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Quantum Groups and Knot Invariants

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Introduction

Knot theory arguably holds claim to the title of the mathematical discipline with the most unusually diverse applications. A *knot* can be defined topologically as an embedding of S^1 in $\mathbb{R}^3$. Naturally, two knots are topologically equivalent if one cannot be smoothly deformed into the other. The question of whether two knots are equivalent is highly non-trivial, and so the question of knot invariants used to distinguish knots has occupied knot theorists for over a century. Knot theory has found application in statistical mechanics [1], symbolic logic and set theory [2], quantum field theory [3], quantum computing [4], etc. This thesis focuses on a connection of knot invariants to a still evolving field: quantum groups. The representation theory of a particular quantum group, $U_q(\mathfrak{sl}_2(\mathbb{C}))$, encodes information that, when expressed via a knot diagram in a well-defined graphical calculus, produces the Jones polynomial, arguably the most famous of knot invariants. Section 1 gives an introduction of this quantum group. Section 2 details the representation theory of $U_q(\mathfrak{sl}_2(\mathbb{C}))$. Section 3 introduces category theory and the category Rep_{U_q}, and shows how Rep_{U_q} can produce the Jones polynomial through an example with the trefoil knot.

I am grateful to Dr. Heather Russell in the completion of this work.

1 Quantum algebra

Our quantum algebra $U_q(\mathfrak{sl}_2(\mathbb{C}))$ arises from "deforming" a classical Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. Definitions from this section closely follow [5].

Definition 1.1. *An algebra A is defined by the triple (A, μ, η), where A is a vector space over field k, μ, η linear maps $\mu : A \times A \to A$, $\eta : k \to A$. Denote μ, η as the multiplication and unit of the algebra, respectively.*

Let $(A, \mu, \eta), (A', \mu', \eta')$ be algebras. A morphism of algebras is defined by a map $f : A \to A'$ such that $\mu' \circ (f \otimes f) = f \circ \mu$, $f \circ \eta = \eta'$ (i.e., f preserves multiplication and unit).

Definition 1.2. *A is a Lie algebra if multiplication is defined by the bilinear Lie bracket $[\ ,\ ] := \mu$, subject to the relations*

$$[a, b] = -[b, a],\ \ [a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$$

$\forall a, b, c \in A$. These relations are known as skew-symmetry and the Jacobi identity, respectively.

1.1 $\mathfrak{sl}_2(\mathbb{C})$

Consider the 2×2 traceless matrices given below:

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.1}$$

In field $\mathbb{C}$, this basis spans the vector space denoted $\mathfrak{sl}_2(\mathbb{C})$. Clearly $\mathfrak{sl}_2(\mathbb{C})$ is not an algebra under matrix multiplication, since

$$eh = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \notin \mathfrak{sl}_2(\mathbb{C})$$

Therefore, we define $[\ ,\ ] : \mathfrak{sl}_2(\mathbb{C}) \times \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{sl}_2(\mathbb{C})$ by $(x, y) \to xy - yx$. This gives $\mathfrak{sl}_2(\mathbb{C})$ a Lie algebra structure with Lie bracket relations

$$[e, f] = h, [h, f] = -2f, [h, e] = 2e \tag{1.2}$$

However, note that the Lie bracket is not associative:

$$[[f + h, h], e] = [[f, h] + [h, h], e] = [2f + 0, e] = [2f, e] = 2[f, e] = -2h$$

$$[f + h, [h, e]] = [f + h, 2e] = [f, 2e] + [h, 2e] = 2[f, e] + 2[h, e] = -2h + 4e$$

To recover associativity, we introduce the universal enveloping algebra $U(\mathfrak{sl}_2(\mathbb{C}))$. Recovering associativity is necessary for our representation theory to work properly.

1.2 $U(\mathfrak{sl}_2(\mathbb{C}))$

For Lie algebra L, let $T(L)$ to be the tensor algebra, generated by all tensor products of the generators of L (that is, elements of the form $e,\ e \otimes f,\ e \otimes e \otimes h$, etc.). Multiplication in $T(L)$ is the concatenation of tensor products. Take the (two-sided) ideal $I(L)$ generated by elements of the form $x \otimes y - y \otimes x - [x, y]$ and define $U(L) = T(L)/I(L)$. Since the tensor algebra is associative, $U(L)$ is associative [5]. If $\{a_i\}$ is the set of generators of L with Lie bracket defined by

$$[a_i, a_j] = \sum_l s_l a_l, s_l \in \mathbb{C},\ i \neq j \tag{1.3}$$

then $U(L)$ has generators $\{a_i\}$ with multiplication given by concatenation modulo the following relations

$$a_i \otimes a_j - a_j \otimes a_i = \sum_l s_l a_l \tag{1.4}$$

In the particular case of $\mathfrak{sl}_2$, $U(\mathfrak{sl}_2)$ with generators e, f, h, we have

$$e \otimes f - f \otimes e = 2h, h \otimes f - f \otimes h = -2f, h \otimes e - e \otimes h = 2e \tag{1.5}$$

A quantum deformation of this algebra is what gives us $U_q(\mathfrak{sl}_2(\mathbb{C}))$ as seen below.

1.3 $U_q(\mathfrak{sl}_2(\mathbb{C}))$

In what follows, we take $\mathbb{C}(q)$ to the be the polynomial ring with indeterminate q and coefficients in $\mathbb{C}$. We can now define the quantum algebra of interest, $U_q(\mathfrak{sl}_2(\mathbb{C}))$ through a one-parameter deformation (this section follows the notation in [6]). Let $q \notin \{0, 1, -1\}$. Our generators are E, F, K, K^{-1} with multiplicative relations

$$KK^{-1} = K^{-1}K = 1,\ KE = q^2 EK,\ KF = q^{-2} FK,\ EF - FE = \frac{K - K^{-1}}{q - q^{-1}} \tag{1.6}$$

In this current presentation, it is difficult to see how $U_q(\mathfrak{sl}_2(\mathbb{C}))$ relates to our enveloping algebra. There is another presentation that illuminates the relationship [5].

Definition 1.3. *The algebra U_q is isomorphic to the algebra U'_q generated by the five variables E, F, K, K^{-1}, L and the relations*

$$KK^{-1} = K^{-1}K = 1, \tag{1.7}$$

$$KEK^{-1} = q^2 E,\ KFK^{-1} = q^{-2} F, \tag{1.8}$$

$$[E, F] = L,\ (q - q^{-1})L = K - K^{-1}, \tag{1.9}$$

$$[L, E] = q(EK + K^{-1}E),\ [L, F] = -q^{-1}(FK + K^{-1}F). \tag{1.10}$$

Theorem 1.1. $U \cong U'_{q=1}/(K - 1)$

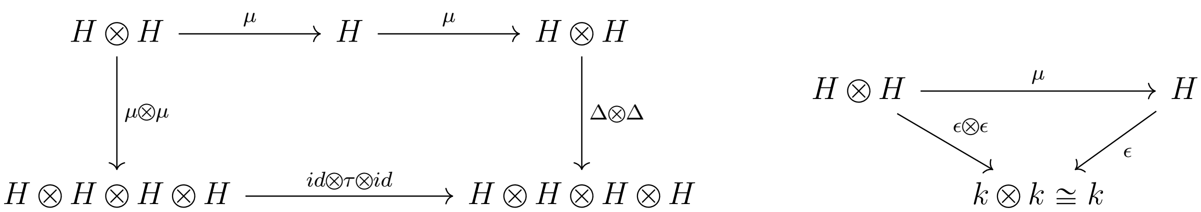
Proof. The projection of $U'_{q=1}$ onto U is given by an isomorphism sending E to e, F to f, K to 1, and L to h, which can be checked against the U generator relations. □

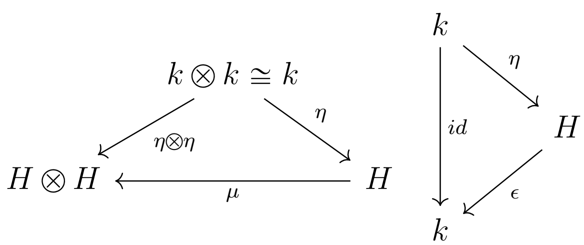
1.4 Bialgebra structure on $U_q(\mathfrak{sl}_2(\mathbb{C}))$

In the sequel, we'll need to talk about how an algebra acts upon tensor products of its *simple modules*, a term we'll introduce in Section 2. To do so, we must introduce the notion of a coalgebra and bialgebra.

