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An Observation About Frostman Shifts

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ABSTRACT. A classical theorem of O. Frostman says that if B is a Blaschke product (or any inner function), then its Frostman shifts $B_w = (B - w)(1 - \overline{w}B)^{-1}$ are Blaschke products for all $|w| < 1$ except possibly for w in a set of logarithmic capacity zero. If B is a Frostman Blaschke product, equivalently an inner multiplier for the space of Cauchy transforms of measures on the unit circle, we show that for all $|w| < 1$, B_w is indeed another Frostman Blaschke product.

1. STATEMENT OF THE MAIN RESULT

If B is a Blaschke product

$$B(z) = e^{i\gamma} z^m \prod_{n=1}^{\infty} \frac{\overline{a_n}}{|a_n|} \frac{a_n - z}{1 - \overline{a_n} z}, \quad |z| < 1,$$

with zeros $(a_n)_{n \geqslant 1}$ in the open unit disk $\mathbb{D} = \{z : |z| < 1\}$ (repeated according to multiplicity) and $w \in \mathbb{D}$, let

$$B_w := \frac{B - w}{1 - \overline{w}B}$$

denote a *Frostman shift* of B. Certainly B_w is an inner function. A famous theorem of O. Frostman [9] [6, p. 37] goes further and says that B_w is a Blaschke product for every $w \in \mathbb{D}$ except possibly for w in a set of logarithmic capacity zero.

The above Blaschke product B is said to satisfy the *uniform Frostman condition* if

$$\sigma(B) := \sup_{|\zeta|=1} \sum_{n=1}^{\infty} \frac{1 - |a_n|^2}{|\zeta - a_n|} < \infty.$$

Blaschke products B for which $\sigma(B) < \infty$ are sometimes called *Frostman Blaschke products* (FBP) and have several remarkable properties. First [10] [6, p. 33], the radial limit

$$\lim_{r \to 1^-} B(r\zeta)$$

of B and all its sub-products exist and are of unit modulus for *every* $\zeta \in \mathbb{T} = \partial\mathbb{D}$ and not just almost everywhere as in Fatou's theorem [6, p. 17]. Second, such Blaschke products are precisely the inner multipliers of the space of Cauchy transforms of measures on $\mathbb{T}$ (see Theorem 4.1 below). Our main theorem about Frostman shifts of FBP is the following.

Theorem 1.1. *If $B \in FBP$, then $B_w \in FBP$ for all $w \in \mathbb{D}$.*

The proof of Theorem 1.1 appears in Section 6 and depends on two facts. The first is a powerful theorem of Hruščev and Vinogradov [14] (see also [5, Chapter 6]) which says that for an inner function ϕ, the co-analytic Toeplitz operator $T_{\overline{\phi}}$ is a bounded operator on H^∞ (the bounded analytic functions on $\mathbb{D}$) if and only if $\phi \in$ FBP. The second fact is the estimate

$$\|T_{\overline{B}}\|_{H^\infty \to H^\infty} \leqslant 1 + \frac{\pi}{2}\sigma(B), \quad B \in \text{FBP},$$

which was outlined by Tolokonnikov [26, Lemma 3.8] (via Vinogradov) and uses a theorem of Pekarskiĭ [23]. Since Tolokonnikov's outline is difficult to follow and uses results from papers not readily accessible such as [8, 25], and, since the estimate is interesting in its own right, we spend some time in Section 5 on the technical details of this estimate. This estimate is also worthwhile since, for one, it yields an alternate proof of one direction of the above mentioned Hruščev-Vinogradov result. Secondly, this estimate is also better than the more well-known estimates found in for example [21, p. 375].

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2. Indestructible Blaschke products

In order to place Theorem 1.1 in a broader context, we give a short survey of indestructible Blaschke products. We refer the reader to the papers [2, 12, 13, 16, 18, 19, 20] for further results and examples. A function $\phi \in H^\infty$, the bounded analytic functions on $\mathbb{D}$, is said to be *inner* if the radial limits

$$\lim_{r \to 1^-} \phi(r\zeta),$$

of ϕ have modulus one for almost every $\zeta \in \mathbb{T}$. It is well known [7, p. 24] that any inner function can be factored as $\phi = BS_\mu$, where B and S_μ are also inner functions,

$$B(z) = e^{i\gamma} z^m \prod_{n=1}^{\infty} \frac{\overline{a_n}}{|a_n|} \frac{a_n - z}{1 - \overline{a_n} z}$$

is the *Blaschke factor*, and

$$S_\mu(z) = \exp\left(-\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta)\right)$$

is the *singular inner* factor. The measure μ is a finite positive measure on $\mathbb{T}$ with $\mu \perp m$. Here $dm = |d\zeta|/2\pi$ is normalized Lebesgue measure on $\mathbb{T}$.

