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The Secretary Problem with a Call Option

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THE SECRETARY PROBLEM WITH A CALL OPTION*

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In addition to accepting or rejecting a candidate arriving at time r, we may consider purchasing an option at a cost cx to recall the candidate at time r+x, but this privilege may be invoked only once. For large sample size, using the best-choice criterion and deducting option costs, the optimal strategy and return are obtained.

Secretary problem, optimal choice, recall

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1. INTRODUCTION AND SUMMARY

The classical model of the secretary problem can be found in Gilbert and Mosteller [3, Section 2a] or De Groot [2, pp. 325-331], and its solution is reviewed briefly in the next section. A restrictive assumption of the model is the impossibility of recalling previously rejected applicants. This restriction was relaxed by Yang [7]: at any stage of the process we may attempt to recall a previously rejected applicant, whose availability is uncertain and (maybe) stochastically decreasing in time. Others have subsequently extended Yang's work; see Corbin [1] or Petruccelli [5]. A different approach to the recall issue is proposed here.

Let n be the population size and suppose that, at some stage m, we observe an applicant which is preferable to the $m-1$ previously sampled, and rejected, applicants. Then, this applicant is a <u>candidate</u> (for the best). If we select (reject) the candidate, then we risk (not) finding a better one among the $n-m$ applicants yet unsampled. In order to mitigate this risk, we may purchase an option granting us the privilege to recall the candidate at stage $m+k$, $k=1,\ldots, n-m$. The cost to acquire such an option is bk, $b\geq 0$. If the option is acquired, we say that we are <u>holding</u> the candidate for k periods; the interval $\{m+1,\ldots, m+k\}$ is called the holding period. The essential operational difference between our model and Yang's is that we must decide at the time of observation whether or not we shall later have the option to recall the candidate.

In Rose [6], the call options were renewable, so it was sufficient to consider only the one-stage holding period, $k=1$. Here, we expressly prohibit renewal so that the duration, k, of the holding period becomes a crucial decision variable. Consequently, if no better applicant is observed during the holding period, the option will be exercised and the candidate selected - clearly, in this context, it is suboptimal to purchase an option and then let it expire. Should a better applicant appear during the holding period, then we might take out another option to recall this new candidate, giving rise to the rather awkward situation in which two, or even more, applicants are being held simultaneously. This phenomenon is under investigation and will be reported on elsewhere; to avoid that complication here, assume that we are permitted to hold only one candidate. Then, a better applicant appearing during the holding period will be accepted or rejected according to the classical procedure. Our goal is to determine at what stages a candidate should be held, and for how long, in order to maximize the probability of selecting the best applicant less the holding costs.

In the next section, letting $n\to\infty$ and $b\to 0$ such that $nb\to c$, the infinite model is derived from the finite problem introduced above. The solution to the infinite model is summarized in the following theorem, whose proof is relegated to Section 3. The function $U(\cdot)$ defined on $[0,1]$ denotes the optimal return function; see (1) and (8). The dependence of U on c is suppressed.

THEOREM I. If $c \geq 1$, then the classical procedure is optimal and $U(r)=\max\{e^{-1}, r\}$.

II. For $0 \leq c < 1$, there exist $s_1(c)$ and $s_2(c)$, with $0 < s_1 \leq s_2 < 1$, such that, for a candidate arriving at time r, it is optimal to:

a. reject , $r \leq s_1$;

b. hold until time r/c, $s_1 < r \leq s_2$;

c. hold until time 1, $s_2 < r < 1$,

provided that no candidate has been held previously. Furthermore,

d. $e^{-1} < r/c < 1$, for $s_1 < r \leq s_2$;

e. there exists $c_1 \doteq .32136$ such that $c_1 e^{-1} < s_1 < e^{-1}$;

f. $s_1(c)=s_2(c)$ if and only if $c \leq c_1$;

g. s_1 and s_2 are increasing in c.