Definition 1.4. *A coalgebra is defined by the triple* (C, Δ, ϵ)*, where* C *is a vector space and* Δ, ϵ *are linear maps,* $\Delta : C \to C \times C, \epsilon : C \to k$*.* Δ, ϵ *are the comultiplication and counit of the coalgebra, respectively.*

Definition 1.5. *Let* $(H, \Delta, \mu, \epsilon, \eta)$ *be both a coalgebra and algebra. Then* H *is a bialgebra if the following diagrams commute*





In $U_q(\mathfrak{sl}_2(\mathbb{C}))$, the comultiplication and counit are given by the following:

$$\Delta(E) = E \otimes 1 + K^{-1} \otimes E \tag{1.11}$$

$$\Delta(F) = F \otimes K + 1 \otimes F \tag{1.12}$$

$$\Delta(K^{\pm 1}) = K^{\pm 1} \otimes K^{\pm 1} \tag{1.13}$$

$$\epsilon(E) = \epsilon(F) = 0, \epsilon(K) = \epsilon(K^{-1}) = 1 \tag{1.14}$$

With our algebra at hand, we now turn to its representation theory.

2 Representation theory of $U_q(\mathfrak{sl}_2(\mathbb{C}))$

Representation theory gives a means of discussing algebras and groups in the context of linear algebra, illustrating an important relationship between vector spaces, linear maps, and groups/algebras.

Definition 2.1. *Let A be an algebra with unit. An A-module is a vector space V together with a bilinear map $(a, v) \mapsto av$ from $A \times V$ to V such that*

$$a(a'v) = (aa')v \text{ and } 1v = v \tag{2.1}$$

for all $a, a' \in A,\ v \in V$.

The action of A on an A-module V defines an algebra morphism ρ from A to $\mathrm{End}(V)$ by

$$\rho(a)(v) = av \tag{2.2}$$

ρ is a *representation* of A on V.

An A-submodule V' of an A-module V is a subspace of V with an A-module structure such that the inclusion of V' into V is A-linear. A *simple* A-module V is one which has no non-trivial submodules. Correspondingly, an *irreducible* representation ρ acts upon a simple module V. When the context is clear, we will denote a representation by ρ or V.

To map from one representation to another, we need what are known as *intertwining* maps.

Definition 2.2. *Let A be an algebra, $(V, \rho_V), (W, \rho_W)$ A-modules, and let $F : V \to W$ an A-linear map. Then F is an intertwining map if $\forall a \in A, v \in V, \rho_W(a)(F(v)) = F(\rho_V(a)v)$.*

The set of intertwining maps between V, W is a linear subspace under composition. The tensor product of intertwining maps is also intertwining.

2.1 Irreducible representations of $U_q(\mathfrak{sl}_2(\mathbb{C}))$

It was shown in [5] that for nonnegative integer n there is a unique irreducible representation of $U_q(\mathfrak{sl}_2)$ of dimension $n + 1$. In [6], these are denoted V_n, with basis

$$\{v^m\}, -n \leqslant m \leqslant n,\ m \equiv n \mod 2$$

To start, we define a representation:

$$Ev^m = \left[\frac{n-m}{2}\right] v^{m+2} \tag{2.3}$$

$$Fv^m = \left[\frac{n+m}{2}\right] v^{m-2} \tag{2.4}$$

$$K^{\pm 1} v^m = q^{\pm m} v^m \tag{2.5}$$

Here

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} \tag{2.6}$$

We also define $v^{n+2}, v^{-n-2} = 0$.

Theorem 2.1. *Equation (1.7-1.10) are respected by the proposed representation (i.e., the representation is well-defined).*

Proof.

$$K^{\pm 1}K^{\mp 1}v^m = q^{\pm m}q^{\mp m}v^m = 1v^m$$

$$KEv^m = K\Big[\frac{n-m}{2}\Big]v^{m+2} = \Big[\frac{n-m}{2}\Big]q^{m+2}v^{m+2} =$$

$$q^{m+2}\frac{q^{(n+m)/2} - q^{-(n+m)/2}}{q-q^{-1}}v^{m+2} = q^2q^m\frac{q^{(n+m)/2} - q^{-(n+m)/2}}{q-q^{-1}}$$

$$= q^2EKv^m$$

$$KF = q^{-2}FK \text{ (same logic as above)}$$

$$[E,F]v^m = EFv^m - FEv^m = \Big[\frac{n+m}{2}\Big]\Big[\frac{n-m+2}{2}\Big]v^m - \Big[\frac{n-m}{2}\Big]\Big[\frac{n+m+2}{2}\Big]v^m$$

$$= [\frac{q^{n+1} - q^{m-1} - q^{-m+1} + q^{-n-1}}{(q-q^{-1})^2} - (\frac{q^{n+1} - q^{-m-1} - q^{m+1} + q^{-n-1}}{(q-q^{-1})^2})]v^m$$

$$= [\frac{-q^{m-1} - q^{-m+1} + q^{-m-1} + q^{m+1}}{(q-q^{-1})^2}]v^m$$

$$= \frac{(q-q^{-1})q^m - (q-q^{-1})q^{-m}}{(q-q^{-1})^2}v^m = \frac{q^m - q^{-m}}{q-q^{-1}}v^m = \frac{K-K^{-1}}{q-q^{-1}}v^m$$

□

We know that for $a \otimes b \in V_n \otimes V_n$, $U_q(\mathfrak{sl}_2(\mathbb{C})) \otimes U_q(\mathfrak{sl}_2(\mathbb{C}))$ acts via $(X \otimes Y)(a \otimes b) = Xa \otimes Yb$ for $X, Y \in U_q(\mathfrak{sl}_2(\mathbb{C}))$. Our bialgebra structure makes $V_n \otimes V_n$ into a $U_q(\mathfrak{sl}_2(\mathbb{C}))$ module via comultiplication:

$$XY(a \otimes b) = \Delta(XY)(a \otimes b) = \Delta(X)\Delta(Y)(a \otimes b) \tag{2.7}$$

Theorem 2.2. *The action described in Equation 2.7 above defines a well-defined representation of $U_q(\mathfrak{sl}_2(\mathbb{C}))$ on $V_n \otimes V_n$.*

Proof. Let $a \otimes b \in V_n \otimes V_n$.