For $a \in \mathbb{D}$, let

$$\tau_a(z) := \frac{z - a}{1 - \overline{a} z}$$

and note that $\tau_a(\mathbb{D}) = \mathbb{D}$ and $\tau_a(\mathbb{T}) = \mathbb{T}$. Thus for any inner function ϕ, the *Frostman shifts*

$$\phi_a := \tau_a \circ \phi, \quad a \in \mathbb{D},$$

are also inner.

Theorem 2.1 (Frostman[1]). *For any inner function ϕ, the Frostman shifts*

$$\phi_w = \frac{\phi - w}{1 - \overline{w}\phi}$$

[1] See [9] or [6, p. 37] [11, p. 79].

are Blaschke products for all $w \in \mathbb{D}$, with the possible exception of w in a set of logarithmic capacity zero.

Using an argument with the uniformizer [6, p. 37], one can show a somewhat lesser known result which says that given a compact set $E \subset \mathbb{D}$, $0 \notin E$, with logarithmic capacity zero, there is an inner function ϕ, indeed a Blaschke product, such that the Frostman shifts ϕ_w are singular inner functions for all $w \in E$ (In fact, one can find an interpolating Blaschke product B such that B_a is an an interpolating Blaschke product for all $a \in \mathbb{D} \setminus E$ while B_a is a singular inner function for all $a \in E$ [12, Theorem 1.1]). The *exceptional sets* for an inner function, those $E \subset \mathbb{D}$ for which ϕ_a, $a \in E$, has a non-trivial singular inner factor, were described in [17] as precisely the F_σ sets of logarithmic capacity zero.

We say a Blaschke product B is *indestructible* if B_a is also a Blaschke product for every $a \in \mathbb{D}$, i.e., there is no exceptional set in Theorem 2.1. The term 'indestructible' was coined by McLaughlin [16] though, in a vague sense, this concept was explored earlier by Heins [13].

Proposition 2.2 (Frostman[2])**.** *Suppose ϕ is an inner function with a non-trivial singular inner factor. Then there is a $\zeta_0 \in \mathbb{T}$ such that*

$$\lim_{r \to 1^-} \phi(r\zeta_0) = 0.$$

Theorem 2.3 (Frostman[3])**.** *For a Blaschke product B with zeros $(a_n)_{n \geqslant 1}$, repeated according to multiplicity, a necessary and sufficient condition that the radial limit of B and all its sub-products have modulus one at $\zeta_0 \in \mathbb{T}$ is*

$$\sum_{n=1}^{\infty} \frac{1 - |a_n|^2}{|\zeta_0 - a_n|} < \infty.$$

Proposition 2.2 says that any Blaschke product whose radial limits exist and are unimodular everywhere on $\mathbb{T}$ is indestructible. Other indestructible Blaschke products are created along these lines [16, Theorem 2]. In particular, by Theorem 2.3, the Frostman Blaschke products are indestructible. Our main theorem (Theorem 1.1) says more.

There have been attempts to characterize the indestructible Blaschke products in terms of their zeros [16, Theorem 1]. However, the condition in the cited theorem is very difficult to apply. Moreover, indestructibility seems to be rather delicate in that there are destructible Blaschke products (B such that B_a is not a Blaschke product for some $a \in \mathbb{D}$) which become indestructible when one of its zeros is removed [18, Proposition 4.1].

The main result of this paper deals not only with the topic of whether or not the FBP are indestructible (they are) but whether or not the class of FBP is preserved under Frostman shifts, i.e., if $B \in \text{FBP}$, is $B_a \in \text{FBP}$ for all $a \in \mathbb{D}$? There have been other studies related to this type of 'class preservation' under Frostman shifts. For example, a paper of A. Nicolau [20] and several others (see the list of references in [19, p. 287]) examined the class

$$\mathcal{P} := \{B \in \text{CN} : B_a \in \text{CN} \ \forall a \in \mathbb{D}\}.$$

[2]See [9] or [22, p. 33].

[3]See [10] or [6, p. 33].

Here CN denotes the set of Carleson-Newman Blaschke products, i.e., those Blaschke products which can be written as a finite product of interpolating Blaschke products (see [19] for the definitions). A paper of Mortini and Nicolau [19] examined the class

$$\mathcal{M} := \{\phi \text{ inner} : \phi_a \in \text{CN } \forall a \in \mathbb{D} \setminus \{0\}\}.$$

Clearly $\mathcal{P} \subset \mathcal{M}$. The inclusion is proper.

3. Frostman Blaschke products

We mention several known results about FBP. First, any finite Blaschke product is a FBP. Second, infinite FBP actually exist! For example, an argument from [28] (see also [5, p. 130]) shows that the Blaschke product whose zeros are

$$a_n = \left(1 - \frac{1}{2^n}\right) \exp\left(i\frac{2^n}{3^n}\right)$$

belongs to FBP. The third fact to mention is that the zeros $(a_n)_{n \geqslant 1}$ of a FBP B must satisfy something stronger than just the Blaschke condition

$$\sum_{n=1}^{\infty} (1 - |a_n|) < \infty.$$

Indeed, Vasjunin [27] showed that

$$\sum_{n=1}^{\infty} (1 - |a_n|) \log \frac{1}{1 - |a_n|} \leqslant c\sigma(B) < \infty.$$

However, the above condition does not characterize FBP.

