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The classical Dirichlet space

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Dedicated to J. A. Cima on the occasion of his 70th birthday and to over 45 years of dedicated service to mathematics.

1. Introduction

In this survey paper, we will present a selection of results concerning the class of analytic functions f on the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ which have finite Dirichlet integral

$$D(f) := \frac{1}{\pi} \int_{\mathbb{D}} |f'|^2 \, dx \, dy.$$

In particular, we will cover the basic structure of these functions - their boundary values and their zeros - along with two important operators that act on this space of functions - the forward and backward shifts. This survey is by no means complete. For example, we will not cover the Toeplitz or Hankel operators on these functions, nor will we cover the important topic of interpolation. These topics are surveyed in a nice paper of Wu [**64**]. In order to make this survey more manageable, we will also restrict ourselves to this space of functions with finite Dirichlet integral and will not try to cover the many related Dirichlet-type spaces. We refer the reader to the papers [**11, 41, 47, 54**] for more on this.

2. Basic definitions

An analytic function f on the open unit disk $\mathbb{D}$ belongs to the classical *Dirichlet space* $\mathcal{D}$ if it has finite *Dirichlet integral*

$$D(f) := \frac{1}{\pi} \int_{\mathbb{D}} |f'|^2 \, dA,$$

where dA is two dimensional Lebesgue area measure. Thinking of f as a mapping from $\mathbb{D}$ to some region $f(\mathbb{D})$, one computes the Jacobian determinant J_f to be $|f'|^2$ and so the two dimensional area of $f(\mathbb{D})$, counting multiplicities, is

$$\int_{f(\mathbb{D})} 1 \, dA = \int_{\mathbb{D}} |J_f| \, dA = \int_{\mathbb{D}} |f'|^2 \, dA.$$

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Thus a function has finite Dirichlet integral exactly when its image has finite area (counting multiplicities).

Another interesting geometric observation to make is that for each $\zeta \in \mathbb{T}$, the quantity

$$L(f, \zeta) := \int_0^1 |f'(r\zeta)|\, dr \tag{2.1}$$

is the length of the curve $\{f(r\zeta) : 0 < r < 1\}$, which is the image of the ray $\{r\zeta : 0 < r < 1\}$ under the mapping f. If $D(f) < \infty$, we can apply the Cauchy-Schwartz inequality to show that

$$\int_0^{2\pi} L(f, e^{i\theta})\, d\theta \leqslant c D(f) < \infty$$

and so $L(f, e^{i\theta}) < \infty$ for almost every θ. A theorem of Beurling (see Theorem 5.5 below) says that $L(f, e^{i\theta}) < \infty$ for quasi-every θ in the sense of logarithmic capacity.

Writing

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

as a power series and setting $z = re^{i\theta}$, one can integrate in polar coordinates to see that

$$D(f) = \sum_{n=0}^{\infty} n|a_n|^2.$$

Closely related to the Dirichlet space is the classical *Hardy space* H^2 of analytic functions f on $\mathbb{D}$ for which

$$\|f\|_{H^2}^2 := \sup_{0<r<1} \int_{\mathbb{T}} |f(r\zeta)|^2\, dm(\zeta) < \infty.$$

Here dm denotes Lebesgue measure on the unit circle $\mathbb{T} = \partial\mathbb{D}$ normalized so that $m(\mathbb{T}) = 1$. Observe that

$$\int_{\mathbb{T}} |f(r\zeta)|^2\, dm(\zeta) = \sum_{n=0}^{\infty} r^{2n} |a_n|^2$$

and so

$$\|f\|_{H^2}^2 = \sum_{n=0}^{\infty} |a_n|^2.$$

Thus $\mathcal{D} \subset H^2$ and so, via Fatou's theorem on radial limits [**22**], functions in $\mathcal{D}$ have radial boundary values almost everywhere on $\mathbb{T}$, that is to say,

$$f(\zeta) := \lim_{r \to 1^-} f(r\zeta) \tag{2.2}$$

exists and is finite for almost every $\zeta \in \mathbb{T}$. It will turn out (see Theorem 5.10 and Theorem 5.12) that functions in $\mathcal{D}$ have much stronger regularity near $\mathbb{T}$ than H^2 functions.

The quantity $D(f)$ is not a norm, since $D(c) = 0$ for any constant function c. However, one can endow $\mathcal{D}$ with the norm $\|f\|$, where

$$\|f\|^2 := \|f\|_{H^2}^2 + D(f) = \sum_{n=0}^{\infty} (1+n)|a_n|^2. \tag{2.3}$$

An easy estimate yields

$$|f(0)|^2 + D(f) \leqslant \|f\|^2 \leqslant 2\left(|f(0)|^2 + D(f)\right)$$

and several authors use the quantity

$$\sqrt{|f(0)|^2 + D(f)}$$

to define an equivalent norm on $\mathcal{D}$ which is sometimes more convenient to use. With the norm $\|\cdot\|$ in eq.(2.3) above, one defines an inner product

$$\langle f, g\rangle := \int_{\mathbb{T}} f\overline{g}\, dm + \frac{1}{\pi}\int_{\mathbb{D}} f'\overline{g'}\, dA.$$

Simple computations show that the quantities

$$\frac{1}{\pi}\int_{\mathbb{D}} (zf)'\overline{(zg)'}\, dA$$

and

$$\sum_{n=0}^{\infty}(n+1)a_n\overline{b_n},$$

where the a_n are the Taylor coefficients of f and the b_n are those for g, are both equal to $\langle f, g\rangle$.

Another power series computation shows that if we define kernels $k_z(w)$ by

$$k_z(w) := \frac{1}{\overline{w}z}\log\frac{1}{1-\overline{w}z}, \quad z, w \in \mathbb{D},$$

then

$$f(z) = \langle f, k_z\rangle.$$

Moreover, by the Cauchy-Schwartz inequality,

$$|f(z)| \leqslant \|f\|\|k_z\| = \|f\|\langle k_z, k_z\rangle^{1/2} = \|f\|k_z(z)^{1/2}$$

and so f satisfies the pointwise estimate

$$|f(z)| \leqslant c\|f\|\left(\log\frac{1}{1-|z|^2}\right)^{1/2}.$$

In particular, this pointwise estimate shows that if $f_n \to f$ in norm of $\mathcal{D}$, then $f_n \to f$ uniformly on compact subsets of $\mathbb{D}$. Hence $\mathcal{D}$ is a reproducing kernel Hilbert space of analytic functions on $\mathbb{D}$. Finally, it is clear from the definition of the norm that if

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

belongs to $\mathcal{D}$, then

$$\left\|\sum_{n=0}^{N} a_n z^n - f\right\|^2 = \sum_{n=N+1}^{\infty}(n+1)|a_n|^2$$

which goes to zero as $N \to \infty$. Thus the polynomials form a dense subset of $\mathcal{D}$.

3. The Douglas and Carleson formulas

In his investigations of the Plateau problem, Jesse Douglas [**20**] proved the following formula for the Dirichlet integral

$$D(f) = \int_{\mathbb{T}} \int_{\mathbb{T}} \left| \frac{f(\zeta) - f(\xi)}{\zeta - \xi} \right|^2 dm(\zeta)\, dm(\xi),$$

where we understand that the function $\zeta \mapsto f(\zeta)$ on $\mathbb{T}$ is the almost everywhere defined boundary function via radial boundary values eq.(2.2). The inner integral

$$D_\zeta(f) := \int_{\mathbb{T}} \left| \frac{f(\zeta) - f(\xi)}{\zeta - \xi} \right|^2 dm(\xi)$$

is called the *local Dirichlet integral* and one can use it to make some interesting observations [**47**]. First notice how

$$\int_{\mathbb{T}} D_\zeta(f)\, dm(\zeta) = D(f)$$

and so for a Dirichlet function, the local Dirichlet integral is finite almost everywhere.

PROPOSITION 3.1. *Suppose $\zeta \in \mathbb{T}$ and $f \in H^2$ with $D_\zeta(f) < \infty$. Then the oricyclic limit of f exists at ζ. Thus any $f \in \mathcal{D}$ has an oricyclic limit almost everywhere.*

Here the oricyclic approach regions with contact point ζ are

$$A_{2,c}(\zeta) := \{z \in \mathbb{D} : |\zeta - z| \leqslant c(1 - |z|)^{1/2}\}, \ \ c > 0.$$

For example, $A_{2,2}(\zeta)$ contains the disk with center $\zeta/2$ and radius $1/2$. We say that f has *oricyclic limit* L at ζ if

$$\lim_{z \to \zeta, z \in A_{2,c}(\zeta)} f(z) = L$$

for every $c > 0$. Compare this to the non-tangential approach regions with contact point ζ

$$A_{1,c}(\zeta) := \{z \in \mathbb{D} : |\zeta - z| \leqslant c(1 - |z|)\}, \ \ c > 0,$$

which are triangle shaped regions with vertex at ζ. We will discuss much stronger results in Section 5. We also mention that whenever $D_\zeta(f) < \infty$, the Fourier series of f converges to $f(\zeta)$.

A function $f \in H^2$ has the standard Nevanlinna factorization [**22**]

$$f = b s_\mu F, \tag{3.2}$$

where

$$b(z) = z^m \prod_{n=1}^{\infty} \frac{\overline{a_n}}{|a_n|} \frac{a_n - z}{1 - \overline{a_n} z}$$

is the Blaschke factor with zeros at $z = 0$ and $(a_n)_{n \geqslant 1} \subset \mathbb{D} \backslash \{0\}$ (repeated according to multiplicity),

$$s_\mu(z) = \exp\left\{ - \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, d\mu(\zeta) \right\},$$

is the singular inner factor with positive singular measure μ on $\mathbb{T}$ (i.e., $\mu \perp m$), and

$$F(z) = \exp\left\{ \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} u(\zeta)\, dm(\zeta) \right\}, \quad u(\zeta) = \log|f(\zeta)|,$$

is the outer factor. The two inner factors b and s_μ are bounded analytic functions on $\mathbb{D}$ with unimodular boundary values almost everywhere. The outer factor F belongs to H^2.