$$\Delta(K^{\pm}K^{\mp})(a \otimes b) = (K^{\pm}K^{\mp} \otimes K^{\pm}K^{\mp})(a \otimes b) = (a \otimes b)$$

$$\Delta(KE)(a \otimes b) = \Delta(K)\Delta(E)(a \otimes b) = (K \otimes K)(E \otimes 1 + K^{-1} \otimes E)(a \otimes b) = (K \otimes K)(Ea \otimes b + K^{-1}a \otimes Eb)$$

$$= KEa \otimes Kb + KK^{-1}a \otimes KEb = q^2EKa \otimes Kb + q^2a \otimes EKb$$

$$= q^2(E \otimes 1 + K^{-1} \otimes E)(K \otimes K)(a \otimes b) = q^{-2}\Delta(EK)(a \otimes b)$$

$$\Delta(KF)(a \otimes b) = q^{-2}\Delta(FK)(a \otimes b) \text{ (same logic as above)}$$

$$\Delta([E,F])(a \otimes b) = \Delta(EF - FE)(a \otimes b) = \Delta(EF)(a \otimes b) - \Delta(FE)(a \otimes b)$$

$$= \Delta(E)\Delta(F)(a \otimes b) - \Delta(F)\Delta(E)(a \otimes b)$$

$$= (E \otimes 1 + K^{-1} \otimes E)(F \otimes K + 1 \otimes F)(a \otimes b) - (F \otimes K + 1 \otimes F)(E \otimes 1 + K^{-1} \otimes E)(a \otimes b)$$

$$= EFa \otimes Kb + Ea \otimes Fb + K^{-1}Fa \otimes EKb + K^{-1}a \otimes EFb - (FEa \otimes Kb + FK^{-1}a \otimes KEb + Ea \otimes Fb + K^{-1}a \otimes FEb)$$

$$= EFa \otimes Kb + Ea \otimes Fb + K^{-1}Fa \otimes EKb + K^{-1}a \otimes EFb - FEa \otimes Kb - FK^{-1}a \otimes KEb - Ea \otimes Fb - K^{-1}a \otimes FEb$$

$$= ((EF - FE)\otimes K)(a\otimes b) + K^{-1}Fa\otimes EKb - FK^{-1}a\otimes KEb + (K^{-1}\otimes(EF-FE))(a\otimes b)$$

$$= \frac{1}{q-q^{-1}}((K-K^{-1})\otimes K)(a\otimes b) + K^{-1}Fa\otimes EKb - FK^{-1}a\otimes KEb + \frac{1}{q-q^{-1}}(K^{-1}\otimes(K-K^{-1}))(a\otimes b)$$

$$= \frac{1}{q-q^{-1}}(Ka\otimes Kb - K^{-1}a\otimes Kb) + K^{-1}Fa\otimes EKb - FK^{-1}a\otimes KEb + \frac{1}{q-q^{-1}}(K^{-1}a\otimes Kb - K^{-1}a\otimes K^{-1}b)$$

$$= \frac{1}{q-q^{-1}}(Ka\otimes Kb) - \frac{1}{q-q^{-1}}(K^{-1}a\otimes K^{-1}b) + K^{-1}Fa\otimes EKb - FK^{-1}a\otimes KEb$$

$$= \frac{1}{q-q^{-1}}(K\otimes K - K^{-1}\otimes K^{-1})(a\otimes b) + ((K^{-1}F - q^2FK^{-1})\otimes KE)(a\otimes b)$$

$$= \frac{1}{q-q^{-1}}(K\otimes K - K^{-1}\otimes K^{-1})(a\otimes b) + (K^{-1}F - q^2q^{-2}K^{-1}F)\otimes KE)(a\otimes b)$$

$$= \frac{1}{q-q^{-1}}(K\otimes K - K^{-1}\otimes K^{-1})(a\otimes b) = \Delta(\frac{K-K^{-1}}{q-q^{-1}})(a\otimes b)$$

□

We call the one-dimensional representation $V_0 \cong \mathbb{C}(q)$ the trivial representation, and the two dimensional representation $V_1 \cong \mathbb{C}(q)v^1 \oplus \mathbb{C}(q)v^{-1}$ the fundamental representation.

Since much of the sequel relies on explicit calculation, we take $n = 1$ and specify the action of $U_q(\mathfrak{sl}_2(\mathbb{C}))$ on V_1 explicitly:

$$Ev^1 = 0, Ev^{-1} = v^1 \tag{2.8}$$

$$Fv^1 = v^{-1}, Fv^{-1} = 0 \tag{2.9}$$

$$K^{\pm 1}v^1 = K^{\pm 1}v^{-1} = q^{\mp 1}v^{-1} \tag{2.10}$$

Theorem 2.3. *$V_0 \otimes V_1 \cong V_1 \otimes V_0 \cong V_1$ as $U_q(\mathfrak{sl}_2(\mathbb{C}))$ modules.*

Proof. Set $v^{\pm 1}\otimes v^0 \cong v^0 \otimes v^{\pm 1} \cong v^{\pm 1}$, and note that $E(v^0) = F(v^0) = 0$, and K is the identity map on V_0. This mapping respects the comultiplication and counit, and thus we are done. □

The following are maps between $V_0,\ V_1\otimes V_1$

$$\epsilon_1 : V_1 \otimes V_1 \to V_0$$

$$\epsilon_1(v^1\otimes v^1) = \epsilon_1(v^{-1}\otimes v^{-1}) = 0, \epsilon_1(v^{-1}\otimes v^1) = 1, \epsilon_1(v^1\otimes v^{-1}) = -q \tag{2.11}$$

$$\delta_1 : V_0 \to V_1 \otimes V_1$$

$$\delta_1(1) = v^1\otimes v^{-1} - q^{-1}v^{-1}\otimes v^1 \tag{2.12}$$

Two other maps, denoted by $R : V_1\otimes V_1 \to V_1\otimes V_1$ and $R^{-1} : V_1\otimes V_1 \to V_1\otimes V_1$, are given below.

$$R := q^{1/2}\delta_1\circ\epsilon_1 + q^{-1/2}id_{V_1\otimes V_1} \tag{2.13}$$

$$R^{-1} := q^{-1/2}\delta_1\circ\epsilon_1 + q^{1/2}id_{V_1\otimes V_1} \tag{2.14}$$

The fact that these two maps are inverses of one another will be proven in Section III.

Theorem 2.4. *$\epsilon_1 : V_1 \otimes V_1 \to V_0, \delta_1 : V_0 \to V_1 \otimes V_1$ are intertwining maps.*

Proof. Since Δ, ϵ are ring homomorphisms, it suffices to check this only for the action of the generators E, F, K, K^{-1} on the bases of $V_0, V_1 \otimes V_1$. For brevity in these proofs, brackets are used to denote which equation depends on which sign choice for the tensor elements.