Unlike the Blaschke products, where the zeros can be a rich subset of the disk, the zeros of a FBP are not. For example, if

$$\Gamma_\alpha(\zeta) = \{z \in \mathbb{D} : |z - \zeta| < \alpha(1 - |z|)\}, \quad \zeta \in \mathbb{T}, \quad \alpha > 1,$$

are the standard Stoltz domains with vertex at ζ, and B is a FBP, then [14] (see also [5, p. 143])

$$\sup_{\zeta \in \mathbb{T}} \operatorname{card}\left(B^{-1}(\{0\}) \cap \Gamma_\alpha(\zeta)\right) \leqslant c_\alpha \sigma(B),$$

where c_α is an absolute constant depending only on α. This says that the number of zeros of a FBP in any Stoltz domain $\Gamma_\alpha(\zeta)$ is finite and uniformly bounded in ζ. A theorem in [15] goes further and shows that the zeros of a FBP product can only accumulate on a nowhere dense subset of $\mathbb{T}$. It is also true [14] (see also [5, p. 140]) that FBP $\subset$ CN. Using Douglas algebra techniques, Tolokonnikov [26] showed that FBP $\subset \mathcal{P}$. Our main theorem says a bit more, namely $B \in \text{FBP} \Rightarrow B_a \in \text{FBP}$ for all $a \in \mathbb{D}$.

4. Multipliers of the space of Cauchy transforms

The proof of Theorem 1.1 depends on a theorem relating FBP and multipliers of the space of Cauchy transforms (see Theorem 4.1 below). Two references for what we say below are [4, 5]. The space of Cauchy transforms

$$\mathcal{K} := \left\{ \int \frac{d\mu(\zeta)}{1 - \bar{\zeta} z} : \mu \in M \right\},$$

where M is the space of finite complex Borel measures on $\mathbb{T}$, can be endowed with the norm

$$\|f\| := \inf\left\{\operatorname{Var}(\mu) : f(z) = \int \frac{d\mu(\zeta)}{1-\overline{\zeta}z}\right\},$$

where $\operatorname{Var}(\mu)$ is the total variation norm of μ. Endowed with this norm, one can prove the following estimate

$$|f(z)| \leqslant \frac{\|f\|}{1-|z|}, \quad z \in \mathbb{D},$$

which shows that $\mathcal{K}$ is a Banach space of analytic functions on $\mathbb{D}$ in that the injection $i : \mathcal{K} \to \operatorname{Hol}(\mathbb{D})$ is continuous. A classical result of Smirnov says that

$$\mathcal{K} \subset \bigcap_{0<p<1} H^p,$$

where for $0 < p < \infty$, H^p is the usual Hardy space of analytic functions f on $\mathbb{D}$ for which

$$\sup_{0<r<1} \int_{\mathbb{T}} |f(r\zeta)|^p dm(\zeta) < \infty.$$

Here $dm = |d\zeta|/2\pi$ is normalized Lebesgue measure on $\mathbb{T}$. When $p = \infty$, H^∞ is the algebra of bounded analytic functions on $\mathbb{D}$ with sup-norm

$$\|f\|_\infty := \sup\{|f(z)| : z \in \mathbb{D}\}.$$

An analytic function ϕ on $\mathbb{D}$ is a *multiplier* of $\mathcal{K}$ if $\phi\mathcal{K} \subset \mathcal{K}$. The space of multipliers $\mathfrak{M}(\mathcal{K})$ can be endowed with the multiplier norm by considering the natural operator norm

$$\|M_\phi\| := \sup\{\|\phi f\| : \|f\| \leqslant 1\}$$

of the multiplication operator $M_\phi f = \phi f$ on $\mathcal{K}$. If $\mathcal{K}_a$ are the Cauchy transforms of $\mu \ll m$, then $\mathfrak{M}(\mathcal{K}) = \mathfrak{M}(\mathcal{K}_a)$ (see Proposition 5.3 below).

One can easily show that $\mathfrak{M}(\mathcal{K}) \subset H^\infty$ and in fact $\|\phi\|_\infty \leqslant \|M_\phi\|$. However, $\mathfrak{M}(\mathcal{K}) \subsetneq H^\infty$ since the multipliers must display some extra regularity near $\mathbb{T}$ in that the radial limit

$$\lim_{r\to 1^-} \phi(r\zeta)$$

must exist and be finite for *every* $\zeta \in \mathbb{T}$, and not just almost every ζ as in Fatou's theorem.

