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Principles of control theory as applied to a thermostat

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PRINCIPLES OF CONTROL THEORY AS APPLIED TO A THERMOSTAT

BY

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INTRODUCTION

The recent development of solid state electronics has opened new possibilities in instrumentation design. One development has been the availability of compact, low cost, high gain, D.C. amplifiers which can be used in the design and construction of sensitive laboratory instruments (11). Using these amplifiers a thermostat capable of precise control can be constructed and its performance compared with the performance expected by an analysis of the closed loop control equations.

HISTORICAL

Some of the earliest examples of process control (4) are found in the biological developments which allow plants to point toward the sun and animals to regulate respiration and heart beat. One of the first man-made control systems was the flyball governor which Watts invented for his steam engine in 1788 (3).

The understanding of control systems began with the theorems of Laplace and Fourier, who in the early 19th century expressed the oscillation and damping of physical systems as differential equations. In the early 20th century these differential equations were applied to the development of control theory. Some of the principal contributors were Routh in stability analysis, Kirckhoff in the analysis of electrical circuits, and Kelvin and Heaviside in the continued development of techniques for the solution of differential equations (3).

The major advances in automatic control practice occurred during World War II. The design of systems such as the servomechanisms required for aircraft controls, radar control of gun fire, and the remote control systems required in the manipulation of radioactive materials, necessitated exact performance data. Designers had to know the instant by instant behavior of the controlled system. Transient response tests were developed to test the system recovery after impulse and step changes in set point or load, while frequency response tests, originally used in radio and telephone work, were used to determine system stability to cyclic distrubances at various control amplifications (3).

Since World War II the application of control theory and process dynamics to chemical processes has become increasingly important. In some cases (i.e. petroleum refineries) without automatic control systems, the process would be impossible to operate (2). As these changes have taken place, it has become necessary for the practicing chemist or engineer to have at least a rudimentary knowledge of the principles of control theory.

The control of temperature is a common problem which is part of most experimental work. Many older controllers were of the simple off-on type. They could be constructed easily but inherently produced an oscillatory control. A typical controller of this type, as reported by J. M. Walsh (16), consisted of an imersion heater with a cartridge thermoregulator to produce the off-on control. With this device precision of $\pm 1°F.$ was reported for baths between 1.5 and 14 liters volume.

The next degree of complexity is proportional control which produces a correcting response proportional to the deviation from the controller set point. With proportional control the oscillation of off-on controllers can be eliminated. An inexpensive proportional control thermostat was devised by R. A. Anderson (1) using either a mercury thermometer or a thermister bridge sensor. The error signal was amplified in a transistorized circuit and the signal used to trigger a silicon controlled rectifier (SCR) into conduction. A larger bridge unbalance causes a SCR to conduct earlier in the AC wave cycle (Figure 1) and thus to conduct a greater average current. Control of ±0.02°C. was reported.

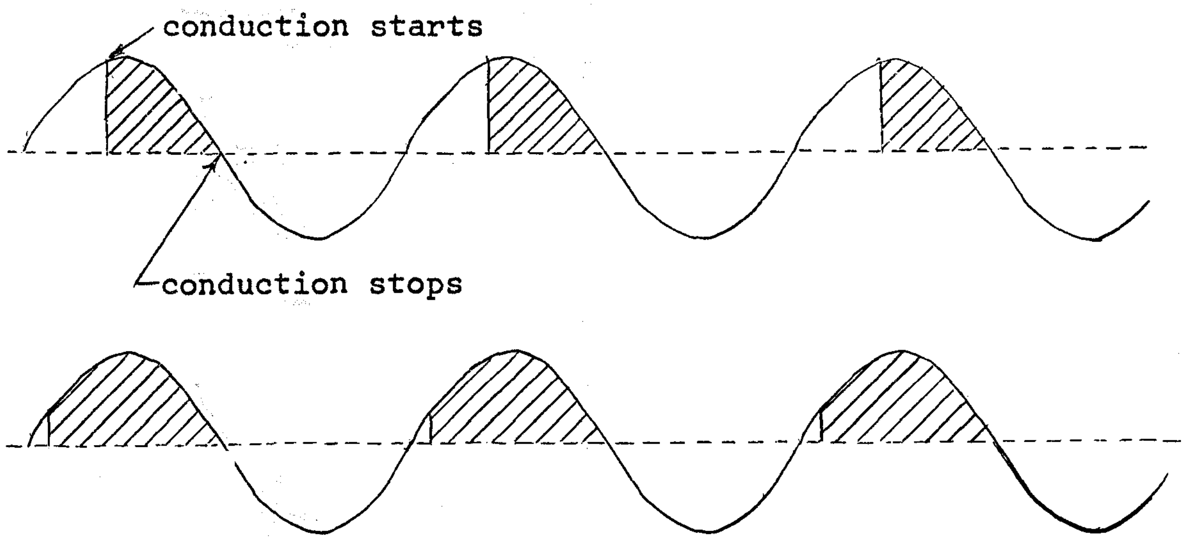


Figure 1. Conduction Cycle of an SCR

C. A. Miller (10) designed a temperature controller using proportional, integral and derivative modes for the operation of a furnace at about 1400°C. [The integral mode generates a control action proportional to the integral of the deviation from the set point, while the derivative mode generates a control action proportional to the rate at which a deviation from the set point is occurring.] A P_t -Rh thermocouple was compared with a reference voltage generated by zener diodes. The error signal was amplified with a D.C. operational amplifier to provide the signal for a derivative and integral control amplifier. A saturable core reactor provided heat to the furnace. (Figure 2)

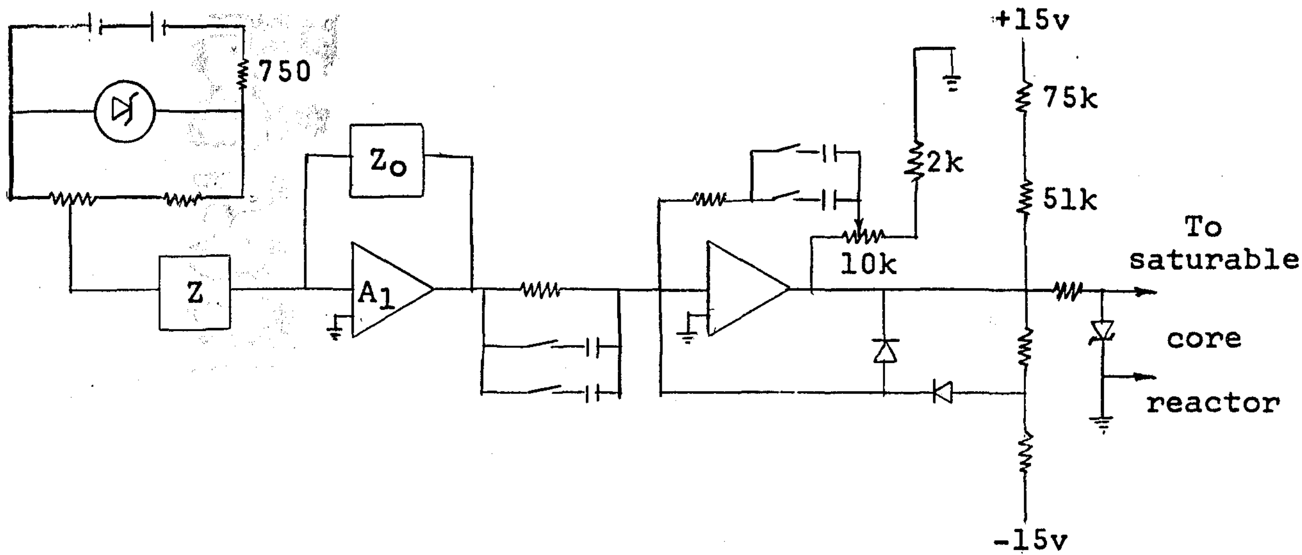


Figure 2. Furnace Temperature Controller

With this device control within ±0.5°C. was reported.

THEORETICAL

In general a control operation consists of the functions of sensing, comparison and correction. A sensor and transducer convert the property which is to be controlled into a control signal. The most common control signals are air pressure, electric voltage and electric current. The control signal is compared with the desired value and a difference or error is determined. This error signal is fed into a controller which determines the amount of corrective action to be taken and sends a signal to the control element, often a valve, so that the system property is changed towards the desired value.

There are several modes of control action. The two types used in this work are proportional control and proportional plus integral control. With proportional control the amount of corrective action taken is proportional to the deviation from the set point. With a proportional control system there must be a deviation from the set point (offset) to produce a change in the final control element. A proportional control system is usually adjusted or set so that at one set of operating conditions the deviation from the set point is zero. For any other combination of operating conditions, there will be some offset between the desired value and the actual value of the controlled variable.

The offset in a proportional controller could be eliminated by resetting the controller for the new conditions. Resetting the controller to eliminate the offset can be done automatically by adding an integrating action to the controller [Integral control is sometimes called reset action]. With integral control the controller output is made proportional to the integral of the deviation from the set point, and the deviation is brought to zero.

The analysis of the dynamic aspects of a control system involves the solution of differential equations. By using Laplace transforms the solution of these equations can proceed in a systematic manner through algebraic manipulation of the terms in the equation. After solving for the dependent variable using the Laplace transform variable s, the transforms are inverted or returned to terms involving the original variable. The operation is analogous to the use of logarithms to replace multiplication and division of numbers by addition and subtraction of their logarithms and then taking the inverse of the logarithm to obtain the numerical result.

One important limitation on the use of the transforms is that their use is restricted to linear equations. This means that

$$L[af_1(t) + bf_2(t)] = aL(f_1(t)) + bLf_2(t)$$

where L is Laplace transform operator, a and b are constants and $f_1(t)$ and $f_2(t)$ are two functions of t. Many real systems are nonlinear (i.e. the dependent variable is of degree other than 0 or 1 in one or more terms in the equation), and in making linear approximations the range over which the mathematical model accurately describes the real system becomes restricted.

In working with the control equations, it is convenient to consider deviations from the set point rather than absolute values. Deviation variables will be used throughout this paper.

As an illustrative example, the heat balance of a vessel will be considered. Liquid of specific heat c_p and temperature T_{in} enters a well mixed vessel, and the liquid leaves the vessel with temperature T_v. (Figure 3) A relationship between the temperature of the vessel contents and the temperature of the entering liquid will be developed.

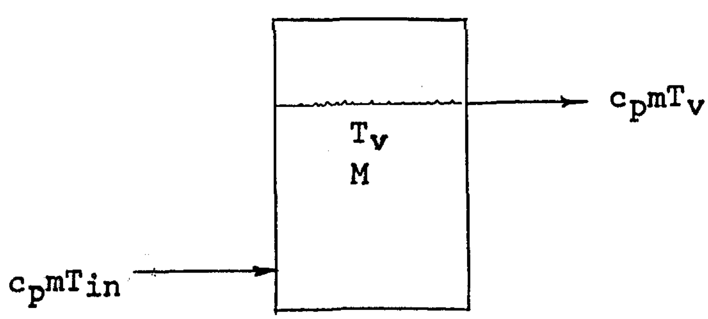


Figure 3. Heat Balance of a Vessel

$$\begin{bmatrix}\text{Rate of} \\ \text{heat entering}\end{bmatrix} - \begin{bmatrix}\text{Rate of} \\ \text{heat leaving}\end{bmatrix} = \begin{bmatrix}\text{Rate of} \\ \text{heat accumulation}\end{bmatrix}$$

$$c_p m T_{in} - c_p m T_v = c_p M \frac{dT_v}{dt} \tag{1}$$

At steady state $T_{in} = T_v = T_{steady\ state}$ and

$$c_p m T_{s.s.} - c_p m T_{s.s.} = c_p M \frac{dT_{s.s.}}{dt} = 0 \tag{2}$$

By subtracting equation (2) from equation (1) the system can be represented in deviations from the steady state.

$$c_p m(T_{in}-T_s) - c_p m(T_v-T_s) = c_p M \frac{dT_v}{dt} \tag{3}$$

Let $(T_{in} - T_s) = T'_{in}$
$(T_v - T_s) = T'_v$

$$c_p m T'_{in} - c_p m T'_v = c_p M \frac{dT'_v}{dt} \tag{4}$$

The Laplace transform of this equation is

$$c_p m T_{in}(s) - c_p m T_v(s) = c_p M s T_v(s) \tag{5}$$

where

$T_{in}(s)$ = Laplace transform of T_{in}
$T_v(s)$ = Laplace transform of T_v

Rearranging terms equation (5) becomes

$$(c_p m + c_p M s) T_v(s) = c_p m T_{in}(s) \tag{6}$$

$$T_v(s) = \frac{c_p m}{c_p M S + c_p m} T_{in}(s) \tag{7}$$

Equation (7) relates the Laplace transform of the temperature in the vessel to the Laplace transform of the entering temperature. The time variation of the vessel temperature can be obtained by inversion of the Laplace transforms back to time variables for a particular change in the entering temperature. For example, the Laplace transform for a unit step change is $\frac{1}{s}$. A unit step change in T_{in} gives

$$T_v(s) = \frac{\frac{c_p m}{c_p M}}{(s + \frac{c_p m}{c_p M})} \cdot \frac{1}{s} \tag{8}$$

The inverse transform of $\frac{a}{s+a} \cdot \frac{1}{s}$ is not found in the tables; however, the inverse transforms of the individual factors are given. The separation of quotients of polynomials into a series of partial fractions is one procedure used to find the inverse transforms. In equation (8), let $c_p m/c_p M$ equal a.

$$\frac{a}{s+a} \cdot \frac{1}{s} = \frac{A}{s+a} + \frac{B}{s} \tag{9}$$

In order to determine the value of A both sides of equation (9) are multiplied by s+a.

$$\frac{a}{s} = A + \frac{B}{s}(s+a) \tag{10}$$

when $s = -a$

$$\frac{a}{-a} = A$$

$$A = -1 \tag{11}$$

Similarly, to determine the value of B, both sides of equation (9) are multiplied by s.

$$\frac{a}{s+a} = \frac{As}{s+a} + B \tag{12}$$

At $s = 0$

$$\frac{a}{a} = B$$

$$B = 1 \tag{13}$$

$$T_v(s) = \frac{a}{s+a} \cdot \frac{1}{s} = -\frac{1}{s+a} + \frac{1}{s} \qquad (14)$$

The inverse transform of $\frac{1}{s+a}$ is e^{-at}, and the inverse transform of $\frac{1}{s}$ is 1.

$$T_v(t) = 1 - e^{-at}$$

$$T_v(t) = 1 - e^{-\frac{c_p m}{c_p M}t} \qquad (15)$$

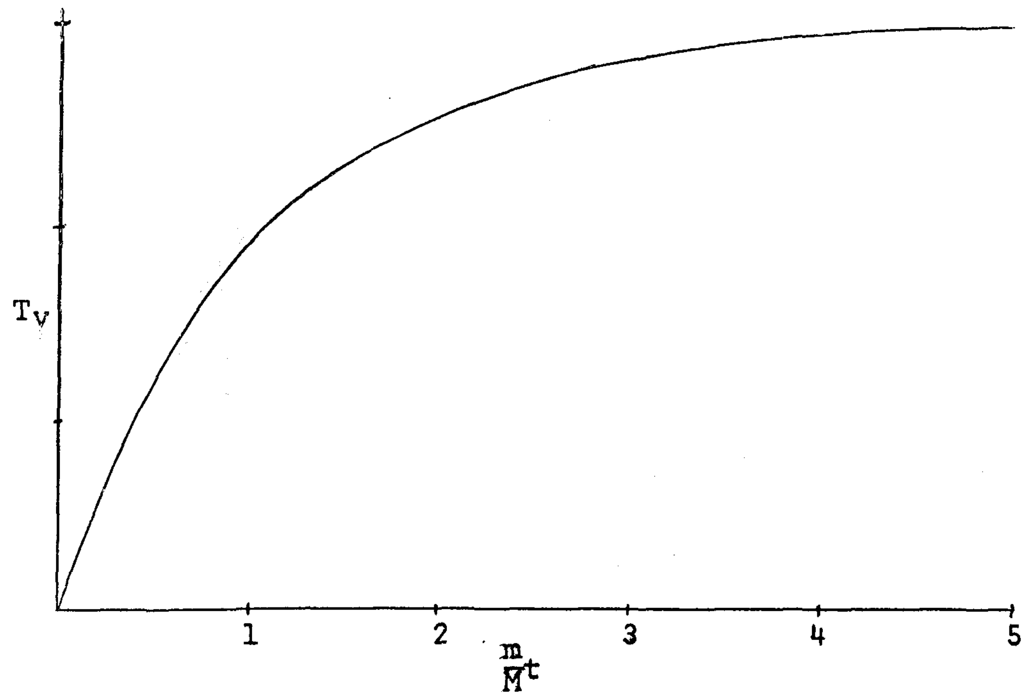


Figure 4. Response of the Vessel Temperature to a Step Change in Inlet Temperature

Equation (15) is shown graphically in Figure 4.

Block Diagram

In combining the process and control components into an integrated system, the block diagram aids in visualizing the system relationships and organizing the calculations. In the development of a block diagram of the system, the individual components are represented by a block which acts on an input to produce an output. The transfer function inputs and outputs are interconnected as they occur in the process to form a control loop.

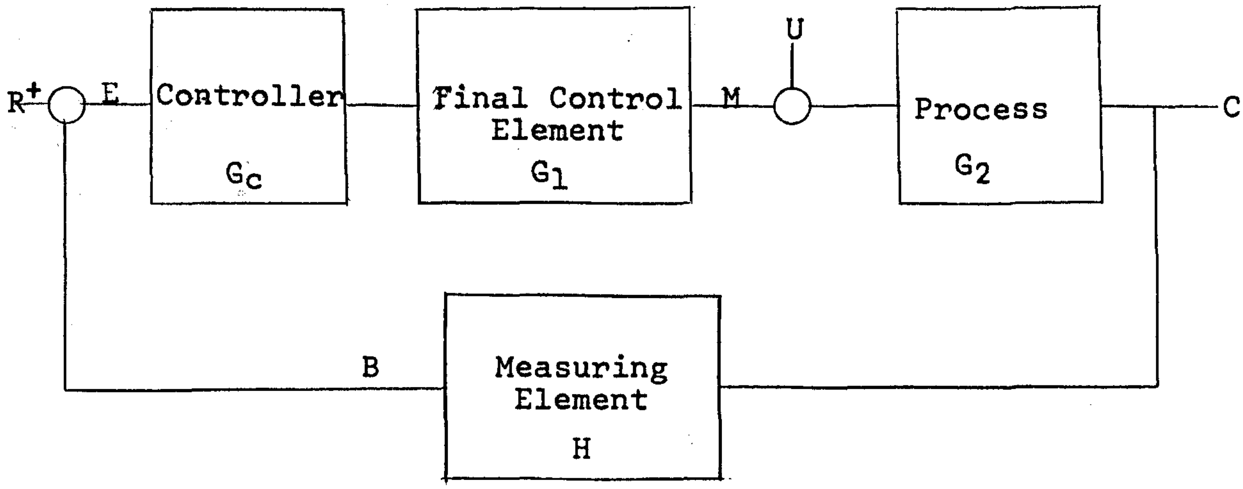


Figure 5. Block Diagram

R = Set Point
E = Error
M = Manipulated Variable
U = Load Variable
C = Controlled Variable
B = Measuring Element Signal
H, G_i = Mathematical function which will convert the input to the output for the i^{th} block

The overall transfer function relates the dependent variable, C, to the independent variables, U and R, for the process it can be determined by reducing the block diagram to a single block representing a single equation. Because the use of the block diagram and its reduction to a single block representing the overall function is necessary to the understanding of the rest of the paper, a brief description of the technique (7,8) follows.

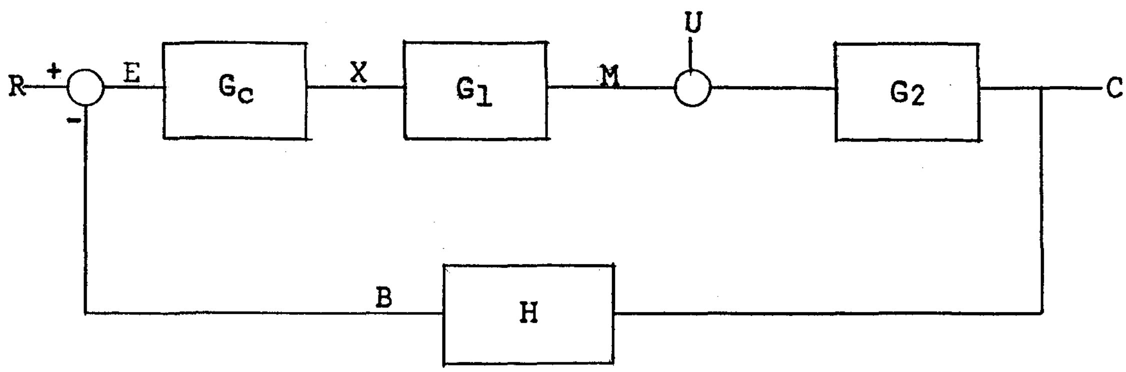


Figure 6. Series Reduction

1. Combining two blocks in series.

$X = G_cE$
$M = G_1X$

Therefore $M = G_1G_cE$

Several blocks in series can be represented by the product of the individual transfer functions.

