 + 0.011750 + 0.005549) = 0.010755 \tag{A17}$$

Determinant of VCOVL-ZBC based on equation (A16):

$$-1 \times (0.010755) = -0.010755 \tag{A18}$$

Based on equation (14), the minimum variance portfolio weight for Security A is:

$$W_A = \det(\text{VCOVL-ZBC}) \div \det(\text{VCOVL-ABC})$$

$$W_A = (-0.010755) \div (-0.012950) = 83.05\% \tag{A19}$$