Frostman's theorem on Blaschke products (Theorem 2.3) implies that not every inner function is a multiplier of $\mathcal{K}$ since there are Blaschke products which do not have radial limits at every point of $\mathbb{T}$. The following powerful theorem of Hruščev and Vinogradov [14] (see also [5, Chapter 6]) characterizes the inner multipliers of $\mathcal{K}$.

Theorem 4.1 (Hruščev/Vinogradov)**.** *An inner function ϕ is a multiplier of $\mathcal{K}$ if and only if $\phi \in FBP$.*

5. Norm of a co-analytic Toeplitz operator

As mentioned earlier, the proof of our main theorem (Theorem 1.1) depends on knowing the norms of certain classes of co-analytic Toeplitz operators. If $\phi \in H^{\infty}$, define the co-analytic Toeplitz operator

$$T_{\overline{\phi}} : H^2 \to H^2, \quad T_{\overline{\phi}} f = (\overline{\phi} f)_+,$$

where for $g \in L^2$ with Fourier coefficients

$$\widehat{g}(n) := \int_{\mathbb{T}} \overline{\zeta}^n g(\zeta) dm(\zeta), \quad n \in \mathbb{Z},$$

the function

$$g_+(z) = \sum_{n=0}^{\infty} \widehat{g}(n) z^n, \quad |z| < 1,$$

is the *Riesz projection* of L^2 onto H^2.

The function

$$g_-(z) = \sum_{n=1}^{\infty} \frac{\widehat{g}(-n)}{z^n}, \quad |z| > 1,$$

is the projection of L^2 onto

$$H^2_0(\mathbb{D}_e) := \{f(1/z) : f \in H^2, f(0) = 0\},$$

the Hardy space of the extended exterior disk $\mathbb{D}_e := \widehat{\mathbb{C}} \setminus \mathbb{D}^-$. There are the following useful integral formulas for g_+ and g_-

$$g_+(z) = \int_{\mathbb{T}} \frac{g(\zeta)}{1 - \overline{\zeta} z} dm(\zeta), \quad |z| < 1, \tag{5.1}$$

$$g_-(z) := -\frac{1}{2\pi i} \oint_{\mathbb{T}} \frac{g(\zeta)}{\zeta - z} d\zeta, \quad |z| > 1, \tag{5.2}$$

where we understand the line integral in eq.(5.2) to be in the counterclockwise direction.

Simple estimates using the continuity of the Riesz projection, show that $T_{\overline{\phi}}$ is continuous on H^2 as well as H^p for $1 < p < \infty$ [7, p. 54]. When $p = \infty$, $T_{\overline{\phi}}$ is not necessarily continuous on H^{∞} nor on A, the disk algebra of continuous functions on $\mathbb{D}^-$ which are also analytic on $\mathbb{D}$. However, we can relate the continuity of $T_{\overline{\phi}}$ on H^{∞} or A with the multipliers of the space of Cauchy transforms $\mathcal{K}$. See [14] and [5, p. 117].

Proposition 5.3. *For $\phi \in H^{\infty}$, the following are equivalent.*

(1) $\phi \in \mathfrak{M}(\mathcal{K})$;
(2) $\phi \in \mathfrak{M}(\mathcal{K}_a)$;
(3) $T_{\overline{\phi}}$ *is bounded on* A;
(4) $T_{\overline{\phi}}$ *is bounded on* H^{∞}.

Moreover, if any of the above conditions hold, then

$$\|T_{\overline{\phi}}\|_{A \to A} = \|M_{\phi}\|_{\mathcal{K} \to \mathcal{K}}, \quad \|T_{\overline{\phi}}\|_{H^{\infty} \to H^{\infty}} = \|M_{\phi}\|_{\mathcal{K}_a \to \mathcal{K}_a}.$$

The main result of this section is the following. The proof is outlined in [26, Lemma 3.8] but for the sake of completeness, and since that outline is difficult to follow, we include it here.

Proposition 5.4 (Vinogradov/Tolokonnikov)**.** *If B is a finite Blaschke product with zeros $\{a_1, \cdots, a_n\}$, repeated according to multiplicity, then*

$$\|M_B\|_{\mathcal{K}_a \to \mathcal{K}_a} = \|T_{\overline{B}}\|_{H^\infty \to H^\infty} \leqslant 1 + \frac{\pi}{2}\sigma(B).$$

The proof of this will require some set up but for now, we state and prove the following corollary (see [26, Lemma 3.8]) which is the key to proving Theorem 1.1.