2. Overall transfer function for a change in set point. (Relationship between C and R.)

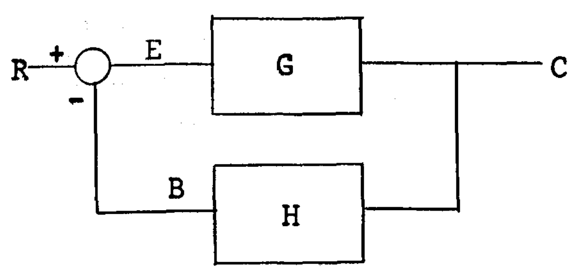


Figure 7. Loop Reduction for Set Point Change

$$C = GE$$
$$E = R - B$$
$$B = HC$$

Therefore

$$E = R - HC$$
$$C = G[R - HC]$$
$$C = \frac{G}{1 + HG} R$$

3. Overall transfer function for a change in load (i.e. R is 0). (Relationship between C and U.)

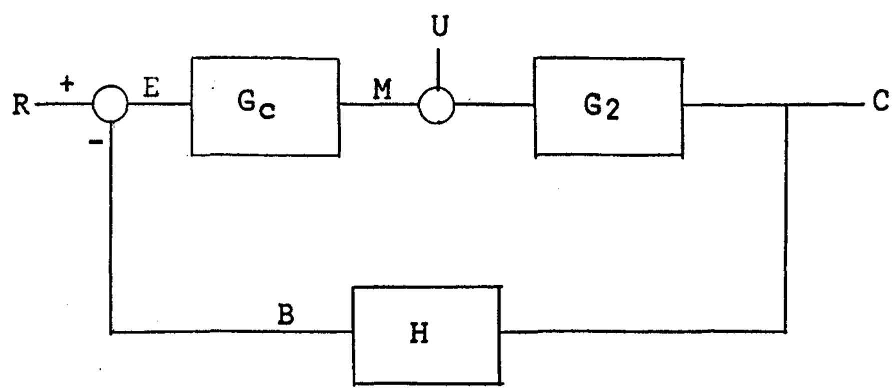


Figure 8. Loop Reduction for Load Change

$$C = G_2(M+U)$$
$$M = G_cE$$
$$E = -B = -HC$$

Therefore

$$C = G_2\ (-G_cHC + U)$$
$$C = G_2U - G_2G_cHC$$
$$C = \frac{G_2}{1 + G_2G_cH} U$$

In all cases of negative feedback the denominator of the closed loop transfer function is 1+ the open loop transfer function G_2G_cH. [The open loop transfer function relates the measured variable B to the set point R if the feedback loop were opened at the error detector. The closed loop transfer function relates a pair of variables with the feedback loop closed.]

<u>Root Locus Method</u>

The differential equations describing the control system, after being written in terms of the Laplace transform variable and being manipulated with the aid of the block diagrams, must be returned to the time variable form in order to determine the time response of the system variable. As discussed earlier, the inversion of a quotient of polynomials into a series of partial fractions is one procedure used to find the inverse transform; however, the use of partial fractions requires that the denominator of the control equation be factored.

The root locus method is a graphical procedure, first introduced by W. R. Evans (9), which can be used to locate the roots of the denominator. With the equation roots determined, the denominator may be factored and the Laplace transform inverted into the time domain by the partial fraction technique.

The plotting of a root locus diagram (7,8) proceeds in the following manner:

The denominator of the feedback control equation 1 + G, when set equal to zero, is called the characteristic equation of the closed loop system. The roots of the characteristic equation determine the form of the time response of the system.

The open loop transfer function, G, may be written in the form

$$G = K \frac{N}{D}$$

where the numerator, N, and denominator, D, are in factored form as derived from the process components.

$$N = (S - Z_1)(S - Z_2)\cdots\cdot(S - Z_m)$$
$$D = (S - P_1)(S - P_2)\cdots(S - P_n)$$

where

$Z_{1,2,\cdots m}$ are zeros of the open loop transfer function (when S = Z the numerator and hence the equation equals zero.)

$P_{1,2\cdots n}$ are poles of the open loop transfer function (when S = P the denominator equals zero and the equation becomes indeterminate.)

The characteristic closed loop equation

$$1 + G = 0$$

can be written

$$G = -1 = K\frac{(S + Z_1)(S + Z_2)\cdots(S + Z_m)}{(S + P_1)(S + P_2)\ldots(S + P_n)} .$$

In terms of a magnitude and phase angle the equation may be written as

$$K\frac{|S - Z_1|\ |S - Z_2|\cdots\ |S - Z_m|}{|S - P_1|\ |S - P_2|\cdots\ |S - P_n|} = 1$$

and

$$\sphericalangle(S - Z_1) + \sphericalangle(S - Z_2) + \cdots + \sphericalangle(S - Z_m)$$
$$- \sphericalangle(S - P_1) - \sphericalangle(S - P_2) - \cdots\cdot - \sphericalangle(S - P_n)$$
$$= (2j + 1)$$

where j is any positive or negative integer or zero.

There are several rules cited by Coughanowr and Koppel (7) which enable the location of the roots at the characteristic closed loop equation at various proportional gains to be plotted rapidly.

1. The number of branches equals the number of open loop poles, P_n.

2. The root loci begin at open loop poles and terminate at open loop zeros. The termination of (n - m) of the loci are at infinity along asymptotes. A multiple order pole or zero will be the beginning or termination of the number of loci equal to its order.

3. The real axis is part of the root locus when the sum of the number of poles and zeros to the right of the point on the real axis is odd. A multiple pole or zero is counted the same number of times as its order.

4. Asymptotes

   There are (n - m) loci which approach (n - m) straight lines radiating from the center of gravity of the poles and zeros. The center of gravity is given by

$$\gamma = \frac{\sum_{j=1}^{n} P_j - \sum_{i=1}^{m} Z_i}{n - m}$$

   The lines make angles of [(2K = 1) (n - m)] with the real axis (K = 0,1,2,···n - m - 1)

5. Breakaway Point

   The point at which two root loci, emerging from adjacent poles or toward adjacent zeros on the real axis, intersect and then leave the real axis is determined by the solution of the equation

$$\sum_{i=1}^{m} \frac{1}{S - Z_i} = \sum_{j=1}^{n} \frac{1}{S - P_j}$$

With the roots of the characteristic equation known, the control equation can be inverted to the time domain by either graphic or algebraic means, and the transient and final response of the system can be calculated. From the location of the roots in relation to the real and imaginary axes, the general characteristics of the response - whether oscillatory or not and the rate at which it will approach a steady state - can be obtained by inspection.

A shortcoming of the root locus method concerns the handling of time delays in the system. The term representing the dead time, $e^{-\tau s}$ can not be expressed in rational form. One method used to circumvent the problem is to use the first two terms of a Taylor expansion of $\frac{e^{-\tau s/2}}{e^{+\tau s/2}} = -\frac{s - 2/\tau}{s + 2/\tau}$.

An example which will illustrate the techniques used is the temperature control of the water out of a heater.

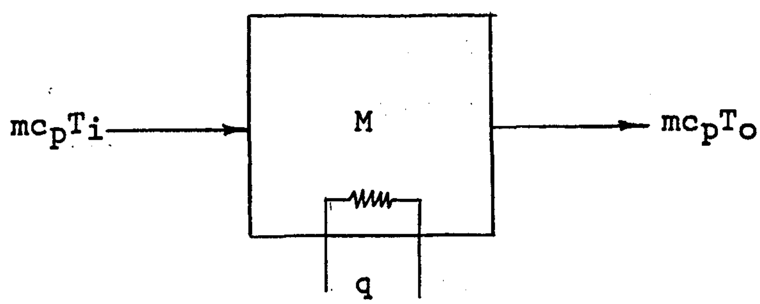


Figure 9. Water Heater System

m = lbs./min. water entering and leaving the heater
c_p = specific heat
T_i = Temperature of the entering water
T_o = Temperature of the leaving water
M = lbs. of water held in the heater
q = heat added to the system BTU/min.

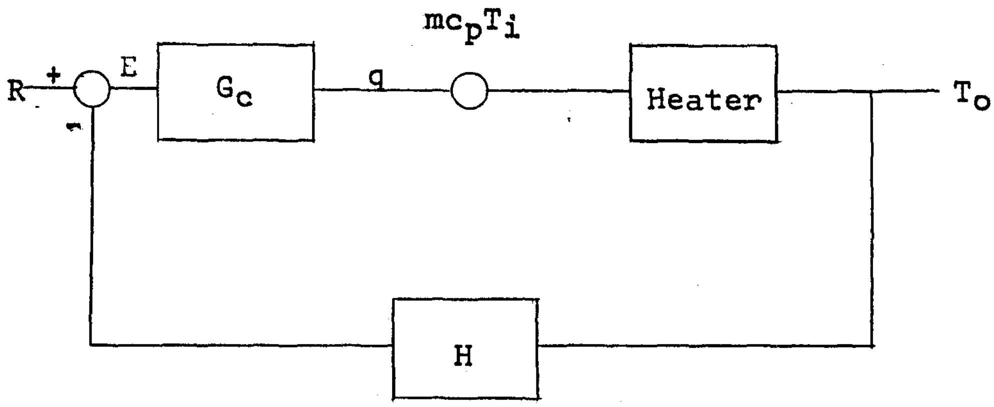


Figure 10. Block Diagram

For this example let

$$G_c = 10$$
$$H = 1$$
$$c_p = 1$$

The transfer function for the heater must be determined using the heat balance equations

$$mT_i = q - mT_o = M\frac{dT_o}{dt} \tag{16}$$

After taking the Laplace transform and using deviation variables, equation (16) becomes

$$mT_i(s) + q(s) - mT_o(s) = MsT_o(s) \tag{17}$$

$$MsT_o(s) + mT_o(s) = mT_i(s) + q(s) \tag{18}$$

Solving equation (18) for $T_o(s)$ in terms of the two independent variables, $T_i(s)$ and $q(s)$, the relationship becomes

$$T_o(s) = \frac{m}{Ms + m} T_i(s) + \frac{1}{Ms + m} q(s). \tag{19}$$

$$T_o(s) = \frac{1/M}{s + m/M} mT_i(s) + \frac{1/M}{s + m/M} q(s) \tag{20}$$

The block diagram can now be redrawn in terms of the transfer functions.

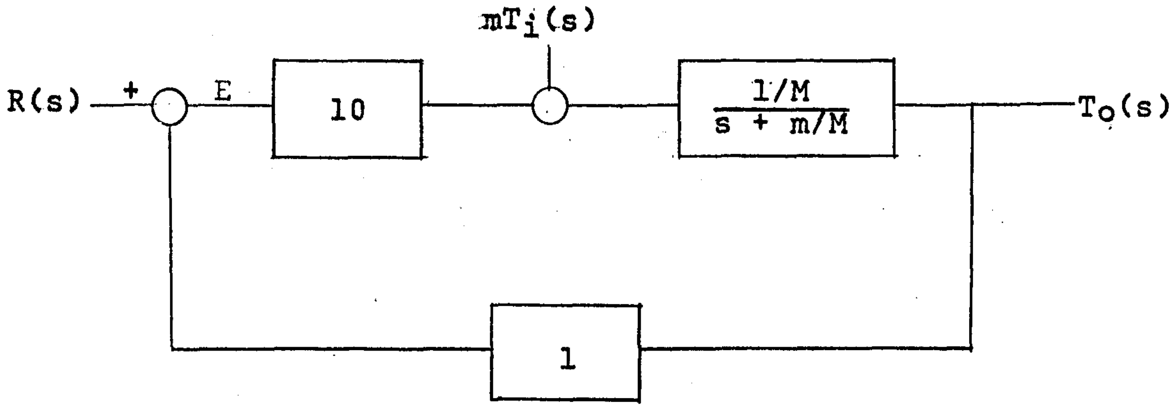


Figure 11. Block Diagram with Transfer Functions for Water Heater

The techniques for reducing a block diagram to a single function were developed earlier. The results are directly applicable to the present case, and the control equation can be written as follows:

$$T_o(s) = \frac{10\frac{1/M}{s + m/M}}{1 + (10)(1)(\frac{1/M}{s + m/M})} R(s) + \frac{\frac{1/M}{s + m/M}}{1 + (10)(1)(\frac{1/M}{s + m/M})} mT_i(s) \qquad (21)$$

In this example the response to a step change in set point at constant entering water temperature ($T_i(s) = 0$) will be considered.

$$T_o(s) = \frac{10\frac{1/M}{s + m/M}}{1 + (10)(1)(\frac{1/M}{s + m/M})} R(s) \qquad (22)$$

For this example let

$m = 3$
$M = 12$

$$T_o(s) = \frac{10 \frac{1/12}{s + 3/12}}{1 + 10 \frac{1/12}{s + 3/12}} R(s) \tag{23}$$

$$T_o(s) = \frac{\frac{10}{12} \frac{1}{s + 1/4}}{1 + \frac{10}{12} (\frac{1}{s + 1/4})} R(s) \tag{24}$$

The inversion from the Laplace transform variable to the time variable can proceed by either the algebraic or the root locus methods. Inversion by algebraic methods proceeds as follows:

$$T_o(s) = \frac{\frac{10}{12} \frac{1}{s + 1/4} \qquad 12(s + 1/4)}{1 + \frac{10}{12} (\frac{1}{s + 1/4}) \quad 12(s + 1/4)} R(s) \tag{25}$$

$$T_o(s) = \frac{10}{12(s + 1/4) + 10} R(s) \tag{26}$$

$$T_o(s) = \frac{10/12}{s + 13/12} R(s) \tag{27}$$

The Laplace transform for a unit step change in set point is $\frac{1}{s}$. Evaluating equation (27) for a unit step change in set point gives

$$T_o(s) = \frac{10/12}{(s + 13/12)} \frac{1}{s} \cdot \tag{28}$$

The quotient may be separated into the sum of its factors by the partial fractions method.

$$\frac{10/12}{(s + 13/12)} \frac{1}{s} = \frac{A}{s} + \frac{B}{s + 13/12} \tag{29}$$

Multiplying by s gives

$$\frac{10/12}{(s + 13/12)} = A + \frac{Bs}{s + 13/12} \tag{30}$$

At s = 0

$$\frac{10/12}{13/12} = A$$

$$A = \frac{10}{13} \tag{31}$$

Multiplying equation (29) by s + 13/12 gives

$$\frac{10/12}{s} = \frac{A(s + 13/12)}{s} + B. \tag{32}$$

At s = -13/12

$$\frac{10/12}{-13/12} = B$$

$$-\frac{10}{13} = B \tag{33}$$

$$T_o(s) = \frac{\frac{10}{13}}{s} \quad \frac{-\frac{10}{13}}{s + 13/12} \tag{34}$$

The inverse transform of $\frac{1}{s}$ is 1, and the inverse transform of $\frac{1}{s + 13/12}$ is $e^{-13t/12}$.

$$T_o(t) = \frac{10}{13}\,(1 - e^{-13t/12}) \tag{35}$$

Notice that even after the exponential term vanishes, the temperature of the heater will not reach the desired set point change.

Returning to equation (24), the same inversion will be made using the root locus procedure.

$$T_o(s) = \frac{\frac{10}{12}\,\frac{1}{s + 1/4}}{1 + \frac{10}{12}\,\frac{1}{s + 1/4}} \tag{24}$$

1. The first step is to locate the open loop poles and zeros from the open loop transfer function.

$$G = K\frac{N}{D} = \frac{K}{12}\frac{1}{(s + 0.25)} \quad (36)$$

The open loop transfer function has a pole at S = -0.25, and there are no open loop zeros in this case.

2. The root locus begins at the open loop pole and terminates at - infinity since there are no open loop zeros.

3. The real axis forms the root locus in this example since the sum of the poles and zeros to the right of the point is odd.

4. The location of the roots as a function of the proportional gain can be obtained from the magnitude criterion equation

$$1 = K\frac{|s - Z_1|\,|s - Z_2|\,|s - Z_m|}{|s - P_1|\,|s - P_2|\,|s - P|} = \frac{K}{12}\frac{1}{|s - (-.25)|} \quad (37)$$

The location of the roots of the characteristic equation can be computed for several values of K from equation (37).

s for root of characteristic equation	$\frac{K}{12}$	K
-1	0.75	9
-13/12	10/12	10
-2	1.75	21
-3	2.75	33

In this case for a proportional gain of 10, the characteristic equation has one root located at -13/12.

Returning to the control equation (24), the denominator can now be expressed in factored form.

$$T_o(s) = \frac{10\,\frac{1/12}{s + 1/4}}{1 + 10\,\frac{1/12}{s + 1/4}}\,R(s) \tag{24}$$

$$T_o(s) = \frac{\frac{10}{12}}{(s + 1/4) + \frac{10}{12}(1)}\,R(s) \tag{38}$$

$$T_o(s) = \frac{\frac{10}{12}}{s - r_1}\,R(s) \tag{39}$$

where $r_1 = -13/12$

$$T_o(s) = \frac{\frac{10}{12}}{s + 13/12} \cdot \frac{1}{s} \tag{40}$$

This equation can be inverted by separating the factors using the partial fractions method and solved as shown in the algebraic solution. The root locus method has the advantage of showing the effect of a change in a process or control variable. The more negative the location of the root, the more rapid will be the decay of the transient terms.

Development of the Control Equations for a Controlled Temperature Bath

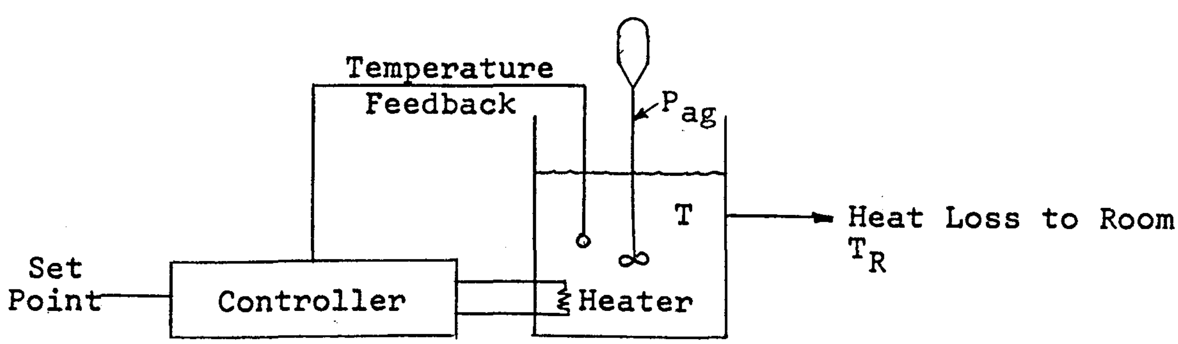


Figure 12. Controlled Temperature Bath

The basic control equations for a controlled temperature bath will be developed from a heat balance around the bath and the Laplace transforms of the control element transfer functions. The heat balance around the bath starts with the equation

Rate Heat in - Rate Heat out = Rate of Accumulation

In this case heat was added by the stirring and by the heater, and heat was lost to the surrounding room.

The heat balance can be written

$$P_{Stirrer} + Q_{Heater} - UA(T - T_R) = c_p m \frac{dT}{dt} \quad (41)$$

P = Power input from stirrer, cal./min.
Q = Power input from heater, cal./min.
UA = Heat transfer coefficient x Area, $\frac{cal.}{cm^2\ min.\ C^o} \cdot cm^2$
T = Temperature of bath, °C.
T_R = Temperature of room, °C.
c_p = Specific heat, $\frac{cal.}{gm}$
m = Mass of bath, gm.

After substituting deviation variables and taking the Laplace transform, equation (41) can be written

$$P_{St}(s) + Q_H(s) - UAT(s) + UAT_R(s) = c_p msT(s) \quad (42)$$

which can be solved for the temperature of the bath as a function of the independent variables.

$$(c_p ms + UA)T(s) = P_{St}(s) + Q_H(s) + UAT_R(s) \quad (43)$$

$$T(s) = \frac{1}{(c_p ms + UA)} P_{St}(s) + \frac{1}{(c_p ms + UA)} Q(s) + \frac{UA}{c_p ms + UA} T_R(s) \quad (44)$$

The independent variable Q(s) is to be controlled in order to achieve control of the bath temperature. The relationship between the heat added to the bath and the set point is determined by the error detector and controller. This can most readily be seen from the block diagram (Figure 13).

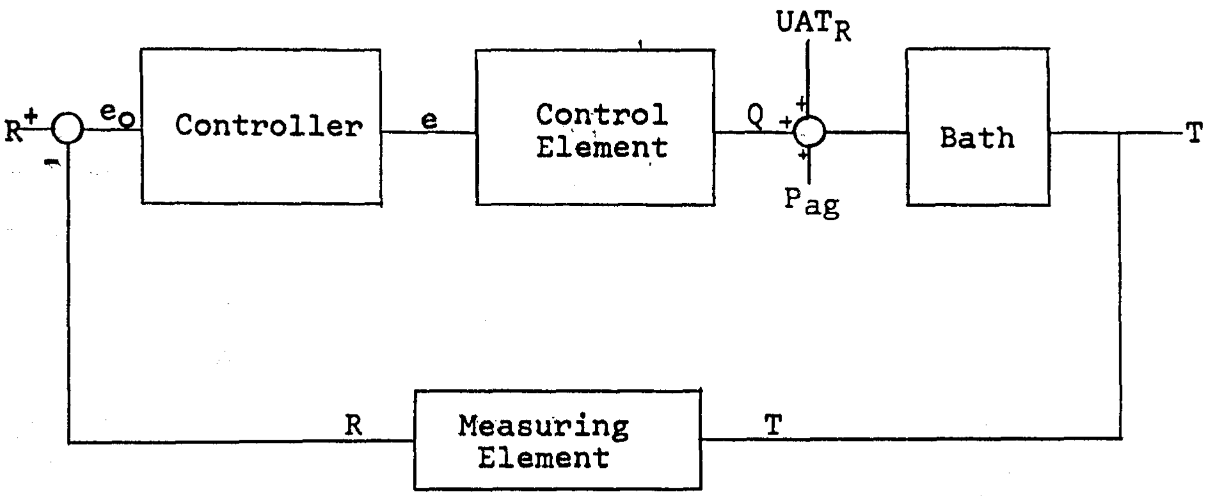


Figure 13. Block Diagram for Controlled Temperature Bath

The particular bridge network used to detect the difference between the set point and bath temperature as shown in Figure 14 had the following relationship

$$e_o = -\frac{[R_D - R_{th}]}{R_{th}} \frac{E_o}{2} \tag{45}$$

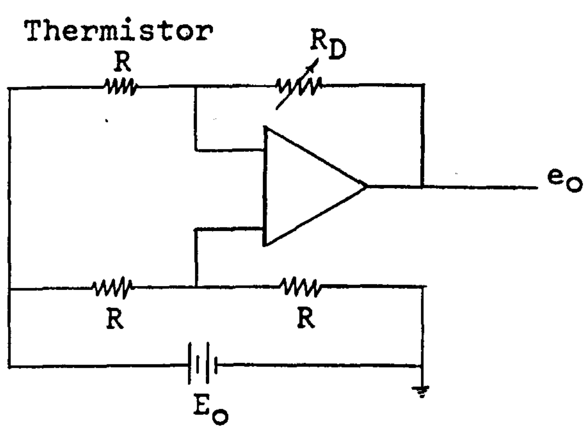


Figure 14. Error Detector

This bridge (11) was chosen because of its linear output for changes in $(R_D - R_{th})$. At steady state $R_D = R_{th}$, and the error signal is zero. For a deviation from steady state, the term $(R_D - R_{th})$ may be represented as the deviation variable R.

$$e_o = -\frac{E_o}{2\ R_{th}} R \tag{46}$$

After taking the Laplace transform, the error bridge transfer function becomes

$$\frac{e_o(s)}{R(s)} = -\frac{E_o}{2\ R_{th}} \ . \tag{47}$$

There will be two types of controllers used in this work - proportional control and proportional plus integral control. The transfer function for proportional control is simply a constant K_c. The transfer function for an integral controller (11) is derived from the integrating amplifier circuit shown in Figure 15.