$$\epsilon_1(\Delta(E)(v^{\pm 1} \otimes v^{\pm 1}))$$
$$= \epsilon_1((E \otimes 1 + K^{-1} \otimes E)(v^{\pm 1} \otimes v^{\pm 1}))$$
$$= \epsilon_1(Ev^{\pm 1} \otimes v^{\pm 1} + K^{-1}v^{\pm 1} \otimes Ev^{\pm 1}) = \left\{ \begin{array}{ll} \epsilon_1(0+0) & + \\ \epsilon_1(v^1 \otimes v^{-1} + q^{-1}(v^{-1} \otimes v^1)) & - \end{array} \right\}$$
$$= 0 = E(0) = E\epsilon_1(v^{\pm 1} \otimes v^{\pm 1})$$

$$\epsilon_1(\Delta(F)(v^{\pm 1} \otimes v^{\pm 1})) = \epsilon((F \otimes K + 1 \otimes F)(v^{\pm 1} \otimes v^{\pm 1}))$$
$$= \epsilon_1(Fv^{\pm 1} \otimes Kv^{\pm 1} + 1v^{\pm 1} \otimes Fv^{\pm 1})$$
$$= \left\{ \begin{array}{ll} \epsilon_1(v^{-1} \otimes qv^1 + v^1 \otimes v^{-1}) & + \\ \epsilon_1(0+0) & - \end{array} \right\}$$
$$= 0 = F(0) = F\epsilon_1(v^1 \otimes v^1)$$

$$\epsilon_1(\Delta(K)(v^{\pm 1} \otimes v^{\pm 1})) = \epsilon_1((K \otimes K)(v^{\pm} \otimes v^{\pm})) = \epsilon_1(2Kv^{\pm 1} \otimes Kv^{\pm 1}) = 2q^{\pm 2}\epsilon_1(v^{\pm 1} \otimes v^{\pm 1})$$
$$= 0 = K(0) = K\epsilon_1(v^{\pm 1} \otimes v^{\pm 1})$$

K^{-1} proceeds exactly like above.

$$\epsilon_1(\Delta(E)(v^{\pm 1} \otimes v^{\mp 1})) = \epsilon_1((E \otimes 1 + K^{-1} \otimes E)(v^{\pm 1} \otimes v^{\mp 1})) = \epsilon_1(Ev^{\pm 1} \otimes v^{\mp 1} + K^{-1}v^{\pm 1} \otimes Ev^{\mp 1})$$
$$= \epsilon_1(0+0) = 0 = E(0) = E\epsilon_1(v^{\pm} \otimes v^{\mp})$$

$$\epsilon_1(\Delta(F)(v^{\pm 1} \otimes v^{\mp 1})) = \epsilon_1((F \otimes K + 1 \otimes F)(v^{\pm 1} \otimes v^{\mp 1})) = \epsilon_1(Fv^{\pm 1} \otimes Kv^{\mp 1} + v^{\pm 1} \otimes Fv^{\mp 1})$$
$$= \epsilon_1(0+0) = 0 = F(0) = F\epsilon_1(v^{\pm} \otimes v^{\mp})$$

$$\epsilon_1(\Delta(K)(v^{\pm 1} \otimes v^{\mp 1})) = \epsilon_1((K \otimes K)(v^{\pm 1} \otimes v^{\mp 1})) =$$
$$\epsilon_1(Kv^{\pm 1} \otimes Kv^{\mp 1}) = \epsilon_1(v^{\pm 1} \otimes v^{\mp 1}) = \left\{ \begin{array}{ll} -q = -qK(1) = K\epsilon_1(v^1 \otimes v^{-1}) & + \\ 1 = 1K(1) = K\epsilon_1(v^{-1} \otimes v^1) & - \end{array} \right\}$$

$$\epsilon_1(\Delta(K^{-1})(v^{\pm 1} \otimes v^{\mp 1})) = \epsilon_1((K^{-1} \otimes K^{-1})(v^{\pm 1} \otimes v^{\mp 1})) =$$
$$\epsilon_1(K^{-1}v^{\pm 1} \otimes K^{-1}v^{\mp 1}) = \epsilon_1(v^{\pm 1} \otimes v^{\mp 1}) = \left\{ \begin{array}{ll} 1 = 1K^{-1}(1) = K^{-1}\epsilon_1(v^{-1} \otimes v^1) & + \\ -q = -qK^{-1}(1) = K^{-1}\epsilon_1(v^1 \otimes v^{-1}) & - \end{array} \right\}$$

Now we check for δ_1:

$$\delta_1(E(1)) = \delta(0) = 0 = \Delta(E)(v^1 \otimes v^{-1} - q^{-1}v^{-1} \otimes v^1) = \Delta(E)(\delta_1(1))$$
$$\delta_1(F(1)) = \delta(0) = 0 = \Delta(F)(v^1 \otimes v^{-1} - q^{-1}v^{-1} \otimes v^1) = \Delta(F)(\delta_1(1))$$

$$\delta_1(K^{\pm 1}(1)) = \delta_1(1) = v^1 \otimes v^{-1} - q^{-1}v^{-1} \otimes v^1 = \Delta(K^{\pm 1})(v^1 \otimes v^{-1} - q^{-1}v^{-1} \otimes v^1) = \Delta(K)(\delta_1(1))$$

□

Corollary 2.1. *$R,\ R^{-1} : V_1 \otimes V_1 \to V_1 \otimes V_1$ are intertwining maps.*

Proof. Follows from Definition 2.2 and Theorem 2.4. □

The representation theory for $U_q(\mathfrak{sl}_2(\mathbb{C}))$ takes on an interesting pictorial interpretation in the context of category theory, described in the next section.

3 Category theory

3.1 Introduction

Category theory offers a means of describing multiple fields of mathematics at once, as well as the connections between them. In our setting and many others, category theory provides a framework for viewing complicated mathematics from a graphical perspective which is both visually appealing and intuitive.

Definition 3.1. *A category C contains a class $ob(C)$ of objects, a class $hom(C)$ of morphisms between objects in $ob(C)$, and the requirement that for every $a, b, c \in ob(C), hom(a, b) \times hom(b, c) \to hom(a, c)$ is an (associative) composition of maps. Moreover,*

$$\forall x \in ob(C), \exists 1_x : x \to x \text{ such that } \forall f, g,\ f : a \to x, g : x \to b,\ 1_x \circ f = f,\ g \circ 1_x = g$$

Definition 3.2. *A (covariant) functor $F : A \to B$ is a homomorphism between categories. That is, it maps objects to objects and morphisms to morphisms, and respects the following*

$$F(1_X) = 1_{F(X)}$$

$$F(f \circ g) = F(f) \circ F(g)$$

$$\forall X \in ob(A), f, g \in Hom(C)$$

A bifunctor for category C is a functor from $C \times C \to C$.

Though categories can seem quite vague and generic, extra constraints can be placed upon the objects and morphisms living within a category.

Definition 3.3. *A monoidal category C is a category with bifunctor $\otimes$, an identity object i, and associativity constraints between n-tuple tensor products of objects in $ob(C)$.*

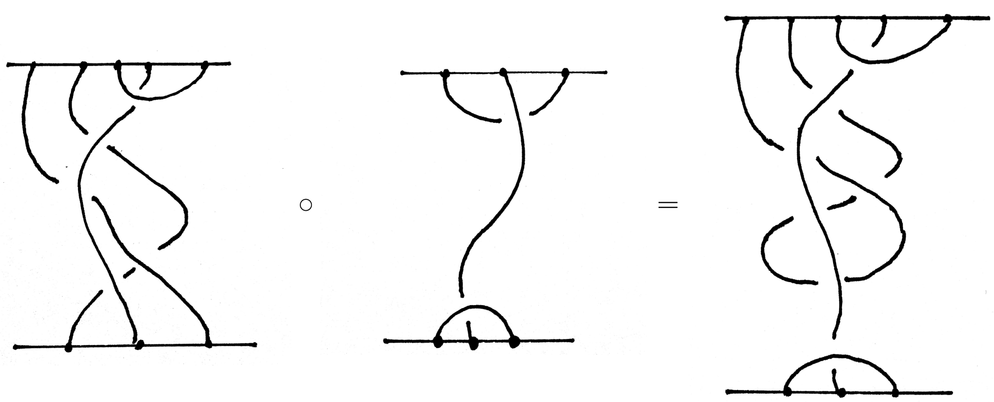
Definition 3.4. *A strict monoidal category is a monoidal category in which the associator is an identity map.*

The idea behind a monoidal category is that it mirrors tensor products of vector spaces. Strictness implies that associativity is an identity map, rather than just an isomorphism. An example of an important non-strict monoidal category is the modular tensor category $Fib(\mathbf{1}, \tau)$ composed of two objects $\mathbf{1}, \tau$ with a direct sum decomposition $\mathbf{1} \otimes \tau = \mathbf{1} \oplus \tau$ [7]. In this category associativity is certainly non-trivial.