If $f \in \mathcal{D}$, then $f \in H^2$ and as such has a factorization $f = bs_\mu F$ as in eq.(3.2). Here, the inner factors b and s_μ do not belong to $\mathcal{D}$ unless b is a finite Blaschke product and $s_\mu \equiv 1$ (i.e., $\mu \equiv 0$). However, the outer factor does belong to $\mathcal{D}$ (see Corollary 3.5 below). This following theorem of Carleson [**14**], which has proven to be a quite a workhorse in the subject, computes $D(f)$ in terms of the Nevanlinna factorization.

Theorem 3.3 (Carleson). *For $f = bs_\mu F \in \mathcal{D}$ as in eq.(3.2),*

$$\begin{aligned}\pi D(f) &= \int_{\mathbb{T}} \left(m + \sum_{n=1}^{\infty} P_{a_n}(\zeta) \right) |f(\zeta)|^2 \, dm(\zeta) + \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{2}{|\zeta - \xi|^2} \, d\mu(\zeta) |f(\xi)|^2 \, dm(\xi) \\ &+ \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{(e^{2u(\zeta)} - e^{2u(\xi)})(u(\zeta) - u(\xi))}{|\zeta - \xi|^2} \, dm(\zeta) \, dm(\xi).\end{aligned}$$

In the above formula,

$$P_a(\zeta) = \frac{1 - |a|^2}{|\zeta - a|^2}, \quad a \in \mathbb{D}, \zeta \in \mathbb{T},$$

is the usual Poisson kernel. We also agree to the understanding that if any of the factors in eq.(3.2) are missing, then those corresponding components of the Carleson formula are zero.

Carleson's formula has some nice consequences. For example, two applications of Fubini's theorem with Carleson's formula yields:

Corollary 3.4. *An inner function bs_μ belongs to $\mathcal{D}$ if and only if $\mu = 0$ and b is a finite Blaschke product.*

One can also use standard facts about inner functions and the Carleson formula to prove the division property for $\mathcal{D}$ (often called the *F-property* for $\mathcal{D}$).

Corollary 3.5. *Suppose $f \in \mathcal{D}$ and ϑ is an inner function that divides the inner factor of f, equivalently $f/\vartheta \in H^2$. Then $f/\vartheta \in \mathcal{D}$. Consequently, if $f \in \mathcal{D}$, its outer factor also belongs to $\mathcal{D}$.*

For a general Banach space of analytic functions $X \subset H^2$ we say that X has the *F-property* if whenever ϑ is inner and $f/\vartheta \in H^2$, then $f/\vartheta \in X$. When $X = H^2$, this is automatic by the Nevanlinna factorization in eq.(3.2). For other spaces of analytic functions, this becomes more involved and sometimes is false [**56**].

Another interesting corollary can be obtained by re-arranging the terms in Carleson's formula.

Corollary 3.6. *For $f \in \mathcal{D}$,*

$$D(f) = \{D(bF) - D(F)\} + \{D(s_\mu F) - D(F)\} + D(F),$$

where each of the individual summands is non-negative.

In [**47**], one finds an analogous decomposition of the local Dirichlet integral, namely,

$$\begin{aligned} D_\zeta(f) &= \left(m + \sum_{n=1}^{\infty} P_{a_n}(\zeta)\right)|f(\zeta)|^2 + \int_{\mathbb{T}} \frac{2}{|\zeta - \xi|^2} d\mu(\xi)|f(\zeta)|^2 \\ &+ \int_{\mathbb{T}} \frac{e^{2u(\xi)} - e^{2u(\zeta)} - 2e^{2u(\zeta)}(u(\xi) - u(\zeta))}{|\zeta - \xi|^2} dm(\xi). \end{aligned}$$

Using the local Dirichlet integral and an argument using cut-off functions, as in [**47**], one can show the following.

THEOREM 3.7. *If $f \in \mathcal{D}$, then $f = g_1/g_2$, where $g_1, g_2 \in H^\infty \cap \mathcal{D}$.*

Here H^∞ denotes the bounded analytic functions on $\mathbb{D}$. Notice how this theorem mimics the well-know fact that any H^2 function can be written as the quotient of two bounded analytic functions [**22**]. Theorem 3.7 was first proved in [**46**] in the more general setting of the Dirichlet space of a general domain. The authors in [**47**] present a new proof of this and obtain some bounds on the local Dirichlet integrals of the functions that comprise the quotient. In fact, they show that g_1 and g_2 can be chosen so that both g_1 and $1/g_2$ belong to $\mathcal{D}$. To push the analogy with H^2 further, notice how H^∞ are the multipliers of H^2 (see definitions of multipliers below) and so every H^2 function can be written an the quotient of two multipliers of H^2. Can every function in $\mathcal{D}$ be written as the quotient of two multipliers of $\mathcal{D}$?

4. Potentials

One can also realize the Dirichlet space as a space of potentials. Let

$$k(\zeta) = |1 - \zeta|^{-1/2}, \quad \zeta \in \mathbb{T},$$

and notice that the Fourier coefficients

$$\widehat{k}(n) := \int_{\mathbb{T}} k(\zeta)\overline{\zeta}^n dm(\zeta), \quad n \in \mathbb{Z},$$

satisfy the estimates

$$\delta(1+n)^{-1/2} \leqslant |\widehat{k}(n)| \leqslant \delta^{-1}(1+n)^{-1/2}, \tag{4.1}$$

for some $\delta > 0$. With the integral convolution

$$(g_1 * g_2)(\zeta) := \int_{\mathbb{T}} g_1(\overline{\xi}\zeta)g_2(\xi)\, dm(\xi),$$

one can form the space of potentials

$$L^2_{1/2} := \left\{k * g : g \in L^2\right\},$$

where L^2 is the standard Lebesgue space of measurable functions g on $\mathbb{T}$ with norm

$$\|g\|_2 := \sqrt{\int_{\mathbb{T}} |g|^2\, dm}.$$

One places a norm on $L^2_{1/2}$ by

$$\|k * g\|_{1/2} := \|g\|_{L^2}.$$

These potentials are closely related to the standard Bessel potentials [**57**].

If $P(k*g)$ denotes the Poisson integral of $k*g$, that is,

$$P(k*g)(z) = \int_{\mathbb{T}} (k*g)(\xi) P_z(\xi)\, dm(\xi),$$

one can compute in polar coordinates to show that for $\zeta \in \mathbb{T}$ and $0 < r < 1$,

$$P(k*g)(r\zeta) = \sum_{n=-\infty}^{\infty} r^{|n|}\zeta^n \widehat{k*g}(n) = \sum_{n=-\infty}^{\infty} r^{|n|}\zeta^n \widehat{k}(n)\widehat{g}(n).$$

If one extends the definition of the Dirichlet integral to harmonic functions u by

$$D(u) := \frac{1}{\pi}\int_{\mathbb{D}} \left(|u_x|^2 + |u_y|^2\right) dA,$$

we can define the *harmonic Dirichlet space* $\mathcal{D}_h$ to be the space of harmonic functions u on $\mathbb{D}$ with finite Dirichlet integral $D(u)$. With $u = P(k*g)$, one can compute $D(u)$ in polar coordinates to get

$$D(u) = \sum_{n=-\infty}^{\infty} |n||\widehat{k*g}(n)|^2$$

and so by eq.(4.1), $D(u) < \infty$.

Proposition 4.2. *For u harmonic on $\mathbb{D}$, the following are equivalent.*

(1) *$D(u) < \infty$;*
(2) *$u = P(k*g)$ for some $g \in L^2$;*
(3) *The boundary function*

$$u(\zeta) = \lim_{r\to 1^-} (r\zeta)$$

exists almost everywhere and satisfies

$$\int_{\mathbb{T}}\int_{\mathbb{T}} \left|\frac{u(\zeta)-u(\xi)}{\zeta-\xi}\right|^2 dm(\zeta)dm(\xi) < \infty.$$

Furthermore,

$$D(u) = \int_{\mathbb{T}}\int_{\mathbb{T}} \left|\frac{u(\zeta)-u(\xi)}{\zeta-\xi}\right|^2 dm(\zeta)dm(\xi).$$

Harmonic functions with finite Dirichlet integral must have almost everywhere defined boundary values that belong to L^2 and so we can place a norm on the harmonic Dirichlet space $\mathcal{D}_h$ by

$$\|u\| = \sqrt{\|u\|_{L^2}^2 + D(u)}.$$

From our estimates above, notice that

$$\|P(k*g)\| \asymp \|g\|_{L^2}.$$

Also observe how this norm is the same as the analytic Dirichlet space norm from eq.(2.3) when u is an analytic function.

Thinking of the Dirichlet space (harmonic or analytic) as a space of potentials allows us to discuss the fine (capacity) properties of the boundary function. We will get to this in a moment. Before doing so, let us mention the following integral representation of Dirichlet functions [**38**].

PROPOSITION 4.3. *An analytic function f belongs to $\mathcal{D}$ if and only if*

$$f(z) = \int_{\mathbb{T}} \frac{g(\zeta)}{(1-\overline{\zeta}z)^{1/2}}\, dm(\zeta)$$

for some $g \in L^2$.