Corollary 5.5 (Vinogradov/Tolokonnikov)**.** *If $B \in FBP$ and $n \in \mathbb{N}$,*

$$\|M_B^n\|_{\mathcal{K}_a \to \mathcal{K}_a} = \|T_{\overline{B^n}}\|_{H^\infty \to H^\infty} \leqslant n\left(1 + \frac{\pi}{2}\right)\sigma(B).$$

Proof. If B_N is the product of the first N terms of the Blaschke product B, we know that $B_N \to B$ pointwise in $\mathbb{D}$ and thus in the weak-$*$ topology of H^∞, that is to say,

$$\lim_{N\to\infty} \int_{\mathbb{T}} B_N(\zeta)h(\zeta)dm(\zeta) = \int_{\mathbb{T}} B(\zeta)h(\zeta)dm(\zeta),$$

for all $h \in L^1$ [3, Prop. 2]. Thus for any $f \in H^\infty$ and any $z \in \mathbb{D}$, we can use eq.(5.1) to get

$$(5.6)\qquad (T_{\overline{B_N}}f)(z) = \int_{\mathbb{T}} \overline{B_N(\zeta)}\frac{f(\zeta)}{1-\bar{\zeta}z}dm(\zeta) \to \int_{\mathbb{T}} \overline{B(\zeta)}\frac{f(\zeta)}{1-\bar{\zeta}z}dm(\zeta) = (T_{\overline{B}}f)(z)$$

as $N \to \infty$.

Thus from Proposition 5.4, we have, for any $z \in \mathbb{D}$,

$$|(T_{\overline{B_N}}f)(z)| \leqslant 1 + \frac{\pi}{2}\sigma(B_N) \leqslant 1 + \frac{\pi}{2}\sigma(B).$$

Now bring in eq.(5.6) to get

$$|(T_{\overline{B}}f)(z)| = \lim_{N\to\infty} |(T_{\overline{B_N}}f)(z)| \leqslant 1 + \frac{\pi}{2}\sigma(B).$$

Hence Proposition 5.4 can be extended to FBP. The result now follows from the facts that $\sigma(B^n) = n\sigma(B)$ and $\sigma(B) \geqslant 1$. □

In order to prove Proposition 5.4, we need to review a theorem of Pekarskiĭ [23, Theorem 3.1] which estimates the derivative of a Cauchy transform. We include a proof.

Theorem 5.7 (Pekarskiĭ)**.** *If $f \in H^\infty$ and B is a finite Blaschke product, then*

$$\left|\frac{d}{dz}\left(f\overline{B}\right)_{-}(z)\right| \leqslant \|f\|_\infty \frac{1-|B(z)|^{-2}}{|z|^2-1}, \quad |z| > 1.$$

The proof of Pekarskiĭ's theorem requires the following technical tool. Write the Blaschke product B as

$$B(z) = \prod_{k=1}^{n} \frac{z-a_k}{1-\overline{a_k}z}.$$

The zeros a_k are repeated as factors according to to their multiplicity and, since B is a finite Blaschke product, we leave off the $-\overline{a_k}/|a_k|$ terms in each of the factors. This B will differ from the traditional finite Blaschke product by a unimodular constant which will not matter in our estimates. Form the Takenaka-Malmquist [8, 25] orthonormal system

$$T_k(z) := \frac{\sqrt{1-|a_k|^2}}{1-\overline{a_k}z}\prod_{j=1}^{k-1}\frac{z-a_j}{1-\overline{a_j}z}, \quad |z|<1, \quad k=1,\cdots,n.$$

$$T_k^*(z) := \frac{\sqrt{1-|a_k|^2}}{z-a_k}\prod_{j=1}^{k-1}\frac{1-\overline{a_j}z}{z-a_j}, \quad |z|>1, \quad k=1,\cdots,n.$$

When $k=1$ we set the product term $\prod_{j=1}^{k-1}$ in the above formulas equal to the constant one. It is easy to show that if $\langle\cdot,\cdot\rangle$ denotes the usual L^2 inner product, then

$$\langle T_k, T_l\rangle = \delta_{k,l}. \tag{5.8}$$

The functions T_k and T_k^* are rational functions with T_k analytic on $\mathbb{D}$ and T_k^* analytic on $\mathbb{D}_e$.

The functions $T_1,\cdots,T_n$ also form an orthonormal basis for the subspace $(BH^2)^\perp$. Indeed for any $h\in H^2$,

$$\begin{aligned}
\langle T_k, Bh\rangle &= \int_{\mathbb{T}} \frac{\sqrt{1-|a_k|^2}}{1-\overline{a_k}\zeta}\prod_{j=k}^{n}\frac{\overline{\zeta}-\overline{a_j}}{1-a_j\overline{\zeta}}\,\overline{h(\zeta)}\,dm(\zeta)\\
&= \int_{\mathbb{T}} \frac{\sqrt{1-|a_k|^2}}{|1-\overline{a_k}\zeta|^2}(\overline{\zeta}-\overline{a_k})\prod_{j=k+1}^{n}\frac{\overline{\zeta}-\overline{a_j}}{1-a_j\overline{\zeta}}\,\overline{h(\zeta)}\,dm(\zeta)\\
&= \int_{\mathbb{T}} \frac{1-|a_k|^2}{|1-\overline{a_k}\zeta|^2}\frac{\overline{\zeta}-\overline{a_k}}{\sqrt{1-|a_k|^2}}\prod_{j=k+1}^{n}\frac{\overline{\zeta}-\overline{a_j}}{1-a_j\overline{\zeta}}\,\overline{h(\zeta)}\,dm(\zeta)\\
&= \int_{\mathbb{T}} \frac{1-|a_k|^2}{|1-\overline{a_k}\zeta|^2}\overline{L(\zeta)}\,dm(\zeta),
\end{aligned}$$

