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# Advanced Portfolio Theory: Why Understanding the Math Matters

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# Advanced Portfolio Theory: Why Understanding The Math Matters

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*The goal of this paper is to motivate the use of efficient set mathematics for portfolio analysis [as seen in Roll, 1977] in the classroom. Many treatments stop at the two asset portfolio case (avoiding the use of matrix algebra) and an alarming number of treatments rely on illustration and templates to provide a heuristic sense of the material without really teaching how efficient portfolios are generated. This is problematic considering that the benefits of understanding efficient set mathematics go beyond portfolio analysis and into such topics as regression analysis (as demonstrated here).*

## INTRODUCTION

Portions of efficient set mathematics exist in many textbooks, but are often reduced to a portfolio of two assets and are somewhat incomplete. This treatment is rather unsatisfying for an undergraduate investments/portfolio theory course, particularly when the students realize that most investment portfolios have much more than two assets. There are educational products that provide the student with the experience of being a fund manager. However, the curiosity of how does one optimize a many asset portfolio is still left unquenched. The purpose of this paper is to produce efficient set mathematics beyond the realm of two assets in a clear direct manner. More specifically, instead of just lecturing about a fund manager dealing with many securities, this treatment teaches the student how to find a solution to the problem the fund manager faces.

Although one could argue that a non-rigorous intuitive approach exists through illustrations and templates, the efficient set mathematics provides further benefits. This paper demonstrates that the same set of mathematics produces regression analysis quite easily. Even if one believes that regression analysis is not for them to teach or introduce, denying students the added value of efficient set mathematics seems in one sense “wrong” and in another sense, “incomplete”. With the appropriate lesson plan and some Excel programming, efficient set mathematics can be introduced in the classroom

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without any great burden to the student or the instructor.

The paper introduces linear algebra as a means of simplifying portfolio statistics in the first section. This theme is expanded in section 2 where a matrix algorithm is developed for finding the minimum return variance portfolio weights of many asset portfolios. In section 3, the matrix algorithm is expanded even further to generate an efficient frontier and a capital allocation line for a given number of securities. In section 4, the mathematics developed in the previous sections produce regression analysis. Section 5 concludes the paper.

## A COMPELLING REASON TO USE LINEAR ALGEBRA

It is assumed that the student is already familiar with the statistical concepts of expectation ( $E(\cdot)$ , mean), variance ( $\text{Var}(\cdot)$ ), standard deviation ( $\text{SD}(\cdot)$ ), covariance ( $\text{Cov}(\cdot)$ ), and correlation [see Arnold and Nail, 2001 for a review in the context of portfolio theory] as it relates to a two asset portfolio. To find the expected return and return variance for a two asset portfolio there are two basic equations ( $W_i$  is the weight of asset "i", the weights of the assets in the portfolio sum to one, and capital letters represent security returns):

$$(1) \quad E[\text{portfolio}] = W_A * E[A] + W_B * E[B]$$

$$(2) \quad \text{Var}[\text{portfolio}] = W_A^2 * \text{Var}[A] + W_B^2 * \text{Var}[B] + 2 * W_A * W_B * \text{Cov}[A, B]$$

These equations are not particularly difficult, but let's produce the same two equations for three assets.

$$(3) \quad E[\text{portfolio}] = W_A * E[A] + W_B * E[B] + W_C * E[C]$$

$$(4) \quad \begin{aligned} \text{Var}[\text{portfolio}] = & W_A^2 * \text{Var}[A] + W_B^2 * \text{Var}[B] + W_C^2 * \text{Var}[C] \\ & + 2 * W_A * W_B * \text{Cov}[A, B] + 2 * W_A * W_C * \text{Cov}[A, C] \\ & + 2 * W_B * W_C * \text{Cov}[B, C] \end{aligned}$$

Now let's produce the same two equations for four assets.

$$(5) \quad E[\text{portfolio}] = W_A * E[A] + W_B * E[B] + W_C * E[C] + W_D * E[D]$$

$$\begin{aligned}
 \text{Var}[\text{portfolio}] &= W_A^2 * \text{Var}[A] + W_B^2 * \text{Var}[B] + W_C^2 * \text{Var}[C] \\
 &+ 2 * W_D^2 * \text{Var}[D] + 2 * W_A * W_B * \text{Cov}[A, B] \\
 &+ 2 * W_A * W_C * \text{Cov}[A, C] + 2 * W_A * W_D * \text{Cov}[A, D] \\
 (6) \quad &+ 2 * W_B * W_C * \text{Cov}[B, C] + 2 * W_B * W_D * \text{Cov}[B, D] \\
 &+ 2 * W_C * W_D * \text{Cov}[C, D]
 \end{aligned}$$

Given the length of the latter equations, it does not take much convincing to demonstrate that a better, more systematic way of generating expected portfolio returns and portfolio return variances is necessary. Memorizing a new set of formulas for each n-asset portfolio just isn't productive.