3.2 Rep_{U_q} and Fr_{tang}

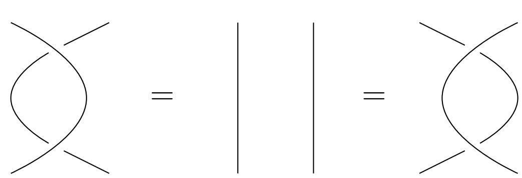
We now introduce a strict monoidal category Rep_{U_q}. $ob(Rep_{U_q})$ is tensor products of the simple modules V_0, V_1 (e.g., $V_1 \otimes V_1,\ V_0 \otimes V_1 \otimes V_1$, etc.). The morphisms are intertwining maps between representations (e.g, $\epsilon_1,\ \delta_1,\ R,\ R^{-1}$, etc.).

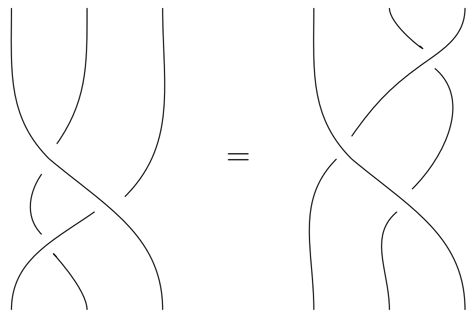
Our final category is the category of un-oriented framed tangles, denoted Fr_{tang}. $ob(Fr_{tang})$ is the set $\mathbb{N}$, and for $a, b \in ob(Fr_{tang})$, $\hom(a, b)$ is the set of $\mathbb{C}$-linear combinations of un-oriented tangle diagrams between two lines, one with a endpoints, the other with b endpoints, modulo *regular isotopy*, described below. Following convention, the two horizontal lines are vertically aligned. Fr_{tang} is also a strict monoidal category. The tensoring of objects is done by placing endpoints next to one another on one of the two lines (that is, for $n, m \in ob(Fr_{tang})$, $n \otimes m$ is represented by $n + m$ points on a line). Composition of morphisms is defined by "vertical stacking": the endpoints of one morphism is identified with the endpoints of the other, as shown below in the composition of morphisms $f \circ g, f \in \hom(3, 5),\ g \in \hom(3, 3)$.

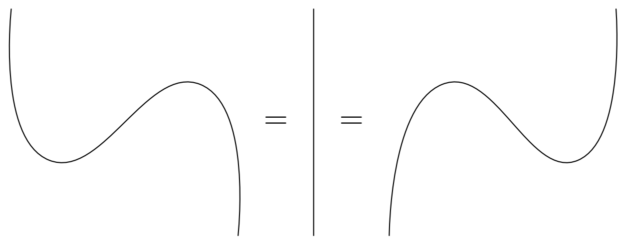


By regular isotopy, we mean tangles are equivalent if and only if they can be smoothly deformed into one another via two local moves that change crossing information, known as Reidemeister II and III, along with planar deformation:

RII:=



RIII:= 

Planar deformation := 

Note that any tangle diagram can be decomposed into a composition of simple tangle diagrams, as shown in Figure 1.

We set a covariant functor $\mathcal{F} : Fr_{tang} \to Rep_{U_q}$, pictorially described in Figure 2. The fact that $\mathcal{F}$ is well-defined is shown in the next subsection.

Notice that by our definition of composition of morphisms in Fr_{tang}, we have the relations shown in Figure 3.

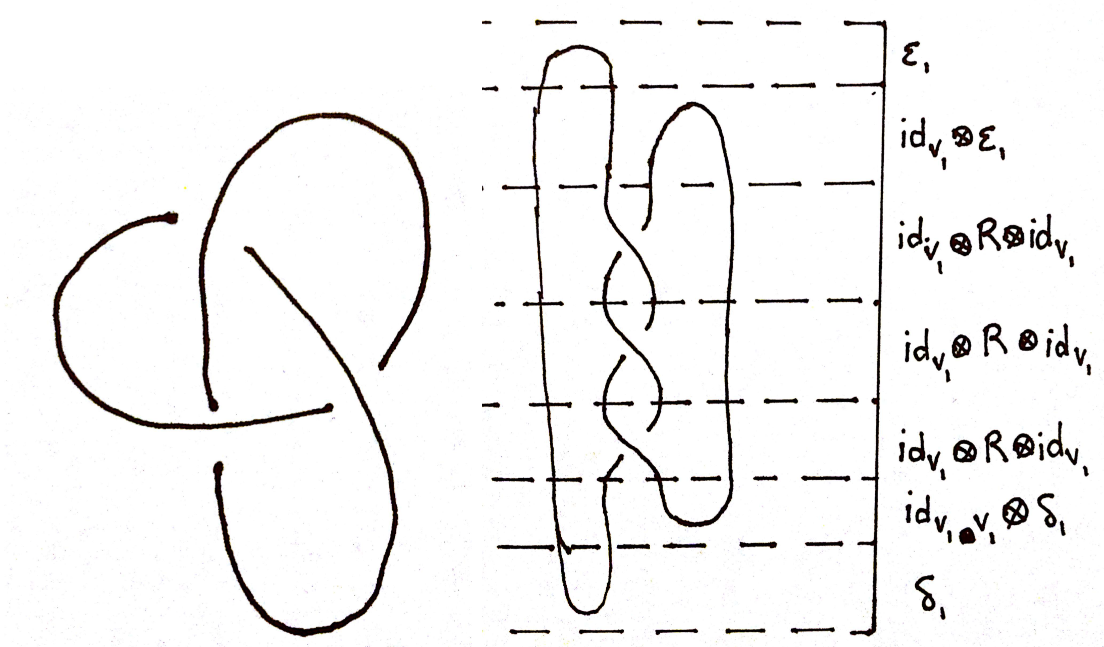
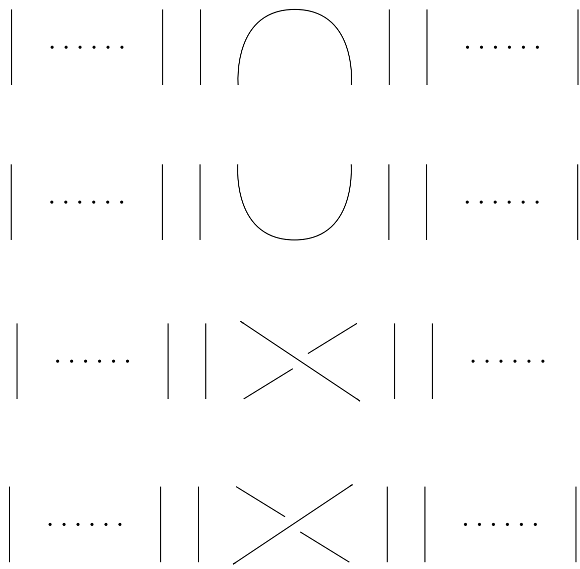