Potentials allow us to define a capacity. The *capacity* of a set $E \subset \mathbb{T}$ is defined to be

$$\gamma(E) := \inf\{\|g\|_{L^2}^2 : g \in L^2, g \geqslant 0, k * g \geqslant 1 \text{ on } E\}.$$

This is often called the *Bessel capacity* and is a monotone, sub-additive set function on $\mathbb{T}$. Moreover, one can show that for an arc $I \subset \mathbb{T}$ (sufficiently small),

$$\gamma(I) \sim \left(\log \frac{1}{m(I)}\right)^{-1}.$$

The capacity γ is generally larger than Lebesgue measure in that there are sets E which have zero measure but positive capacity. We say a property holds *quasi-everywhere* if it holds except possibly on a set of capacity zero. Note that if a property holds quasi-everywhere, it holds almost everywhere. We could go on at length about the Bessel capacity but we will not need it in this presentation. We refer the reader to [**3**] for a thorough treatment of all this.

We would like to mention an alternative definition and older definition of capacity which, fortunately, is equivalent to the square root of the above (Bessel) capacity. It is called the logarithmic capacity. Let M_+ denote the positive finite Borel measures on $\mathbb{T}$. For $\mu \in M_+$, define the logarithmic potential on $\mathbb{C}$ by

$$u_\mu(z) = \int_{\mathbb{T}} \log \frac{e}{|1-\overline{\zeta}z|}\, d\mu(\zeta).$$

A computation with power series shows that

$$u_\mu(z) = \mu(\mathbb{T}) + \frac{1}{2}\sum_{n=1}^{\infty} \frac{\widehat{\mu}(n)}{n} z^n + \frac{1}{2}\sum_{n=1}^{\infty} \frac{\widehat{\mu}(-n)}{n} \overline{z}^n, \quad z \in \mathbb{D},$$

where

$$\widehat{\mu}(n) := \int \overline{\zeta}^n d\mu(\zeta), \quad n \in \mathbb{Z},$$

are the Fourier coefficients of μ. Notice that u_μ is a non-negative harmonic function on $\mathbb{D}$. Define the *energy* of μ to be

$$E(\mu) := \int_{\mathbb{T}} u_\mu(\zeta) d\mu(\zeta). \tag{4.4}$$

Remembering that μ is a positive measure and so

$$\widehat{\mu}(-n) = \overline{\widehat{\mu}(n)},$$

we can compute $E(\mu)$ to be

$$E(\mu) = \mu(\mathbb{T})^2 + \sum_{n=1}^{\infty} \frac{|\widehat{\mu}(n)|^2}{n}.$$

We set

$$\mathcal{E} := \{\mu \in M_+ : E(\mu) < \infty\}$$

to be the measures of finite energy.

If $\widetilde{u_\mu}$ is the conjugate function for u_μ, normalized so that $\widetilde{u_\mu}(0) = 0$, then

$$f_\mu(z) := u_\mu(z) + i\widetilde{u_\mu}(z) = \mu(\mathbb{T}) + \sum_{n=1}^{\infty} \frac{\widehat{\mu}(n)}{n} z^n. \tag{4.5}$$

We also have the nice identity,

$$|f_\mu(0)|^2 + D(f_\mu) = \mu(\mathbb{T})^2 + \sum_{n=1}^{\infty} n \left| \frac{\widehat{\mu}(n)}{n} \right|^2 = E(\mu).$$

Hence $f_\mu \in \mathcal{D}$ exactly when $\mu \in \mathcal{E}$. It is also interesting to note that

$$f'_\mu(z) = \int_{\mathbb{T}} \frac{1}{\zeta - z} \, d\mu(\zeta),$$

the Cauchy transform of μ.

For a compact set $F \subset \mathbb{T}$, the *logarithmic capacity* of F is defined to be

$$c(F) := \sup\{\mu(F) : \mu \in \mathcal{E}, \operatorname{supp}(\mu) \subset F, u_\mu|F \leqslant 1\}.$$

Extend this definition to any set $E \subset \mathbb{T}$ by

$$c(E) := \sup\{c(F) : F \subset E, \, F \text{ compact}\}.$$

One can show that

$$c(E)^2 \asymp \gamma(E)$$

and so the notion of quasi-everywhere (i.e., 'except for a set of capacity zero') is the same for these capacities. Depending on the setting, various authors use different definitions of logarithmic capacity for their particular application. Fortunately, these are essentially the same.

5. Boundary values

From the observation $\mathcal{D} \subset H^2$, we know that the boundary function

$$f(\zeta) := \lim_{r \to 1^-} f(r\zeta)$$

for $f \in \mathcal{D}$ exists almost everywhere. Can we say more? The answer is a resounding yes.

If $u \in \mathcal{D}_h$, the harmonic Dirichlet space, then $u = P(k * g)$ for some $g \in L^2$ and moreover, the radial boundary function for u is $k * g$ almost everywhere. This next results begins to unpack the finer relationship between the boundary values of u and the potential $k * g$.

Theorem 5.1. *Let $u = P(k*g)$ for some $g \in L^2$. For fixed $\theta \in [0, 2\pi)$, consider the following four limits:*

(1)
$$\lim_{r \to 1^-} u(re^{i\theta}),$$
the radial limit of u;

(2)
$$\lim_{N \to \infty} \sum_{n=-N}^{N} \widehat{k * g}(n) e^{in\theta},$$
the limit of the partial sums of the Fourier series for $k * g$;

(3)
$$\lim_{h\to 0^+}\frac{1}{2h}\int_{\theta-h}^{\theta+h}(k*g)(e^{it})\,dt;$$
(4)
$$P.V.(k*g)(e^{i\theta}).$$
If one of them exists and is finite, they all do and they are equal.

Remark 5.2. The equivalence of (1) and (2) is Abel's theorem and an old result of Landau [**31**, p. 65 - 66]. The equivalence of (2) and (3) was pointed out by Beurling in [**6**] and uses some old results dating back to Fatou and Fejér. The equivalence of (1) and (4) can be found in [**44**]. Unfortunately, there are cases where $(k*g)(re^{i\theta})$ as a finite limit as $r\to 1$ but $(k*|g|)(e^{i\theta})=\infty$ [**44**].

Combine the previous result with this next result of Beurling to complete the picture.

Theorem 5.3 (Beurling [**6**]). *If $u\in\mathcal{D}_h$, then there is a set of $W\subset\mathbb{T}$ of capacity zero such that*
$$u(\zeta):=\lim_{r\to 1^-}u(r\zeta)$$
exists and is finite for every $\zeta\in\mathbb{T}\setminus W$. Thus, the four limits in Theorem 5.1 exist and are equal for these ζ.

This theorem is sharp.

Theorem 5.4 (Carleson [**15**]). *Given any closed set $F\subset\mathbb{T}$ of capacity zero, there is an $f\in\mathcal{D}$ such that*
$$\lim_{r\to 1^-}f(r\zeta)$$
does not exist for all $\zeta\in F$.

Recall from eq.(2.1) the quantity $L(f,\zeta)$ which is the length of the arc $\{f(r\zeta): 0<r<1\}$. We noted earlier that for $f\in\mathcal{D}$, that $L(f,\zeta)<\infty$ for almost every $\zeta\in\mathbb{T}$.

Theorem 5.5 (Beurling [**6**]). *For $f\in\mathcal{D}$, $L(f,\zeta)<\infty$ for quasi-every $\zeta\in\mathbb{T}$.*

There are similar results for other classes of functions [**60**]. We point out the special nature of this result since there are examples of $f\in H^2$ for which $L(f,\zeta)=\infty$ for *every* $\zeta\in\mathbb{T}$ [**23**]. We also point out that the exceptional set in Theorem 5.5 can not be made any smaller. Indeed if $f\in\mathcal{D}$ and $L(f,\zeta)<\infty$, one can use the identity
$$f(s\zeta)-f(0)=\int_0^s\zeta f'(r\zeta)\,dr,$$
to show that $f(r\zeta)$ has a finite limit as $r\to 1$. However by Theorem 5.4, given any closet set F of capacity zero, there is an $f\in\mathcal{D}$ such that f does not have a radial limit for every $\zeta\in F$. This function must then satisfy $L(f,\zeta)=\infty$ for all $\zeta\in F$.

Before moving on to talk about other types of limits (non-tangential, oricyclic, etc.) we want to mention a nice relationship between $\mathcal{E}$, the measures of finite energy from eq.(4.4), and the inner product on $\mathcal{D}$. We already know that for f_μ defined in eq.(4.5),
$$f_\mu\in\mathcal{D}\Leftrightarrow\mu\in\mathcal{E}.$$

If we use the alternate inner product

$$(f, g) := f(0)\overline{g(0)} + \sum_{n=1}^{\infty} n a_n \overline{b_n},$$

where the a_n's are the Taylor coefficients of f and the b_n's are those for g, one can show that if p is an analytic polynomial and $\mu \in \mathcal{E}$ then

$$(p, f_\mu) = \int p(\zeta)\, d\mu(\zeta).$$

Since

$$(f_\mu, f_\mu) = |f_\mu(0)|^2 + D(f_\mu) = \mu(\mathbb{T})^2 + E(\mu),$$

we see that the linear functional

$$p \mapsto \int p(\zeta) d\mu(\zeta)$$

extends to be continuous on $\mathcal{D}$. Furthermore, we also have the following maximal-type theorem [**15**].