where $L\in H^2$ and $L(a_k)=0$. By the Poisson integral formula [7, p. 41], this last integral is equal to zero. From eq.(5.8), the T_k's are orthogonal and it is well-known that $(BH^2)^\perp$ is n-dimensional.

Since the reproducing kernel for $(BH^2)^\perp$ is

$$\frac{1-B(z)\overline{B(w)}}{1-z\overline{w}}, \quad w,z\in\mathbb{D},$$

(see [5, p. 186]) we know, from basic properties of kernel functions [1, p. 9], that

$$\frac{1-B(z)\overline{B(w)}}{1-z\overline{w}} = \sum_{k=1}^{n} T_k(z)\overline{T_k(w)}, \quad w,z\in\mathbb{D}.$$

Letting $z=w$, we get the identity

$$\frac{1-|B(z)|^2}{1-|z|^2} = \sum_{k=1}^{n}|T_k(z)|^2, \quad |z|<1. \tag{5.9}$$

Using the identity

$$B(z) = \frac{1}{\overline{B(1/\overline{z})}}, \quad |z|>1,$$

we get, for $|z| > 1$,

$$\begin{aligned} \frac{1-|B(z)|^{-2}}{|z|^2-1} &= \frac{1}{|z|^2}\frac{1-|B(1/\overline{z})|^2}{1-\frac{1}{|z|^2}} \\ &= \frac{1}{|z|^2}\sum_{k=1}^{n}|T_k(1/\overline{z})|^2 \quad \text{(by eq.(5.9))} \\ &= \frac{1}{|z|^2}\sum_{k=1}^{n}\left|\overline{zT_k^*(z)}\right|^2. \end{aligned}$$

This gives us the useful formula

$$\sum_{k=1}^{n}|T_k^*(z)|^2 = \frac{1-|B(z)|^{-2}}{|z|^2-1}, \quad |z|>1. \tag{5.10}$$

For fixed $|z| > 1$, let $\chi_z : \mathbb{T} \to \mathbb{C}$ be defined by

$$\chi_z(\zeta) := \frac{1}{\zeta - z}$$

and observe that $\chi_z(\zeta)$ has an analytic extension (as a function of ζ) to $\mathbb{D}$. Let $F : \mathbb{T} \to \mathbb{C}$ be the 'Fourier series' of χ_z with respect to the orthonormal system $T_1, \cdots, T_n$, i.e.,

$$F(\zeta) := \sum_{k=1}^{n}\langle\chi_z, T_k\rangle T_k(\zeta). \tag{5.11}$$

Notice that $F(\zeta)$ is a rational function of ζ and has an analytic extension to $\mathbb{D}$ (since the T_k's have this same property). A computation with the facts that

$$\overline{T_k(\zeta)} = \zeta T_k^*(\zeta), \quad \zeta\in\mathbb{T},$$

and that T_k^* is analytic on the extended exterior disk $\mathbb{D}_e$, show that

$$\langle\chi_z, T_k\rangle = \int_{\mathbb{T}}\frac{1}{\zeta-z}\overline{T_k(\zeta)}\,dm(\zeta) = \oint_{\mathbb{T}}\frac{T_k^*(\zeta)}{\zeta-z}\frac{d\zeta}{2\pi i} = -T_k^*(z), \quad |z|>1.$$

Thus

$$F(\zeta) = -\sum_{k=1}^{n}T_k(\zeta)T_k^*(z), \quad |\zeta|=1, \quad |z|>1. \tag{5.12}$$

Since F is also the orthogonal projection of χ_z on to $(BH^2)^\perp$ (see eq.(5.11)), we can decompose χ_z with respect to the orthogonal sum $H^2 = (BH^2)^\perp \oplus BH^2$ to get

$$\chi_z(\zeta) = F(\zeta) + B(\zeta)Q(z,\zeta), \tag{5.13}$$

where, for fixed $|z| > 1$, $Q(z, \zeta)$ is a rational function of ζ and has an analytic extension to $\mathbb{D}$.

See [8] for a more computational way of thinking about the above Takenaka-Malmquist system and the associated formulas.