Figure 1: Deforming trefoil knot (left) into vertical composition of simple tangle diagrams (right)

$$\mathcal{F}(n) = \begin{cases} V_0, \ n = 0 \\ \bigotimes^n V_1, n \neq 0 \end{cases}$$



$$\xrightarrow{\mathcal{F}} id_{V_1} \otimes \cdots \otimes \epsilon_1 \otimes \cdots \otimes id_{V_1}$$

$$\xrightarrow{\mathcal{F}} id_{V_1} \otimes \cdots \otimes \delta_1 \otimes \cdots \otimes id_{V_1}$$

$$\xrightarrow{\mathcal{F}} id_{V_1} \otimes \cdots \otimes R \otimes \cdots \otimes id_{V_1}$$

$$\xrightarrow{\mathcal{F}} id_{V_1} \otimes \cdots \otimes R^{-1} \otimes \cdots \otimes id_{V_1}$$

Figure 2: The covariant functor $\mathcal{F}$

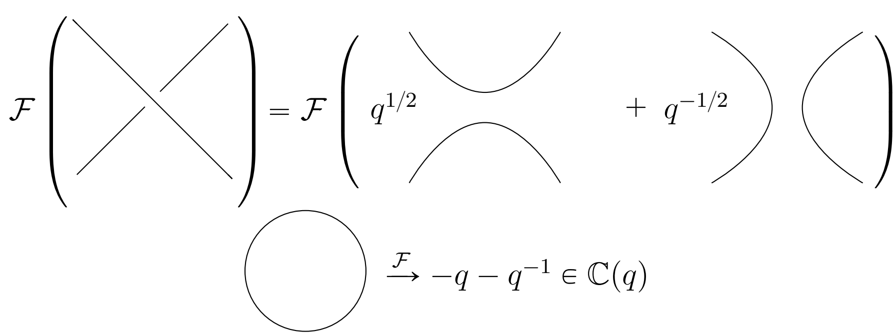
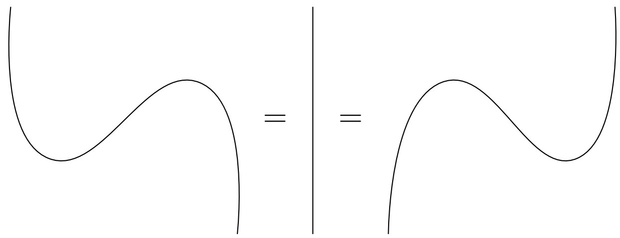


Figure 3: Skein and trace relation

3.3 Checking $\mathcal{F}$ respects planar deformation and Reidemeister II, III

We first verify that $\mathcal{F}$ is well-defined under smooth planar deformation.



In Rep_{U_q}, the left hand side of the figure above corresponds to

$$V_1 \xrightarrow{\cong} V_1 \otimes V_0 \xrightarrow{id_{V_1} \otimes \delta_1} V_1 \otimes (V_1 \otimes V_1) \xrightarrow{\cong} (V_1 \otimes V_1) \otimes V_1 \xrightarrow{\epsilon_1 \otimes id_V} V_0 \otimes V_1 \xrightarrow{\cong} V_1$$

The middle equality corresponds to

$$V_1 \xrightarrow{\cong} V_0 \otimes V_1 \xrightarrow{\delta_1 \otimes id_V} (V_1 \otimes V_1) \otimes V_1 \xrightarrow{\cong} V_1 \otimes (V_1 \otimes V_1) \xrightarrow{id_V \otimes \epsilon_1} V_1 \otimes V_0 \xrightarrow{\cong} V_1$$

Checking with linear maps:

$$v^{\pm 1} \xrightarrow{\cong} v^{\pm 1} \otimes v^0 \xrightarrow{id_{V_1} \otimes \delta_1} v^{\pm 1} \otimes (v^1 \otimes v^{-1} - q^{-1} v^{-1} \otimes v^1) = v^{\pm 1} \otimes v^1 \otimes v^{-1} - q^{-1} v^{\pm 1} \otimes v^{-1} \otimes v^1$$

$$\xrightarrow{\epsilon_1 \otimes id_V} 0 + v^{\pm 1} = v^{\pm 1}$$

For the middle equality,

$$v^{\pm 1} \xrightarrow{\cong} v^0 \otimes v^{\pm} \xrightarrow{\delta_1 \otimes id_V} (v^1 \otimes v^{-1} - q^{-1} v^{-1} \otimes v^1) \otimes v^{\pm 1} = v^1 \otimes v^{-1} \otimes v^{\pm 1} - q^{-1} v^{-1} \otimes v^1 \otimes v^{\pm 1}$$

$$\xrightarrow{id_V \otimes \epsilon_1} v^1 \otimes v^0 \xrightarrow{\cong} v^{\pm 1}$$

Algebraically, Reidemeister II corresponds to the following:

$$V_1 \otimes V_1 \xrightarrow{R} V_1 \otimes V_1 \xrightarrow{R^{-1}} V_1 \otimes V_1$$

We check with linear maps on each of the basis elements of $V_1 \otimes V_1$ on the left hand side of the equality:

$$v^1 \otimes v^{-1} \xrightarrow{R} q^{1/2} \delta_1(\epsilon_1(v^1 \otimes v^{-1})) + q^{-1/2}(v^1 \otimes v^{-1}) = -q^{3/2} \delta_1(1) + q^{-1/2}(v^1 \otimes v^{-1})$$

$$= -q^{3/2}(v^1 \otimes v^{-1} - q^{-1} v^{-1} \otimes v^1) + q^{-1/2}(v^1 \otimes v^{-1})$$

$$-q^{3/2} v^1 \otimes v^{-1} + q^{1/2} v^{-1} \otimes v^1 + q^{-1/2} v^1 \otimes v^{-1} \xrightarrow{R^{-1}}$$

$$-q\delta_1(-q) + \delta_1(1) + q^{-1}\delta_1(-q) - q^2 v^1 \otimes v^{-1} + q^1 v^{-1} \otimes v^1 + v^1 \otimes v^{-1}$$

$$= q^2(v^1 \otimes v^{-1} - q^{-1} v^{-1} \otimes v^1) - q^2 v^1 \otimes v^{-1} + q v^{-1} \otimes v^1 + v^1 \otimes v^{-1} = v^1 \otimes v^{-1}$$

$$v^{-1}\otimes v^1 \xrightarrow{R} q^{1/2}(v^1\otimes v^{-1} - q^{-1}v^{-1}\otimes v^1) + q^{-1/2}v^{-1}\otimes v^1$$
$$= q^{1/2}v^1\otimes v^{-1} - q^{-1/2}v^{-1}\otimes v^1 + q^{-1/2}v^{-1}\otimes v^1 = q^{1/2}v^1\otimes v^{-1} \xrightarrow{R^{-1}}$$
$$-q(v^1\otimes v^{-1} - q^{-1}v^{-1}\otimes v^1) + q^1v^1\otimes v^{-1} = v^{-1}\otimes v^1$$

For the case of $v^{\pm1}\otimes v^{\pm1}$, since both are in the kernel of ϵ_1, both elements will be sent to $0\in V_0$, hence we need only worry about the identity map. Since $q^{1/2}q^{-1/2} = 1$, we have equality.