Theorem 5.6. *For $f \in \mathcal{D}$ and $\zeta \in \mathbb{T}$, let*

$$(Mf)(\zeta) := \sup\{|f(r\zeta)| : 0 < r < 1\}$$

be the radial maximal function. For $\mu \in \mathcal{E}$,

$$\left(\int (Mf)(\zeta)\, d\mu(\zeta) \right)^2 \leqslant c(f_\mu, f_\mu) E(\mu) < \infty.$$

As a corollary to this maximal theorem, we can now prove this useful identity: For $g \in \mathcal{D}$ and $\mu \in \mathcal{E}$,

$$(g, f_\mu) = \int g(\zeta)\, d\mu(\zeta). \tag{5.7}$$

We now move on to other types of limits considered in [**29, 38, 61**]. Consider the approach regions

$$A_{\gamma,c}(\zeta) := \{z \in \mathbb{D} : |\zeta - z| < c(1 - |z|)^{1/\gamma}\}, \;\; \zeta \in \mathbb{T}, c, \gamma > 0.$$

These are the γ order contact regions. With a little geometry, one can see that when $\gamma = 1$, these regions are triangle shaped with vertex at ζ and are called the non-tangential approach regions. When $\gamma = 2$, these regions become (essentially) circles tangent to $\mathbb{T}$ at ζ and are called oricyclic approach regions. Observe how that when $\gamma > 1$, these domains are tangent to the circle and the degree of tangency increases as γ increases. We say, for an analytic function f on $\mathbb{D}$, that f has an *A_γ-limit* L at ζ if $f(z) \to L$ as $z \to \zeta$ within $A_{\gamma,c}(\zeta)$ for every $c > 0$.

Theorem 5.8 (Beurling). *Every $f \in \mathcal{D}$ has a finite A_1-limit at quasi-every point of $\mathbb{T}$.*

From Proposition 3.1, every $f \in \mathcal{D}$ has oricyclic A_2-limits at almost every point of $\mathbb{T}$. This next result [**29**] improves this to higher order contact.

Theorem 5.9 (Kinney). *If $\gamma > 0$ and $f \in \mathcal{D}$, then f has finite A_γ-limits for almost every point of $\mathbb{T}$.*

Twomey [**61**] improves 'almost everywhere' in the previous theorem.

THEOREM 5.10 (Twomey). *If $\gamma > 0$ and $f \in \mathcal{D}$, then f has finite A_γ-limits for quasi-every point of $\mathbb{T}$.*

The state of the art here involves the exponential contact regions

$$E_{\gamma,c}(\zeta) := \left\{ z \in \mathbb{D} : |\zeta - z| < c \left(\log \frac{1}{1-|z|} \right)^{-1/\gamma} \right\}, \quad \zeta \in \mathbb{T}, c > 0, \gamma > 0. \tag{5.11}$$

THEOREM 5.12 (Nagel, Rudin, J. Shapiro [**38**]). *Every $f \in \mathcal{D}$ has an E_1-limit for almost every point of $\mathbb{T}$.*

Twomey improves this to the following.

THEOREM 5.13 (Twomey). *If $f \in \mathcal{D}$ and $0 < \gamma < 1$, there is a set W_γ of γ-dimensional Hausdorff content zero such that f has a finite E_γ-limit on $\mathbb{T} \setminus W_\gamma$.*

See the references in [**61**] for more on this. Twomey also points out that these results are in a sense sharp (see Theorems 6 and 7 in [**61**]). We end this section with the following theorem that points out the special nature of the existence of limits of Dirichlet function in tangential contact regions.

THEOREM 5.14. *Let C be any curve in $\mathbb{D}$ that approaches the point 1 tangent to the unit circle. Then there is a bounded analytic function f on $\mathbb{D}$ whose limit along the curve ζC does not exist for any $\zeta \in \mathbb{T}$.*

Littlewood [**32**] proved the 'almost everywhere' version of this theorem while Lohwater and Piranian [**33**] proved that f could be a Blaschke product. Aikawa [**4**] proved that f could be a bounded outer function.

6. Zeros

In this section, we address the following question: Given a sequence $Z = (z_n)_{n\geqslant 1} \subset \mathbb{D}$, what are necessary and sufficient conditions for Z to be the zeros Z_f of a function $f \in \mathcal{D} \setminus \{0\}$? To place our discussion in broader context, we review some well-known theorems about the zeros of functions from other classes.

Recall our earlier notation that H^2 denotes the classical Hardy space and H^∞ denotes the bounded analytic functions on $\mathbb{D}$. A well-known theorem of Blaschke says that if $Z = (z_n)_{n\geqslant 1}$ is a sequence of points in $\mathbb{D}$ that satisfies the *Blaschke condition*

$$\sum_{n=1}^{\infty} (1 - |z_n|) < \infty,$$

then the Blaschke product

$$b(z) = \prod_{n=1}^{\infty} \frac{\overline{z_n}}{|z_n|} \frac{z_n - z}{1 - \overline{z_n} z}$$

converges uniformly on compact subsets of $\mathbb{D}$ and forms an H^∞ function whose zeros are precisely Z. Conversely, an argument using Jensen's inequality shows that the zeros Z_f of an $f \in H^\infty \setminus \{0\}$ must satisfy the Blaschke condition. From here one can use the fact that every H^2 function is the quotient of two H^∞ functions to show that the Blaschke condition is both necessary and sufficient for a sequence $Z \subset \mathbb{D}$ to be the zeros of a function from $H^2 \setminus \{0\}$.

To look at the zeros functions which are 'smoother' up to the boundary than bounded analytic functions, we recall an old theorem of Riesz [**22**, p. 17].

THEOREM 6.1 (Riesz). *If* $f \in H^\infty \setminus \{0\}$*, then*

$$\int_{\mathbb{T}} \log |f| dm > -\infty.$$

For the space of analytic functions A on $\mathbb{D}$ which extend to be continuous on $\mathbb{D}^-$, called the *disk algebra*, the zeros (in $\mathbb{D}$) of an $f \in A \setminus \{0\}$ must certainly satisfy the Blaschke condition. However, if these zeros accumulate on a set $E \subset \mathbb{T}$ of positive measure then, by the continuity of f on $\mathbb{D}^-$, $f|E = 0$ making the integral

$$\int_{\mathbb{T}} \log |f| dm$$

divergent. Thus the zeros Z_f in $\mathbb{D}$ of an $f \in A \setminus \{0\}$ must satisfy the Blaschke condition as well as satisfy $m(Z_f^- \cap \mathbb{T}) = 0$. A theorem of Fatou [**27**, p. 80] says that if $K \subset \mathbb{T}$ is compact and of measure zero, then there is a $g \in A \setminus \{0\}$ for which $g|K = 0$. Hence, if $(z_n)_{n \geqslant 1} \subset \mathbb{D}$ satisfies

$$\sum_{n=1}^{\infty} (1 - |z_n|) < \infty \quad \text{and} \quad m\left((z_n)_{n \geqslant 1}^- \cap \mathbb{T}\right) = 0,$$

then $f := gb$, where b is the Blaschke product with zeros $(z_n)_{n \geqslant 1}$ and g is the function from Fatou's result, belongs to $A \setminus \{0\}$ with the desired zeros.

When one moves to spaces of even smoother functions, say the Lipschitz classes Λ_α, $0 < \alpha < 1$, of analytic f on $\mathbb{D}$ for which

$$\sup \left\{ \frac{|f(\zeta) - f(\xi)|}{|\zeta - \xi|^\alpha} : \zeta, \xi \in \mathbb{T}, \zeta \neq \xi \right\} < \infty,$$

the situation is even more delicate. We first notice that $f \in \Lambda_\alpha$ if and only if

$$\sup \left\{ \frac{|f(z) - f(w)|}{|z - w|^\alpha} : z, w \in \mathbb{D}^-, z \neq w \right\} < \infty.$$

Thus

$$|f(z) - f(\zeta)| \leqslant C_f |z - \zeta|^\alpha, \quad z \in \mathbb{D}, \zeta \in \mathbb{T}.$$

Taking logarithms of both sides of the above equation and replacing z by one of the zeros of f in $\mathbb{D}$ we see that

$$\log |f(\zeta)| \leqslant C_f + \alpha \log \operatorname{dist}(\zeta, Z_f).$$

By integrating both sides and using the Riesz theorem (Theorem 6.1) we see that the zeros $Z_f \subset \mathbb{D}$ of an $f \in \Lambda_\alpha \setminus \{0\}$ must satisfy

$$\int_{\mathbb{T}} \log \operatorname{dist}(\zeta, Z_f) dm(\zeta) > -\infty$$

as well as the Blaschke condition. A deep theorem of Taylor and Williams [**58**] (discovered independently by others [**17, 40**]) says the conditions

$$\sum_{z \in Z} (1 - |z|) < \infty \quad \text{and} \quad \int_{\mathbb{T}} \log \operatorname{dist}(\zeta, Z) dm(\zeta) > -\infty \tag{6.2}$$

on a sequence $Z \subset \mathbb{D}$ are both necessary and sufficient to be the zeros of an $f \in \Lambda_\alpha \setminus \{0\}$. In fact, if the two conditions in eq.(6.2) are satisfied, there is a function f such that $f^{(n)} \in A$ for all $n \in \mathbb{N}_0$ and $Z_f = Z$.