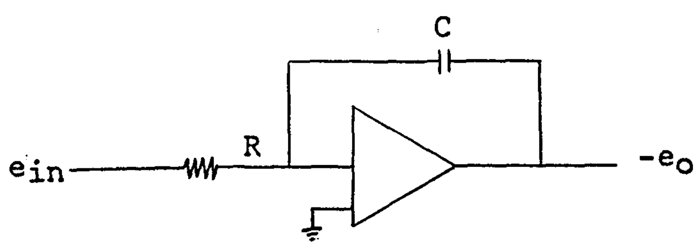


Figure 15. Integral Amplifier Circuit

$$e_o = -\frac{1}{RC}\int e_{in}dt \tag{48}$$

Taking the Laplace transform and using deviation variables, equation (48) becomes

$$e_o(s) = -\frac{1}{RC}\frac{1}{s}e_{in}(s)$$

$$\frac{e_o(s)}{e_{in}(s)} = -\frac{1}{RCs} \tag{49}$$

The final control element can have many types of transfer function, not all of them necessarily linear. The transfer function of the transistor heater system used in this work was determined experimentally. At this point the transfer function will be represented symbolically by the symbol K_F.

A thermistor was used as the feedback measuring element. A thermistor is a temperature sensitive semiconductor which is used to convert temperature measurement into electrical resistance. The transfer function may be developed from the equations relating resistance and temperature of a thermistor (6)

$$\ln R = \ln R_s + A\left[\frac{1}{T} - \frac{1}{T_s}\right] \tag{50}$$

where

R = Thermistor resistance
R_s = Thermistor resistance at temperature T_s
T = Temperature °K
A = Temperature coefficient of thermistor

The relationship between R and T is not linear; therefore, a linear approximation of equation (50) must be made. A Taylor's series expansion of equation (48) can be used to make the linearization (13).

$$R = R_s e^{A\left(\frac{1}{T} - \frac{1}{T_s}\right)} \tag{51}$$

$$R = a_o + a_1 (T - T_s + a_2 (T - T_s)^2 + \cdots \tag{52}$$

where

$$a_o = f(T_s) = R_s$$

$$a_1 = f'(T_s) = - \frac{AR_s}{T_s^2}$$

$$a_2 = \frac{f''(T_s)}{2!} = - \frac{AR_s^2}{T_s^3}$$

Keeping only the linear terms, the result is

$$R = R_s - \frac{AR_s}{T_s^2} (T - T_s). \tag{53}$$

Utilizing deviation variables and taking the Laplace transform

$$R(s) = - \frac{AR_s}{T_s^2} T(s) \tag{54}$$

For the derivation of the control equation, the transfer function for the thermistor feedback loop will be represented in deviation variables as K_H. The value of K_H will be determined in the experimental section using experimental data and equation (54).

Combining the elements of the control loop, the control equation expressed in deviation variables is obtained by substitution in equation (44). This gives

$$T(s) = \frac{\left(\frac{1}{c_p m s + UA}\right) \qquad P_{ag}(s)}{1 + \left(\frac{E_o}{2\ R_{th}}\right)(K_c)(K_f)(K_H)\left(\frac{1}{c_p m s + UA}\right)} + \frac{\left(\frac{E_o}{2\ R_{th}}\right)(K_c)(K_f)\left(\frac{1}{c_p m s + UA}\right) \qquad R(s)}{1 + \left(\frac{E_o}{2\ R_{th}}\right)(K_c)(K_f)(K_H)\left(\frac{1}{c_p m s + UA}\right)} + \frac{\left(\frac{UA}{c_p m s + UA}\right) \qquad T_R(s)}{1 + \left(\frac{E_o}{2\ R_{th}}\right)(K_c)(K_f)(K_H)\left(\frac{1}{c_p m s + UA}\right)} \qquad (55)$$

The normal procedure in evaluating the response characteristics is to let all but one of the deviation variables equal zero. The remaining variable is given the type of forcing function to be evaluated, and the resulting temperature response is then calculated.

EXPERIMENTAL

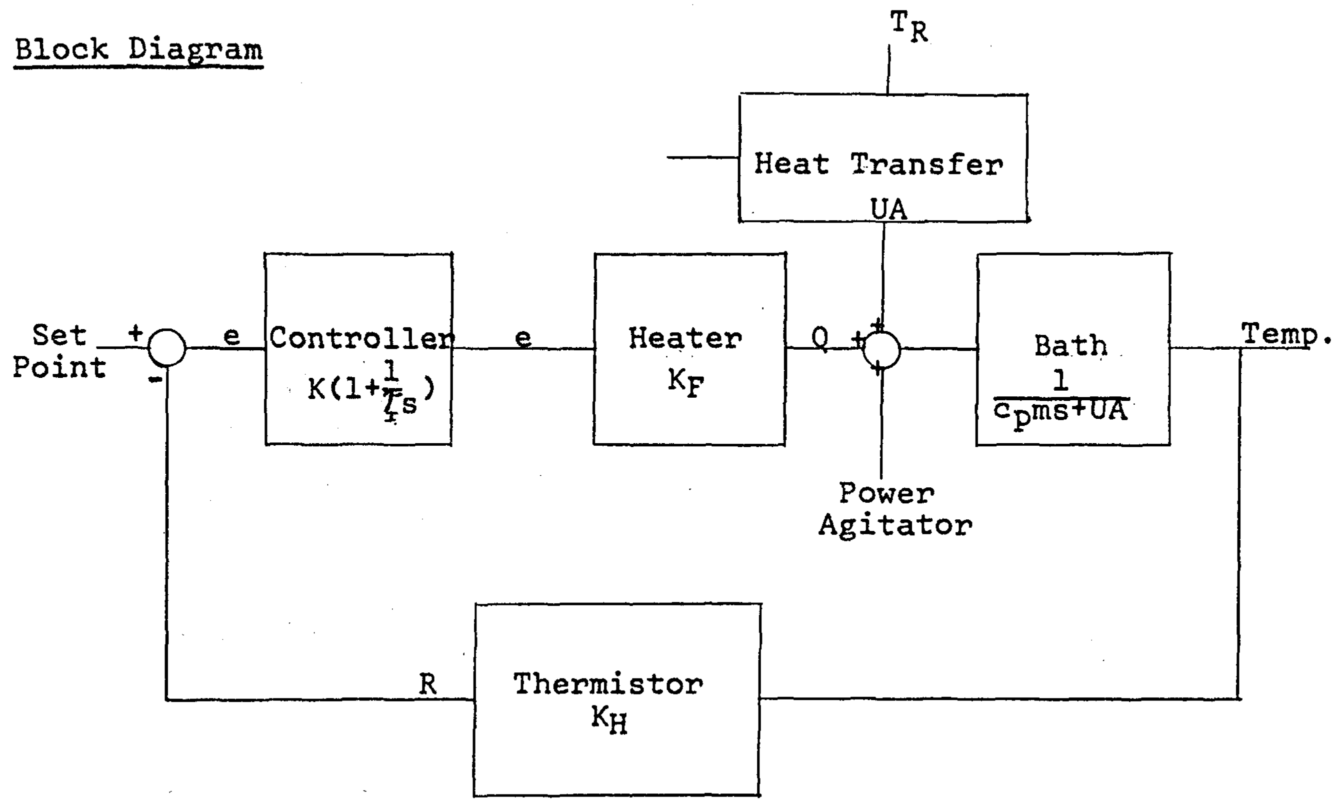
The bath was a 2,000 ml. beaker filled with 1.8L of stabilized bath oil. The beaker was lightly insulated to control heat loss to the room. The bath was agitated with a variable speed motor and serrated disk agitator. By varying the speed of the agitator, a change in the process (load) variable could be made (Figure 16).

The linear error bridge, shown in figure 14, and control circuit, the integral part of which is shown in figure 15, were constructed using model P-85AU Philbrick high gain D.C. amplifiers (11). Precision resistors and a 2 microfarad mylar capacitor were used to construct the error bridge and control circuit. Shielded cable was required in the error bridge and integrator circuit to prevent excessive noise in the signal.

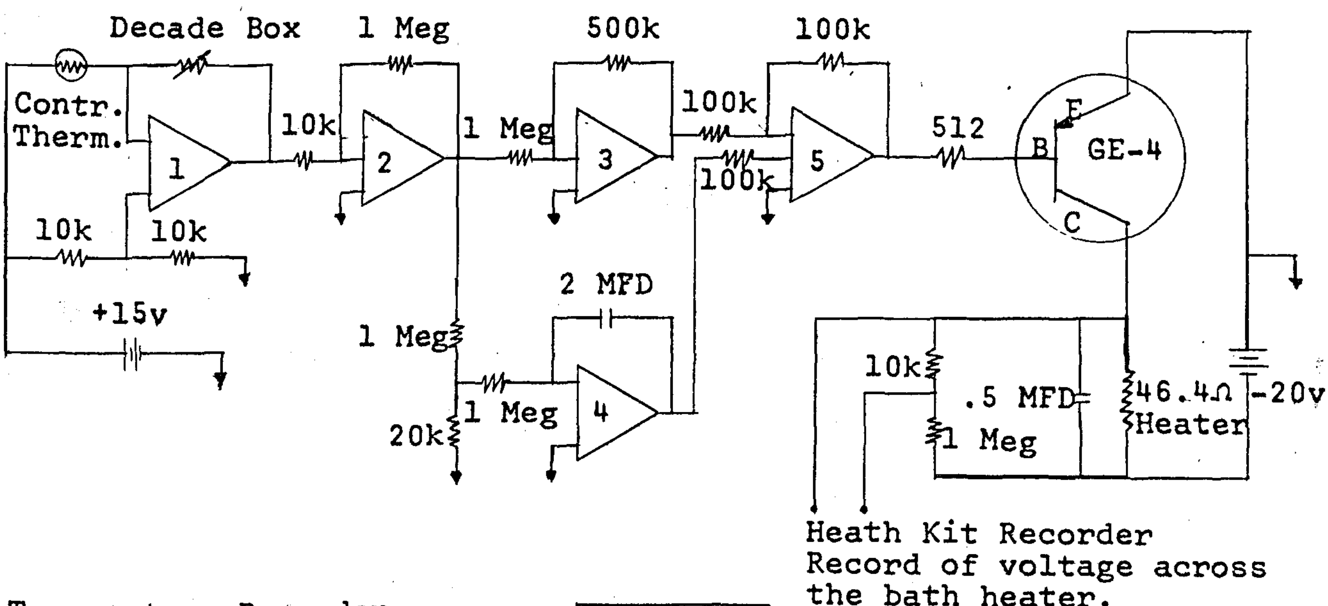
The final amplifier in the control circuit, shown in figure 16, was a model P-45AU because of its larger current output of 20ma. The final control element was a GE-4 6822 PNP power transistor mounted on a heat exchanger which was used to control the current to a 46.4 ohm immersion resistance heater. A Heathkit D.C. constant voltage supply set at 20 volts was used to supply power to the transistor - resistance heater circuit.

Temperature feedback to the error bridge was provided by a thermistor sensor. Bath oil was used in the temperature

Block Diagram



Electrical Diagram



Temperature Recorder

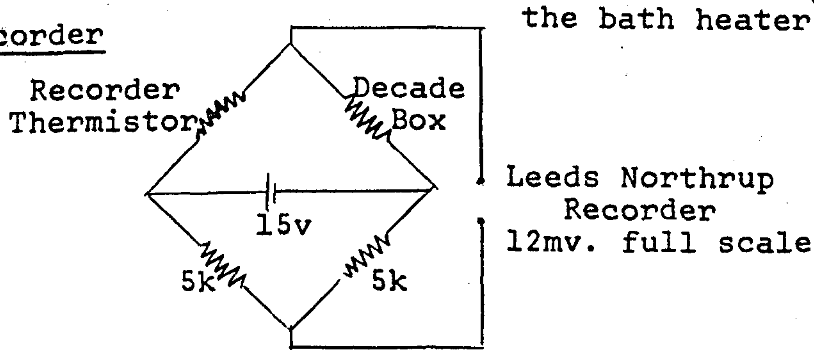


Figure 16. System Schematics

bath in order to eliminate electrolytic corrosion of the immersion heater. This also eliminated heat losses due to evaporation, which simplified the heat balance equations (Figure 16).

A separate system was used to obtain a temperature record of the bath. A Leeds and Northrup model R820-1 recorder was used to record the unbalance of a thermistor bridge circuit (Figure 16).

In order to analyze the control system, the value of several constants had to be determined. These included the proportional gain constant, the change in thermistor resistance with temperature, the integrator time constant, the transistor - heater power transfer function, and the heat loss to the room.

Thermistor Transfer Function K_H

As discussed earlier, the transfer function for the thermistor follows the relationship

$$R(s) = -\frac{AR_s}{T_s2} T(s) \qquad (54)$$

The value of A can be determined from the least squares slope of a plot of lnR vs. 1/T. As shown in Figures 17 and 18 and Table I, the transfer function for the control bath thermistor is -(2610 $\pm$120) ohms/°C., and the transfer function for the recorder thermistor is -(2870 $\pm$60) ohms/°C.

TABLE I

Thermistor Transfer Functions

Recorder Thermistor

RkΩ	T°k	$10^3/T$	Log R
85.8	302.5	3.306	4.9335
80.9	303.8	3.292	4.9080
76.2	305.2	3.277	4.8820
71.1	306.8	3.259	4.8519
67.0	308.1	3.246	4.8261
65.9	308.5	3.241	4.8189
63.1	309.5	3.231	4.8000
60.1	310.6	3.220	4.7789

Slope = $(1.78 \pm .04)\ 10^3$ k°

$$\frac{R(s)}{T(s)}_{@35^\circ C.} = \frac{-2.303\times(-1.78\pm.04)10^3\times(66{,}800\pm200)}{(308.2\pm.1)}$$

$$\frac{R(s)}{T(s)}_{@35^\circ C.} = -2870\pm60 \text{ ohm/C.}^\circ$$

Controller Thermistor

RkΩ	T°k	$10^3/T$	Log R
62.0	307.0	3.257	4.7924
60.2	307.6	3.251	4.7796
57.2	308.8	3.238	4.7574
54.0	310.1	3.225	4.7324
49.6	310.9	3.216	4.7185
52.3	312.0	3.205	4.6955

Slope = $(1.83 \pm .08)\ 10^3$ k°

$$\frac{R(s)}{T(s)}_{@35^\circ C.} = \frac{-(2.303)(1.83\pm.08)10^3(58.8\pm.2)10^3}{(308.2\pm.1)^2}$$

$$\frac{R(s)}{T(s)}_{@35^\circ C.} = -2610\pm120 \text{ ohm/C.}^\circ$$

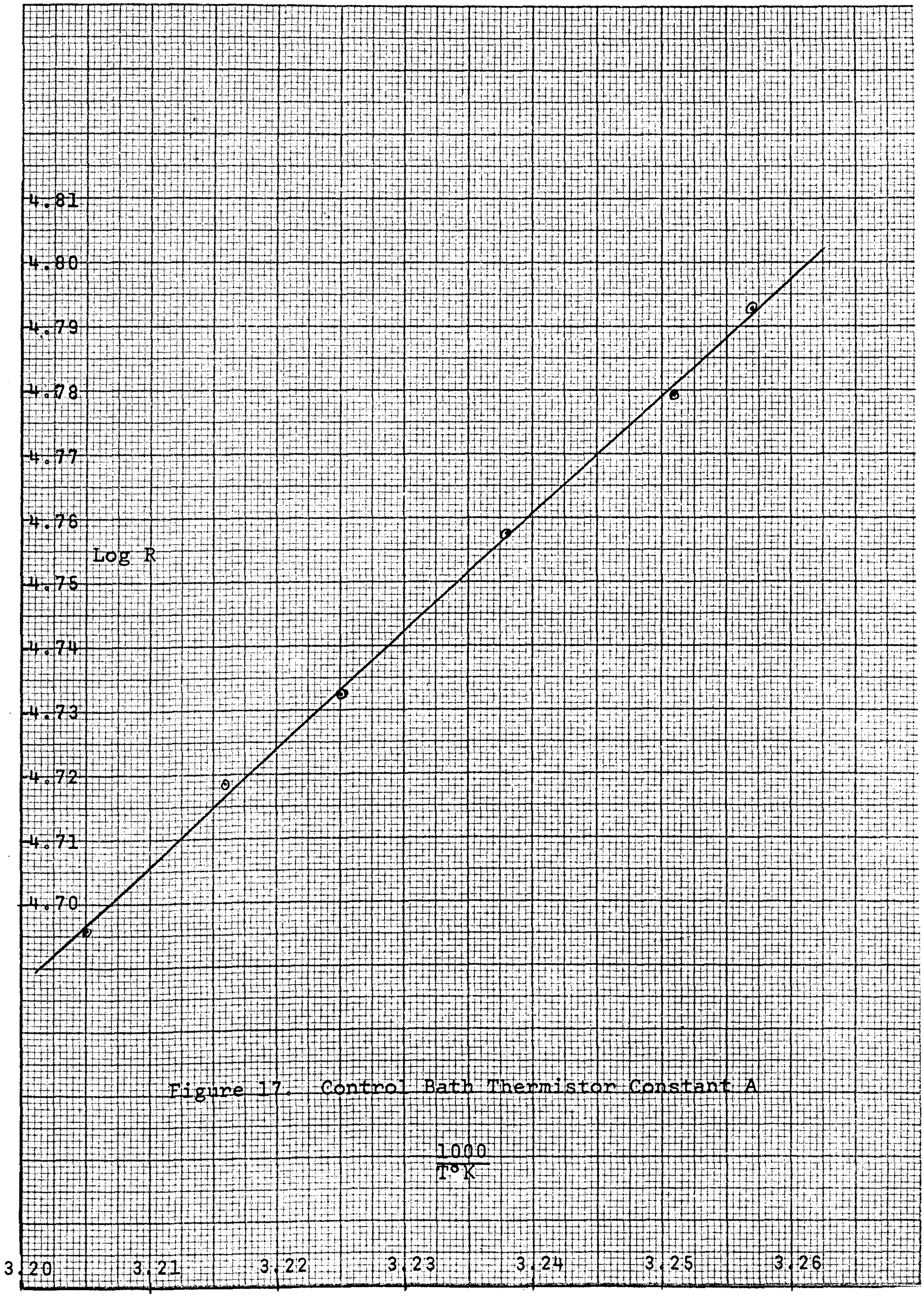


Figure 17. Control Bath Thermistor Constant A

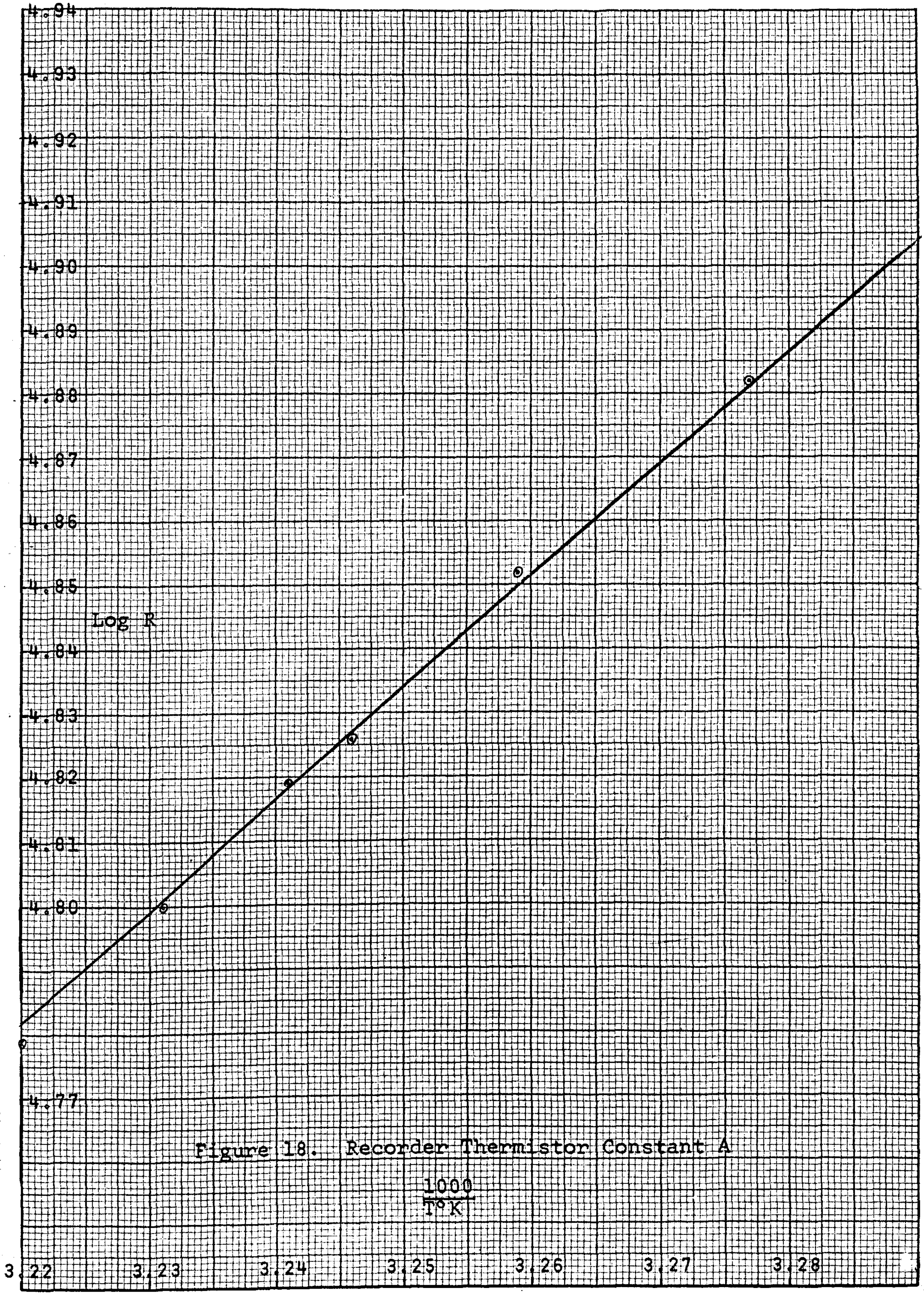


Figure 18. Recorder Thermistor Constant A

Control Element Transfer Function K_F

The power dissipated in the heater was calculated from the voltage drop across the heater and the heater resistance. The voltage drop across the heater was measured at specific voltage inputs to the transistor base circuit with the transistor wired in its common emitter configuration as used. A plot was made of the heat supplied to the bath in calories per minute vs. the voltage to the transistor base (Figure 19). The slope was calculated using a least squares fit over the linear region of the curve. At a heater dissipation above 75 cal./min. and at some point below 10 cal./min., the data could no longer be fitted by a straight line. (The thermostat was operated within the linear region of the curve.)