By using matrices for these calculations, a generic algorithm emerges that never changes. The matrix contents adjust to the number of securities under consideration, however, even these adjustments occur in a systematic fashion. By producing the matrix equivalent of these equations for the two asset case, extensions to the algorithm produce minimum variance portfolios, an efficient frontier, and a capital allocation line. Although not much matrix algebra is necessary, chapter 2 of Greene [2000] provides a nice review for the interested reader.

To produce the matrix equivalents of the portfolio expected return and the portfolio return variance for a two asset portfolio, two column matrices and one square matrix are necessary. Any column matrix can be transposed to become a row matrix. The transposition of a matrix is symbolized with a superscript "T" (or a prime can be used). "M" will be the column matrix containing the expected returns for the individual securities. "VCOV" will be the square matrix containing the return variance of the individual securities along its diagonal with the appropriate return covariances along the off-diagonal (i.e. VCOV is the variance-covariance matrix). Finally, "W" will be the column matrix containing the individual security portfolio weights (remember, the weights must sum to one).

$$(7) \quad M = \begin{bmatrix} E[A] \\ E[B] \end{bmatrix}; VCOV = \begin{bmatrix} \text{Var}[A] & \text{Cov}[A, B] \\ \text{Cov}[A, B] & \text{Var}[B] \end{bmatrix}; W = \begin{bmatrix} W_A \\ W_B \end{bmatrix}$$

To find the portfolio expected return, we simply pre-multiply W by M<sup>T</sup> (i.e. M<sup>T</sup>\*W):

$$\begin{aligned}
 (8) \quad E[\text{portfolio}] &= M^T * W = [E[A] \ E[B]] * \begin{bmatrix} W_A \\ W_B \end{bmatrix} \\
 &= W_A * E[A] + W_B * E[B]
 \end{aligned}$$

Technically, the result of the matrix multiplication is a 1x1 matrix, but the matrix notation is dropped when a 1x1 matrix is the result. When performing this operation in Excel, one uses the =TRANSPOSE() function to transpose a matrix and the MMULT() function to multiply matrices. However, Excel is somewhat peculiar in its execution of matrix multiplication. First, one must highlight the area for the matrix that is the result of a matrix multiplication. Next, enter the ranges of the matrices to be multiplied within the MMULT() function (do not hit the "ENTER" key!!). Finally, simultaneously hit "CTRL-SHIFT-ENTER" keys to perform the matrix multiplication. When Excel equations produce a 1x1 matrix, the final step is required, however, it is good practice to get use to it. If you are wondering, you must start over completely should you forget to finish with CTRL-SHIFT-ENTER. Thus, the Excel equation for the portfolio mean is:

$$=MMULT(TRANSPOSE(\text{range for matrix M}), \text{range for matrix W})$$

The portfolio return variance is a multiplication of three matrices:  $W^T \cdot VCOV \cdot W$ . We produce the multiplication in steps to produce a clear visualization:

$$\begin{aligned}
 \text{Var}[\text{portfolio}] &= W^T \cdot VCOV \cdot W \\
 &= \begin{bmatrix} W_A & W_B \end{bmatrix} * \begin{bmatrix} \text{Var}[A] & \text{Cov}[A, B] \\ \text{Cov}[A, B] & \text{Var}[B] \end{bmatrix} * \begin{bmatrix} W_A \\ W_B \end{bmatrix} \\
 (9) \quad &= \begin{bmatrix} W_A & W_B \end{bmatrix} * \begin{bmatrix} W_A * \text{Var}[A] + \text{Cov}[A, B] \\ W_A * \text{Cov}[A, B] + W_B * \text{Var}[B] \end{bmatrix} \\
 &= W_A^2 * \text{Var}[A] + W_A * W_B * \text{Cov}[A, B] + W_B^2 * \text{Var}[B] + W_A * W_B * \text{Cov}[A, B] \\
 &= W_A^2 * \text{Var}[A] + W_B^2 * \text{Var}[B] + 2 * W_A * W_B * \text{Cov}[A, B]
 \end{aligned}$$

The corresponding Excel equations are:

$$\begin{aligned}
 &=MMULT(TRANSPOSE(\text{range of matrix W}), \text{range of matrix VCOV}) \\
 &=MMULT(\text{matrix from the previous calculation}, \text{range of matrix W})
 \end{aligned}$$

Note: the MMULT function is limited to the multiplication of two matrices only.

Table 1 summarizes the execution of the Excel commands for a two asset portfolio with actual numbers. Replicating the steps illustrated in the table allows the user to become comfortable with the Excel commands and is highly recommended.

The matrix multiplication equivalents to the portfolio expected return and the portfolio return variance are the same no matter how many assets are in the portfolio.