The complete classification of the zero sets for Dirichlet functions is still unresolved. Lokki [**34**] mistakenly claimed that the Blaschke condition was both necessary and sufficient to be a Dirichlet zero set. But in fact, there are radii $(r_n)_{n\geqslant 1} \subset [0,1)$ which satisfy the Blaschke condition

$$\sum_{n=1}^{\infty}(1-r_n) < \infty$$

but for which there are angles θ_n such that $z_n = r_n e^{i\theta_n}$ are not the zeros of any $f \in \mathcal{D} \setminus \{0\}$ [**16**]. There are also $f \in \mathcal{D} \setminus \{0\}$ whose zeros $Z_f \subset \mathbb{D}$ satisfy

$$\operatorname{dist}(\zeta, Z_f) = 0 \text{ for all } \zeta \in \mathbb{T}.$$

Thus the two conditions in eq.(6.2) are sufficient but not necessary. There is, however, this observation of Carleson [**12**]: If $(r_n)_{n\geqslant 1} \subset [0,1)$ satisfy the Blaschke condition, then the Blaschke product

$$b(z) = \prod_{n=1}^{\infty} \frac{r_n - z}{1 - r_n z}$$

satisfies

$$|b'(z)| \leqslant \sum_{n=1}^{\infty} \frac{1-r_n}{|1-r_n z|^2} \leqslant \frac{C}{|1-z|^2}.$$

Hence the function

$$f(z) = (1-z)^2 b(z)$$

belongs to $\mathcal{D} \setminus \{0\}$ and has the r_n's as its zeros. A recent result of Bogdan [**8**] extends this observation.

THEOREM 6.3 (Bogdan). *A necessary and sufficient condition on a set $W \subset \mathbb{D}$ to have the property that every Blaschke sequence $(z_n)_{n\geqslant 1} \subset W$ is a Dirichlet zero sequence is*

$$\int_{\mathbb{T}} \log \mathit{dist}(\zeta, W)\, dm(\zeta) > -\infty. \tag{6.4}$$

Furthermore, if eq.(6.4) holds, then there is an outer function $F \in \mathcal{D}$ such that $bF \in \mathcal{D}$ for every Blaschke product whose zeros lie in W.

For example, some simple estimates show that any Blaschke sequence lying in a single non-tangential or even a finite order contact region $A_{\gamma,c}(\zeta)$ is a Dirichlet zero set. The same is true for certain (but not all) exponential contact regions $E_{\gamma,c}(\zeta)$.

If one is hoping for a necessary and sufficient condition on the radii r_n for a sequence $z_n = r_n e^{i\theta_n}$ to be a Dirichlet zero sequence (as the Blaschke condition is for bounded analytic functions), there does not seem to be such a result. Carleson [**12**] proved that the radii r_n for a Dirichlet zero set must satisfy

$$\sum_{n=1}^{\infty} \left\{ \log \frac{1}{1-r_n} \right\}^{-1-\varepsilon} < \infty \text{ for every } \varepsilon > 0. \tag{6.5}$$

Moreover if

$$\sum_{n=1}^{\infty} \left\{ \log \frac{1}{1-r_n} \right\}^{-1+\varepsilon} < \infty \text{ for some } \varepsilon > 0 \tag{6.6}$$

then $z_n = r_n e^{i\theta_n}$ is a Dirichlet zero set for every choice of angles θ_n. The conditions in eq.(6.5) and eq.(6.6) do not characterize the Dirichlet zero sets in that for any continuous function h on $\mathbb{R}_+$ with $h(0) = 0$ and $h(x) > 0$ for $x > 0$, and radii $(r_n)_{n\geqslant 1}$ satisfying

$$\sum_{n=1}^{\infty} \left\{ \log \frac{1}{1-r_n} \right\}^{-1} h(1-r_n) < \infty,$$

there is a sequence of angles θ_n so that $z_n = r_n e^{i\theta_n}$ is not a Dirichlet zero set [**53**]. An extension of eq.(6.5) by H. S. Shapiro and A. Shields [**53**] says that if

$$\sum_{n=1}^{\infty} \left\{ \log \frac{1}{1-r_n} \right\}^{-1} < \infty, \tag{6.7}$$

then $z_n = r_n e^{i\theta_n}$ is a Dirichlet zero set for any choice of angles θ_n.

Unfortunately, even this sharpened condition in eq.(6.7) is not necessary. Using the exponential contact result in Theorem 5.12 one can, as was observed in [**38**], produce a counterexample: Beginning at $\zeta = 1$, lay down arcs $I_n \subset \mathbb{T}$ of length

$$\left\{ \log \frac{1}{1-r_n} \right\}^{-1}$$

end-to-end (repeatedly traversing the unit circle). If we assume that

$$\sum_{n=1}^{\infty} \left\{ \log \frac{1}{1-r_n} \right\}^{-1} = \infty, \tag{6.8}$$

we observe that

$$\sum_{n=1}^{\infty} m(I_n) = \infty,$$

and so each $\zeta \in \mathbb{T}$ will be contained in infinitely many of the arcs $(I_n)_{n\geqslant 1}$. Let $e^{i\theta_n}$ be the center of the arc I_n and note that simple geometry shows that for every $e^{i\theta}$, the exponential contact region $E_{1,1}(e^{i\theta})$ (see eq.(5.11)) contains infinitely many of the points $r_n e^{i\theta_n}$. Thus if $f \in \mathcal{D}$ and $f(r_n e^{i\theta_n}) = 0$ for all n, Theorem 5.12 says that the boundary function for f will vanish almost everywhere on $\mathbb{T}$, forcing f to be identically zero (see Theorem 6.1). This argument actually shows that the sequence $(r_n e^{i\theta_n})_{n\geqslant 1}$ just created can not be the zeros of any $u \in \mathcal{D}_h \setminus \{0\}$, the harmonic Dirichlet space. A relatively recent result [**45**] shows that one does not even have to make the zeros z_n accumulate at every point of $\mathbb{T}$, as in the above counterexample. Assuming eq.(6.8), one can arrange the angles θ_n so that the zeros $z_n = r_n e^{i\theta_n} \to 1$. In a much earlier result, Caughren [**16**] proved that one can have a Blaschke sequence that converges to a single point on the boundary that is not a Dirichlet zero sequence. There is also the following probabilistic version of all this [**8**].

Theorem 6.9 (Bogdan). *Let $(\theta_n)_{n\geqslant 1}$ be a sequence of independent random variables uniformly distributed on $(-\pi, \pi]$ and $(r_n)_{n\geqslant 1} \subset [0, 1)$. If*

$$\sum_{n=1}^{\infty} \left\{ \log \frac{1}{1-r_n} \right\}^{-1} = \infty,$$

then almost surely the sequence $(r_n e^{i\theta})_{n\geqslant 1}$ is not a Dirichlet zero sequence.

In another line of thought, we can also talk about boundary zero sets. If $f \in A \setminus \{0\}$, then, by Riesz's theorem (Theorem 6.1), the boundary zeros

$$E_f := \{\zeta \in \mathbb{T} : f(\zeta) = 0\}$$

must be a set of measure zero. Conversely, by Fatou's theorem mentioned earlier [**27**, p. 80], if E is a closed subset of $\mathbb{T}$ with $m(E) = 0$, then there is an $f \in A \setminus \{0\}$ such that $E_f = E$. For the Lipschitz classes Λ_α a necessary and sufficient condition for a closed subset of $\mathbb{T}$ to satisfy $E = E_f$ for some $f \in \Lambda_\alpha \setminus \{0\}$ is the following:

$$m(E) = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} m(I_n) \log \frac{1}{m(I_n)} < \infty,$$

where I_n are the complimentary arcs of E.

For $f \in \mathcal{D}$, we know from Beurling's theorem (Theorem 5.3) that the radial limit

$$f(\zeta) = \lim_{r \to 1^-} f(r\zeta)$$

exists and is finite quasi-everywhere. For a set $E \subset \mathbb{T}$, one can define the space

$$\mathcal{D}_E := \{f \in \mathcal{D} : f|E = 0 \text{ quasi-everywhere}\}.$$

The standard capacity estimate

$$\gamma(\{\zeta \in \mathbb{T} : |f(\zeta)| > \lambda\}) \leqslant \frac{c}{\lambda^2} \|f\|^2$$

shows that $\mathcal{D}_E$ is a closed subspace of $\mathcal{D}$ [**11**]. Using the identity in eq.(5.7), one can show that

$$\mathcal{D}_E^{\perp} = \bigvee \{f_\mu : \mu \in \mathcal{E}, \operatorname{supp}(\mu) \subset E, u_\mu | E \leqslant 1\}.$$

The question now is: For what sets $E \subset \mathbb{T}$, is $\mathcal{D}_E = (0)$? Such sets are called *sets of uniqueness*. Certainly if $m(E) > 0$, Riesz's theorem (Theorem 6.1) says that $\mathcal{D}_E = (0)$. Carleson [**13**] showed that if $\gamma(E) = 0$, then $\mathcal{D}_E \neq (0)$. In fact, if E is also closed, then $\mathcal{D}_E$ contains outer functions that also belong to the disk algebra [**10**]. The problem is very delicate since there are sets with $\gamma(E) > 0$ and $m(E) = 0$ but $\mathcal{D}_E = (0)$. There is a complete characterization of the sets of uniqueness due to Malliavin [**35**] but the necessary and sufficient condition involves the 'modified logarithmic capacity' and is quite difficult to apply to particular situations. Other partial results can be found in [**26**]. The reference [**28**] contains a survey of the sets of uniqueness for several other classes of analytic functions.

7. Forward shift invariant subspaces

In this section we wish to study the forward shift operator $S : \mathcal{D} \to \mathcal{D}$

$$(Sf)(z) := zf(z).$$

In particular, we focus our attention on the invariant subspaces of S, that is, those closed linear manifolds $\mathcal{M} \subset \mathcal{D}$ such that $S\mathcal{M} \subset \mathcal{M}$. We denote the collection of these invariant subspaces by $\operatorname{Lat}(S, \mathcal{D})$. Though $\operatorname{Lat}(S, \mathcal{D})$ is not completely understood, there has been quite a lot of work on this subject. In order to place these results in some context, we mention a few classical theorems. The first, and probably one that always needs to be mentioned when talking about the shift on spaces of analytic functions, is Beurling's theorem [**7, 22**].