Table II

Control Element Transfer Function

Volts to Base	Cal./Min.
-0.43	12.0
-0.65	27.5
-0.70	33.5
-0.90	46.0
-1.00	55.5
-1.20	70.5

Slope = $-(75.9 \pm 5)$ cal./volt·min.

Heat Loss to Room

In order to calculate the heat transfer to the room from the bath, UA, an energy balance was made around the bath. This is illustrated by the following equation:

Rate Heat In - Rate Heat Out = Rate Accumulation.

There were two sources of energy into the bath - the heater

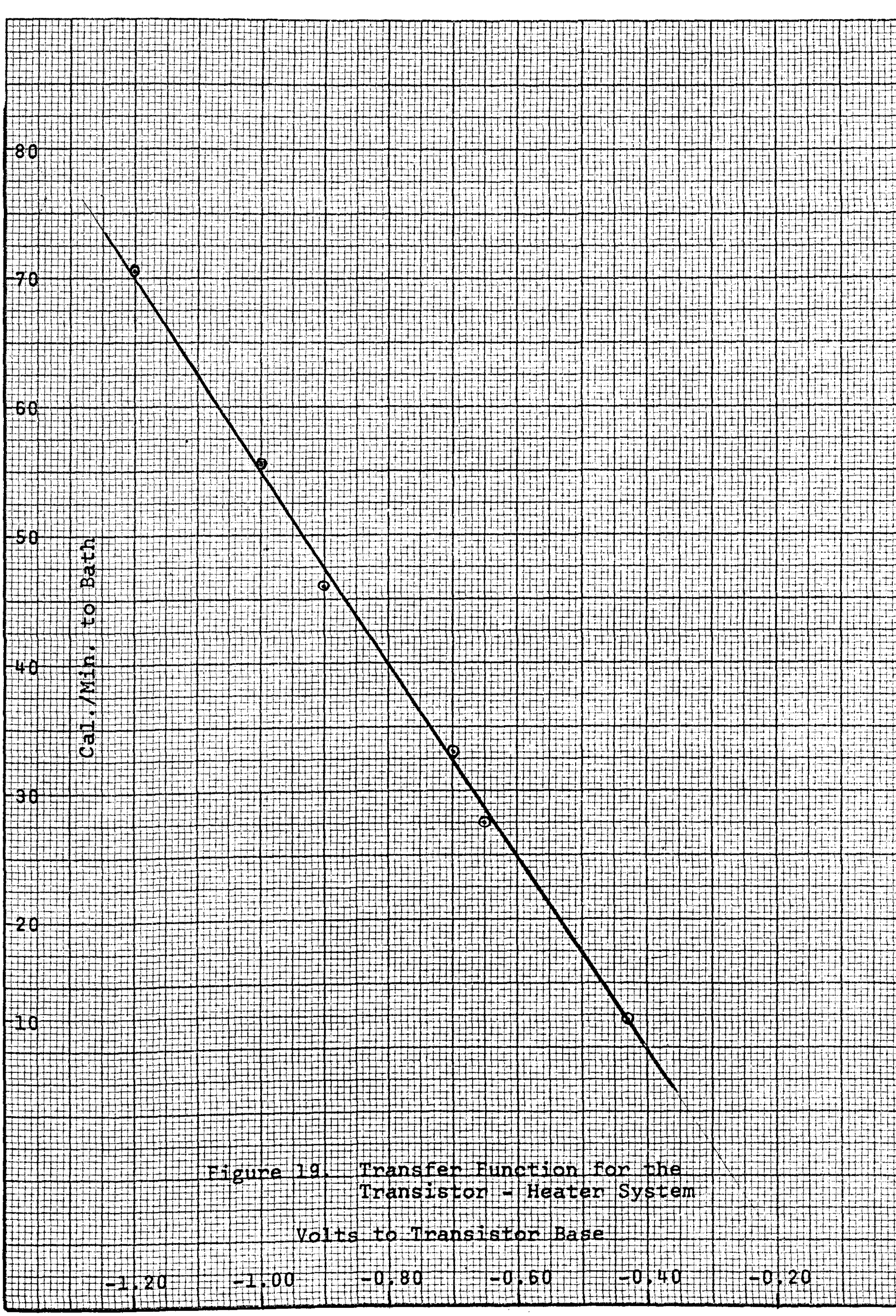


Figure 19. Transfer Function for the Transistor - Heater System

and the stirrer. The heat loss to the surroundings is dependent on the relationship

$$Q = UA\ (T_{Bath} - T_{Surround}). \qquad (56)$$

The rate of heat accumulation in the bath is

$$Q = c_p m \frac{dT}{dt}\ . \qquad (57)$$

In order to determine the value of the heat transfer coefficient-area term, UA, the difference in heat supplied to the bath at two different surrounding temperatures was determined. Using simultaneous equations the heat generated by the stirrer could be eliminated. A second determination was made by using the rate of cooling with only the agitator running and by using the steady state conditions with only the agitator running. Again by using simultaneous equations the agitator power could be eliminated.

$$\text{Rate Heat In} - \text{Rate Heat Out} = \text{Rate Accumulation} \qquad (58)$$

$$P_{Agitator} + P_{Heater} - UA(T_{Bath} - T_{Room}) = c_p m \frac{dT}{dt} \qquad (59)$$

$$P_{ag} + 21.7\ \text{cal./min.} - UA\ (33.9 - 24.4) = 0 \qquad (60)$$

$$P_{ag} + 44.9\ \text{cal./min.} - UA\ (33.9 - 20.7)^{\circ}\text{C.} = 0 \qquad (61)$$

$$23.2\ \text{cal./min.} - UA\ (3.7^{\circ}\text{C.}) = 0 \qquad (61)-(60)$$

$$UA = 6.3\ \frac{\text{cal.}}{\text{min.}\ ^{\circ}\text{C.}} \qquad (62)$$

$$P_{ag} + 0\ \text{cal./min.} - UA\ (34.75 - 20.45)^{\circ}\text{C.} = 840\ (\frac{-0.59^{\circ}\text{C.}}{10\ \text{min.}}) \qquad (63)$$

$$P_{ag} - UA\ (14.3^{\circ}\text{C.}) = -49.7\ \frac{\text{cal.}}{\text{min.}}$$

$$P_{ag} + 0 \text{ cal./min.} - UA\,(26.4 - 19.8)^{\circ}C. = 0 \qquad (64)$$

$$P_{ag} - UA\,(6.6^{\circ}C.) = 0$$

$$7.7^{\circ}C.\ UA = 49.7\ \frac{\text{cal.}}{\text{min.}} \qquad (64)-(63)$$

$$UA = 6.4\ \frac{\text{cal.}}{\text{min. }^{\circ}C.} \qquad (65)$$

$$P_{ag} + 51.8\ \frac{\text{cal.}}{\text{min.}} - UA\,(34.9 - 20.5)^{\circ}C. = 0 \qquad (66)$$

$$P_{ag} = UA\,(6.6^{\circ}C.) \qquad (67)$$

$$UA\,(6.6^{\circ}C.) + 51.8\ \frac{\text{cal.}}{\text{min.}} - UA\,(14.4^{\circ}C.) = 0 \qquad (68)$$

$$UA\ 7.8^{\circ}C. = 51.8\ \frac{\text{cal.}}{\text{min.}}$$

$$UA = 6.6\ \frac{\text{cal.}}{\text{min. }^{\circ}C.} \qquad (69)$$

The average value of the heat transfer coefficient-area term, UA, was $6.4 \pm 0.2\ \frac{\text{cal.}}{\text{min. }^{\circ}C.}$.

Integrating Amplifier Time Constant

After assembling the apparatus the operational amplifiers were balanced. With the integrator circuit the above was accomplished by grounding the input end of the resistance tee network and adjusting the amplifier offset bias until the integrator output remained constant.

The integrator circuit gain was checked by timing the output voltage change for constant input voltages. The time required for the change in output voltage to equal the input voltage, the integrator time constant, was checked in the positive and negative direction and found to be equal and within 4% of the value calculated from the nominal component values.

$$E = \frac{1}{RC} \int_{t=o}^{t_1} e\,dt \tag{70}$$

$$E = \frac{1}{RC} e\,(t_1 - t_o) \tag{71}$$

$$E = e\,\frac{t}{RC} \tag{72}$$

when $t = RC \quad E = e$

where $RC = (52 \times 10^6)(2 \times 10^{-6})$

$$= 104 \text{ sec.} \tag{73}$$

The proportional control amplifier was originally set at a gain of 10. It was found that at gains of 10 and later 5, the control action was off-on rather than proportional. A 0.5 gain factor was used in the proportional control circuit because it gave a good balance between the speed of response and the oscillation produced by the integrator.

System Time Delays

The stability of any control system is affected by the time delays in the system. This is the time it takes the control system to sense the need and translate the need into control action by the final control element. The time delay in the thermostat system was determined as follows: The bath was allowed to warm due to the heat supplied by the stirrer alone until a constant rate of rise in temperature was observed. Then a large change in set point was made so that the controller immediately went from zero power to the bath to full power to the bath (Figure 20, point A). The time required for the temperature recorder to reach a new rate of rise was 0.2 minutes. The process was repeated in reverse (from full power on to power off), and a new rate of temperature rise was again observed in 0.2 minutes (Figure 20, point B).

Because the time delays were short in comparison with the other time constants in the system and in comparison with the frequency of changes in the system, the effect of the time delay can be neglected. If the delay were approximated by the term $(1 - e^{-at})$, the exponential decay factor would be 23. Terms introduced by this factor would rapidly vanish.

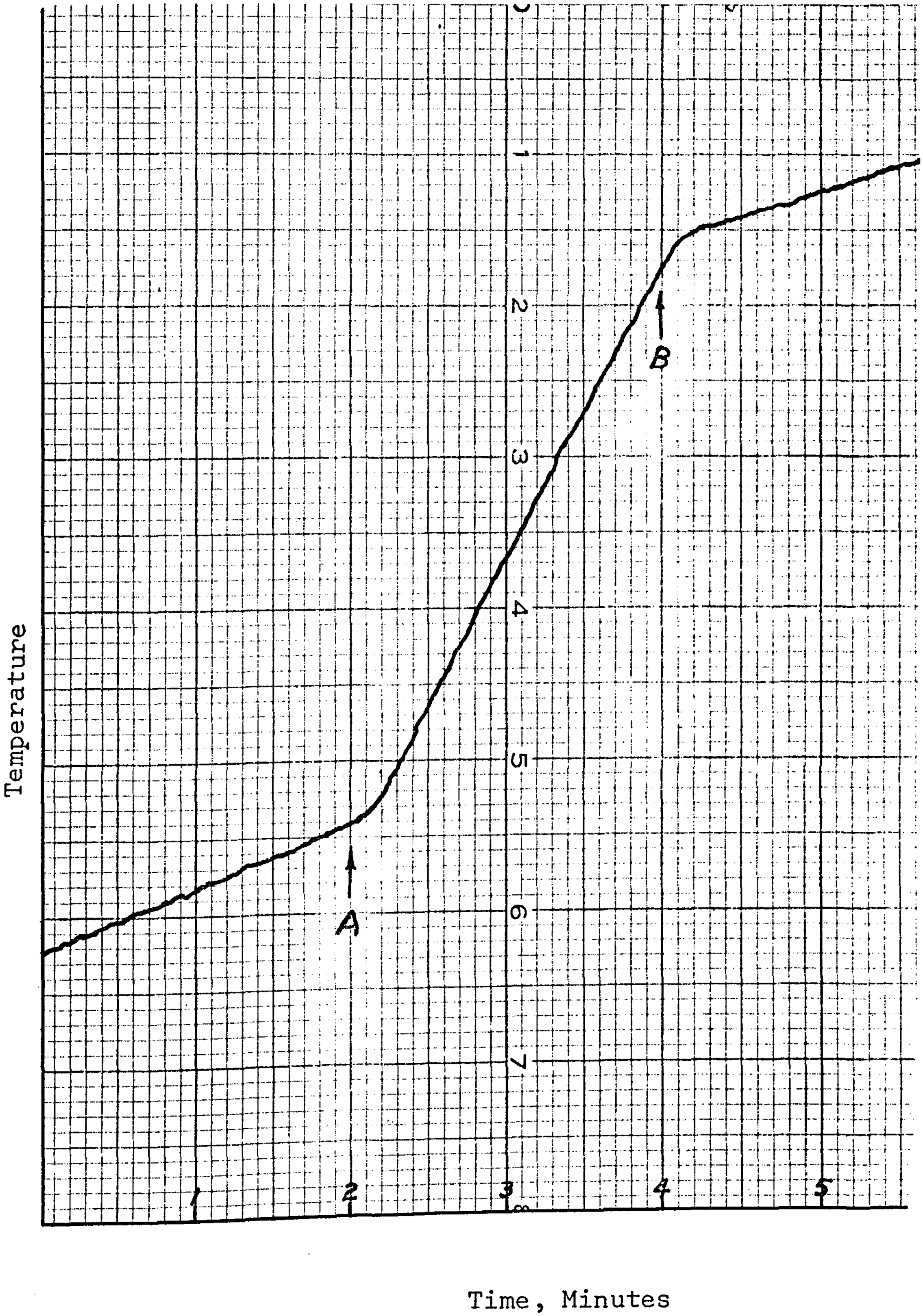


Figure 20. Thermostat System Time Delays

CALCULATIONS

Using the relationships and constants developed in the Theoretical and Experimental sections, the transient response of the constant temperature bath to changes in set point and load can be calculated. The numerical value of the error detector and controller transfer functions must first be determined from their components. Figure 16 gives a schematic electrical diagram and a block diagram of the system.

The transfer function for the error bridge was given as

$$\frac{e(s)}{R(s)} = -\frac{E_o}{2\ R_{th}} \tag{47}$$

in equation (47). In the temperature range under investigation R_{th} was 58.8 kohm, and a 15 volt energy source was used. The transfer function was then

$$\frac{e(s)}{R(s)} = -\frac{15}{2(58,800)}$$

$$= -1.27 \times 10^{-4}\ \frac{\text{volt}}{\text{ohm}} \tag{74}$$

Amplifier number two served to multiply the error signal output by a factor of -100, making the combined transfer function for the error detector

$$\frac{e(s)}{R(s)} = +1.27 \times 10^{-2}\ \text{volt/ohm} \tag{75}$$

High gain D.C. amplifiers were used to obtain proportional gain and integral gain transfer functions. Amplifier number three was used to provide a proportional gain of -0.5. Amplifier number four was used to provide integral control. The transfer function for the integrating amplifier was given as

$$\frac{e4(s)}{e2(s)} = -\frac{1}{RCs} \tag{49}$$

in equation (49). A resistance tee network was used to increase the equivalent resistance and obtain a longer integrator time constant.

The equivalent resistance of the tee network (Figure 21) is

$$R_{eq} = R_1 + R_2 + \frac{R_1 R_2}{R_3} \tag{76}$$

$$= 10^6 + 10^6 + \frac{10^6 . 10^6}{2 \times 10^4} \tag{77}$$

$$R_{eq} = 52 \times 10^6 \text{ ohms} \tag{78}$$

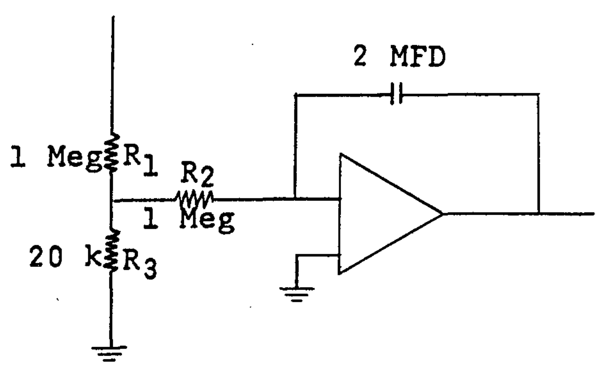


Figure 21. Resistance Tee Network

The transfer function for the integrator can be written as

$$\frac{e_4(s)}{e_2(s)} = -\frac{1}{52 \times 10^6 \times 2 \times 10^6} \frac{1}{s} \quad (79)$$

with the time in seconds. Converting this into minutes gives

$$\frac{e_4(s)}{e_2(s)} = -\frac{1}{1.74s} \quad . \quad (80)$$

Amplifier number five was used to sum the output of the proportional gain amplifier and the integral control amplifiers. A unity gain was used on the summing amplifier. The transfer function for the combined proportional gain and integrating amplifiers was

$$\frac{e_5(s)}{e_2(s)} = -(\frac{100}{100})(-0.50) -(\frac{100}{100})(-\frac{1}{1.74s})$$

$$\frac{e_5(s)}{e_2(s)} = +0.50 \ (1 + \frac{1}{0.87s}) \quad . \quad (81)$$

Combining the transfer function for the error detector and the control amplifiers, the output of the controller can be related to the set point

$$\frac{e_5(s)}{e_2(s)} \cdot \frac{e_2(s)}{R(s)} = (1.27 \times 10^{-2})(.50)(1 + \frac{1}{.87s}) \quad (82)$$

$$\frac{e_5(s)}{R(s)} = 0.635 \times 10^2 \ (1 + \frac{1}{.87s}) \ \frac{\text{volts}}{\text{ohm}} \quad (83)$$

The determination of the transistor heater system transfer function, as reported in the experimental section, was $-75.9 \ \frac{\text{cal.}}{\text{volt}}$

The transfer function for the thermistor temperature feedback system, G_H, was experimentally determined to be -2610 ohms/C°. (reported in the experimental section).

The transfer function for the bath was given in equation (44) as

$$T(s) = \frac{1}{c_p ms + UA} P_{St}(s) + \frac{1}{c_p ms + UA} Q(s) + \frac{UA}{c_p ms + UA} T_R(s) \quad (44)$$

$$T(s) = \frac{1}{c_p m} \frac{1}{(s + UA/c_p m)} P_{St}(s) + \frac{1}{c_p m} \frac{1}{(s + UA/c_p m)} Q(s) + \frac{UA}{c_p m} \frac{1}{(s + UA/c_p m)} T_R(s) \quad (84)$$

$Q(s)$ = Deviation in heat entering through the heater
$P_{St}(s)$ = Deviation in heat entering due to the stirrer
$T_R(s)$ = Deviation in the temperature of the room
UA = Heat transfer coefficient from vessel area, $6.4 \frac{\text{cal.}}{\text{min. °C.}}$
$c_p m$ = Specific heat x mass of vessel, 840 cal./°C.
$T(s)$ = Deviation in the temperature of the bath.

$$T(s) = \frac{1}{840} \frac{1}{(s + \frac{6.4}{840})} P_{St}(s) + \frac{1}{840} \frac{1}{(s + \frac{6.4}{840})} Q(s) + \frac{6.4}{840} \frac{1}{(s + \frac{6.4}{840})} T_R(s) \quad (85)$$

Temperature Recorder

The temperature recording was made using a Leeds Northrup model R820-1 recorder to monitor the unbalance of a thermistor bridge circuit (Figure 22). The relationship between a change in the chart reading and a change in temperature is derived below.

$$\Delta E = \frac{E_o A}{2A} \frac{R - R_{th}}{R + R_{th}} \quad (86)$$

E_o = Bridge voltage 15 volt
A = 5,000 ohms
R_{th} = Resistance of thermistor
R = Resistance of decade box

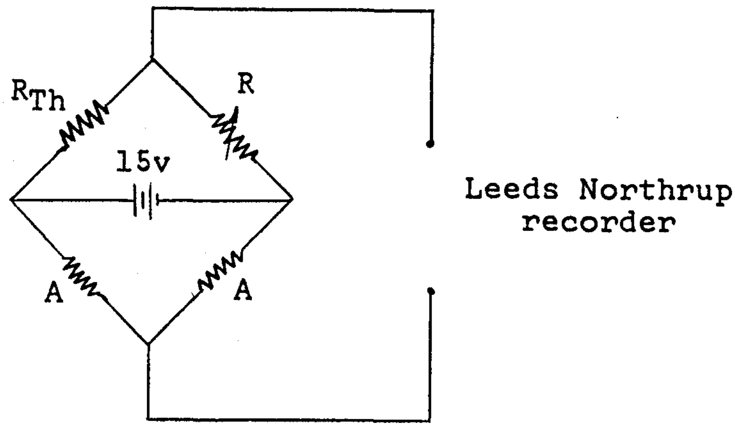


Figure 22. Thermistor Bridge Circuit

For small changes in temperature near the bridge balance point, equation (86) may be written

$$\Delta E = \frac{E_o}{2} \frac{\Delta R}{2R} \tag{87}$$

Evaluating equation (87) at 35°C., R was 66.8 K ohms and

$$\Delta E = \frac{15}{2 \cdot 2\ (66.8 \times 10^3)} \Delta R \tag{88}$$

$$\Delta E = (5.60 \times 10^{-5} \text{ volts/ohm})\ \Delta R \tag{89}$$

The recorder had a full scale reading of 12 millivolts for 100 scale divisions so

$$\Delta E = \left(\frac{12 \times 10^{-3}}{100} \text{ volts/scale division}\right) \Delta \text{ scale divisions} \tag{90}$$

Δ scale divisions

$$= \left(5.60 \times 10^{-5} \frac{100}{12 \times 10^{-3}} \frac{\text{scale divisions}}{\text{ohm}}\right) \Delta R \tag{91}$$

The relationship between a change in resistance and a change in temperature for the recorder thermistor as determined in the experimental section was

$$\Delta R = -(2870 \text{ ohm/°C.}) \Delta T$$

Δ scale divisions

$$= 5.60 \times 10^{-5} \times \frac{100}{12 \times 10^{-3}} \times -2870 \;\Delta T \tag{92}$$

$$\Delta T = -(7.4 \times 10^{-4} \frac{\text{°C.}}{\text{scale division}}) \;\Delta \text{ scale divisions} \tag{93}$$

<u>Calculated Response for a Step Change in Set Point</u>

Now that the system constants and differential equations have been developed, the control equations for a set point change can be written for the proportional control and for proportional plus integral control cases. Taking the proportional control equations first, the block diagram shown in Figure 23 yields equation (94).