**Table 1. Matrix Multiplication Techniques in Excel to Find Portfolio Mean and Variance**

<i>Panel A: Necessary Matrices Displayed in Excel Format</i>							
	A	B	C	D	E	F	G
1	Matrix "M":		Matrix "VCOV":			Matrix "W":	
2	0.07		0.1296	0.0936		0.45	
3	0.09		0.0936	0.2704		0.55	
4							
<i>Panel B: Matrix Solution for the Portfolio Mean in Excel Format</i>							
	A	B	C	D	E	F	G
5	0.081	←Relevant Cell Formula: =MMULT(TRANPOSE(A2:A3),F2:F3) {C-S-E}					
<i>Panel C: Matrix Solution for the Portfolio Variance in Excel Format</i>							
-Step 1: highlight the area for $W^T \cdot VCOV$ (a 1 x 2 matrix occupying cells A7:B7)							
-Step 2: Perform the matrix multiplication in Excel							
-Step 3: Take the resulting 1 x 2 matrix and matrix multiply it by W							
	A	B	C	D	E	F	G
6							
7	0.1098	0.1098	←Relevant Cell Formula: =MMULT(TRANPOSE(F2:F3),C2:D3) {C-S-E}				
8							
9	0.1544	← Relevant Cell Formula: =MMULT(A7:B7,F2:F3) {C-S-E}					

- Matrix "M" is the column matrix of expected future return for two securities
- Matrix "VCOV" is the variance-covariance matrix relevant to the two securities
- Matrix "W" is the column matrix of portfolio weights for the two securities
- {C-S-E} means hit the keys CTRL-SHIFT-ENTER simultaneously to enter the matrix multiplication command

Thus, the matrix approach is very systematic in this respect. As the portfolio size increases to "n" assets, the contents of the three matrices change. W and M remain column matrices but with "n" elements in each. VCOV remains a square matrix, but with "n\*n" elements. Students tend not to have difficulty with M and W as the number of assets in a portfolio rise, but they do have a difficulty in visualizing how VCOV changes. Before continuing, an intuitive means of producing VCOV as the number of securities in a portfolio increases is useful.

The first step is to view VCOV as a table of covariances in which each column and row represents a given security and where a given column and row intersect, take the covariance between the two relevant securities. This is performed in Panel A of Table 2 for a three security portfolio. In Panel B of Table 2, use the fact that the covariance

**Table 2. The Construction of VCOV in a Table of Covariances Format for Three Securities**

<i>Panel A: Covariance Between Securities Based on Row and Column Position</i>			
	Security A:	Security B:	Security C:
Security A:	Cov(A,A)	Cov(A,B)	Cov(A,C)
Security B:	Cov(A,B)	Cov.(B,B)	Cov(B,C)
Security C:	Cov(A,C)	Cov(B,C)	Cov(C,C)
<i>Panel B: Reproduce Panel A with Variances</i>			
	Security A:	Security B:	Security C:
Security A:	Var(A)	Cov(A,B)	Cov(A,C)
Security B:	Cov(A,B)	Var(B)	Cov(B,C)
Security C:	Cov(A,C)	Cov(B,C)	Var(C)

\*Cov(i,j) = Cov(j,i); The shaded area in Panel B is the VCOV matrix

of a given security with itself is equal to the variance of the given security.

The shaded area in Table 2 Panel B is the VCOV matrix. Whenever, the portfolio increases by one security, the VCOV gains another row and column.

Although a matrix algorithm for finding the portfolio expected return and the portfolio return variance for “n” securities now exists, the student still does not have the full story. Hiding in the background is the assumption that the individual security weights in the portfolio sum to one. This is an easy condition to fulfill in the two asset portfolio by setting the weight of the second security to one minus the weight of the first security. A set of portfolio weights that produce the minimum variance portfolio [see Arnold and Nail, 2001] can easily be generated. When there are more than two securities, the ability to minimize the portfolio variance while keeping the portfolio weights summed to one is a complicated task without the use of matrices. In the next section, a matrix algorithm is produced to perform this task.

## MINIMUM VARIANCE PORTFOLIOS

The matrix algorithm developed in this section is applied to a two asset portfolio case to keep the notation simple. However, the applicability of the result is for “n” asset portfolios. First, the constraint that the portfolio weights sum to one must be considered. By introducing a Lagrange Multiplier [Chiang, 1984, pp.372 – 375] the constraint will be incorporated. In very simple terms, a Lagrange Multiplier introduces a constraint by creating a zero condition. When the constraint is met, the zero

condition is equal to zero. Thus, a zero condition must be created by the constraint that  $W_A$  plus  $W_B$  must equal one. By realizing that one minus  $W_A$  minus  $W_B$  is equal to zero, the zero condition is created. The multiplier itself is symbolized (usually) by a greek letter ( $\lambda$  is used here).  $\lambda$  is multiplied by the zero condition and is added to the portfolio return variance equation. To simplify notation, this new equation is referred to as "V."

$$(10) \quad V = W_A^2 * Var[A] + W_B^2 * Var[B] + 2 * W_A * W_B * Cov[A, B] + \lambda * (1 - W_A - W_B)$$

To find the individual security weights that minimize the portfolio return variance, take the first derivative with respect to  $W_A$ ,  $W_B$ , and  $\lambda$ . Each derivative is set to zero and then solved for the portfolio weights. Three equations emerge that need to be solved simultaneously.