Proof of Theorem 5.7. With these preliminaries in place, we are now ready to estimate

$$\left|\frac{d}{dz}\left(f\overline{B}\right)_-(z)\right|, \quad |z|>1.$$

Indeed, from eq.(5.2) we have, for $|z| > 1$,

$$\begin{aligned}
\frac{d}{dz}(f\overline{B})_{-}(z) &= -\frac{1}{2\pi i}\oint_{\mathbb{T}} \frac{f(\zeta)\overline{B(\zeta)}}{(\zeta - z)^2}\, d\zeta \\
&= -\frac{1}{2\pi i}\oint_{\mathbb{T}} f(\zeta)\overline{B(\zeta)}\left\{F(\zeta) + B(\zeta)Q(z,\zeta)\right\}^2 d\zeta \quad \text{(by eq.(5.13))} \\
&= -\frac{1}{2\pi i}\oint_{\mathbb{T}} f(\zeta)\overline{B(\zeta)}F(\zeta)^2\, d\zeta.
\end{aligned}$$

Notice in the last integral computation how we use the fact that B, F, $Q(z,\cdot)$, and f belong to H^∞.

Thus, for $|z| > 1$, we can use the previous computation along with the fact from eq.(5.8) that $\{T_1, \cdots, T_n\}$ is an orthonormal system to get

$$\begin{aligned}
\left|\frac{d}{dz}\left(f\overline{B}\right)_{-}(z)\right| &\leqslant \|f\|_\infty \int_{\mathbb{T}} F(\zeta)\overline{F(\zeta)}\, dm(\zeta) \\
&= \|f\|_\infty \int_{\mathbb{T}} \left(\sum_{k=1}^{n} T_k(\zeta)T_k^*(z)\right)\left(\sum_{l=1}^{n} \overline{T_l(\zeta)T_l^*(z)}\right) dm(\zeta) \\
&= \|f\|_\infty \sum_{k=1}^{n} |T_k^*(z)|^2 \\
&= \|f\|_\infty \frac{1 - |B(z)|^{-2}}{|z|^2 - 1} \quad \text{(from eq.(5.10))}.
\end{aligned}$$

□

The proof of Proposition 5.4 also requires two additional technical lemmas.

Lemma 5.14. *If B is a finite Blaschke product with zeros $\{a_1, \cdots, a_n\}$, repeated according to multiplicity, then*

$$\frac{1 - |B(z)|^{-2}}{|z|^2 - 1} \leqslant \sum_{k=1}^{n} \frac{1 - |a_k|^2}{|z - a_k|^2}, \quad |z| > 1.$$

Proof. We begin with the following simple observation: If $\epsilon_1, \cdots, \epsilon_n \in (0,1)$, then

$$1 - \prod_{k=1}^{n} \epsilon_k < \sum_{k=1}^{n} (1 - \epsilon_k). \tag{5.15}$$

To see this inequality, it is suffices, via induction, to verify it when $n = 2$. Indeed

$$0 < (1 - \epsilon_1)(1 - \epsilon_2) = 1 - \epsilon_1 - \epsilon_2 + \epsilon_1\epsilon_2.$$

Rearranging this, we get

$$1 - \epsilon_1\epsilon_2 < 1 - \epsilon_1 + 1 - \epsilon_2$$

as desired.

Next we observe the identity

$$\frac{1 - \left|\frac{a-z}{1-\overline{a}z}\right|^{-2}}{|z|^2 - 1} = \frac{1 - |a|^2}{|z - a|^2}, \quad |z| > 1, \quad |a| < 1. \tag{5.16}$$

Finally, if

$$B(z)=\prod_{k=1}^{n}\frac{a_k-z}{1-\overline{a_k}z},$$

$1/B$ is an inner function on $\mathbb{D}_e$. Thus eq.(5.15) applies to give us

$$\frac{1-|B(z)|^{-2}}{|z|^2-1}\leqslant\sum_{k=1}^{n}\frac{1-|b_{a_k}(z)|^{-2}}{|z|^2-1},\quad |z|>1,$$

where

$$b_a(z):=\frac{a-z}{1-\overline{a}z}.$$

The identity in eq.(5.16) finishes the proof. □

Lemma 5.17. *If $a\in\mathbb{D}$ then*

$$\int_1^\infty\frac{1}{|t-a|^2}\,dt\leqslant\frac{\pi}{2}\frac{1}{|1-a|}.$$

Proof. If a is a real number, the result is obvious so assume $a=a_1+ia_2$ with $a_1,a_2\in\mathbb{R}$ and $a_2\neq 0$. Then

$$\begin{aligned}\int_1^\infty\frac{1}{|t-a|^2}\,dt&=\int_1^\infty\frac{1}{(t-a_1)^2+a_2^2}\,dt\\&=\frac{\pi}{2}\frac{1}{|a_2|}-\frac{1}{|a_2|}\tan^{-1}\left(\frac{1-a_1}{|a_2|}\right).\end{aligned}$$