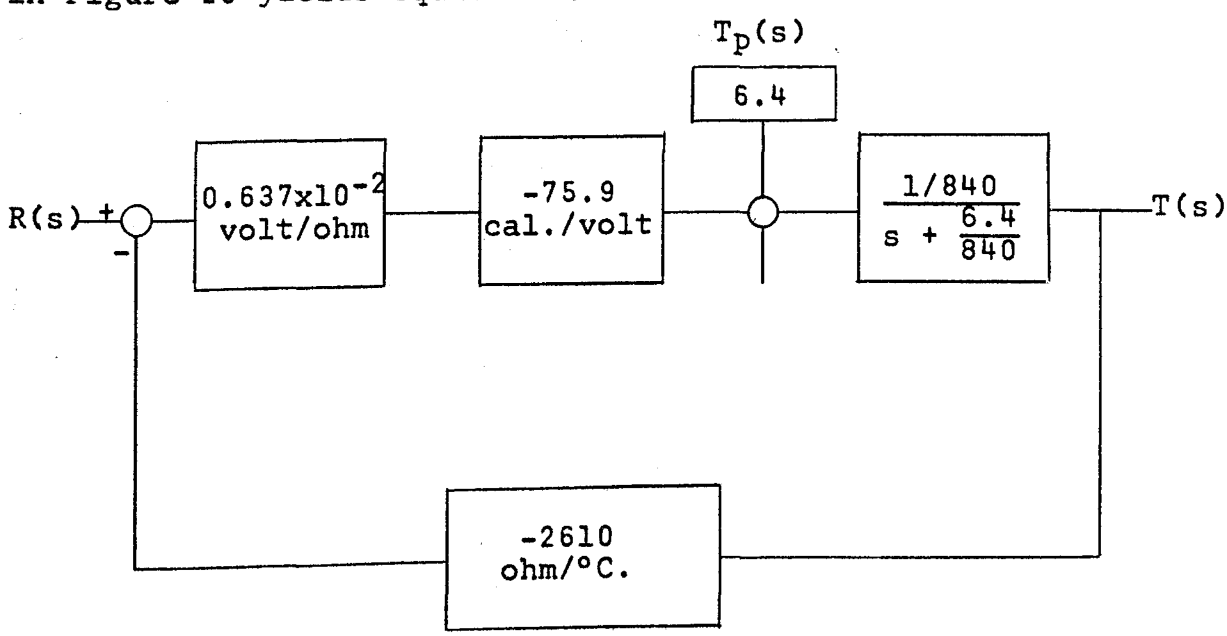


Figure 23. Block Diagram for Proportional Control

$$\frac{T(s)}{R(s)} = \frac{(0.637 \times 10^{-2})(-75.9)(\frac{1}{840})(\frac{1}{s + \frac{6.4}{840}})}{1 + (0.637 \times 10^{-2})(-75.9)(-2610)(\frac{1}{840})(\frac{1}{s + \frac{6.4}{840}})} \quad . \qquad (94)$$

Consolidating factors gives

$$\frac{T(s)}{R(s)} = \frac{-5.76 \times 10^{-2} \left(\frac{1}{s + 0.0076}\right)}{1 + 150 \left(\frac{1}{s + 0.0076}\right)} \qquad (95)$$

Multiplying numerator and denominator by (s + 0.0076) gives

$$\frac{T(s)}{R(s)} = \frac{-5.76 \times 10^{-4}}{s + 1.508} \qquad (96)$$

This equation may be evaluated for a unit step change in set point by letting $R(s) = \frac{1}{s}$.

$$T(s) = \frac{-5.76 \times 10^{-4}}{s\,(s + 1.508)} \qquad (97)$$

The inverse transform may be obtained

from $\qquad \int f(t) = \frac{1}{s}Lf(t) = \frac{1}{s}f(s);$

therefore, $L^{-1}\,\frac{1}{s}f(s) = \int_0^t f(t).$

The inverse transform of $\frac{1}{s + a} = e^{-at}$.

$$T(t) = -5.76 \times 10^{-4} \int_0^t e^{-1.508t} \qquad (98)$$

$$= -5.76 \times 10^{-4} \left[\frac{1}{1.508} \left(e^{-1.508t} - e^{o}\right)\right] \qquad (99)$$

$$= -\frac{5.76 \times 10^{-4}}{1.508} \left(1 - e^{-1.508t}\right) \qquad (100)$$

$$T(t) = -3.81 \times 10^{-4} \left(1 - e^{-1.508t}\right) \frac{°C.}{\sqcap \text{Set Point Change}} \qquad (101)$$

This is the desired result - a relation between a step change in set point and the temperature of the bath at any time after the set point change.

The corresponding equation for proportional plus integral control will be developed next. The block diagram is the same as the proportional control diagram except for the inclusion of the term $(1 + \frac{1}{.87s})$ representing the integral control action.

$$\frac{T(s)}{R(s)} = \frac{(0.637 \times 10^{-2})(1 + \frac{1}{.87s})(-75.9)(\frac{1}{840})(\frac{1}{s + \frac{6.4}{840}})}{1 + (0.637 \times 10^{-2})(1 + \frac{1}{.87s})(-75.9)(-2610)(\frac{1}{840})(\frac{1}{s + \frac{6.4}{840}})} \qquad (102)$$

Consolidating terms gives

$$\frac{T(s)}{R(s)} \quad \frac{-5.76 \times 10^{-4}(1 + \frac{1}{.87s})(\frac{1}{s + \frac{6.4}{840}})}{1 + 1.50\ (1 + \frac{1}{.87s})(\frac{1}{s + \frac{6.4}{840}})} \qquad (103)$$

$$\frac{T(s)}{R(s)} = \frac{-5.76 \times 10^{-4}\ (\frac{s + 1.15}{s})(\frac{1}{s + 0.0076})}{1 + 1.50\ (\frac{s + 1.15}{s})(\frac{1}{s + 0.0076})} \qquad (104)$$

Multiplying numerator and denominator by s (s + 0.0076) gives

$$\frac{T(s)}{R(s)} = \frac{-5.76 \times 10^{-4}\ (s + 1.15)}{s\ (s + 0.0076) + 1.50\ (s + 1.15)} \qquad (105)$$

$$\frac{T(s)}{R(s)} = \frac{-5.76 \times 10^{-4}\ (s + 1.15)}{s^2 + 1.508s + (1.50)(1.15)} \qquad (106)$$

For a unit step change in set point $R(s) = \frac{1}{s}$, and equation (106) yields

$$T(s) = \frac{-5.76 \times 10^{-4}\ (s + 1.15)}{s\ (s^2 + 1.508s + 1.725)} \qquad (107)$$

The inversion from Laplace transform variable to the time variable will be accomplished using the partial fractions method. As a first step the quadratic term is factored.

$$T(s) = \frac{-5.76 \times 10^{-4} (s + 1.15)}{s (s + 0.754 + 1.08j)(s + 0.754 - 1.08j)} \quad (108)$$

The equation is then separated into a series of fractions.

$$\frac{-5.76 \times 10^{-4} (s + 1.15)}{s (s + 0.754 + 1.08j)(s + 0.754 - 1.08j)}$$

$$= \frac{A}{s} + \frac{B}{s + 0.754 + 1.08j} + \frac{C}{s + 0.754 - 1.08j} \quad (109)$$

To determine A, both sides of the equation will be multiplied by s, and the resulting equation will be solved at s = 0.

$$\frac{-5.76 \times 10^{-4} (s + 1.15)}{(s + 0.754 + 1.08j)(s + 0.754 - 1.08j)}$$

$$= A + \frac{Bs}{s + 0.754 + 1.08j} + \frac{Cs}{s + 0.754 - 1.08j} \quad (110)$$

$$\frac{-5.76 \times 10^{-4} (1.15)}{(0.754 + 1.08j)(0.754 - 1.08j)} = A \quad (111)$$

$$A = \frac{-5.76 \times 10^{-4} (1.15)}{1.725}$$

$$A = -3.83 \times 10^{-4} \quad (112)$$

To evaluate B, equation (109) will be multiplied by (s + 0.754 + 1.08j), and the resulting equation will be solved at s = -754 - 1.08j.

$$\frac{-5.76 \times 10^{-4} (s + 1.15)}{s (s + 0.754 - 1.08j)}$$

$$= \frac{A (s + 0.754 + 1.08j)}{s} + B$$

$$+ \frac{C (s + 0.754 + 1.08j)}{(s + 0.754 - 1.08j)} \quad (113)$$

$$\frac{-5.76 \times 10^{-4} (-.754 - 1.08j + 1.15)}{(-754 - 1.08j)(-.754 - 1.08j + .754 - 1.08j)} = B \qquad (114)$$

$$\frac{-5.76 \times 10^{-4} (0.396 - 1.08j)}{(-0.754 - 1.08j)(-2.16j)} = B \qquad (115)$$

$$\frac{-5.76 \times 10^{-4} (0.396 - 1.08j)}{2.33j^2 + 1.63j} = B \qquad (116)$$

$$\frac{-5.76 \times 10^{-4} (0.396 - 1.08j)(-2.33 - 1.63j)}{(-2.33 + 1.63j)(-2.33 - 1.63j)} = B \qquad (117)$$

$$\frac{-5.76 \times 10^{-4} (-2.682 + 1.875j)}{+5.43 - 2.63j^2} = B \qquad (118)$$

$$(+1.915 - 1.34j)\ 10^{-4} = B \qquad (119)$$

By similar calculations C can be shown to be $(+1.915 + 1.34j)\ 10^{-4}$. Equation (109) can be rewritten

$$T(s) = \frac{-3.83 \times 10^{-4}}{s} + \frac{(1.915 - 1.34j)\ 10^{-4}}{s + 0.754 + 1.08j} + \frac{(1.915 + 1.34j)\ 10^{-4}}{s + 0.754 - 1.08j} \qquad (120)$$

The inverse transform of equation (120) is

$$T(t) = -3.83 \times 10^{-4} + e^{-0.765t} \times 10^{-4} (3.83\cos 1.08t - 2.68\sin 1.08t)\ ^\circ C. \qquad (121)$$

The sin cos portion of this equation may be simplified by using polar coordinates.

$$p \cos A + q \sin A = r \sin (A + \theta)$$

$$r = (p^2 + q^2)^{1/2} \qquad = \tan^{-1} \frac{p}{q}$$

$$T(t) = -3.83 \times 10^{-4} + e^{-0.754t}\ 4.67 \times 10^{-4} \sin \left(\frac{1.08\ (360)}{2\pi} t + 125^\circ\right) \qquad (122)$$

This is the desired relationship for the time response of the temperature to a step change in the set point.

[The negative sign in equations (101) and (122) is caused by the nature of a thermistor. A decrease in thermistor resistance corresponds to an increase in temperature. A decrease of 20 ohms in set point will result in an increase in the thermostat temperature.]

Calculated Response for a Step Change in Load

The dynamic response of the system to a change in load was investigated by making a step change in the stirring speed. The control equations can be written for proportional control and proportional plus integral control cases. Starting with the block diagram for proportional control (Figure 24),

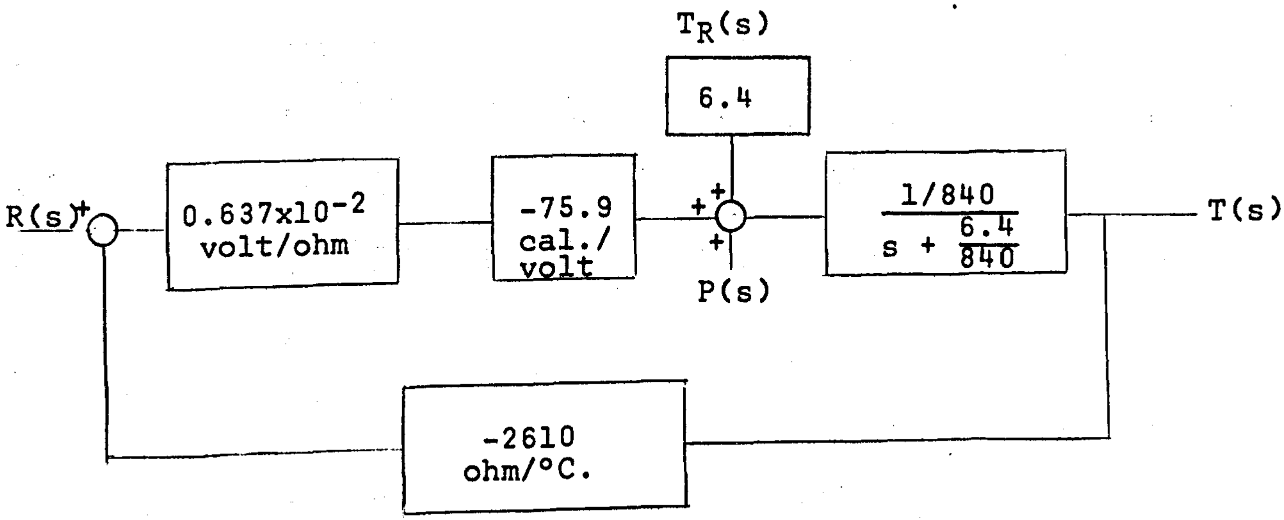


Figure 24. Block Diagram for Proportional Control

the control equation for a change in load, P(s), can be written

$$\frac{T(s)}{P(s)} = \frac{\frac{1}{840}\left(\frac{1}{s + \frac{.064}{840}}\right)}{1 + 0.637 \times 10^{-2} \times (-75.9)(-2610)\left(\frac{1}{840}\right)\left(\frac{1}{s + \frac{6.4}{840}}\right)} \qquad (123)$$

Consolidating terms gives

$$\frac{T(s)}{P(s)} = \frac{\frac{1}{840}\left(\frac{1}{s + 0.0076}\right)}{1 + 1.50\left(\frac{1}{s + 0.0076}\right)} \quad . \qquad (124)$$

Multiplying by (s + 0.0076) in the numerator and denominator, equation (124) becomes

$$\frac{T(s)}{P(s)} = \frac{0.119 \times 10^{-2}}{s + 0.0076 = 1.50} \qquad (125)$$

Equation (125) can be evaluated for a unit step change in load by setting the forcing function $P(s) = \frac{1}{s}$.

$$T(s) = \frac{0.119 \times 10^{-2}}{s\,(s + 1.508)} \qquad (126)$$

The inversion of equation (126) can proceed along lines similar to the inversion of equation (97).

$$T(t) = 0.119 \times 10^{-2} \int_{0}^{t} e^{-1.508t} \, dt \qquad (127)$$

$$T(t) = -\frac{0.119}{1.508} \times 10^{-2} \left(e^{-1.508t}\right|_{0}^{t} \qquad (128)$$

$$T(t) = 7.9 \times 10^{4} \, (1 - e^{-1.508t}) \; {}^{\circ}C. \qquad (129)$$

This is the temperature response of the proportional controlled thermostat to a step change in load of 1 calorie/min.

The thermostat temperature response for a step change in load will now be derived for the proportional plus integral control case. With proportional plus integral control, the controller transfer function is modified to include the term $(1 + \frac{1}{0.87s})$ representing the integral control.

$$\frac{T(s)}{P(s)} = \frac{\dfrac{1/840}{(s + \frac{6.4}{840})}}{1 + .637 \times 10^{-2} \times (1 + \frac{1}{.87s})(-75.9)(-2610)(\frac{1}{840})(\dfrac{1}{s + \frac{6.4}{840}})} \qquad (130)$$

Consolidating factors gives

$$\frac{T(s)}{P(s)} = \frac{\frac{1}{840} (\frac{1}{s + .0076})}{1 + 1.50 (\frac{s + 1.15}{s})(\frac{1}{s + 0.0076})} \; . \qquad (131)$$

Multiplying the numerator and denominator by s (s + 0.0076)

gives

$$\frac{T(s)}{P(s)} = \frac{\frac{1}{840} s}{s (s + 0.0076) + 1.50s + (1.50)(1.15)} \; . \qquad (132)$$

$$\frac{T(s)}{P(s)} = \frac{0.119 \times 10^{-2} s}{s^2 + 1.508s + 1.725} \qquad (133)$$

For a unit step change in load $P(s) = \frac{1}{s}$.

$$T(s) + \frac{0.119 \times 10^{-2} \not{s}}{\not{s} (s^2 + 1.508s + 1.725)} \qquad (134)$$

Inversion of equation (134) may proceed through the partial fractions method. Factoring the denominator, equation (134) gives

$$T(s) \quad \frac{0.119 \times 10^{-2}}{(s + 0.754 + 1.08j)(s + 0.754 - 1.08j)} \qquad (135)$$

$$\frac{0.119 \times 10^{-2}}{(s + 0.754 + 1.08j)(s + 0.754 - 1.08j)}$$

$$= \frac{A}{(s + 0.754 + 1.08j)}$$

$$+ \frac{B}{(s + 0.754 - 1.08j)} \qquad (136)$$

Solving for A by multiplying by (s + 0.754 + 1.08j) gives

$$\frac{0.119 \times 10^{-2}}{(s + 0.754 - 1.08j)} = A + \frac{B\,(s + 0.754 + 1.08j)}{(s + 0.754 - 1.08j)} \quad (137)$$

At $s = -0.754 - 1.08j$

$$\frac{0.119 \times 10^{-2}}{-0.754 - 1.08j + 0.754 - 1.08j} = A \quad (138)$$

$$\frac{0.119 \times 10^{-2}}{-2.16j} = A \quad (139)$$

By a similar process B was found equal to $-0.055 \times 10^{-2}j$. Equation (136) can be written

$$T(s) = \frac{.055 \times 10^{-2}j}{s + 0.754 + 1.08j} + \frac{-0.055 \times 10^{-2}j}{s + 0.754 - 1.08j} \quad (141)$$

The inversion of equation (141) is

$$T(t) = e^{-0.754t}\,(.110 \times 10^{-2})\sin(1.08 \cdot \frac{360}{2\pi}\, t) \quad (142)$$

This is the desired response of the thermostat to a step change in load with proportional plus integral control.

RESULTS

Long Term Stability of Bath

On two occasions the bath temperature was recorded for 16 hour periods, the recorder calibration having been checked with a Beckman Thermometer. In both instances temperatures were controlled within 0.005°C. of the set point and ±0.0025°C. of the median value (Figure 25).

Observation of the Transient Response Characteristics

The control system response to step set point changes and to step load changes was observed experimentally for a proportional control and a proportional plus integral control system and compared with the computed control curves. Proportional control was obtained by placing a jumper wire around the capacitor in the integrator controller, making the output of amplifier number 4 in Figure 16 equal to zero. The first step in the procedure was to bring the system to stable operating conditions. A step change in set point was then made, and the change in temperature was observed over a period of time.

A separate plot was made of the voltage across the bath heater. Samples of the temperature recorder charts are shown in Figures 26 and 27.