THEOREM 7.1 (Beurling). (1) *A subspace* $\mathcal{M} \neq \{0\}$ *belongs to* $Lat(S, H^2)$ *if and only if* $\mathcal{M} = \vartheta H^2$, *where* ϑ *is an inner function.*

(2) *If* $f \in H^2$, *then*

$$[f]_S := \bigvee \{S^n f : n \in \mathbb{N}_0\} = \vartheta_f H^2,$$

where ϑ_f *is the inner factor of* f. *Thus* f *is* S*-cyclic, that is* $[f]_S = H^2$, *if and only if* f *is an outer function.*

Let us take a moment to review the description of $\mathrm{Lat}(S, X)$ for some other well-known Banach spaces of analytic functions X. When X is the disk algebra A, discussed earlier, one can show, by approximating every $f \in A$ with its Cesàro polynomials, that every $\mathcal{I} \in \mathrm{Lat}(S, A)$ is a closed ideal of A and by a result of Rudin [**52**] (see also [**27**]) takes the form $\mathcal{I} = \mathcal{I}(E, \vartheta)$, where $E \subset \mathbb{T}$ is closed with $m(E) = 0$ and ϑ is inner such that

$$\underline{Z}(\vartheta) := \left\{ z \in \mathbb{D}^- : \varliminf_{\lambda \to z} |\vartheta(\lambda)| = 0 \right\}$$

satisfies $\underline{Z}(\vartheta) \cap \mathbb{T} \subset E$, and

$$\mathcal{I}(E, \vartheta) := \{f \in A : f/\vartheta \in A, f|E = 0\}.$$

Moreover, every $\mathcal{I}(E, \vartheta)$ is a non-zero closed ideal of A. From here Korenblum [**30**] developed techniques, used by many others (for example [**36, 56**]), to discuss the closed ideals of several other spaces of analytic functions that are smooth up to the boundary. The results are similar to Rudin's result except that the types of closed sets E and the inner functions ϑ have further restrictions on them. Furthermore, in some cases, the boundary zeros of the derivatives come into play.

The Dirichlet space $\mathcal{D}$ is not an algebra of analytic functions and the functions in the Dirichlet space need not have continuous boundary values (see Theorem 5.4). Knowing that Dirichlet functions have radial boundary values quasi-everywhere (Theorem 5.3), one conjectures that every $\mathcal{M} \in \mathrm{Lat}(S, \mathcal{D})$ should take the form $\mathcal{M}(E, \vartheta)$, the space of $f \in \mathcal{D}$ such that

$$f/\vartheta \in \mathcal{D} \quad \text{and} \quad \lim_{r \to 1^-} f(r\zeta) = 0 \text{ for quasi-every } \zeta \in E.$$

Moreover, it should be the case that $f \in \mathcal{D}$ is S-cyclic, that is $[f]_S = \mathcal{D}$, if and only if f is an outer function and the set of boundary zeros has capacity zero. We will say more about this conjecture in a moment. Though this type of result is unknown, there are many other things one can say.

Before discussing these results, let us first say a few words about the operator S on $\mathcal{D}$. On the Hardy space, S is an isometry. On the Dirichlet space, S is a two-isometry

$$\|S^2 f\|^2 - 2\|Sf\|^2 + \|f\|^2 = 0, \ \forall f \in \mathcal{D} \text{ and } \bigcap_{n=0}^{\infty} S^n \mathcal{D} = \{0\}.$$

Moreover, Richter [**43**] showed that every analytic, cyclic, two-isometry on a Hilbert space is unitarily equivalent to S on some local Dirichlet type space $\mathcal{D}(\mu)$.

By Beurling's theorem, the shift S on H^2 is *cellular indecomposable* in that if $\mathcal{M}, \mathcal{N} \in \mathrm{Lat}(S, H^2) \setminus \{0\}$, then $\mathcal{M} \cap \mathcal{N} \neq \{0\}$. Furthermore, $\mathcal{M} \cap H^\infty \neq \{0\}$, and the subspace

$$\mathcal{M} \ominus S\mathcal{M} := \mathcal{M} \cap (S\mathcal{M})^{\perp}$$

is one-dimensional. Richter and Shields [**46**] generalize this to the Dirichlet space.

THEOREM 7.2 (Richter-Shields). *Let* $\mathcal{M}, \mathcal{N} \in \mathit{Lat}(S, \mathcal{D}) \setminus \{0\}$. *Then*

(1) $\mathcal{M} \cap H^{\infty} \neq \{0\}$.
(2) $\mathcal{M} \cap \mathcal{N} \neq \{0\}$.
(3) $\mathcal{M} \ominus S\mathcal{M}$ *is one-dimensional.*

One way to prove Beurling's theorem is to first show that whenever $\mathcal{M} \in \operatorname{Lat}(S, H^2) \setminus \{0\}$, then $\mathcal{M} \ominus S\mathcal{M}$ is one dimensional and $[\mathcal{M} \ominus S\mathcal{M}]_S = \mathcal{M}$. Here for a set Y, $[Y]_S$ is the smallest S-invariant subspace containing Y, or equivalently

$$[Y]_S := \bigvee \{S^n g : n \in \mathbb{N}_0, g \in Y\}.$$

As it turns out, the same technique works for the Dirichlet space [**42**].

THEOREM 7.3 (Richter). *If* $\mathcal{M} \in \mathit{Lat}(S, \mathcal{D})$, *then*

$$[\mathcal{M} \ominus S\mathcal{M}]_S = \mathcal{M}.$$

In H^2 setting, a function $\phi \in \mathcal{M} \ominus S\mathcal{M}$ is a solution (assuming that $\mathcal{M}$ contains a function that does not vanish at the origin) to the extremal problem

$$\inf \left\{ \frac{\|g\|_{H^2}}{|g(0)|} : g \in \mathcal{M} \right\}$$

and solutions to this extremal problem are constant multiplies of inner functions. Inner functions ϕ have the property that $|\phi(\zeta)| = 1$ for almost every $\zeta \in \mathbb{T}$. Furthermore, the formula

$$\phi(z) = z^m \prod_{n=1}^{\infty} \frac{\overline{z_n}}{|z_n|} \frac{z_n - z}{1 - \overline{z_n} z} \exp\left\{ -\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \, d\mu(\zeta) \right\},$$

defining an inner function on $\mathbb{D}$ is valid as a meromorphic function $\widetilde{\phi}$ on $\mathbb{D}_e := \widehat{\mathbb{C}} \setminus \mathbb{D}^-$. In fact, it is not difficult to see that

$$\widetilde{\phi}(z) = \frac{1}{\overline{\phi(1/\overline{z})}}, \quad z \in \mathbb{D}_e \setminus W,$$

where $W = \{1/\overline{z} : \phi(z) = 0\}$. Also observe how ϕ and $\widetilde{\phi}$ are pseudocontinuations of each other in that

$$\lim_{r \to 1^-} \phi(r\zeta) = \lim_{r \to 1^-} \widetilde{\phi}(\zeta/r)$$

for almost every $\zeta \in \mathbb{T}$. We will say more about pseudocontinuations in the next section. By a Morera-type theorem, one can also show that if I is an arc in $\mathbb{T} \setminus \underline{Z}(\phi)$, then ϕ and $\widetilde{\phi}$ are analytic continuations of each other across I. Of course, for example by taking ϕ to be a Blaschke product whose zeros accumulate on all of $\mathbb{T}$, one can have $\underline{Z}(\phi) \supset \mathbb{T}$ and so ϕ need not have an analytic continuation across any arc of $\mathbb{T}$.

For the Dirichlet space, every $\phi \in \mathcal{M} \ominus S\mathcal{M}$ is a solution to the extremal problem

$$\inf \left\{ \frac{\|g\|_{\mathcal{D}}}{|g(0)|} : g \in \mathcal{M} \right\}$$

and has some extra regularity properties near the boundary. Here however, it is the derivative of ϕ that has the pseudo and analytic continuation properties. We start with the following.

THEOREM 7.4 (Richter-Sundberg [**48**]). *If $\phi \in \mathcal{M} \ominus S\mathcal{M}$, then ϕ is a multiplier of $\mathcal{D}$.*

Here ψ is a *multiplier* of $\mathcal{D}$ if $\psi\mathcal{D} \subset \mathcal{D}$. An easy application of the closed graph theorem shows that a multiplier ψ defines a bounded linear operator on $\mathcal{D}$ by $f \mapsto \psi f$. Moreover, multipliers are bounded analytic functions on $\mathbb{D}$. In fact

$$\sup\{|\psi(z)| : z \in \mathbb{D}\} \leqslant \sup\{\|\psi g\| : \|g\| \leqslant 1\}. \tag{7.5}$$

This last inequality says that the H^∞ norm of a multiplier is bounded by the norm of the multiplication operator $f \mapsto \psi f$ on $\mathcal{D}$. A description of the multipliers of $\mathcal{D}$ can be found in a paper of Stegenga [**57**]. Another result of Richter and Sundberg [**49**] refines the standard estimate for multipliers in eq.(7.5) (standard in the sense that this estimate holds for most Banach spaces of analytic functions) and says that if $\phi \in \mathcal{M} \ominus S\mathcal{M}$, then

$$\sup\{|\phi(z)| : z \in \mathbb{D}\} \leqslant \|\phi\|.$$

Thus the sup norm of an extremal function is bounded by the actual norm of ϕ (which is smaller than the multiplier norm).