$$(11) \quad \frac{\partial V}{\partial W_A} = 2 * W_A * Var[A] + 2 * W_B * Cov[A, B] - \lambda = 0$$

$$\therefore W_A * Var[A] + W_B * Cov[A, B] - \frac{\lambda}{2} = 0$$

$$(12) \quad \frac{\partial V}{\partial W_B} = 2 * W_B * Var[B] + 2 * W_A * Cov[A, B] - \lambda = 0$$

$$\therefore W_A * Cov[A, B] + W_B * Var[B] - \frac{\lambda}{2} = 0$$

$$(13) \quad \frac{\partial V}{\partial \lambda} = 1 - W_A - W_B = 0$$

$$\therefore W_A + W_B = 1$$

To get to the matrix algorithm, expand equations 11 through 13 to view them in terms of  $W_A$ ,  $W_B$ , and  $(-\lambda/2)$ .

$$(14) \quad W_A * Var[A] + W_B * Cov[A, B] + \left(-\frac{\lambda}{2}\right) * 1 = 0$$

$$(15) \quad W_A * Cov[A, B] + W_B * Var[B] + \left(-\frac{\lambda}{2}\right) * 1 = 0$$

$$(16) \quad W_A * 1 + W_B * 1 + \left(-\frac{\lambda}{2}\right) * 0 = 1$$

Now, express the simultaneous equations in terms of matrices.

$$(17) \quad \begin{bmatrix} \text{Var}[A] & \text{Cov}[A,B] & 1 \\ \text{Cov}[A,B] & \text{Var}[B] & 1 \\ 1 & 1 & 0 \end{bmatrix} * \begin{bmatrix} W_A \\ W_B \\ -\lambda / 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The first matrix contains VCOV surrounded by ones with a zero in the bottom right corner. For convenience, the matrix is referred to as “MINVCOV”. The second matrix is W with a Lagrange Multiplier term on the bottom. Refer to this matrix as “MINW”. The third matrix simply has zeroes with a one on the bottom. Refer to this matrix as “L”.

The minimum return variance portfolio weights are produced upon finding the solution for MINW. Given the solution for MINW, the top two elements in MINW are the respective minimum return variance portfolio weights. To find the solution, the matrix inverse of MINVCOV (symbolized as MINVCOV<sup>-1</sup>) is necessary.

The inverse of a square matrix is another square matrix that when multiplied by the initial square matrix produces the corresponding identity matrix. Unfortunately, it is a rather rare case in which the inverse of a matrix is just simply the inverse of the elements within the initial matrix. There are “pen and paper” techniques to finding an inverse matrix but Excel’s MINVERSE() function is quite useful in this situation. First, the matrix algorithm (solution) for MINW is MINVCOV<sup>-1</sup>\*L:

$$(18) \quad \begin{bmatrix} \text{Var}[A] & \text{Cov}[A,B] & 1 \\ \text{Cov}[A,B] & \text{Var}[B] & 1 \\ 1 & 1 & 0 \end{bmatrix}^{-1} * \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} W_A \\ W_B \\ -\lambda / 2 \end{bmatrix}$$

The Excel solution for MINW is as follows:

1. Highlight the area for an “n+1” column matrix (assuming an n-asset portfolio; in our current example, n+1 is three) for matrix MINW
2. Type in =MMULT(MINVERSE(range for matrix MINVCOV), range for matrix L)
3. CTRL-SHIFT-ENTER

**Table 3: Generic Technique for the Matrix Multiplication of the Inverse of a Square Matrix by a Column Matrix in Excel**

<i>Panel A: Display of a 3 x 3 Square Matrix which is to be Inverted and Multiplied by a Column Matrix in Excel Format</i>							
	A	B	C	D	E	F	G
1	Square Matrix "A":				Column Matrix "B":		
2	10	20	30		100		
3	40	50	60		200		
4	70	80	90		300		
5							
<i>Panel B: Matrix Solution for <math>A^{-1} \cdot B</math> in Excel Format</i>							
-Step 1: highlight the area for the result of the multiplication (a 3 x 1 matrix occupying cells A6:A8)							
-Step 2: Perform the matrix multiplication in Excel							
	A	B	C	D	E	F	G
6	0	← Relevant Cell Formula :					
7	-8	=MMULT(MINVERSE(A2:C4),E2:E4) {C-S-E}					
8	4						
- {C-S-E} means hit the keys CTRL-SHIFT-ENTER simultaneously to enter the matrix multiplication command							

A generic Excel technique for multiplying the inverse of a square matrix by a column matrix is provided in Table 3 with actual numbers. Again, the user is encouraged to replicate the technique in Table 3 to become comfortable with the Excel matrix commands.

This solution for MINW ( $=\text{MINVCOV}^{-1} \cdot L$ ) applies for all "n" asset portfolios. Again, the only changes are the elements in the respective matrices. The forms of the matrices are not difficult to remember. L has "n" zero elements with a one on the bottom. MINW has "n" minimum return variance portfolio weight elements with a Lagrange Multiplier term on the bottom. MINVCOV has "(n+1)\*(n+1)" elements which are the VCOV matrix surrounded by ones and a zero in the lower right corner element.