In order to remain in the linear portion of the transfer function, the changes were restricted to less than 40 ohms,

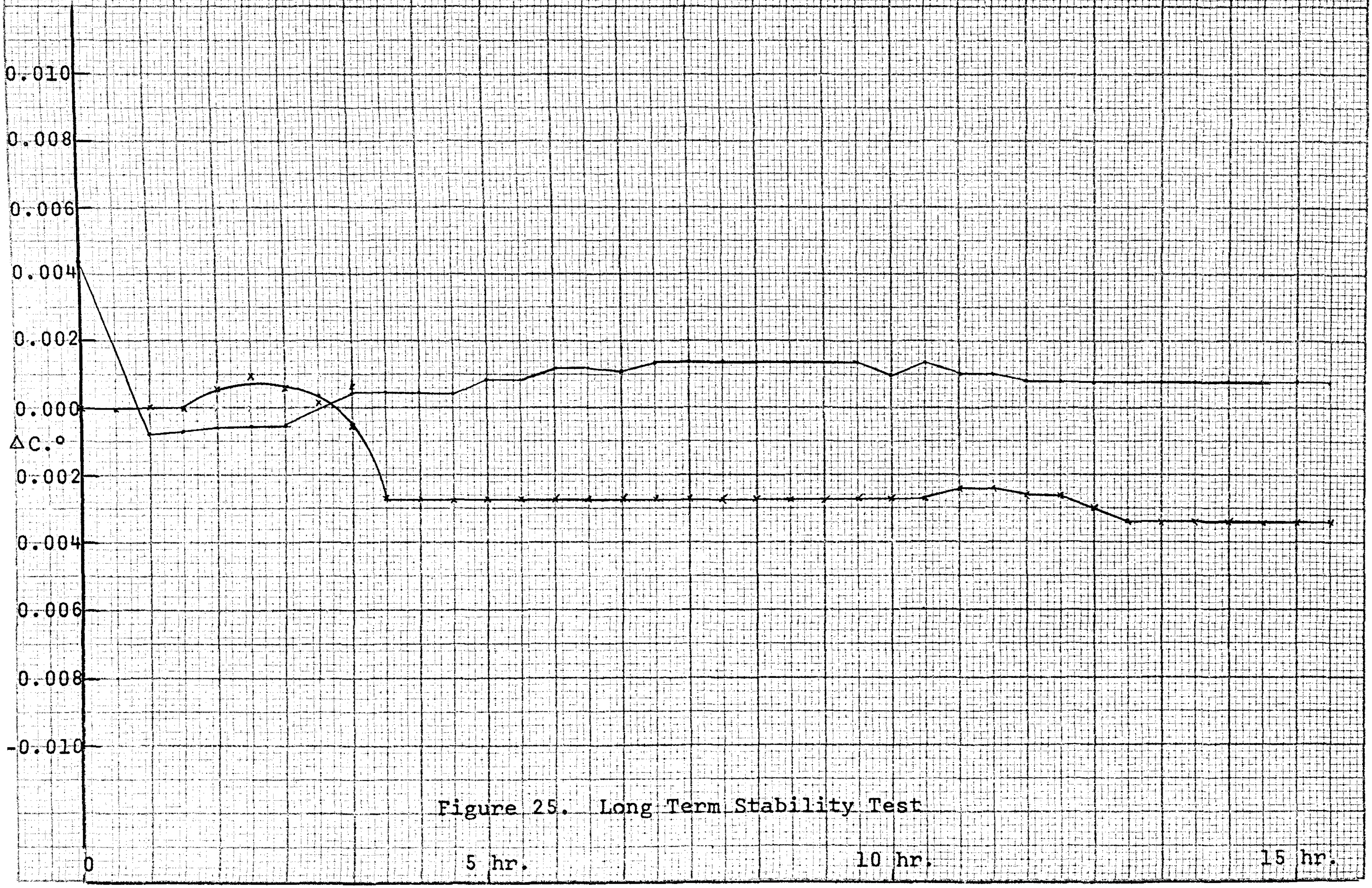
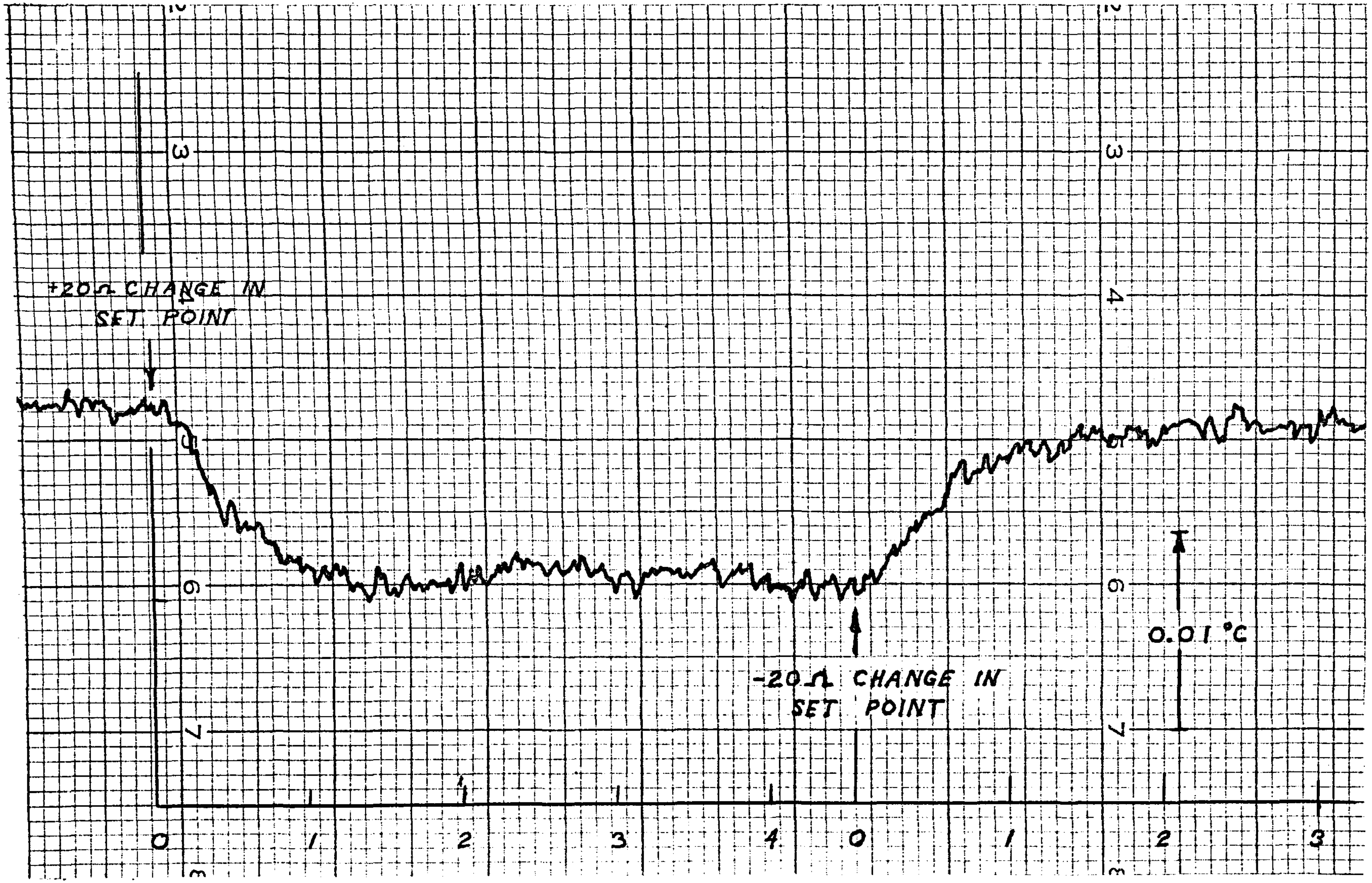
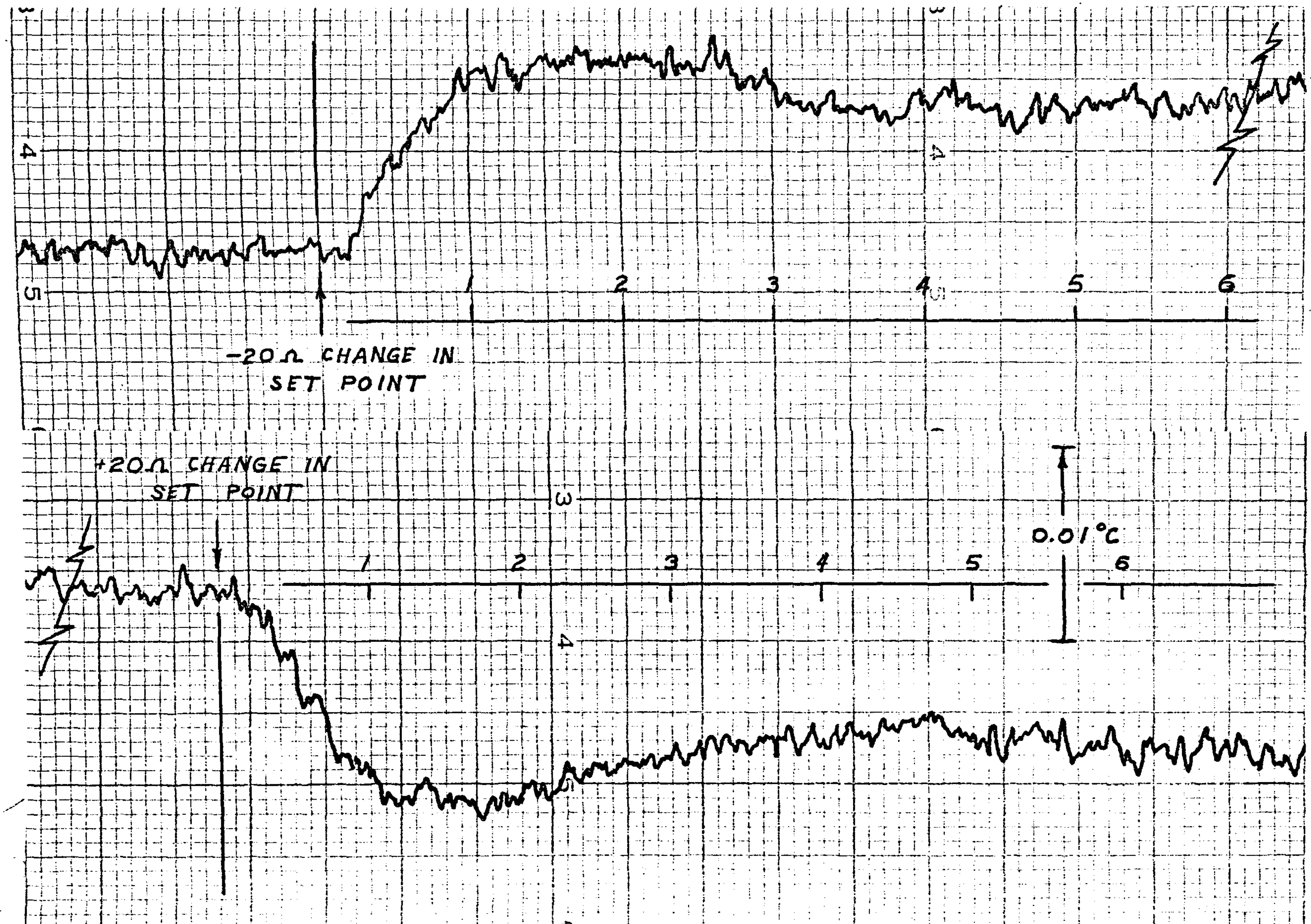


Figure 25. Long Term Stability Test



Time, Minutes

Figure 26. Transient Response to a Change in Set Point
Proportional control temperature chart

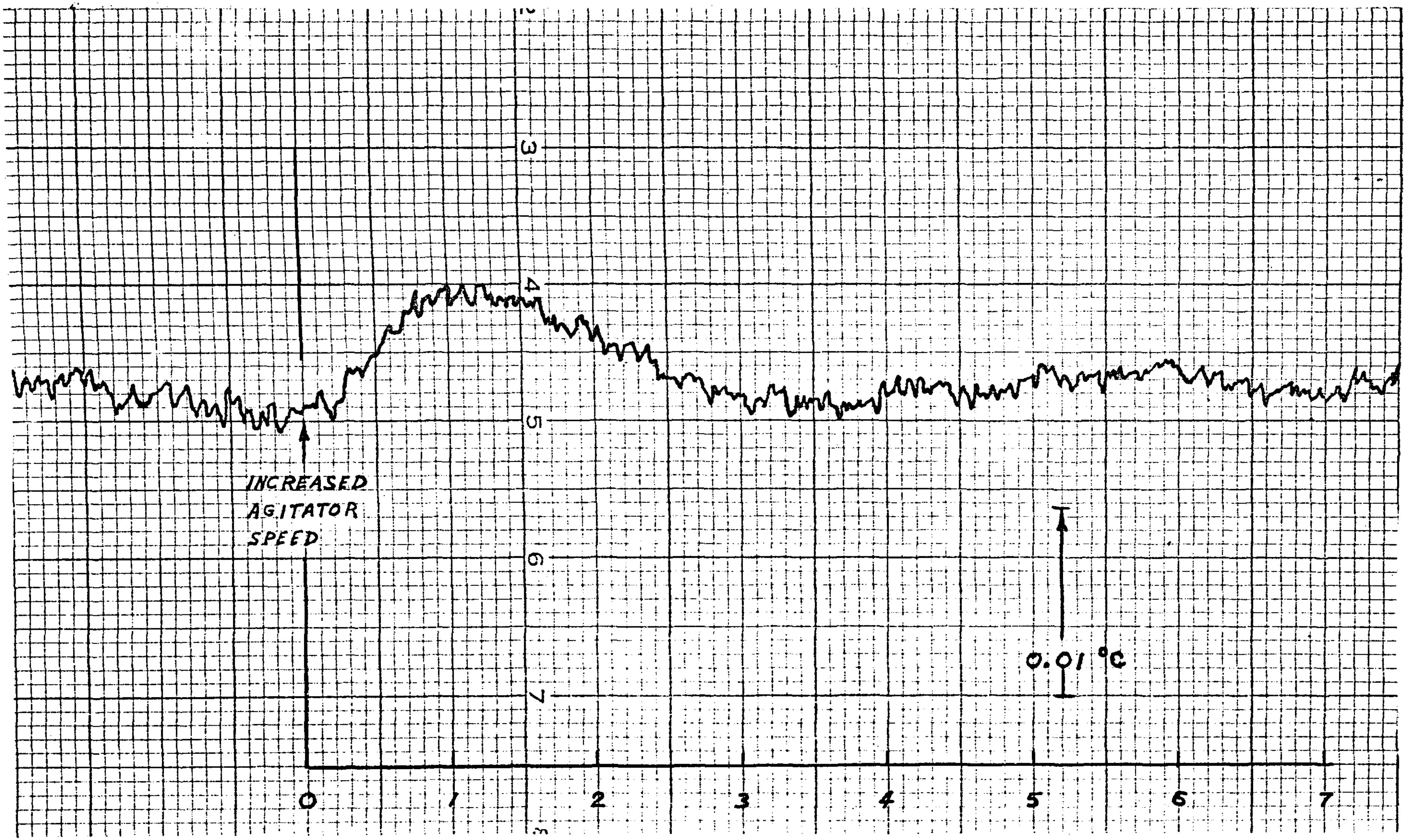


Time, Minutes

Figure 27. Transient Response to Change in Set Point
Proportional plus integral control

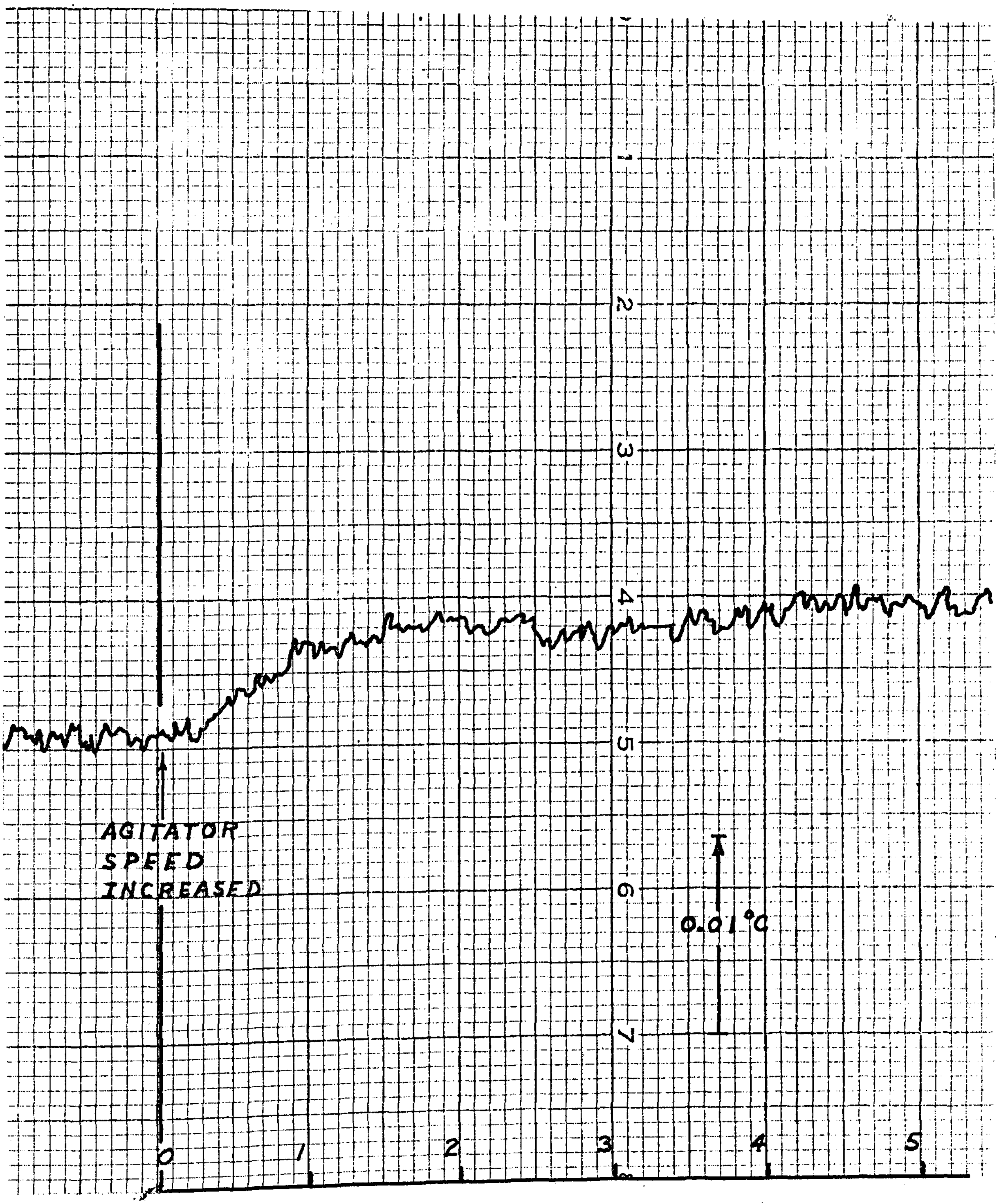
which correspond to a change of $\leq 0.016°C$. Larger step changes caused the output of the heater to enter the nonlinear portion of the transfer function.

A second type of control problem occurs when a change is made in the input condition (a load change). Load change observations were made by making step changes in the stirring speed and recording the change in the bath temperature and bath heater voltage. Samples of the temperature charts are shown in Figures 28 and 29.



Time, Minutes

Figure 28. Transient Response to Change in Load
Proportional plus integral control



Time, Minutes

Figure 29. Transient Response to a Change in Load
Proportional control

DISCUSSION OF RESULTS

Uncertainty of the Calculated Responses

Are the calculated and observed responses of the thermostat in agreement within the uncertainties of the system constants? This question will be broken into several phases. The first concerns the temperature of the thermostat after the transient disturbances have vanished following step changes in set point and load. The constants which have the greatest effect upon the final temperature can be determined by using the final value theorem for the Laplace transform of the control equation. This theorem states that the $\lim_{t \to \infty} [f(t)] = \lim_{s \to 0} [sf(s)]$.

The final value theorem will be applied to the control equation for a set point change with proportional plus integral control.

$$T(s) = \frac{-5.76 \times 10^{-4}\ (s + 1.15)}{s\ (s^2 + 1.508s + 1.15 \times 1.50)} \qquad (143)$$

From the final value theorem the

$$\lim_{t \to \infty} T(t) = \lim_{s \to 0} s\,T(s)\ .$$

$$sT(s) = \frac{-5.76 \times 10^{-4}\ (s + 1.15)\ \cancel{s}}{\cancel{s}\ (s^2 + 1.508s + 1.15 \times 1.50)} \qquad (144)$$

$$\lim_{s \to 0} sT(s) = \frac{-5.76 \times 10^{-4}\ \cancel{(1.15)}}{\cancel{1.15} \times 1.50} \qquad (145)$$

In order to determine the most important transfer functions, equation (145) must be expanded to show the individual factors making up the numerator and denominator.

$$\lim_{s \to 0} sT(s) = \frac{.637 \times 10^{-2} \times (-75.9)(\frac{1}{840})}{.637 \times 10^{-2} \times (-75.9)(-2610)(\frac{1}{840})}$$

$$= \frac{-5.76 \times 10^{-4}}{1.50} \qquad (146)$$

$$\lim_{s \to 0} sT(s) = \frac{1}{-2610} = -3.83 \times 10^{-4}\ ^{\circ}C. \qquad (147)$$

$$\lim_{s \to 0} sT(s) = \frac{1}{K_H} \qquad (148)$$

where K_H = thermistor feedback transfer function

The important result is that with proportional plus integral control the final temperature for a step change in load is entirely determined by the thermistor feedback transfer function. The feed back fraction was determined by the equation $\frac{R(s)}{T(s)} = -\frac{AR}{T^2}$. As discussed in the experimental section, the value of A was determined by a least squares slope of lnR vs $\frac{1}{T}$. The uncertainty of the slope and the uncertanties of the other factors will be used to calculate the uncertainty in the calculated temperature change.

$$\frac{R(s)}{T(s)} = -\frac{AR}{T^2} \qquad (149)$$

$A = (1.83 \pm .08)10^3\ ^{\circ}K$
$R = 58.8 \times 10^3 \pm 0.2$ ohms
$T = 308.2 \pm .1\ ^{\circ}K$

In combining several terms of varying precision, the variances of the individual measurements may be combined.

The general relationship for the variance of Y where

$$Y = f(x_i)$$

is

$$\sigma^2 Y = \Sigma \left(\frac{\partial f}{\partial x_i}\right)^2 \sigma^2 x_i \quad . \tag{150}$$

The uncertainty in Y is then the square root of the sum of the variances (15). As an example if

$$Y = AX + BZ \tag{151}$$

then

$$Y \pm y = [(A)^2(\sigma x)^2 + (X)^2(\sigma A)^2 + (B)^2(\sigma Z)^2 + (Z)^2(\sigma B)^2]^{1/2} \tag{152}$$

This procedure takes into account the probability that some of the uncertainties will be of opposite sign and hence will have cancelling effects.

The uncertainty in the calculated temperature change was

$$\lim_{t \to \infty} T(t) = \frac{1}{K_H} \Delta R = \frac{1}{\frac{R(s)}{T(s)}} \Delta R = \frac{R(s)}{T(s)} \Delta R \tag{153}$$

$$= -\frac{(308.2 \pm .1)^2 \qquad \Delta R \pm 1\%}{2.303\ (1.83 \pm .08)\ 10^3[(58.8 \pm 0.2) \times 10^3]} \tag{154}$$

$$= -(3.83 \pm 5\%)\, 10^{-4}\ °C./ohm \tag{155}$$

$$= -(3.83 \pm 0.2)\ 10^{-4}\ °C./ohm \tag{156}$$

for a 40 ohm change in set point

$$\lim_{t \to \infty} T(t) = -(3.83 \pm 0.2)\ 10^{-4}\ (40) \tag{157}$$

$$\lim_{t \to \infty} T(t) = -(1.53 \pm 0.08)\ 10^{-2}\ °C. \tag{158}$$

A similar analysis of the control equation for a unit step set point change with proportional control alone shows the final value to be dependent upon

$$\lim_{t \to \infty} T(t) = \lim_{s \to 0} sT(s) = \frac{1}{K_H + \frac{UA}{K_c K_f}} = \frac{1}{-2610 - 15} \qquad (159)$$

$$\lim_{t \to \infty} T(t) = -3.80 \times 10^{-4} \ ^\circ C./ohm \qquad (160)$$

Because of the relatively small magnitude of the term $\frac{UA}{K_c K_f}$ in comparison with the term K_H, the uncertainty in the final temperature is, as was found in the proportional plus integral control case, $(-3.80 \pm 0.2)\ 10^{-4}$ °C./ohm.