III. For $0 \leq c < 1$, there exist $t_1(c)$, $t_2(c)$, $t_3(c)$, $t_4(c)$, with $0 < t_1 \leq t_2 \leq t_3 \leq t_4 < 1$, such that

a. $U(r)=U(t_1)$, $r \leq t_1$;

b. $U(r)=cr+e^{-1}+r\log r/c$, $t_1 < r \leq t_2$;

c. $U(r)=cr+e^{-1}+r-c$, $t_2 < r \leq t_3$;

d. $U(r)=cr-r\log c$, $t_3 < r \leq t_4$;

e. $U(r)=cr-r\log r+r-c$, $t_4 < r \leq 1$;

f. $U(0) \leq e^{-1}+e^{-3/2}$.

Note that, from (II.a,e), applicants arriving prior to $c_1 e^{-1}$, or approximately 12% of the population, will be rejected automatically, even if the option cost, c, vanishes. Holding a candidate too soon, we may encounter a better applicant before time

e^{-1}, in which case nothing has been gained - the new applicant will be rejected according to the classical procedure. There would be no such lower bound on $s_1(c)$ if several applicants could be held simultaneously. Even when c=0, the maximal return is only $e^{-1}+e^{-3/2}$, the upper bound in (III.f), which is a relative improvement of about 60% over the classical return, e^{-1}. Again, if options could be taken on several applicants, then we should get U(0)=1 for c=0.

Also, consider (II.b,d,f). If c is not too small, then there are times, r, for which it is optimal to hold for a period r/c-r, which is only part of the time remaining, 1-r. Contrast this result with those of [1 ; Special Case C-Additively Decreasing Recall, p. 21], in which it is optimal to observe the entire population before attempting recall, or of [6], in which it is optimal to renew the option until the end of the process and only then to exercise it. According to (II.d,e), we should hold a candidate appearing somewhat earlier than e^{-1} until some time later than e^{-1}, which is the critical cut-off time for the classical problem (see (7)). Finally, it is implied in (II.a,b,c) that a candidate is accepted immediately upon its arrival only if the call option alternative was used to hold an earlier applicant.

Numerical solutions of the finite problem

are displayed in Table 1. Fortunately, these results are fairly well approximated by the solution of the infinite model as given in the Theorem.

2. FINITE AND INFINITE MODELS

The optimality equations follow from the customary backward induction argument of dynamic programming. Keep in mind that our (net) return is 1 or 0 minus option price. Define two events: A_m={candidate appears at stage m}, and B_m={first m-1 applicants rejected}, m=1,..., n. Let $U_n(m)$ denote the maximum expected return given A_m and B_m. Also conditioning on A_m, let $V_n(m,k)$ be the probability that the best applicant is chosen, given that the candidate is held for k periods. Finally, let $v_n(m)$ denote the maximum expected return, given B_{m+1}. Then,

$$U_n(m) = \max_{k=1,\dots,n-m} \{m/n,\ v_n(m),\ V_n(m,k)-bk\}, \tag{1}$$

where the three bracketed terms in (1) are the returns expected from accepting, rejecting, and holding (for k periods) the candidate, respectively. The second and third terms are computed recursively:

$$v_n(m) = [U_n(m+1)+mv_n(m+1)]/(m+1), \quad (v_n(n)=0) \tag{2}$$

$$V_n(m,k)=[U_n^o(m+1)+mV_n(m+1,k-1)]/(m+1), \quad (V_n(m,0)=m/n) \tag{3}$$

k=1,...,n-m, where U_n^o is the optimal return function under the classical model (see (6) below). The interpretation of (3) should be clear. With probability 1/(m+1) a better applicant appears at stage m+1 and we apply the classical procedure. Otherwise, we still hold the candidate, but for only k-1 periods beyond stage m+1. The condition $V_n(m,0)$=m/n means merely that an expired option should be exercised.