When shown this solution technique, students pick up on the algorithm rather quickly. To review the techniques from section one, have the students compute the portfolio expected return and the portfolio return variance using the minimum return variance portfolio weights. Admittedly, students will tend to ignore the Lagrange Multiplier optimization and just form the component matrices to perform the matrix algorithm solution. However, the underlying math is still there for their benefit.

Now address the students with a logical problem: Suppose I want an expected return greater than the expected return for the minimum return variance portfolio. How do I get this expected return with the least amount of portfolio return variance? Essentially, a solution for a given point on the efficient frontier generated by "n" assets is being requested. The matrix algorithm solution is provided in the next section. Before leaving this section, it is also informative to note that the efficient frontier begins at the minimum variance portfolio.

## THE EFFICIENT FRONTIER AND THE CAPITAL ALLOCATION LINE

First, address the initial problem of finding the minimum return variance portfolio weights given a desired expected return of "k". Again, the matrix algorithm is performed for the two asset case as a matter of convenience. To address the problem, a new constraint must be recognized (i.e.  $k = W_A * E[A] + W_B * E[B]$ ). The constraint is altered into a zero condition ( $0 = k - W_A * E[A] - W_B * E[B]$ ) producing a new Lagrange Multiplier, "δ", which is multiplied by the zero condition and added to the "V" equation from the previous section. Refer to this new equation as "Q".

$$\begin{aligned}
 (19) \quad Q = & W_A^2 * Var[A] + W_B^2 * Var[B] \\
 & + 2 * W_A * W_B * Cov[A, B] \\
 & + \lambda * (1 - W_A - W_B) \\
 & + \delta * (k - W_A * E[A] - W_B * E[B])
 \end{aligned}$$

As in the previous section, take the derivative of Q with respect to  $W_A$ ,  $W_B$ ,  $\lambda$ , and  $\delta$ . Set the derivatives to zero and solve the four equations simultaneously.

$$(20) \quad \frac{\partial Q}{\partial W_A} = 2 * W_A * Var[A] + 2 * W_B * Cov[A, B] - \lambda - \delta * E[A] = 0$$

$$\therefore W_A * Var[A] + W_B * Cov[A, B] - \frac{\lambda}{2} - E[A] * \frac{\delta}{2} = 0$$

$$(21) \quad \frac{\partial Q}{\partial W_B} = 2 * W_B * Var[B] + 2 * W_A * Cov[A, B] - \lambda - \delta * E[B] = 0$$

$$\therefore W_B * Var[B] + W_A * Cov[A, B] - \frac{\lambda}{2} - E[B] * \frac{\delta}{2} = 0$$

$$(22) \quad \frac{\partial Q}{\partial \lambda} = 1 - W_A - W_B = 0$$

$$\therefore W_A + W_B = 1$$

$$(23) \quad \frac{\partial Q}{\partial \delta} = k - W_A * E[A] - W_B * E[B] = 0$$

$$\therefore W_A * E[A] + W_B * E[B] = k$$

The equations are re-express relative to  $W_A$ ,  $W_B$ ,  $(-\lambda/2)$ , and  $(-\delta/2)$ .

$$(24) \quad W_A * Var[A] + W_B * Cov[A, B] + \left(-\frac{\lambda}{2}\right) * 1 + \left(-\frac{\delta}{2}\right) * E[A] = 0$$

$$(25) \quad W_A * Cov[A, B] + W_B * Var[B] + \left(-\frac{\lambda}{2}\right) * 1 + \left(-\frac{\delta}{2}\right) * E[B] = 0$$

$$(26) \quad W_A * 1 + W_B * 1 + \left(-\frac{\lambda}{2}\right) * 0 + \left(\frac{\delta}{2}\right) * 0 = 1$$

$$(27) \quad W_A * E[A] + W_B * E[B] + \left(-\frac{\lambda}{2}\right) * 0 + \left(-\frac{\delta}{2}\right) * 0 = k$$

As in the last section, reproduce the set of simultaneous equations into matrices.

$$(28) \quad \begin{bmatrix} Var[A] & Cov[A, B] & 1 & E[A] \\ Cov[A, B] & Var[B] & 1 & E[B] \\ 1 & 1 & 0 & 0 \\ E[A] & E[B] & 0 & 0 \end{bmatrix} * \begin{bmatrix} W_A \\ W_B \\ -\lambda/2 \\ -\delta/2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ k \end{bmatrix}$$

These three matrices are very similar to MINVCOV, MINW, and L. This comes as no surprise since equation "Q" is equation "V" with an additional constraint. The first matrix contains the MINVCOV matrix bordered with the expected returns of the individual portfolio securities and zeroes. Refer to this matrix as "EFFVCOV". The second matrix is MINW with an additional Lagrange Multiplier term. Refer to this matrix as "EFFW". The third matrix is L with the addition of the desired expected return, "k". Refer to this matrix as "K".