When the final value theorem is applied to a load change, the basic difference between proportional plus integral control and proportional control can be seen. For proportional plus integral control (161)

$$\lim_{t \to \infty} T(t) = \lim_{s \to 0} sT(s) = \frac{\not{s}\ 0.119 \times 10^{-2}\ s}{\not{s}\ [s^2 + 1.508s + (1.50)(1.15)]} \Delta P$$

$$\lim_{t \to \infty} T(t) = 0 \qquad (162)$$

For proportional control alone the response to a load change is

$$\lim_{t \to \infty} T(t) = \lim_{s \to 0} sT(s) = \frac{\not{s}\ (0.119 \times 10^{-2})\ \Delta P}{\not{s}\ (s + 1.508)} \qquad (163)$$

$$\lim_{s \to 0} sT(s) = \frac{0.119 \times 10^{-2}}{1.508} \Delta P \qquad (164)$$

$$\lim_{t \to \infty} T(t) = 7.9 \times 10^{-4} \ ^\circ C./cal./min. \ \Delta P \qquad (165)$$

The system components which have an effect on the final value can be found by expanding equation (165) to show the individual transfer functions.

$$\lim_{t\to\infty} T(t) = \lim_{s\to o} sT(s) = \frac{\frac{1}{c_p m}}{\frac{1}{c_p m} K_c K_f K_H + \frac{UA}{c_p m}} \Delta P \qquad (166)$$

$$= \frac{1}{K_c K_f K_H + UA} \Delta P \qquad (167)$$

Inserting the appropriate values for the individual terms gives

$$\lim_{t} T(t) = \frac{1}{(0.64 \pm .04)(-75.9 \pm 5)(2610 \pm 100) + (6.4 \pm .4)} \Delta P \qquad (168)$$

Because of its relatively small magnitude, the second term may be dropped.

$$\lim_{t\to\infty} T(t) = 7.9 \times 10^{-4} \pm[(6.2\%)^2 + 6.3\%)^2 + 4.4\%)^2]^{1/2} [\Delta P \pm 6\%] \qquad (169)$$

$$\lim_{t} T(t) = 7.9 \times 10^{-4} \Delta P \pm [38 + 40 + 19 + 36]^{1/2} \qquad (170)$$

$$\lim_{t} T(t) = [7.9 \times 10^{-4} \Delta P] \pm 11\% \qquad (171)$$

Uncertainty of Temperature Recording

An additional source of uncertainty in the thermostat temperature is associated with the temperature recorder. The uncertainty in the recording includes the uncertainties connected with the bridge circuit and the noise in the record chart recording.

The uncertainty in the bridge circuit relationship between ΔT and the Δ scale divisions is determined by the resistor uncertainties and by the uncertainty in the thermistor resistance change with temperature.

In the calculations section, the relationship between the change in temperature and the change in chart reading was shown to be

$$\Delta T = -(7.4 \times 10^{-4} \frac{°C.}{\text{scale division}})(\Delta \text{scale divisions}) \tag{172}$$

Expanding the constant so that the individual factors and their uncertainties are shown, equation (172) gives

$$\Delta T = \frac{2(5000 \pm 50 \text{ ohm})(66800 \pm 700 \text{ ohm})2(12 \times 10^{-3} \text{ volt})}{(15 \pm .1 \text{ volt})(5000 \pm 50 \text{ ohm}) \quad (100)(-2870 \pm 60 \text{ ohms})} \times (\Delta \text{scale division}) \tag{173}$$

$$\Delta T = [-7.4 \times 10^{-4} \pm((1\%)^2 (1\%)^2 + (1\%)^2 +(2.1\%)^2)^{1/2}] \times [\Delta \text{scale divisions}] \tag{174}$$

$$\Delta T = [-7.4 \times 10^{-4} \pm (8.4\%)^{1/2}] [\Delta \text{ scale divisions}] \tag{175}$$

$$\Delta T = [-7.4 \times 10^{-4} \pm 3\%] [\Delta \text{scale divisions}] \tag{176}$$

$$\Delta T = [(-7.4 \pm .2) 10^{-4}] [\Delta \text{scale divisions}] \tag{177}$$

In addition to the component uncertainties, the temperature recording had a random variation of ± 1 scale division which had the appearance of electrical noise. The average change in the chart reading was 15 scale divisions making the uncertainty $\pm 6.6\%$ with a minimum uncertainty of $\pm 7 \times 10^{-4}$ °C.

The uncertainty in the temperature recording is

$$\Delta T = -7.4 \times 10^{-4} \times \Delta \text{scale divisions} \pm ((3\%)^2 + (6.6\%)^2))^{1/2} \tag{178}$$

$$\Delta T = [-7.4 \times 10^{-4} \pm (9 + 44)^{1/2}] \times [\Delta \text{ scale divisions}] \tag{179}$$

$$\Delta T = [-7.4 \times 10^{-4} \pm 7] \quad \Delta \text{ scale divisions} \tag{180}$$

$$\Delta T = [-(7.4 \pm 0.5) \times 10^{4}\ ^{\circ}C.] \quad \Delta \text{ scale divisions} \tag{181}$$

The results of the uncertainty analyses are summarized in the following table.

TABLE III

Change	Control System	Calculated Response Dependent On	Calculated Response	Observed Response	Are values within experimental uncertainty?
40 ohm Set Point	Proportional plus Integral	$\frac{1}{K_H}$	$-(1.53 \pm .08)\ 10^{-2}\ ^{\circ}C.$	$-(1.63 \pm .11)\ 10^{-2}\ ^{\circ}C.$	Yes
40 ohm Set Point	Proportional	$\frac{1}{K_H + \frac{UA}{K_c K_f}}$	$-(1.52 \pm .08)\ 10^{-2}\ ^{\circ}C.$	$-(1.63 \pm .11)\ 10^{-2}\ ^{\circ}C.$	Yes
13.5 cal./min. Load Change	Proportional plus Integral	No long term deviation from set point	0	$(0.0 \pm .07)\ 10^{-2}\ ^{\circ}C.$	Yes
10 cal./min. Load Change	Proportional	$\frac{1}{K_c K_f K_H + UA}$	$(.79 \pm .09)\ 10^{-2}\ ^{\circ}C.$	$(.65 \pm .07)\ 10^{-2}\ ^{\circ}C.$	Yes

Transient Response Decay and Period of Oscillation for Proportional plus Integral Control

The transient response decay and period of oscillation, as calculated from the system control equation, were $e^{-0.75t}$ and 5.8 minutes respectively. The effect of changes in the proportional gain and integrator time constant can be shown graphically with a root locus plot. As discussed earlier when the denominator of the closed loop system equation, 1 + G, is set equal to zero, the result is called the characteristic equation of the closed loop system. The roots of the characteristic equation determine the form of the system response to a forcing function. A root locus plot is a graphical display of the value of the roots with changing proportional gain. A root locus plot for the closed loop characteristic equation of the thermostat is shown in Figure 30. The plot was developed for proportional plus integral control in the following manner:

1. The open loop zeros and poles are determined from the open loop transfer function (equation 182)

$$G = K \frac{N}{D} = \frac{K\,(s - Z_1)(s - Z_2)(s - Z_m)}{(s - p_1)(s - p_2)(s - p_m)} \tag{182}$$

$$G = \frac{K\,(s + 1.15)}{s\,(s + 0.0076)} \tag{183}$$

There is a zero located at -1.15 and poles located at zero and -0.0076.

2. Because there are two poles, there are two branches to the location plot.

3. The number of asymptotes is equal to the number of poles minus the number of zeros, or in this case, one. The angle the asymptote makes with the real axis is

$$\measuredangle = \frac{\pi(2K + 1}{n - m} \tag{184}$$

$(K = 0, 1, 2 \cdots, n-m-1)$

$$\measuredangle = \frac{\pi}{1} \tag{185}$$

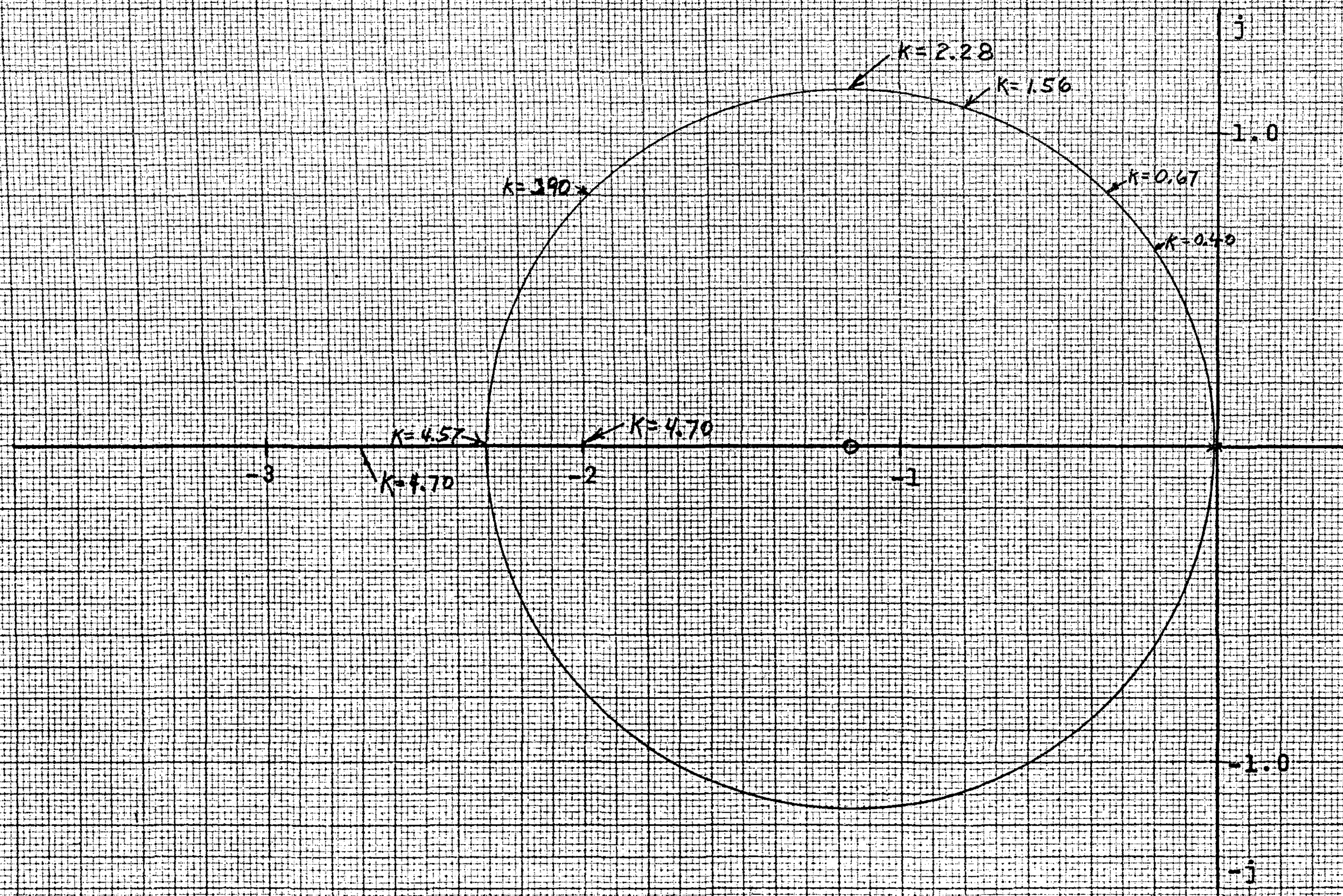


Figure 30. Root Locus Diagram

4. The real axis is a portion of the locus between 0 and -0.0076 and from -1.15 to minus infinity.

5. The locus breaks away from the real axis at

$$\sum \frac{1}{s - Z} = \sum \frac{1}{s - P} \tag{186}$$

$$\frac{1}{s + 1.15} = \frac{1}{s - o} + \frac{1}{s + .0076} \tag{187}$$

$$s^2 + .0076 = s^2 + 1.157s + .087 + s\,(s + 1.15) \tag{188}$$

$$o = s^2 + 2.30s + .0087 \tag{189}$$

$$s = \frac{-2.30 \pm (5.29 - (4)(.0087))^{1/2}}{2} \tag{190}$$

$$s = \frac{-2.30 \pm (5.255)^{1/2}}{2} \tag{191}$$

$$s = \frac{-2.30 \mp 2.293}{2} \tag{192}$$

$$s - -2.296, -.003 \tag{193}$$

6. The location of the locus as it leaves the real axis is determined by selecting a trial point and determining if the phase angle equation is satisfied.

$$\begin{aligned} &\measuredangle(s - Z_1) + \measuredangle(s - Z_2) + \cdots + \measuredangle(s - Z_m) \\ &- [\, \measuredangle(s - P_1) + \measuredangle(s - P_2) + \cdots + \measuredangle(s - P_n) \\ &= (2i + 1)\pi \end{aligned} \tag{194}$$

where

$\measuredangle$ is measured from the real axis to the line connecting the trial point and the pole or zero.

i is any positive or negative integer including zero.

For trial point -0.2, + 0.64j

$$34 - [106 + 108] \overset{?}{=} (2i + 1)\pi \quad (195)$$

$$34 - 214 \overset{?}{=} (2i + 1)\pi \quad (196)$$

$$-180° = (2i + 1)\pi \text{ for } i = -1 \quad (197)$$

After several points on the root locus have been plotted, the root locus can be sketched. The proportional gain at any point on the locus can be found from the magnitude criteria

$$K \frac{|s - Z_1|\ |s - Z_2| \cdots |s - Z_m|}{|s - P_1|\ |s - P_2| \cdots |s - P_m|} = 1 \quad (198)$$

The distances may be measured directly with a ruler in units consistent with those used on the graph axis.

For the point -.2, +.64j (199)

$$K \frac{1.15}{.68 \quad .68} = 1 \quad (200)$$

$$K = \frac{(.68)(.68)}{(1.15)} \quad (201)$$

$$K = 0.40 \quad (202$$

The pole at zero is inherent with integral control. The location of the pole at 0.0076 was determined by $\frac{UA}{c_p m}$, and the location of the zero at 1.15 was determined by the reciprocal of the integrator time constant.

From the root locus diagram it can be seen that only a large change in the proportional control constant would appreciably change the exponential decay (position on x axis) or cycle time (position on y axis).

The uncertainty in the exponential decay factor and cycle time can be determined from the factored control equation.

A Laplace transform of the form

$$\frac{a + jb}{s + K_1 + jK_2} + \frac{a - jb}{s + K_1 - jK_2} \tag{203}$$

inverts to an equation of the form

$$e^{-K_1 t} [-2a\cos K_2 t + 2bj\sin K_2 t] . \tag{204}$$

The exponential decay factor is determined by K_1, and the cycle time is determined by K_2.

In the present case K_1 and K_2 were determined by factoring the equation

$$s^2 + K_f K_c K_H \frac{1}{c_p m} s + K_f K_c K_H \frac{1}{c_p m} \frac{1}{\tau_I} = 0 \tag{205}$$

where

K_f = final control element constant

$= -75.9 \pm 5 \frac{\text{cal.}}{\text{volt}}$

K_c = controller constant

$= (0.64 \pm .04)\, 10^{-2} \frac{\text{volt}}{\text{ohm}}$

K_H = feed back constant

$= (-2610 \pm 100) \frac{\text{ohm}}{°\text{C.}}$

$\frac{1}{c_p m}$ = [(specific heat)(mass of bath)]$^{-1}$

$= \frac{1}{840 \pm 120}$

$\frac{1}{\tau_I}$ = Integrator time constant

$= \frac{1}{0.87 \pm .03}$.

The term K_1 is equal to $\frac{1}{2} (K_f K_c K_H \frac{1}{c_p m})$. The uncertainty in K_1 is

$$K_1 = \frac{1}{2}(-75.9 \pm 5)(0.64 \pm .04)\ 10^{-2}$$
$$(-2610 \pm 100)(840 \pm 85)^{-1} \quad (206)$$

$$K_1 = \frac{1}{2}(75.9)(.64 \times 10^{-2})(2610)(\frac{1}{840}) \quad (207)$$
$$\pm [(6.6\%)^2 + (6.2\%)^2 + (3.8\%)^2 + (10\%)^2]^{1/2}$$

$$K_1 = .754 \pm 14\% \quad (208)$$

$$K_1 = .754 \pm .11\ . \quad (209)$$

The uncertainty in the exponential decay is then as follows:

$$\text{exponential decay} = Y = e^{-K_1 t} \quad (210)$$

$$\text{uncertainty in } Y = \pm y = [(\frac{\partial e^{-K_1 t}}{\partial K_i})^2(\sigma^2 K)]^{1/2}$$

with $K_1 = 0.754$, $\sigma K_1 = \pm 0.11$ (211)

$$\pm y = e^{-0.754t}(-t)(\pm .11) \quad (212)$$

at t = 1 minute

$$\text{exponential decay} = 0.47 \pm 0.05 \quad (213)$$

at t = 3 minutes

$$\text{exponential decay} = 0.104 \pm 0.03\ . \quad (214)$$

The period of the oscillation is determined by K_2 in equation (204). The value of K_2 was determined by factoring the equation

$$s^2 + K_f K_c K_H \frac{1}{cpm} s + K_f K_c K_H \frac{1}{cpm} \frac{1}{\tau_I} = 0 \quad (215)$$

$$K_2 = \frac{\pm \sqrt{b^2 - 4C}}{2} \quad (216)$$

where

$$b = K_f K_c K_H \frac{1}{cpm}$$

$$c = K_f K_c K_H \frac{1}{c_p m} \frac{1}{\tau_I}$$

$$c = b \frac{1}{\tau_I} \; .$$

The change in K_2 with changes in b and τ_I is

$$dK_2 = \frac{\partial K_2}{\partial b}\partial b + \frac{\partial K_2}{\partial \tau_I}\partial \tau I \tag{217}$$

The uncertainty in K_2 would again be the square root of the squared partial derivatives.

$$\text{uncertainty} = [(\frac{\partial K}{\partial b}\partial b)^2 + (\frac{\partial K2}{\partial b}\partial b)^2]^{1/2} \tag{218}$$

From the previous case

$$b = 1.50 \pm 14\% \tag{219}$$

$$b = 1.50 \pm .21 \tag{220}$$

$$\frac{1}{\tau_I} = 1.15 \pm 3.5\% \tag{221}$$

$$= 1.15 \pm .04 \tag{222}$$

$$dK_2 = \frac{\partial K_2}{\partial b}\partial b + \frac{\partial K_2}{\partial \tau_I}\partial \tau_I \tag{223}$$

$$dK_2 = \pm\frac{1}{2}[\frac{1}{2}(b^2 - 4b\;(\tau_I)^{-1})^{-1/2}$$
$$(2b - 4\;(\tau_I)^{-1})\partial b$$
$$+ \frac{1}{2}(b^2 - 4b\;(\tau_I)^{-1})^{-1/2}$$
$$(-4b)\partial \tau_I{}^{-1}] \tag{224}$$

$$dK_2 = \pm\frac{1}{2}[\frac{1}{2}[1.50^2 - 4\ (1.50)(1.15)]^{-1/2}$$
$$(2\cdot 1.50 - 4\cdot 1.15)(\pm.21)$$
$$+ \frac{1}{2}[1.50^2 - 4\ (1.50)(1.15)]^{-1/2}$$
$$(-4\cdot 1.50)(\pm.04) \quad (225)$$

$$dK_2 = \pm\frac{1}{2}[\frac{1}{2}(\frac{1}{2.25 - 6.90})^{1/2}$$
$$(3.0 - 4.60)\ \pm.21$$
$$+ \frac{1}{2}(\frac{1}{2.25 - 6.90})^{1/2}$$
$$(-6.00)(\pm.04)] \quad (226)$$

$$dK_2 = \pm\frac{1}{2}[\frac{1}{2}(\frac{1}{-4.65})^{1/2}\ (-1.60)(\pm.21)$$
$$+ \frac{1}{2}(\frac{1}{-4.65})^{1/2}\ (-6.00)(\pm.04)] \quad (227)$$

$$dK_2 = [(-.215)^{1/2}\ (-.40)(\pm.21)$$
$$+ (-.215)^{1/2}\ (+.50)(\pm.04)] \quad (228)$$

