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# ON THE PRESENCE OF n-ODS AND INFINITE-ODS

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Abstract. An $n$-od (respectively, infinite-od) is a continuum $X$ which has a subcontinuum $K$ such that $X\backslash K$ has $n$ components (respectively, infinitely many components). In 1944, Sorgenfrey proved that if a continuum $X$ is the union of three subcontinua with a point in common and such that no one of the subcontinua is contained in the union of the other two, then $X$ contains a triod. In this note a single simple proof is given for the obvious generalization of Sorgenfrey's theorem to $n$-ods and infinite-ods.

In this note a *continuum* is a compact, connected metric space. For a positive integer $n$, an $n$-*od* is a continuum $X$ which contains a subcontinuum $K$, called the *core*, such that $X\backslash K$ has $n$ components. An *infinite-od* is a continuum which contains a subcontinuum $K$ such that $X\backslash K$ has infinitely many components.

Theorem 1.8 of [1] can be restated in the following way: if $X$ is a continuum which is the union of three subcontinua with nonempty intersection, and such that no one of the subcontinua is contained in the union of the other two, then $X$ contains a triod. This theorem is only a little more difficult to prove than it appears, but it is indispensable in the study of atriodic continua. Recently, the author has been concerned with continua which do not contain an $n$-od and continua which do not contain an infinite-od. The following theorem is also very useful when working with these continua, and the proof is at least as simple as Sorgengrey's proof of the $n = 3$ case.

THEOREM. *Suppose $X$ is a continuum which is the union of a collection $\Phi$ of either a finite or countable number of continua, such that $\cap\Phi \neq \phi$, and such that each continuum in $\Phi$ contains a point not in the closure of the union of the other members of $\Phi$. Then, if $\alpha$ is the number of continua in $\Phi$, $X$ contains an $\alpha$-od.*

PROOF: Let $p$ be a point in $\cap\Phi$. Let $N$ be the union of all intersections $F_i \cap F_j$, where $F_i$ and $F_j$ are different elements of $\Phi$.

Consider two cases. In the first case, assume that there is a continuum $F_i$ in $\Phi$ such that $\overline{F_i \cap N}$ has infinitely many components. Note that an infinite number of these components must intersect $N$. Let $U_1, U_2, U_2, \ldots$ be a collection of open sets with disjoint closures such that, for each $j$, $U_j$ contains a component $C_j$ of $\overline{F_i \cap N}$ such that $C_j$ intersects $N$, and such that $U_j \subset N_{1/j}(C_j)$. For each $j$, since $C_j \cap N \neq \emptyset$, there is an integer $n_j \neq i$ such that $C_j$ contains a component of $F_i \cap F_{n_j}$. For each $j$, let $L_j$ be the closure of some component of $U_j \cap F_{n_j}$ which intersects $C_j$. For each $j$, since $F_{n_j}$ is not contained in $F_i$, $L_j$ is not contained in $F_i$. Also, for $j \neq k$, $L_j \cap L_k = \phi$. The set $I = U(L_j \cup F_i)$ is an infinite-od with core $F_i$. Since an infinite-od must contain an $\alpha$-od, where $\alpha$ is the number of continua in $\Phi$, the theorem is proved for this case.

In the second case, assume that, for each positive integer $i \leq \alpha$, $\overline{F_i \cap N}$ has a finite number of components. Let $C$ be the component of $\overline{N}$ which contains $p$. For each positive integer $i$, let $C_i$ be the component of $C \cap F_i$ which contains $p$. Note that each component of $\overline{N \cap F_i}$ which intersects $C_i$ is contained in $C_i$. For each positive integer $i$, since $\overline{N \cap F_i}$ has a finite number of components, there is an open set $U_i$ which contains $C_i$ such that each component of $\overline{N \cap F_i}$ which intersects $\overline{U_i}$ is contained in $C_i$, and such that $U_i \subset N_{1/i}(C_i)$. For each $i$, let $L_i$ be the closure of the component of $U_i \cap F_i$ which contains $C_i$. For each $i$, since $F_i$ is not contained in $\overline{N}$, $L_i$ is not contained in $C$. Also, since $C_i$ contains each component of $\overline{N \cap F_i}$ which intersects $\overline{U_i}$, we have $L_i \backslash C \subset F_i \backslash N$. So, if $i \neq j$, then $(L_i \backslash C) \cap (L_j \backslash C) = \phi$. The number of components of

$$\bigcup_{i \leq \alpha} (L_i \cup C) \backslash C$$

is at least $\alpha$. Therefore,

$$\bigcup_{i \leq \alpha} (L_i \cup C)$$

contains an $\alpha$-od with core $C$.

## References

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