Extremal functions $\phi \in \mathcal{M} \ominus S\mathcal{M}$ in the Dirichlet space have even further regularity properties. Carleson [**12**] proved that if E is a 'thin set' in $\mathbb{T}$ and $\phi \in \mathcal{D}_E \ominus S\mathcal{D}_E$, then $(z\phi)'$ has an analytic continuation to $\mathbb{C} \setminus E$. Actually, Carleson proved a slightly different result since he was using another inner product on $\mathcal{D}$. Richter and Sundberg [**12**] extended this in the following way.

THEOREM 7.6 (Richter-Sundberg). *Let $\mathcal{M} \in Lat(S, \mathcal{D})$ and $\phi \in \mathcal{M} \ominus S\mathcal{M}$. Then*

(1) *$(z\phi)'$ can be written as the quotient of two bounded analytic functions on $\mathbb{D}$, that is to say, $(z\phi)'$ is a function of 'bounded type';*

(2) *$(z\phi)'$ has a meromorphic pseudocontinuation G of bounded type in $\mathbb{D}_e := \widehat{\mathbb{C}} \setminus \mathbb{D}^-$. By this we mean that*

$$\lim_{r \to 1^-} (z\phi)'(r\zeta) = \lim_{r \to 1^-} G(\zeta/r)$$

for almost every $\zeta \in \mathbb{T}$;

(3) *$\psi(z) := |\phi(z)|$ satisfies*

  (a) *for every $\zeta \in \mathbb{T}$,*

$$\psi(\zeta) := \lim_{r \to 1^-} \psi(r\zeta)$$

  exists.

  (b) *ψ is upper semicontinuous on $\mathbb{T}$ and for all $\zeta \in \mathbb{T}$,*

$$\varlimsup_{z \to \zeta} |\phi(z)| = \psi(\zeta).$$

(4) *$(z\phi)'$ has an analytic continuation across $\mathbb{T} \setminus \underline{Z}(\phi)$.*

We will be saying more about pseudocontinuations in the next section.

REMARK 7.7. (1) Statement (1) of the above theorem is significant in a more subtle way. The derivatives of Dirichlet functions belong to the Bergman space and it is well-known that Bergman functions need not have finite radial limits almost everywhere [**22**, p. 86]. The fact that $(z\phi)'$

belongs to the Bergman space yet has radial limits makes this extremal function distinctive.

(2) In statement (4), there are extremal functions ϕ such that $\underline{Z}(\phi) \supset \mathbb{T}$ [**49**, Thm. 4.3].

Before moving on to discuss the cyclic vectors for S, we make one final remark about multipliers. If $\mathcal{M} = \vartheta H^2$, where ϑ is inner, then $P_{\mathcal{M}}$, the projection of H^2 onto $\mathcal{M}$, is given by the formula

$$P_{\mathcal{M}} = M_{\vartheta} M_{\vartheta}^*,$$

where $M_{\vartheta} f = \vartheta f$. For the Dirichlet space, we have the following extension [**25, 37**].

Theorem 7.8. *Suppose $\mathcal{M} \in Lat(S, \mathcal{D})$. Then there is a sequence $(\phi_n)_{n \geqslant 1} \subset \mathcal{M}$ such that each ϕ_n is a multiplier of $\mathcal{D}$ and*

$$P_{\mathcal{M}} = \sum_{n=1}^{\infty} M_{\phi_n} M_{\phi_n}^*,$$

where the convergence above is in the strong operator topology. Furthermore,

$$\lim_{r \to 1^-} \sum_{n=1}^{\infty} |\phi_n(r\zeta)| = 1$$

for almost every $\zeta \in \mathbb{T}$.

We end this section with a discussion of the cyclic vectors for S on $\mathcal{D}$. Recall from Beurling's theorem (Theorem 7.1) that $f \in H^2$ is cyclic for S on H^2 if and only if f is an outer function. Suppose that f is cyclic for S on $\mathcal{D}$, then there is a sequence of polynomials $(p_n)_{n \geqslant 1}$ such that $p_n f \to 1$ in $\mathcal{D}$. But since the $\mathcal{D}$ norm dominates the H^2 norm, then $p_n f \to 1$ in the H^2 norm. Now apply Beurling's theorem to say the following.

Proposition 7.9. *If $f \in \mathcal{D}$ is cyclic for S, then f is outer.*

Unfortunately there are outer functions which are not cyclic [**49**, Theorem 4.3] and so being an outer function does not guarantee cyclicity in the Dirichlet space. Indeed, one can find a set $E \subset \mathbb{T}$ so that $\mathcal{D}_E \neq \{0\}$ and such spaces always contain outer functions.

Proposition 7.10. *Suppose $f \in \mathcal{D}$ and*

$$E_f := \left\{ \zeta \in \mathbb{T} : \lim_{r \to 1^-} f(r\zeta) = 0 \right\}.$$

If f is cyclic for S on $\mathcal{D}$, then E_f must have capacity zero.

Let us outline a proof of this result since it brings in the potential function f_μ mentioned earlier in eq.(4.5). Recall that with the inner product

$$(f, g) := f(0)\overline{g(0)} + \sum_{n=1}^{\infty} n a_n \overline{b_n},$$

where the a_n's are the Taylor coefficients of f and the b_n's are those for g, one can show that $f_\mu \in \mathcal{D}$, whenever $\mu \in \mathcal{E}$, and

$$(g, f_\mu) = \int g(\zeta)\, d\mu(\zeta).$$

Thus whenever $g \in \mathcal{D}$ and E_g has positive capacity, there is a compact subset $F \subset E_g$ with positive capacity and a measure $\mu \in \mathcal{E}$ supported on F. Notice that

$$(S^n g, f_\mu) = \int \zeta^n g(\zeta) d\mu(\zeta) = 0 \ \ \forall n \in \mathbb{N}_0$$

and so g is not cyclic.

It is also worth pointing out here that if E has zero capacity then certainly $\mathcal{D}_E = \mathcal{D}$. A construction of Brown and Cohn [**10**] says that if E is a closed set of capacity zero then there is an $f \in \mathcal{D}$ such that f is outer, $f \in A$, $E_f = E$, and f is cyclic. The main conjecture that has remained open for quite some time is the following:

Conjecture: A function $f \in \mathcal{D}$ is cyclic for S if and only if f is outer and E_f has capacity zero.

There are several partial results here (see [**9, 11, 47, 48**] for some examples) that support this conjecture. We mention a two of them.

Theorem 7.11. (1) *Suppose $f, g \in \mathcal{D}$ and $|f(z)| \geqslant |g(z)|$ for all $z \in \mathbb{D}$. Then $[f]_S \supset [g]_S$. In particular if g is cyclic, then f is cyclic.*
(2) *If f and $1/f$ belong to $\mathcal{D}$, then f is cyclic.*

Remark 7.12. Notice how when we replace $\mathcal{D}$ by H^2 in the above theorem how f must be an outer function, and hence cyclic for S on H^2.

Suppose that $f \in \mathcal{D}$ is univalent and cyclic. Then f can have no zeros on $\mathbb{D}$ and so by [**22**, Theorem 3.17] f is outer. Moreover [**6**], E_f has logarithmic capacity zero. One can prove [**48**] this definitive result.

Corollary 7.13 (Richter-Sundberg). *If $f \in \mathcal{D}$ is univalent, then f is cyclic for S on $\mathcal{D}$ if and only if $f(z) \neq 0$ for all $z \in \mathbb{D}$.*

We began this section asking whether or not every $\mathcal{M} \in \mathrm{Lat}(S, \mathcal{D})$ is of the form $\mathcal{M}(E, \vartheta)$. This next theorem [**48**] very much supports this conjecture.

Theorem 7.14 (Richter-Sundberg). *Let $\mathcal{M} \in Lat(S, \mathcal{D}) \setminus \{0\}$ and ϑ be the greatest common divisor of the inner factors of $\mathcal{M}$. Then, there is an outer function $F \in \mathcal{D}$ such that $F, \vartheta F$ are multipliers of $\mathcal{D}$ and*

$$\mathcal{M} = [\vartheta F]_S = [F]_S \cap \vartheta H^2.$$

For any outer function $f \in \mathcal{D}$, is $[f]_S = \mathcal{D}_{E_f}$?

8. Backward shift invariant subspaces

The backward shift operator

$$Bf = \frac{f - f(0)}{z}$$

is a well-studied operator on H^2 and its invariant subspaces and cyclic vectors are known [**21**]. We will be more specific in a moment. For the Dirichlet space however, there is much work to be done. Suppose that f is non-cyclic for the backward shift on $\mathcal{D}$, that is to say,

$$[f]_B = \bigvee \{B^n f : n \in \mathbb{N}_0\} \neq \mathcal{D}.$$

Choose an

$$L \in [f]_B^{\perp} \setminus \{0\}.$$

Let us say a brief word about notation. Certainly $\mathcal{D}$ is a Hilbert space and so, via the Riesz representation theorem, all linear functionals are identified with unique elements of $\mathcal{D}$. However, the approach we are taking here from [**51**] to examine non-cyclic vectors, works in the general setting of Banach spaces of analytic functions where identifying the dual space is more complicated. With the annihilating L from above, form the meromorphic function

$$f_L(z) := L\left(\frac{f}{w-z}\right)/L\left(\frac{1}{w-z}\right), \quad z \in \mathbb{D}_e := \widehat{\mathbb{C}} \setminus \mathbb{D}^-.$$

In the setting of H^2, the above function f_L can be written as the quotient of two Cauchy integrals and so, via some well-known facts such as Fatou's jump theorem and the F. and M. Riesz theorem, one can show that the non-tangential limits of f (from $\mathbb{D}$) and f_L (from $\mathbb{D}_e$) both exist and are equal almost everywhere. One says that f and f_L are *pseudocontinuations* of each other [**5**] [**18**, p. 85] (see below). In fact, f_L is a pseudocontinuation of f whenever f is a non-cyclic vector for B on many of the Bergman-type spaces [**5**].