Now we'll check the middle equality.

$$v^1\otimes v^{-1} \xrightarrow{R^{-1}} q^{-1/2}\delta_1(\epsilon_1(v^1\otimes v^{-1})) + q^{1/2}(v^1\otimes v^{-1}) = -q^{1/2}\delta_1(1) + q^{1/2}(v^1\otimes v^{-1})$$
$$= -q^{1/2}(v^1\otimes v^{-1} - q^{-1}v^{-1}\otimes v^1) + q^{1/2}(v^1\otimes v^{-1})$$
$$-q^{1/2}v^1\otimes v^{-1} + q^{-1/2}v^{-1}\otimes v^1 + q^{1/2}v^1\otimes v^{-1} \xrightarrow{R^1}$$
$$-q^1\delta_1(-q) + \delta_1(1) + q^1\delta_1(-q) - v^1\otimes v^{-1} + q^{-1}v^{-1}\otimes v^1 + v^1\otimes v^{-1}$$
$$= v^1\otimes v^{-1}$$
$$v^{-1}\otimes v^1 \xrightarrow{R^{-1}} q^{-1/2}(v^1\otimes v^{-1} - q^{-1}v^{-1}\otimes v^1) + q^{1/2}v^{-1}\otimes v^1$$
$$= q^{-1/2}v^1\otimes v^{-1} - q^{-1/2}v^{-1}\otimes v^1 + q^{1/2}v^{-1}\otimes v^1 = q^{1/2}v^1\otimes v^{-1} \xrightarrow{R}$$
$$-q(v^1\otimes v^{-1} - q^{-1}v^{-1}\otimes v^1) - v^1\otimes v^{-1} + q^{-1}v^{-1}\otimes v^1 + q(v^1\otimes v^{-1} - q^{-1}v^{-1}\otimes v^1)$$
$$+q^{-1}v^1\otimes v^{-1} - q^{-1}v^{-1}\otimes v^1 + v^{-1}\otimes v^1$$
$$= v^{-1}\otimes v^1$$

For the case of $v^{\pm1}\otimes v^{\pm1}$, the logic is the same as before. Thus, we've shown that the composition of maps $R^{\pm1}\circ R^{\mp1}$ equates to $id_{V_1\otimes V_1}$, which implies R, R^{-1} are inverses of one another.

The final move to check is Reidemeister III. Algebraically, it corresponds to

$$R\otimes id_{V_1}\circ id_{V_1}\otimes R\circ R^{-1}\otimes id_{V_1} = id_{V_1}\otimes R^{-1}\circ R\otimes id_{V_1}\circ id_{V_1}\otimes R$$

Here we'll demonstrate checking Reidemeister III in the case of $v\otimes v\otimes v^{-1}$ (Reidemeister could also be checked by comparing the composition of maps via Mathematica).

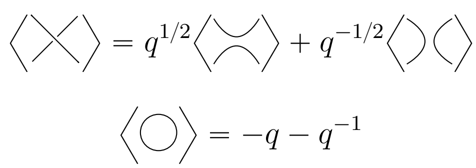
$$v\otimes v\otimes v^{-1} \xrightarrow{R^{-1}\otimes id_{V_1}} q^{1/2}v\otimes v\otimes v^{-1} \xrightarrow{id_{V_1}\otimes R} q^{1/2}\left[(q^{-1/2} - q^{3/2})v\otimes v\otimes v^{-1} + q^{1/2}v\otimes v^{-1}\otimes v\right]$$
$$\xrightarrow{R\otimes id_{V_1}} (q^{-1/2} - q^{3/2})v\otimes v\otimes v^{-1} + q(q^{-1/2} - q^{3/2})v\otimes v^{-1}\otimes v + q^{3/2}v^{-1}\otimes v\otimes v$$

From the RHS:

$$v\otimes v\otimes v^{-1} \xrightarrow{id_{V_1}\otimes R} (q^{-1/2} - q^{3/2})v\otimes v\otimes v^{-1} + q^{1/2}v\otimes v^{-1}\otimes v$$
$$\xrightarrow{R\otimes id_{V_1}} q^{-1/2}(q^{-1/2} - q^{3/2})v\otimes v\otimes v^{-1} + q^{1/2}[(q^{-1/2} - q^{3/2})v\otimes v^{-1}\otimes v + q^{1/2}v^{-1}\otimes v\otimes v]$$
$$\xrightarrow{id_{V_1}\otimes R^{-1}} (q^{-1/2} - q^{3/2})v\otimes v\otimes v^{-1} + q(q^{-1/2} - q^{3/2})v\otimes v^{-1}\otimes v + q^{3/2}v^{-1}\otimes v\otimes v$$

3.4 Example: trefoil knot

Knots come with knot invariants, one of the most famous being the Jones polynomial. Interestingly, the Jones polynomial can be formally derived via the representation theory of $U_q(\mathfrak{sl}_2(\mathbb{C}))$, made possible by the covariant functor from Section 3. In particular, hom(0, 0) in Fr_{tang} is the set of un-oriented knots. Since $\mathcal{F}$ respects regular isotopy, if two knots K, K' are equivalent under regular isotopy, then $\mathcal{F}$ maps them to the same composition of intertwining maps mapping to and from $\mathbb{C}(q)$; i.e., a polynomial known as the Kauffman bracket and denoted by $\langle K \rangle$. With a suitable renormalization, the Kauffman bracket gives the Jones polynomial [8]. Formally, for a knot K, the Kauffman bracket $\langle\ \rangle$ is defined via two relations:



These are precisely the relations found in the previous subsection, given by $\mathcal{F}$. We use the representation theory to determine the Kauffman bracket in the case of the trefoil knot, which has Kauffman bracket $-q^{9/2} + q^{-3/2} + q^{1/2} + q^{-7/2}$. Figure 1 shows the trefoil knot, with "critical regions" (i.e., intertwining maps) identified between dashed lines.

Via $\mathcal{F}$, this diagram corresponds to the composition of intertwining maps given by

$$\epsilon_1 \otimes \epsilon_1 \circ id_{V_1} \otimes R \otimes id_{V_1} \circ id_{V_1} \otimes R \otimes id_{V_1} \circ id_{V_1} \otimes R \otimes id_{V_1} \circ \delta_1 \otimes \delta_1 \tag{3.1}$$

This corresponds to (suppressing tensor symbol $\otimes$ on module elements for brevity)

$$v_0 \cong v_0 v_0 \xrightarrow{\delta_1 \otimes \delta_1} (vv^{-1} - q^{-1}v^{-1}v)(vv^{-1} - q^{-1}v^{-1}v)$$

$$= vv^{-1}v^{-1}v - q^{-1}vv^{-1}v^{-1}v - q^{-1}v^{-1}vvv^{-1} + q^{-2}v^{-1}vv^{-1}v$$

$$\xrightarrow{id_{V_1} \otimes R \otimes id_{V_1}} q^{1/2}vvv^{-1}v^{-1} - q^{-3/2}vv^{-1}v^{-1}v - q^{-3/2}v^{-1}vvv^{-1} - q^{-3/2}vv^{-1}v^{-1}v$$
$$+q^{-2}\left[(q^{-1/2} - q^{3/2})v^{-1}vv^{-1}v + q^{1/2}v^{-1}v^{-1}vv\right]$$