Upon solving for EFFW (=EFFVCOV<sup>-1</sup>\*K), the first "n" elements are the portfolio weights (for an n-asset portfolio) that yield the desired expected return with the least

amount of portfolio return variance.

$$(29) \quad \begin{bmatrix} \text{Var}[A] & \text{Cov}[A,B] & 1 & E[A] \\ \text{Cov}[A,B] & \text{Var}[B] & 1 & E[B] \\ 1 & 1 & 0 & 0 \\ E[A] & E[B] & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \\ k \end{bmatrix} = \begin{bmatrix} W_A \\ W_B \\ -\lambda/2 \\ -\delta/2 \end{bmatrix}$$

The Excel procedure to find EFFW is similar to that for MINW in the previous section (and illustrated in a generic fashion in Table 3):

1. Highlight the area for an “n+2” column matrix (assuming an n-asset portfolio; in our current example, n+2 is four) for matrix EFFW
2. Type in =MMULT(MINVERSE(range for matrix EFFVCOV), range for matrix K)
3. CTRL-SHIFT-ENTER

To map out the efficient frontier for “n” assets, simply graph the minimized portfolio return standard deviation (square root of the minimized portfolio return variance) on the x-axis versus “k” (the desired expected return) on the y-axis. One cannot use an Excel data table in this instance by varying the value of “k” because “k” is a part of matrix K. Unfortunately, no portion of a matrix can be altered in Excel to create a data table. Fortunately, one can enter different values for “k” and record the corresponding minimized portfolio return standard deviation to produce the data pairs for such a graph.

Although the mapping of the efficient frontier is certainly appealing, the procedure to find EFFW is not much different than finding MINW in the previous section. This creates a nice continuity between the two sections and may even make EFFW seem somewhat of an “afterthought”. To prevent an under-appreciation of EFFW, we introduce a risk free asset into the portfolio. Let’s assume it has a return, “ $R_F$ ”, and that it is the second security (“B”) in our two asset portfolio example. By definition, the risk free security will have a zero variance and its covariance with any other security is also zero. Thus, the solution for EFFW changes in a distinct manner assuming “k” is greater than the risk free rate.

$$(30) \quad \begin{bmatrix} \text{Var}[A] & 0 & 1 & E[A] \\ 0 & 0 & 1 & R_F \\ 1 & 1 & 0 & 0 \\ E[A] & R_F & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \\ k \end{bmatrix} = \begin{bmatrix} W_A \\ W_{R_F} \\ -\lambda/2 \\ -\delta/2 \end{bmatrix}$$

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Graphing the expected return versus the portfolio return variance under these circumstances generates a straight line. In fact, for an “n” asset portfolio, the entire portfolio will be composed of really two sets of funds. One fund is the risk free security and the other fund is a “tangent portfolio” of the efficient frontier composed of all of the risky assets. To find the weights of the tangent portfolio: for a given value of “k”, one simply divides each risky asset portfolio weight by (1 – portfolio weight in the risk free security). The tangent portfolio compositions will not change with “k”. This is an example of two fund separation [corollary 5 in the appendix of Roll, 1977].

The straight-line graph itself is a Capital Allocation Line based on the “n” securities. One can very easily segue into the development of the Capital Asset Pricing Model [Sharpe, 1964] or even towards regression analysis as seen in the next section. Thus, the matrix algorithm for EFFW provides more than just an extension of section 2.

## A SEGUE TOWARDS REGRESSION

The matrix mathematics demonstrated in the previous two sections are remarkably similar to what one tries to accomplish in regression analysis. In regression analysis, one is essentially trying to describe the behavior of one variable (the dependent variable, usually symbolized by “Y”) using other variables (independent variables, usually symbolized as “X<sub>i</sub>”). The goal is to select coefficients, “β<sub>i</sub>”, for the independent variables such that the dependent variable less the independent variables with associated coefficients has: a) a mean of zero and b) has as little variance (i.e. squared error) as possible. This process of solving for coefficients can be viewed in a portfolio context.

Visualize a portfolio that is composed of the dependent variable less all of the independent variables. The weight assigned to the dependent variable is 100% and the weights of the independent variables are the associated coefficients (β<sub>i</sub>s). These latter weights must be calculated. Essentially, the goal is to find the minimum variance portfolio for this specific case. However, the minimization is to be calculated with the mean of this portfolio equal to zero. Thus, a Lagrange condition exists that is similar to the condition introduced in the last section by setting “k” to E[Y].

In other words, with the appropriate selection of coefficients (weights) on the independent variables, a model can be found that describes the behavior of the dependent variable. This model, when subtracted from the dependent variable, will have zero error on average (i.e. the mean of our “visualized” portfolio is zero) and minimal summed squared error (i.e. the variance of our “visualized” portfolio is minimized).