For meromorphic functions g on $\mathbb{D}$ and G on $\mathbb{D}_e$ we say they are *pseudocontinuations* of each other if the non-tangential limits of g and G exist and are equal almost everywhere. The following theorem of Privalov [**19**] implies that if g has a pseudocontinuation G, it must be unique.

THEOREM 8.1 (Privalov's uniqueness theorem). *Suppose that f is meromorphic on $\mathbb{D}$ and that the non-tangential limits of f vanish on a set of positive measure in $\mathbb{T}$. Then f must be identically zero.*

Thus, in the Hardy space setting, f_L is a pseudocontinuation of f and is independent of the annihilating L. Douglas, Shapiro, and Shields [**21**] completely characterize the non-cyclic vectors for B on H^2.

THEOREM 8.2 (Douglas-Shapiro-Shields). *A necessary and sufficient condition that $f \in H^2$ be non-cyclic for B is that f has a pseudocontinuation that can be written as the quotient of two bounded analytic functions on $\mathbb{D}_e$.*

Though the existence of a pseudocontinuation may seem somewhat mysterious, the fact that they are unique does give some specific information. For example, functions in H^2 which have isolated winding points, something like $f(z) = \sqrt{1-z}$, must be cyclic vectors for B on H^2. Indeed, if they were not, then f would have a pseudocontinuation f_L. However, f has an analytic continuation across any arc not meeting the point $z = 1$. By Privalov's uniqueness theorem, the analytic and pseudocontinuations must be one in the same, at least in some neighborhood of the arc. This would place a branch cut in the domain of analyticity of f_L, which is impossible (since f_L is meromorphic on $\mathbb{D}_e$). We refer the reader to [**18, 21, 50**] for more about the backward shift on the classical Hardy spaces.

In the Dirichlet space, the situation is very different. For one, the meromorphic function f_L is no longer a pseudocontinuation of f. In fact pseudocontinuations seem to have nothing to do with non-cyclic vectors on $\mathcal{D}$ [**5**].

THEOREM 8.3. *There is a non-cyclic vector f for B on $\mathcal{D}$ which does not have a pseudocontinuation across any set of positive measure in $\mathbb{T}$. More specifically, there is no set $E \subset \mathbb{T}$ of positive measure and no meromorphic G on $\mathbb{D}_e$ such that the non-tangential limits of f and G exist and are equal on E.*

The situation gets more complicated by the fact that, unlike the H^2 case (where f_L is a pseudocontinuation of f and as such, via Privalov's uniqueness theorem, is independent of the annihilating L), f_L depends on the annihilating L [**50**].

THEOREM 8.4. *There is a non-cyclic $f \in \mathcal{D}$ and $L_1, L_2 \in [f]_B^{\perp} \setminus \{0\}$ such that f_{L_1} is a pseudocontinuation of f while f_{L_2} is not.*

There are some positive results that seem to indicate that f_L can be regarded as a 'continuation' of f even though it is not a pseudocontinuation of f. We mention two of them. The first is from [**5**] while the second is from [**50, 51**].

PROPOSITION 8.5. *For non-cyclic $f \in \mathcal{D}$, and $L \in [f]_B^{\perp} \setminus \{0\}$, the non-tangential limit as $z \to \zeta$ of*

$$L\left(\frac{1}{w - z}\right)\{f_L(z) - f(\zeta)\}$$

is equal to zero for almost every $\zeta \in \mathbb{T}$.

In particular, this theorem says that if the function $z \to L((w-z)^{-1})$ has finite non-tangential limits almost everywhere, then f_L is a pseudocontinuation of f.

THEOREM 8.6. *Suppose $f \in \mathcal{D}$ is non-cyclic for B and has an analytic continuation to an open neighborhood U_ζ of $\zeta \in \mathbb{T}$. Then for any $L \in [f]_B^{\perp} \setminus \{0\}$, f_L agrees with f on U_ζ.*

A nice corollary, as was the case for H^2, is that the function $f(z) = (1-z)^{3/2}$ is a cyclic vector for B on $\mathcal{D}$.

COROLLARY 8.7. *Any $f \in \mathcal{D}$ with an isolated winding point on $\mathbb{T}$ must be cyclic for B on $\mathcal{D}$.*

We also wish to make some remarks about the possible linear structure on the set of non-cyclic vectors. Using the Douglas, Shapiro, Shields characterization of the non-cyclic vectors for B on H^2 (Theorem 8.2), one can prove that the sum of two non-cyclic vectors must be non-cyclic. Indeed, if f_1, f_2 are non-cyclic, then f_1 and f_2 have pseudocontinuations F_1/G_1 and F_2/G_2 respectively where F_j, G_j are bounded analytic functions on $\mathbb{D}_e$. The sum $f_1 + f_2$ will have $F_1/G_1 + F_2/G_2$ as a pseudocontinuation. For the Dirichlet space, we have the following curious pathology.

THEOREM 8.8. *There are two non-cyclic vectors f, g for B on $\mathcal{D}$, such that $f + g$ is cyclic.*

This phenomenon was originally discovered by S. Walsh [**63**]. Abakumov [**1, 2**] proved the some result in a more general setting by a gap series argument. There is another proof in [**50**] that uses some old spectral synthesis results of Beurling.

So far, we have discussed cyclic vectors for B on $\mathcal{D}$, at least as well as we could. What about a description of $\mathrm{Lat}(B, \mathcal{D})$? The B-invariant subspaces of H^2 are known. Indeed suppose $\mathcal{M} \in \mathrm{Lat}(B, H^2)$. Then, since $B = S^*$ on H^2, we know that $\mathcal{M}^{\perp}$ is S-invariant. By Beurling's theorem (Theorem 7.1) $\mathcal{M}^{\perp} = \vartheta H^2$, where ϑ is inner, and so $\mathcal{M} = (\vartheta H^2)^{\perp}$. A well-known characterization of Douglas, Shapiro, and Shields [**21**] better describes $(\vartheta H^2)^{\perp}$.

THEOREM 8.9 (Douglas-Shapiro-Shields). *For an inner function ϑ, the following are equivalent for $f \in H^2$.*

(1) *$f \in (\vartheta H^2)^{\perp}$.*
(2) *f/ϑ has a pseudocontinuation G such that $G(1/z) \in H^2$ and vanishes at $z = 0$.*
(3) *There is a $g \in H^2$ with $g(0) = 0$ such that $f = \vartheta \overline{g}$ almost everywhere on $\mathbb{T}$.*

Unfortunately, there is no similar type of theorem for $\mathrm{Lat}(B, \mathcal{D})$. However, if one is willing to recast the problem in terms of approximation by rational functions, there is something to be said. Here is the set up. For each $n \in \mathbb{N}$, choose a finite sequence

$$E_n := \{z_{n,1}, \cdots, z_{n,N(n)}\}$$

of points of $\mathbb{D}$ (multiplicities are allowed) to create the tableau $\mathcal{S}$

$$\begin{gathered} z_{1,1}, z_{1,2}, \cdots, z_{1,N(1)} \\ z_{2,1}, z_{2,2}, \cdots, z_{2,N(2)} \\ \vdots \end{gathered}$$

For each n, create the finite dimensional B-invariant subspace of $X = H^2$ (or $\mathcal{D}$)

$$R_n := \bigvee \left\{ \frac{1}{(1 - \overline{z_{n,j}} z)^s} : j = 1, \cdots, N(n), s = 1, \cdots, \mathrm{mult}(z_{n,j}) \right\}, \tag{8.10}$$

where $\mathrm{mult}(z_{n,j})$ is the number of times $z_{n,j}$ appears in E_n, the n-th row of the tableau. If $z_{n,j} = 0$ with multiplicity k, then the functions $1, z, z^2, \cdots, z^k$ are added to the spanning set for R_n. One can now form the 'liminf space' associated with the tableau $\mathcal{S}$ by

$$R(\mathcal{S}) := \underline{\lim}\, R_n = \left\{ f \in X : \lim_{n \to \infty} \mathrm{dist}(f, R_n) = 0 \right\}. \tag{8.11}$$

When $X = H^2$, there is a condition that determines when $R(\mathcal{S}) \neq H^2$ [**59, 62**]: If

$$\beta(E_n) := \sum_{j=1}^{N(n)} (1 - |z_{n,j}|),$$

then

$$R(\mathcal{S}) \neq H^2 \quad \Leftrightarrow \quad \underline{\lim}_{n \to \infty} \beta(E_n) < \infty. \tag{8.12}$$

In the Dirichlet space, the quantity $\beta(E_n)$ is replaced by another quantity suitable for the Dirichlet space [**39**].

THEOREM 8.13. *Let X be either H^2 or $\mathcal{D}$. If $\mathcal{M} \in Lat(B, X)$ with $\mathcal{M} \neq X$, then there is a tableau $\mathcal{S}$ so that*

$$\mathcal{M} = R(\mathcal{S}).$$

The H^2 case was done by Tumarkin [**59**] while the $\mathcal{D}$ case was done recently by Shimorin [**55**]. A description of $\mathrm{Lat}(B, \mathcal{D})$ in terms of the 'continuation' properties of the function, as was done with pseudocontinuations in the H^2 case, is very much an open problem worth of study.

We end this section with a final remark from [**24**] which says that the B-invariant subspaces of $\mathcal{D}$ have the F-property.

PROPOSITION 8.14. *If $f \in \mathcal{D}$ and ϑ is inner with $f/\vartheta \in H^2$, then $f/\vartheta \in [f]_B$.*

Notice from Theorem 8.9 how the B-invariant subspaces of H^2 have the F-property.

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