Now notice that

$$\begin{aligned}&|1-a|\left\{\frac{\pi}{2}\frac{1}{|a_2|}-\frac{1}{|a_2|}\tan^{-1}\left(\frac{1-a_1}{|a_2|}\right)\right\}\\&=\sqrt{\left(\frac{1-a_1}{a_2}\right)^2+1}\,\left\{\frac{\pi}{2}-\tan^{-1}\left(\frac{1-a_1}{|a_2|}\right)\right\}.\end{aligned}$$

A little calculus will show that the function

$$x\mapsto\sqrt{x^2+1}\left(\frac{\pi}{2}-\tan^{-1}x\right)$$

is a positive decreasing function on $[0,\infty]$ and is equal to $\pi/2$ when $x=0$. The result now follows. □

Proof of Proposition 5.4. For $f\in\operatorname{ball}(H^\infty)$ and almost every $\zeta\in\mathbb{T}$ we have

$$\begin{aligned}|\left(f\overline{B}\right)_-(\zeta)|&=\left|\int_1^\infty((f\overline{B})_-)'(t\zeta)\,dt\right|\quad\text{(fund. thm. of calculus)}\\&\leqslant\int_1^\infty\left|((f\overline{B})_-)'(t\zeta)\right|\,dt\\&\leqslant\int_1^\infty\frac{1-|B(t\zeta)|^{-2}}{|t\zeta|^2-1}\,dt\quad\text{(by Theorem 5.7)}\\&\leqslant\int_1^\infty\sum_{k=1}^{n}\frac{1-|a_k|^2}{|\zeta t-a_k|^2}\,dt\quad\text{(by Lemma 5.14)}\\&\leqslant\frac{\pi}{2}\sum_{k=1}^{n}\frac{1-|a_k|^2}{|\zeta-a_k|}\quad\text{(by Lemma 5.17).}\end{aligned}$$

Thus
$$\| \left(f\overline{B}\right)_{-} \|_{\infty} \leqslant \frac{\pi}{2}\sigma(B), \quad f \in \operatorname{ball}(H^{\infty}).$$
From the definition of the norm of $T_{\overline{B}}$ on H^{∞} and writing
$$f(\zeta)\overline{B(\zeta)} = (f\overline{B})_{+}(\zeta) + (f\overline{B})_{-}(\zeta), \quad \text{a.e. } \zeta \in \mathbb{T},$$
(see [7, p. 39]) we obtain
$$\|T_{\overline{B}}\|_{H^{\infty} \to H^{\infty}} \leqslant 1 + \frac{\pi}{2}\sigma(B).$$
□

6. Proof of the main theorem

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. By Corollary 5.5,
$$\lim_{n \to \infty} \sqrt[n]{\|M_B^n\|_{\mathcal{K}_a \to \mathcal{K}_a}} \leqslant 1$$
and so, by the spectral radius formula [24, p. 253], the spectral radius of M_B is at most one. It follows that for each $w \in \mathbb{D}$, the function $(1 - \overline{w}B)^{-1}$ is a multiplier of $\mathcal{K}_a$ and so
$$B_w = \frac{B - w}{1 - \overline{w}B}$$
is also a multiplier of $\mathcal{K}_a$. Now apply Theorem 4.1 to see that $B_w \in \text{FBP}$. □

Remark 6.1. In our proof, one can also show, by writing
$$B_w = (B - w)\sum_{n=0}^{\infty} \overline{w}^n B^n$$
(where the convergence above is in the multiplier norm) and using the identity $\|M_1\| = 1$, that
$$\|M_{B_w}\|_{\mathcal{K}_a \to \mathcal{K}_a} \leqslant \left((1 + \frac{\pi}{2})\sigma(B) + |w|\right) \frac{(1 + \frac{\pi}{2})\sigma(B)|w|}{(1 - |w|)^2}.$$
Can one obtain a sharper estimate for $\|M_{B_w}\|$?

We mention that unlike the class CN, where one can produce an inner function $\phi \notin \text{CN}$ with $\phi_a \in \text{CN}$ for all $a \in \mathbb{D} \setminus \{0\}$, one can not 'Frostman shift' ones way into FBP without being a FBP in the first place.

Corollary 6.2. *If ϕ is an inner function, the following are equivalent.*
 (1) *$\phi \in FBP$;*
 (2) *$\phi_a \in FBP$ for all $a \in \mathbb{D} \setminus \{0\}$;*
 (3) *$\phi_a \in FBP$ for some $a \in \mathbb{D} \setminus \{0\}$.*

Proof. Use Theorem 1.1 along with the observation that $(\phi_a)_{-a} = \phi$. □

If B is a Blaschke product with zeros $(a_n)_{n \geqslant 1}$, define $f_B : \mathbb{T} \to (0, \infty]$ by
$$f_B(\zeta) := \sum_{n=1}^{\infty} \frac{1 - |a_n|^2}{|\zeta - a_n|}.$$
Our main theorem says that the class of Blaschke products B such that f_B is a bounded function, is preserved under Frostman shifts. What about the class of

Blaschke products B for which f_B belongs to some other class of functions like, say, L^1.

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