$$\xrightarrow{id_{V_1} \otimes R \otimes id_{V_1}} q^{1/2}\left[(q^{-1/2} - q^{3/2})vvv^{-1}v^{-1} + q^{1/2}vv^{-1}vv^{-1}\right] - q^{-2}vv^{-1}v^{-1}v - q^{-2}vv^{-1}v^{-1}v - q^{-2}v^{-1}vvv^{-1}$$

$$+q^{-2}\left[(q^{-1/2} - q^{3/2})^2v^{-1}vv^{-1}v + (q^{-1/2} - q^{3/2})q^{1/2}v^{-1}v^{-1}vv + qv^{-1}vv^{-1}v\right] \xrightarrow{id_{V_1} \otimes R \otimes id_{V_1}}$$

$$q^{1/2}\left[(q^{-1/2} - q^{3/2})^2vvv^{-1}v^{-1} + (q^{-1/2} - q^{3/2})q^{1/2}vv^{-1}vv^{-1} + qvvv^{-1}v^{-1}\right] - q^{-5/2}(vv^{-1}v^{-1}v + v^{-1}vvv^{-1})$$

$$+q^{-2}\left[(q^{-1/2} - q^{3/2})^3v^{-1}vv^{-1}v + (q^{-1/2} - q^{3/2})v^{-1}v^{-1}vv + 2(q^{-1/2} - q^{3/2})qv^{-1}vv^{-1}v + q^{3/2}v^{-1}v^{-1}vv\right]$$

$$\xrightarrow{\epsilon_1 \otimes \epsilon_1} q^3(q^{-1/2} - q^{3/2}) + 2q^{-3/2} + q^{-2}(q^{-1/2} - q^{3/2})(q^{-1} - 2q + q^3) + 2q^{-1}(q^{-1/2} - q^{3/2}) = -q^{9/2} + q^{-3/2} + q^{1/2} + q^{-7/2}$$

Figure 5 shows the Kauffman bracket computed via the skein relation.

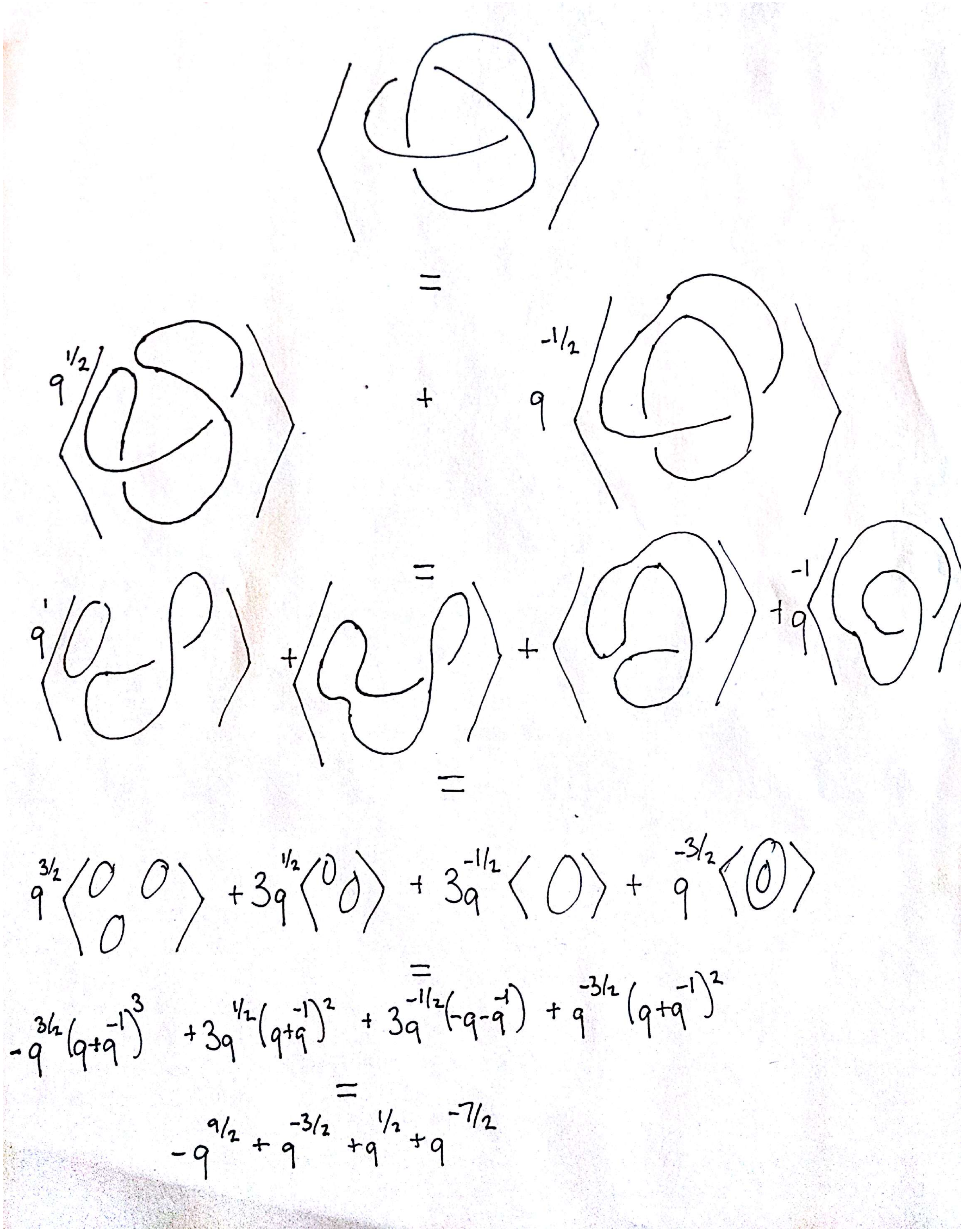


Figure 4: Kauffman bracket via skein relation

Conclusion

The representation theory of $U_q(\mathfrak{sl}_2(\mathbb{C}))$ has a graphical interpretation as tangle diagrams, allowing knots, links, and other objects to be associated with compositions of intertwining maps. Thus, knots can be mapped to polynomials like the Jones polynomial and Kauffman bracket that are invariant under some or all of the Reidemeister moves, meaning that they are knot invariants.

Quantum groups/algebras offer an interesting means of generating knot invariants. Generalizing to $\mathfrak{sl}_n(\mathbb{C})$ leads to more knot polynomials [9]. There is a larger story of categorifying knot invariants like the Jones polynomial; in that particular case, the categorification involves associating smoothings of knot projections with vector spaces, generating chain groups and setting a differential, and then computing homology groups. This invariant, known as Khovanov homology, is *stronger* than many invariants in the sense that it distinguishes more knots [10].

As a physicist, the story of quantum groups becomes highly relevant to me in the context of topological and conformal field theory. (2+1) dimensional systems exhibit statistical properties and topological ordering unseen in (3+1) dimensions, mainly because the relevant symmetry group for particle interchange is not the symmetric group, but the *braid* group [4]. The ground state for such systems has a characteristic topological degeneracy, and the collective excitations, known as *anyons*, can exhibit fractional statistics and potentially be used for fault-tolerant topological quantum computing [11, 4, 12]. The categorical data to describe these particles and their interactions is given by quantum groups [7]. A prescient reader might look at knot projections and imagine particles *intertwining* with one another. A knot with singularities might even represent particles *interacting* with one another. A Feynman diagram is a graphical picture of precisely this, particles interacting with each other. What's more, a Feynman diagram is a picture of intertwining maps between irreducible representations (particles) of a symmetry group. These connections make knot theory, quantum groups, and category theory pivotal to continuing research in theoretical physics.

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