The uncertainty in K_2 is then

$$\text{uncertainty } K_2 = [.215\ (.16)(.04)$$
$$+ .215\ (2.25)(.0016)]^{1/2} \quad (229)$$

$$\text{uncertainty } K_2 = [.0016 + .0008]^{1/2} \quad (230)$$

$$\text{uncertainty } K_2 = \pm[.05 \text{ radians}] \quad (231)$$

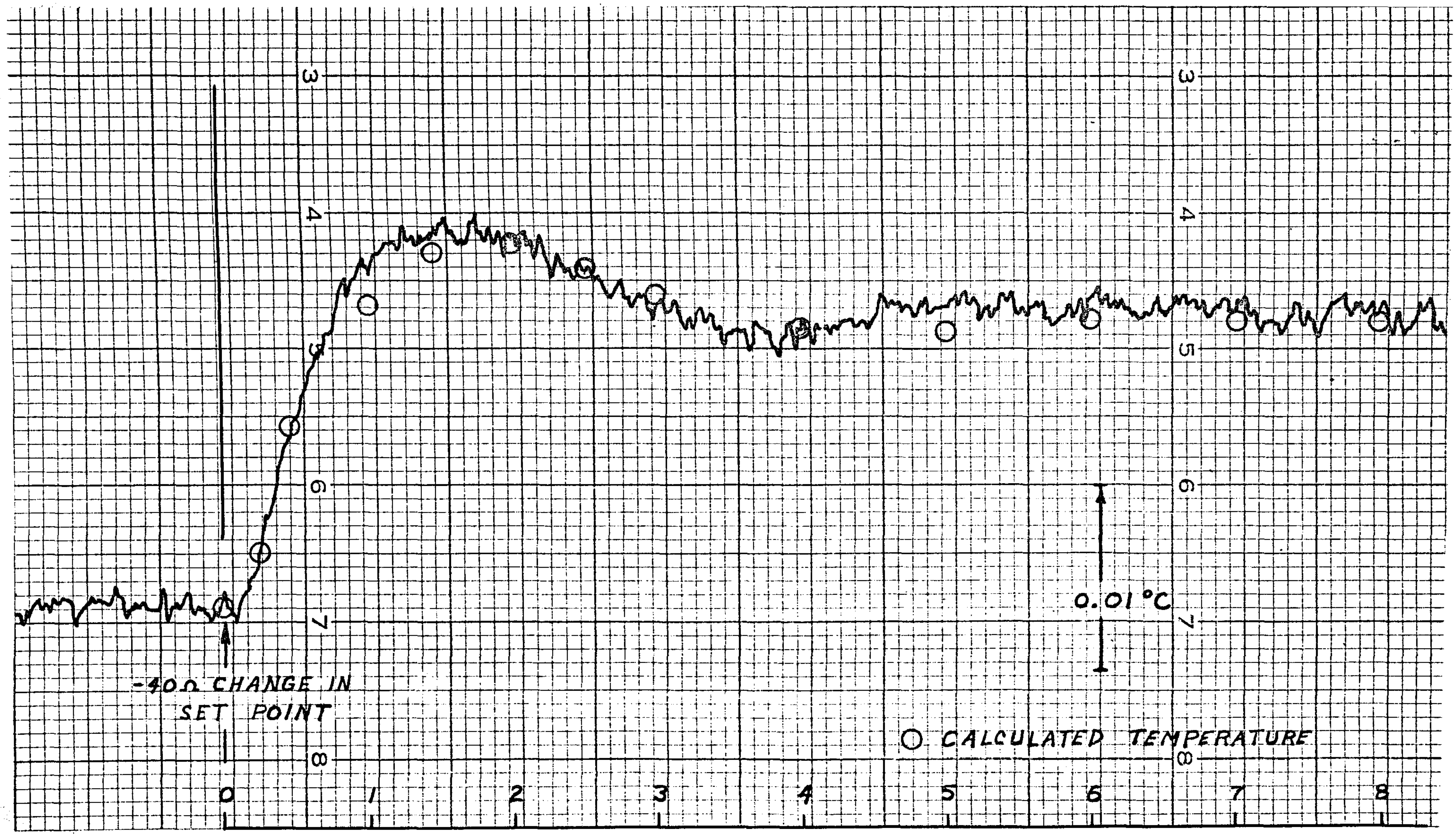
$$K_2 = 1.08 \pm.05 \text{ radians} \quad (232)$$

The effect of this uncertainty on the period of the oscillation is determined by the change made in the time required by the angle function to increase 2π radians.

$$(1.08 \pm.05)t = 2\pi \quad (233)$$

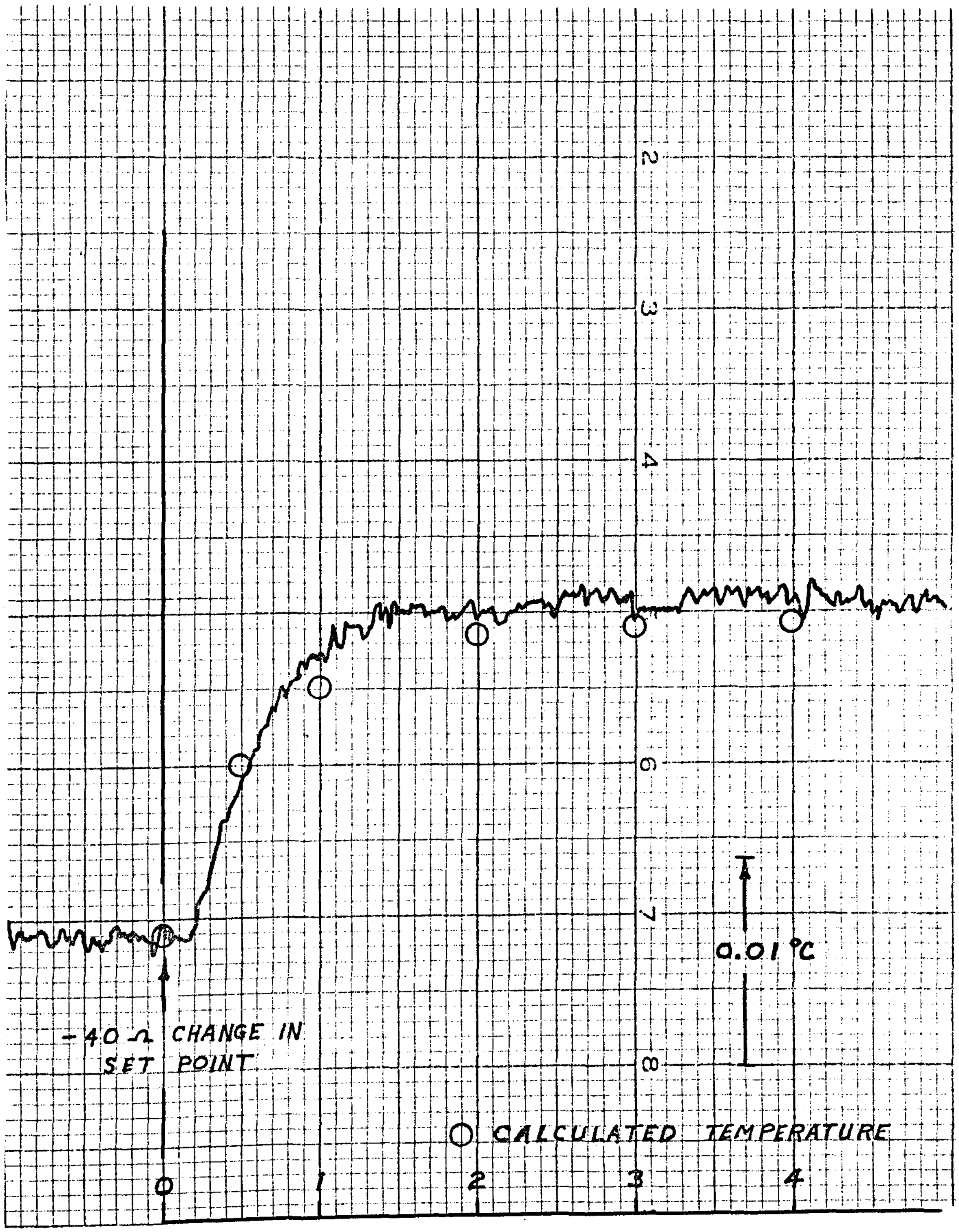
$$t = \frac{2\pi}{1.08} \pm 4.6\% \quad (234)$$

$$t = 5.8 \pm.3 \text{ minutes} \quad (235)$$



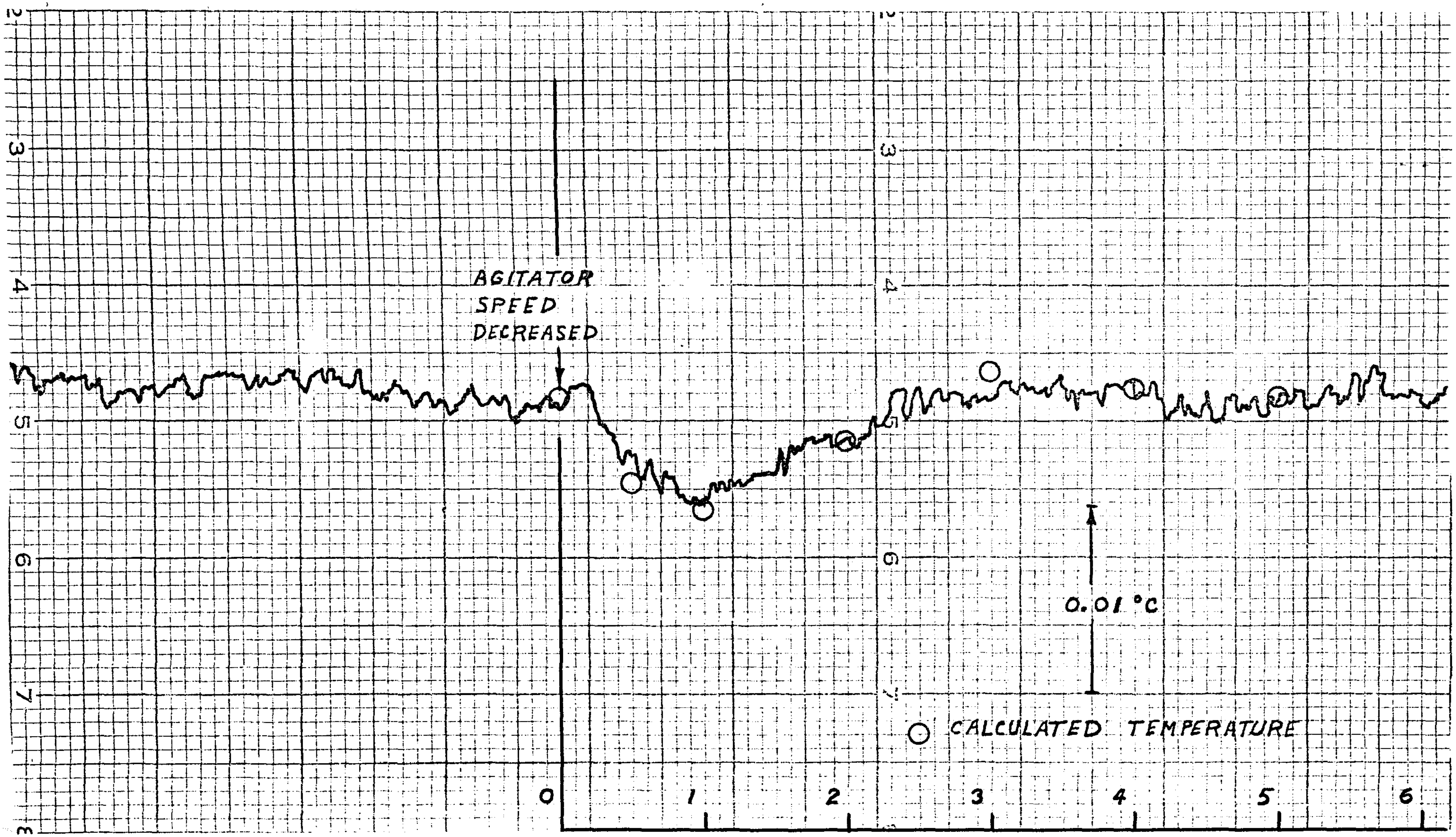
Time, Minutes

Figure 31. Transient Response to Change in Set Point
Proportional plus integral control temperature chart
compared with the calculated temperature



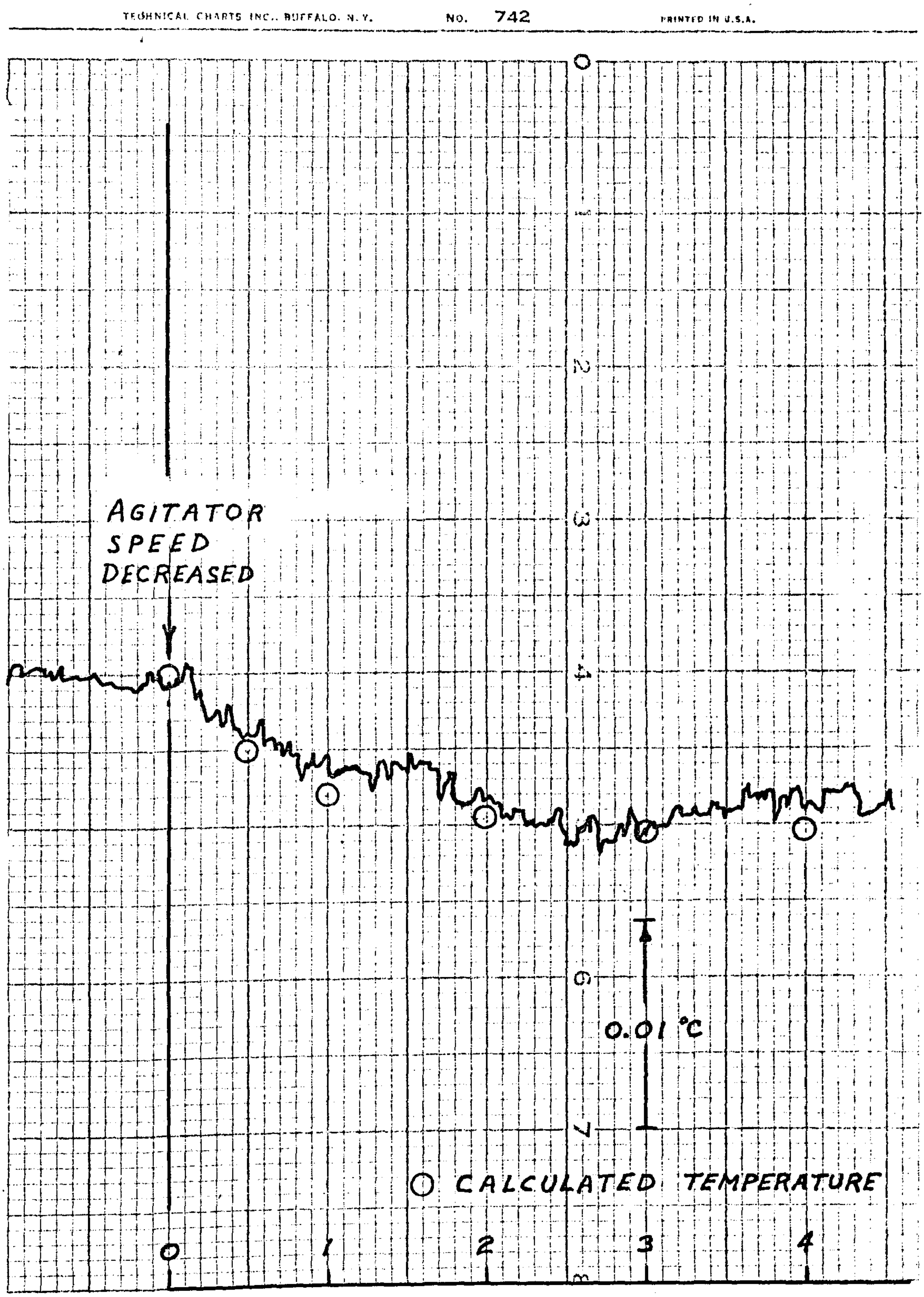
Time, Minutes

Figure 32. Transient Response to Change in Set Point
Proportional control temperature chart
compared with calculated temperature



Time, Minutes

Figure 33. Transient Response to a Change in Load
Proportional plus integral control temperature chart compared with the calculated temperature



Time, Minutes

Figure 34. Transient Response to Change in Load
Proportional control temperature chart
compared with calculated temperature

A graphical comparison of the observed and calculated temperature response curves is shown in Figures 31 through 34.

Use as a Calorimeter

One possible application for the temperature controller is as a device for determining heats of reaction or heats of physical changes. As a demonstration of the feasibility of the procedure, the heat required to melt ice and raise its temperature to the temperature of the bath was determined.

A 25 ml. flask was placed in the oil bath and the bath allowed to come to steady state. While the power to the heater was being recorded, a small, weighed quantity of ice was added to the flask and the change in power requirements was recorded. The integral of the difference between the steady state power requirement and the power required to return the bath to the steady state condition was equal to the heats of fusion plus the sensible heat required.

Three experimental determinations were made to determine the accuracy of the method. The calculations and experimental data are shown in Table IV.

Results

The experimental and calculated heat requirements were as follows:

Trial	Calculated	Experimental	% Difference
1	85.0 cal.	83.3 cal.	-2.0%
2	153.0	149.3	-2.4
3	106.0	87.4	-8.0

The average experimental result was 4.1% below the calculated requirement. The calculated results are biased to overestimate the heat required since there was no compensation taken for any of the weight being present as surface moisture.

TABLE IV

Use as a Calorimeter

Trial 1

Time	Cal./Min. To Bath	ΔCal./Min. From Steady State	Cal.
0	29.0	0.0	0
0.2	31.0	2.0	0.4
0.4	37.0	8.0	1.6
0.6	76.0	47.0	9.4
0.8	97.0	68.0	13.6
1.0	110.0	81.0	16.2
1.2	113.5	84.5	16.9
1.4	113.5	84.5	16.9
1.6	113.5	84.5	16.9
1.8	106.0	77.0	15.4
2.0	94.0	65.0	13.0
2.2	75.5	46.5	9.3
2.4	63.0	34.0	6.8
2.6	52.0	23.0	4.6
2.8	42.0	13.0	2.6
3.0	36.0	7.0	1.4
3.2	32.0	3.0	0.6
3.4	27.0	-2.0	-0.4
3.6	26.0	-3.0	-0.6
3.8	26.0	-3.0	-0.6
4.0	27.5	-1.5	-0.3
4.2	27.5	-1.5	-0.3
4.4	27.5	-1.5	-0.3
4.6	27.5	-1.5	-0.3
4.8	29.0	0	0
5.0	30.0	1.0	0.2
5.2	31.0	2.0	0.4
5.4	32.0	3.0	0.6
5.6	31.0	2.0	0.4
5.8	31.0	2.0	0.4
6.0	31.0	2.0	0.4
6.2	32.0	2.0	0.4
6.4	32.0	2.0	0.4
6.6	32.5	3.5	0.7
6.8	33.0	4.0	0.8
7.0	32.5	3.5	0.7
7.2	31.0	2.0	0.4
7.4	30.5	1.5	0.3
7.6	30.5	1.5	0.3
7.8	29.5	0.5	0.1
8.0	29.0	0	0
			149.3 cal.

gm. ice to Bath 1.33, Bath @ 35°C.

Cal. required 1.33 x 80 = 106.5

1.33 x 35 = 46.5

153.0 cal.

TABLE IV - (Continued)

Trial 2

Time	Cal./Min. To Bath	ΔCal./Min. From Steady State	Cal.
0.0	31.0	0	0
0.2	35.0	4.0	0.8
0.4	46.0	15.0	3.0
0.6	73.0	42.0	8.4
0.8	85.0	54.0	10.8
1.0	90.0	59.0	11.8
1.2	87.5	56.5	11.3
1.4	81.5	50.5	10.1
1.6	70.0	39.0	7.8
1.8	60.0	29.0	5.8
2.0	54.0	23.0	4.6
2.2	50.0	19.0	3.8
2.4	47.5	16.5	3.3
2.6	40.0	9.0	1.8
2.8	35.0	4.0	0.8
3.0	32.0	1.0	0.2
3.2	27.5	-3.5	-0.7
3.4	26.0	-5.0	-1.0
3.6	26.0	-5.0	-1.0
3.8	27.5	-3.5	-0.7
4.0	27.5	-3.5	-0.7
4.2	29.0	-2.0	-0.4
4.4	29.0	-2.0	-0.4
4.6	29.0	-2.0	-0.4
4.8	29.0	-2.0	-0.4
5.0	29.5	-1.5	-0.3
5.2	31.0	0	0
5.4	31.0	0	0
5.6	32.0	1.0	-0.2
5.8	32.5	1.5	-0.3
6.0	33.0	2.0	-0.4
6.2	33.5	2.5	0.5
6.4	32.5	1.5	0.3
6.6	33.0	2.0	0.4
6.8	33.5	2.5	0.5
7.0	34.0	3.0	0.6
7.2	34.5	3.5	0.7
7.4	33.5	2.5	0.5
7.6	33.0	2.0	0.4
7.8	32.0	1.0	0.2
8.0	31.0	0	0
			83.3 cal.

gm. ice to Bath 0.74 gm., Bath @ 35°C.

Cal. required 0.74 x 80 = 59.0

0.74 x 35 = 26.0

85.0 cal.

TABLE IV - (Continued)

Trial 3

Time	Cal./Min. To Bath	Δ Cal./Min. From Steady State	Cal.
0.0	31.0	0	0
0.2	36.5	5.5	1.1
0.4	51.5	20.5	4.1
0.6	70.0	39.0	7.8
0.8	87.5	56.5	11.3
1.0	97.0	66.0	13.2
1.2	94.0	63.0	12.6
1.4	87.5	56.5	11.3
1.6	81.5	50.5	10.1
1.8	70.0	39.0	7.8
2.0	60.0	29.0	5.8
2.2	51.5	20.5	4.1
2.4	48.0	17.0	3.4
2.6	44.5	13.5	2.7
2.8	44.5	13.5	2.7
3.0	40.0	9.0	1.8
3.2	35.0	4.0	0.8
3.4	32.0	1.0	0.2
3.6	29.0	-2.0	-0.4
3.8	27.5	-3.5	-0.7
4.0	27.5	-3.5	-0.7
4.2	27.5	-3.5	-0.7
4.4	27.5	-3.5	-0.7
4.6	29.0	-2.0	-0.4
4.8	30.0	-1.0	-0.2
5.0	30.0	-1.0	-0.2
5.2	30.0	-1.0	-0.2
5.4	31.0	0	0
5.6	32.0	1.0	0.2
5.8	32.0	1.0	0.2
6.0	32.0	1.0	0.2
6.2	32.0	1.0	0.2
6.4	31.0	0	0
6.6	31.0	0	0
6.8	31.0	0	0
			97.4 cal.

gm. ice to Bath 0.93 gm., Bath @ 35°C.

Cal. required 0.93 x 80 = 74.0

0.93 x 35 = 32.0

106.0 cal.

SUMMARY

The thermostat was shown to be capable of control to within 0.005°C. of the set point over a 16 hr. test. The control experienced over 30 minute periods, while investigating transient response characteristics, was within ±0.0015°C. With proportional plus integral control, the thermostat was able to return to ±0.0015°C. of the original set point after varying the heat added to the bath through the stirrer (change in load or regulatory control). During the study of set point changes (servomechanism control), the repeatability of the temperature control at the same set point was also within ±0.0015°C.

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AUTOBIOGRAPHY

Joseph Peter Stidham was born September 27, 1938 in Newport, Rhode Island. In 1956 he graduated from Druid Hills High School in Atlanta Georgia.

At Lehigh University, Bethlehem, Pennsylvania, he was elected to Tau Beta Pi, National Engineering Honorary Society and was graduated in 1960 with a B.S. in Chemical Engineering.

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