From the recursions (2) and (3) we solve for v_n and $V_n(\cdot,k)$ in terms of U_n and U_n^o respectively, obtaining

$$v_n(m) = m \sum_{i=m+1}^{n} U_n(i)/i(i-1), \tag{4}$$

$$V_n(m,k) = m \sum_{i=m+1}^{m+k} U_n^o(i)/i(i-1) + m/n. \tag{5}$$

Now, we need to review briefly the solution to the classical problem. There exists a positive integer, r_n^*, such that the first r_n^*-1 applicants are rejected and the first candidate, if any, observed thereafter is selected. The optimal return function is

$$U_n^o(m) = \begin{cases} m/n, \; m \geq r_n^* \\ (r_n^*-1)/n \sum_{i=r_n^*-1}^{n-1} 1/i \;, \; m < r_n^* \;. \end{cases} \tag{6}$$

In the limit, $r_n^*/n \to e^{-1}$ and $U_n^o([rn]^+) \to U^o(r)$ as $n\to\infty$, where $[p]^+$ denotes the smallest positive integer equal to or greater than p, and

$$U^o(r) = \begin{cases} r, \; r > e^{-1} \\ e^{-1}, \; r \leq e^{-1} \end{cases} \;, \; 0 \leq r \leq 1. \tag{7}$$

Follow the method given in Mucci [4] to obtain the infinite model. To standardize and extend the return functions on the unit interval, write $f_n(r)=U_n([rn]^+)$, $h_n(r)=v_n([rn]^+)$, and $g_n(r,x)=V_n([rn]^+, [xn]^+)$. Then (1) is rewritten as $f_n(r)= \max_{0<x\leq 1-[rn]^+/n} \{[rn]^+/n, h_n(r), g_n(r,x)-b[xn]^+\}$, and the expressions (4) and (5) can be appropriately rewritten, too. Now, take limits as $n\to\infty$. The option alternative is dominated unless $b\to 0$; to keep the

holding cost linear, suppose that $nb \to c \geq 0$ as $n \to \infty$. Thus, $f_n \to U$, $h_n \to v$, and $g_n \to V$ where

$$U(r) = \max_{0 \leq x \leq 1-r} \{r,\ v(r),\ V(r,x)-cx\}, \tag{8}$$

$$v(r) = r \int_r^1 U(y)\,dy/y^2 \,, \tag{9}$$

$$V(r,x) = r \int_r^{x+r} U^0(y)\,dy/y^2 + r, \tag{10}$$

$0 \leq r \leq 1$. From the continuity of $V(r,\cdot)$, use of "max" in (8) is permissible; closing the interval with $x=0$ is a trivial matter, since $V(r,0)=r$. The next section is devoted to solving (8)-(10).

3. PROOF OF THEOREM

Some abbreviated notation will be helpful. Let $w(r,x)=V(r,x)-cx$, let $x^*=x^*(r)$ maximize $w(r,x)$ on $[0,1-r]$, and let $W(r)=w(r,x^*)$.

LEMMA 3.1 If $U(r)=W(r)$ for $r<e^{-1}$, then $r+x^* \geq e^{-1}$.

PROOF Obviously, $U(r) \geq U^0(r)=e^{-1}$. Suppose $r+x^*<e^{-1}$. Substituting (7) into (10) yields $V(r,x^*)<V(r,e^{-1}-r)=e^{-1}$, so $W(r)<e^{-1}$. □

LEMMA 3.2 Let $c<1$ and suppose $U(r)=W(r)$. Then $x^*=1-r$ on $r>c$ and $x^*=r/c-r$ on $r \leq c$. If $c \geq 1$, then $U(r)=U^0(r)$ and (I) holds.

PROOF Using (7) and (10) and holding $r+x \geq e^{-1}$, we obtain $w_x(r,x)=r/(x+r)-c$. For $r>c$, $w_x(r,x)>0$ and $x^*=1-r$; and for $r \leq c$, the maximum is achieved at $x^*=r/c-r$. Now, suppose $c \geq 1$ and that $U(r)>U^0(r)$ for some r. Then $U(r)=W(r)$, so $r+x^* \geq e^{-1}$. However,

$w_x(r,x)\le 0$ on $r+x\ge e^{-1}$. Either $r\ge e^{-1}$ and $x^*=0$, in which case $W(r)=V(r,0)=r=U^0(r)$, or $r<e^{-1}$ and $x^*=e^{-1}-r$, in which case $W(r)<V(r,e^{-1}-r)=e^{-1}=U^0(r)$. □

Henceforth, assume $c<1$. Also, as the next lemma shows, we may eliminate the "r" term from (8) and just compare the reject and hold decisions.

LEMMA 3.3 For $r<1$, $U(r)>r$.