For example, assume the dependent variable, “Y”, is to be characterized by the independent variables “X<sub>1</sub>” and “X<sub>2</sub>”. The equation (call it “R”) for the variance of the (three asset) portfolio is:

$$\begin{aligned}
 R = & (100\%)^2 * Var[Y] + \beta_1^2 * Var[X_1] + \beta_2^2 * Var[X_2] \\
 & - 2 * (100\%) * \beta_1 * Cov[Y, X_1] \\
 & - 2 * (100\%) * \beta_2 * Cov[Y, X_2] \\
 & + 2 * \beta_1 * \beta_2 * Cov[X_1, X_2]
 \end{aligned}
 \tag{31}$$

In addition, the mean of the portfolio is zero, i.e. the model is to match the dependent variable on average. A zero condition of  $(E(Y) - \beta_1 * E(X_1) - \beta_2 * E(X_2))$  can be used, but we may lose some flexibility in allowing the variances of the two independent variables to explain the variance of the dependent variable (however, there are cases in which this is desired). Instead, introduce a new variable “ $\alpha$ ” which behaves as a constant and simply equals the average difference between  $E(Y)$  and  $(\beta_1 * E(X_1) + \beta_2 * E(X_2))$ . This constant term has no variance (similar to the risk-free security in the previous section) and is considered the “intercept” term in the regression. Consequently, the dependent variable, “Y”, is explained by an intercept term “ $\alpha$ ” and two independent variables “ $X_1$ ” and “ $X_2$ ”. The zero condition desired is:  $(E(Y) - \alpha - \beta_1 * E(X_1) - \beta_2 * E(X_2))$ . The Lagrange Multiplier is symbolized with “ $\pi$ ” and allows the creation of equation, “RG”, which contains the previous equation “R”.

$$\begin{aligned}
 RG = & (100\%)^2 * Var[Y] + \beta_1^2 * Var[X_1] + \beta_2^2 * Var[X_2] \\
 & - 2 * (100\%) * \beta_1 * Cov[Y, X_1] - 2 * (100\%) * \beta_2 * Cov[Y, X_2] \\
 & + 2 * \beta_1 * \beta_2 * Cov[X_1, X_2] \\
 & + \pi * (E[Y] - \alpha - \beta_1 * E[X_1] - \beta_2 * E[X_2])
 \end{aligned}
 \tag{32}$$

To solve for  $\alpha$ ,  $\beta_1$ , and  $\beta_2$ , take the derivative of RG relative to  $\beta_1$ ,  $\beta_2$ ,  $\alpha$ , and  $\pi$ . Set each derivative to zero and solve the four equations simultaneously.

$$\begin{aligned}
 \frac{\partial RG}{\partial \beta_1} = & 2 * \beta_1 * Var[X_1] + 2 * \beta_2 * Cov[X_1, X_2] \\
 & - 2 * Cov[Y, X_2] - \pi * E[X_1] = 0
 \end{aligned}
 \tag{33}$$

$$\therefore \beta_1 * Var[X_1] + \beta_2 * Cov[X_1, X_2] - E[X_1] * \frac{\pi}{2} = Cov[Y, X_1]$$

$$(34) \quad \frac{\partial RG}{\partial \beta_2} = 2 * \beta_2 * Var[X_2] + 2 * \beta_1 * Cov[X_1, X_2] - 2 * Cov[Y, X_2] - \pi * E[X_2] = 0$$

$$\therefore \beta_2 * Var[X_2] + \beta_1 * Cov[X_1, X_2] - E[X_2] \frac{\pi}{2} = Cov[Y, X_2]$$

$$(35) \quad \frac{\partial RG}{\partial \alpha} = -\pi = 0$$

$$\therefore \frac{\pi}{2} = 0$$

$$(36) \quad \frac{\partial RG}{\partial \pi} = E[Y] - \alpha + \beta_1 * E[X_1] - \beta_2 * E[X_2] = 0$$

$$\therefore \beta_1 * E[X_1] + \beta_2 * E[X_2] + \alpha = E[Y]$$

Re-express each equation relative to  $\beta_1$ ,  $\beta_2$ ,  $\alpha$ , and  $(-\pi/2)$ .

$$(37) \quad \beta_1 * Var[X_1] + \beta_2 * Cov[X_1, X_2] + \alpha * 0 + (-\pi/2) * E[X_1] = Cov[X_1, Y]$$

$$(38) \quad \beta_1 * Cov[X_1, X_2] + \beta_2 * Var[X_2] + \alpha * 0 + (-\pi/2) * E[X_2] = Cov[X_2, Y]$$

$$(39) \quad \beta_1 * 0 + \beta_2 * 0 + \alpha * 0 + (-\pi/2) * 1 = 0$$

$$(40) \quad \beta_1 * E[X_1] + \beta_2 * E[X_2] + \alpha * 1 + (-\pi/2) * 0 = E[Y]$$

As before, the equations can be expressed in a matrix format.

$$(41) \quad \begin{bmatrix} Var[X_1] & Cov[X_1, X_2] & 0 & E[X_1] \\ Cov[X_1, X_2] & Var[X_2] & 0 & E[X_2] \\ 0 & 0 & 0 & 1 \\ E[X_1] & E[X_2] & 1 & 0 \end{bmatrix} * \begin{bmatrix} \beta_1 \\ \beta_2 \\ \alpha \\ -\pi/2 \end{bmatrix} = \begin{bmatrix} Cov[Y, X_1] \\ Cov[Y, X_2] \\ 0 \\ E[Y] \end{bmatrix}$$

In the current framework, the portfolio solution created is termed an Ordinary Least Squares (OLS) regression. The “least squares” portion of the expression refers to the minimization of the portfolio variance. However, in regression analysis, there is no such thing as a portfolio variance. Rather, the term “squared standard errors” is used and is similar, but not the same as the VCOV portion of the above matrices. Further, any variance remaining in the portfolio (after minimization) is considered the “residual squared error” for the regression. As performed in the previous sections, a solution for  $\beta_1$ ,  $\beta_2$ ,  $\alpha$ , and  $(-\pi/2)$  can be found by taking the inverse of the first matrix and multiplying it by the third matrix.