PROOF If $r<e^{-1}$, then $U(r)\ge U^0(r)=e^{-1}$. Suppose $r\ge e^{-1}$. Then, $U(r)\ge w(r,x)=r+r\log(1+x/r)-cx$, from (7) and (10). For x sufficiently small, $\log(1+x/r)>cx/r$, so $U(r)>r$. □

Finally, we state a lemma which will be proved several times for different values of r and c.

LEMMA 3.4 If $W(r)>v(r)$, then $W(s)>v(s)$, $s>r$.

The constant c_1 appearing in (II.e) is the solution of

$$t = \exp(-3/2(t+1)). \tag{11}$$

The role of this equation becomes apparent subsequently. It is convenient now to consider separately the three cases: $c\ge e^{-1}$, $c_1<c<e^{-1}$, and $c\le c_1$.

3.1 THE CASE $c\ge e^{-1}$. First, hold $r>c$. By Lemma 3.2, $x^*=1-r$, so $W(r)=w(r,1-r)=cr-r\log r+r-c$.

PROOF of Lemma 3.4: Let $D(r)=W(r)-v(r)$, where v is given in (9). Then, $d/dr[D(r)/r]=-1/r+c/r^2+U(r)/r^2\geq 0$, since $U(r)\geq r$. It follows that $D(r)$ is increasing. □

Now, let r be such that $W(r)>v(r)$. By Lemma 3.4, we may substitute $W(\cdot)$ for $U(\cdot)$ in (9), obtaining $v(r)=W(r)-r-cr\log r+(r/2)\log^2 r$. It is easy to show that $1/2\log^2 r-c\log r-1<0$, so $v(r)<W(r)$. Thus, we have established (II.c) and (III.e), with $s_2=t_4=c$.

Next, keep $e^{-1}<r\leq c$ and follow the approach used in the preceding paragraph. Here, $x^*=r/c-r$ and $W(r)=cr-r\log c$. The proof of Lemma 3.4 is even simpler: $d/dr[D(r)/r]=U(r)/r^2>0$. Assume $W(r)>v(r)$. On $(e^{-1},c]$ substitute $W(\cdot)$ for $U(\cdot)$, while on $(c,1]$ use $U(\cdot)$ given in (III.e), thereby obtaining from (9), $v(r)=W(r)-r/2\cdot\log^2 c+r\log c\log r-cr\log r-r$. Let $f(c)=(1+1/2\log^2 c)/(\log c-c)$. Then, $v(r)<W(r)$ provided $\log r>f(c)$. Because $r>e^{-1}$, it suffices to show $f(c)<-1$, which is readily obtained. Hence, (III.d) is proved with $t_3=e^{-1}$.

Finally, let $r\leq e^{-1}$ and repeat the same sort of argument. The optimal holding period is still $x^*=r/c-r$ and, in evaluating $V(r,x^*)$, use Lemma 3.1 to get $W(r)=cr+e^{-1}+r\log r/c$. Also by Lemma 3.1, if $r<ce^{-1}$, then $v(r)>W(r)$. To verify Lemma 3.4, we get $d/dr[D(r)/r]=-e^{-1}/r^2+1/r+U(r)/r^2>0$, since $U(r)\geq e^{-1}$. Again, assume $W(r)>v(r)$. In the computation of $v(r)$, $U(\cdot)$ is given by $W(\cdot)$ on $(r,e^{-1}]$, by (III.d) on $(e^{-1},c]$, and by (III.e) on $(c,1]$. We get $v(r)=W(r)-rg(r)$, where $g(r)$ is quadratic in $\log r$: $g(r)=1/2\log^2 r+(1+c-\log c)\log r+1/2\log^2 c+3/2$. Hence, $v(r)<W(r)$ if and only if

$g(r)>0$. Let r_1 denote the larger root of $g(r)=0$, so $r_1=r_1(c)=\exp(-(1+c)+\log c+\sqrt{d})$, where $d=d(c)=(1+c)^2-2(1+c)\log c-3$. It is fairly straightforward to show that $ce^{-1}<r_1<e^{-1}$ and that $g(r)$ is strictly increasing on $ce^{-1}\le r<r_1$. Hence, we have now verified (II.a,b,d,e,f,g) and (III.b), where $s_1=t_1=r_1$ and $t_2=e^{-1}$.

Because $t_2=t_3$, (III.c) is trivial. To verify (III.a), note that $U(r)=v(r)$ on $r\le r_1$. Then, substitute the appropriate values for $U(\cdot)$, as calculated above, into (9) and solve the resulting differential equation for $v(r)$, given $v(r_1)=cr_1+e^{-1}+r_1\log r_1/c$. Finally, note that $v(r_1)$ is decreasing in c, so $U(0)=v(r_1)\le e^{-2}+e^{-1}$, and (III.f) holds.

The remaining two cases are handled similarly. For the sake of parsimony, we omit most of the argument and concentrate on the rough spots. Once the s and t values are known, the results can be verified directly by substitution into (8)-(10), anyway.

3.2 THE CASE $c_1<c<e^{-1}$. Here, $t_4=t_3=e^{-1}$ and, on $r>e^{-1}$, $U(r)=W(r)$ is given by (III.e), as in the previous section. For $c<r\le e^{-1}$, $x^*=1-r$, $W(r)=cr+e^{-1}+r-c$, and Lemma 3.4 holds. If $U(s)=W(s)$ on $s>r$, then $v(r)=W(r)-r(3/2+(c+1)\log r)$. Thus, $v(r)<W(r)$ if and only if $r>h(c)$, where $h(c)=\exp(-3/2(c+1))$. On $(0,e^{-1})$, h is concave increasing, with $h(0)>0$ and $h(e^{-1})<e^{-1}$. Because $c>c_1$, as given by (11), it follows that $h(c)<c$. Therefore, $r>h(c)$ and (II.c) and (III.c) are established with $s_2=t_2=c$. Finally, if $r\le c$, then (II.a,b,f,g) and (III.a,b,f) are readily obtained as before, with $s_1=t_1=r_1$. Also, $ce^{-1}<r_1<c$, so (II.d,e) hold.

3.3 THE CASE $c \le c_1$. The previous results are applicable here, for $r > e^{-1}$. For $c < r \le e^{-1}$, we still get $v(r) < W(r)$ if and only if $r > h(c)$. However, since $c \le c_1$, $h(c) \ge c$. Thus (II.c) and (III.c) hold now with $s_2 = t_2 = h(c)$. Furthermore, $U(r) = v(r) > W(r)$, $c < r < h(c)$, and it remains to show that $U(r) = v(r)$ on $r \le c$. To this end, substitute (III.c,e) for $U(\cdot)$ into (9) and solve the differential equation for v, obtaining $v(r) = v(s_2) = W(s_2) = cs_2 + e^{-1} + s_2 - c$, a constant. For $r \le c$, $W(r) = cr + e^{-1} + r\log r/c < W(s_2)$, and the result is established. That also proves (II.a,d,g) and (III.a), with $s_1 = t_1 = h(c)$, and finally establishes the "if" part of (II.f). Finally, $s_1 = s_2 = h(c) > h(0) = e^{-3/2} > c_1 e^{-1}$, so (II.e) and (III.f) are verified.

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Table 1

Optimal Procedure $(s_1/n, s_2/n)$ and Maximal Return $(U_n(0))$ for $b=c/n$[1]

Cost (c)	Population Size (n) 25	100	500	∞
0.0	.24,.24,.6121	.23,.23,.5962	.224,.224,.5920	.2231,.2231,.5910
.2	.32,.32,.5322	.29,.29,.5167	.288,.288,.5127	.2865,.2865,.5117
.35	.36,.36,.4823	.34,.35,.4671	.330,.350,.4633	.3287,.3500,.4623
.5	.40,.48,.4402	.36,.50,.4266	.354,.500,.4229	.3534,.5000,.4220
.9	.40,.88,.3842	.38,.90,.3733	.370,.900,.3705	.3678,.9000,.3698

(1) For a candidate arriving at stage m, the optimal procedure prescribes rejecting if $m<s_1$, holding for a period $k<n-m$ if $s_1 \le m < s_2$, and holding for a period $n-m$ if $m \ge s_2$.