However, solving directly for the coefficients is rather instructive. The solutions for  $\alpha$ ,  $\beta_1$ , and  $\beta_2$  are shown below.

$$(42) \quad \alpha = E[Y] - \beta_1 * E[X_1] - \beta_2 * E[X_2]$$

$$(43) \quad \beta_1 = \frac{\{Cov[Y, X_1] * Var[X_2] - Cov[Y, X_2] * Cov[X_1, X_2]\}}{\{Var[X_1] * Var[X_2] - Cov[X_1, X_2] * Cov[X_1, X_2]\}}$$

$$(44) \quad \beta_2 = \frac{\{Cov[Y, X_2] * Var[X_1] - Cov[Y, X_1] * Cov[X_1, X_2]\}}{\{Var[X_1] * Var[X_2] - Cov[X_1, X_2] * Cov[X_1, X_2]\}}$$

The denominator for the  $\beta$  coefficients is the “determinant” of the VCOV matrix, which is a calculation that is instrumental for finding the inverse of the matrix. Since our matrix algorithms depend on the ability to find inverse matrices, this is certainly a point of concern.

If the two independent variables are exactly correlated positively or negatively (i.e. a correlation coefficient of one or negative one), the denominators of the  $\beta$  coefficients become zero and the coefficients are consequently unsolvable. Regarding the matrix algorithm, the matrix containing the VCOV matrix does not have a corresponding inverse matrix causing the matrix algorithm to fail as well.

This is a case of “multicollinearity” within the independent variables. An in-depth analysis of multicollinearity is not necessary (nor the point of this paper), however, it is worth mentioning that if two independent variables have a highly positive (close to one) or highly negative (close to negative one) correlation, calculation of the coefficients is still problematic. In essence, the denominators of the  $\beta$  coefficients approach zero causing the coefficients to become rather large. Fortunately, in the calculation of optimal portfolio weights, multicollinearity is not an issue.

If the independent variables are uncorrelated (i.e. their covariance is zero), the solution for the  $\beta$  coefficients is greatly simplified.

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$$(45) \quad \beta_1 = \frac{\text{Cov}[Y, X_1]}{\text{Var}[X_1]}$$

$$(46) \quad \beta_2 = \frac{\text{Cov}[Y, X_2]}{\text{Var}[X_2]}$$

One can continue along this vein to demonstrate the testing of individual hypotheses. However, the demonstration of how regression analysis is similar to portfolio analysis is complete. Through the brief discussion of multicollinearity, it is further demonstrated how regression analysis involves different issues relative to portfolio analysis. Perhaps the key difference between the two analyses is that portfolio weights need to sum to one and regression coefficients do not need to sum to one. Consequently, the Lagrange condition from section 2 was not needed to perform the regression analysis. For further connections between the two analyses, the interested reader is referred to the work of Britten-Jones [1999].

## CONCLUSION

Although much of the results in the first three sections are expansions from the appendix of Roll [1977] and are consequently attributed to Roll, Roll provides a number of citations to others for efficient set mathematics. The purpose of this paper is to take efficient set mathematics beyond two asset portfolios and bring it into the classroom. The linear algebra introduced is minimal and does not require extensive preparation. In fact, the matrix algorithms introduced provide the impetus to take advantage of the tools of linear algebra.

Although many of the examples are produced with two asset portfolios, the “matrix algorithms” apply to many asset portfolios. Particularly with the aid of a spreadsheet program, multiple asset portfolios are not beyond the comprehension of undergraduate students. This provides a more satisfying course and allows the student to realize that the job of a portfolio manager is not beyond his/her capabilities.

To make a complete treatment, it is recommended that the algorithms produced in this paper be applied to an actual time series of data. An extension of this concept is to produce portfolios based on data for the same set of securities over two different time periods. The two sets of portfolio weights can then be compared. Further, one can have the students set portfolio weights for a given expected return (with minimized return variance) and then measure the portfolio tracking error through time.

The regression analysis segue is not integral to the portfolio management portion of this article. However, regression analysis fits in very nicely with the mathematics

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developed in the earlier sections and demonstrates the additional benefits of teaching efficient set mathematics instead of only a heuristic non-rigorous discussion of the fund manager's problem in generating a portfolio. Allowing the students to work with the efficient set mathematics provides the potential for many other applications and the confidence of being able to use what is believed to be very complicated mathematics. Regression analysis is just one example.

Essentially, this paper provides a set of matrix algorithms that enable the student to manage a multiple asset portfolio and provides the student with the necessary tools for much more. Depending on the lecturer's agenda, many topics and assignments can extend easily from the basic concepts provided by this paper.

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