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Irreducible k-to-1 Maps Onto Grids

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[1]Under the direction of Prof. Van C. Nall

Abstract

In this paper we explore the existence of exactly k-to-1 continuous functions between graphs, and more specifically 2-to-1 continuous function between graphs that are irreducibly 2-to-1, meaning that no restriction of the function to a subgraph is 2-to-1. We show how to construct such functions in some general cases, and then more specifically onto rectangular grids. We have in mind an application to distributed networks and signal verification.

This paper is part of the requirements for honors in mathematics. The signatures below, by the advisor, a departmental reader, and a representative of the departmental honors committee, demonstrate that Susan M. Parker has met all the requirements needed to receive honors in mathematics.

(advisor)

(reader)

(honors committee representative)

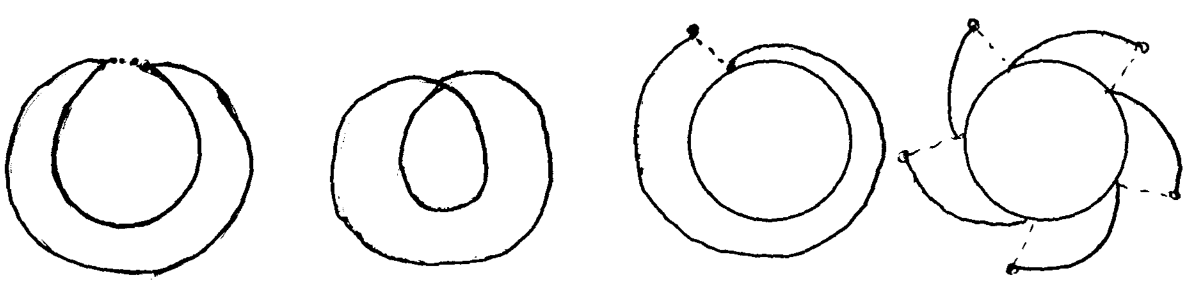
Susan M. Parker

April 15, 1997

Introduction

By a k-to-1 map, we mean a map f : X→Y such that for each y ∈ Y of f, $f^{-1}(y)$ has cardinality k. A map is defined to be a continuous function. This paper will be concerned with k-to-1 maps between graphs, and specifically with which graphs can be the image of a k-to-1 map. Not all graphs can be the image of k-to-1 maps; for example, O.G. Harrold showed in [1] that there does not exist a k-to-1 map onto an arc. In fact, the range must contain a simple closed curve. Furthermore, every 2-to-1 map onto a circle must be irreducibly 2-to-1 in the sense that no restriction of the map to a connected set is 2-to-1. So, one might ask what conditions are necessary to guarantee the existence of an irreducible 2-to-1 map onto a graph. This paper focuses on this question.

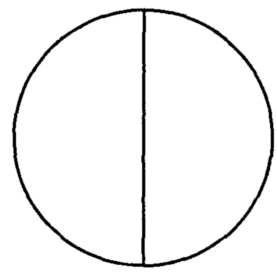
The following examples illustrate a few irreducible 2-to-1 maps onto a circle.



In the first example, the two nodes map onto one node in the range while the two edges map 2-to-1 onto one edge in the range. The second example is a circle wrapped back into itself, thus, the apparent intersection is actually two points on the circle that map 2-to-1 onto one point in the range with the remainder of the graph collapsing down 2-to-1 onto an edge in the range. Similarly, the other examples map 2-to-1 onto a circle.

Examples

We shall call a finite subset C of a graph G a k-to-1 cut set if G\C has at least $|C| \cdot k$ components. The only points of G that need to be considered as part of a k-to-1 cut set are the branch points of G. Branch points are defined to be those points in a graph which have local order > 3. It is easy to see that if $y \in C$ is not a branch point then $C \setminus \{y\}$ is still a k-to-1 cut set unless C has only one element which would have to be a cut point in which case either adjacent node to the cut point would be a k-to-1 cut set. Therefore, we can assume that all k-to-1 cut sets are contained in the node set of the graph. To understand how k-to-1 cut sets are related to irreducible k-to-1 maps, it is helpful to explore a few examples. In addition to the previous examples of irreducible 2-to-1 maps onto a circle, the following graph is the image of an irreducible 2-to-1 map.

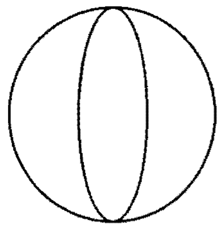


If one begins with any Euler path in this graph and doubles the edges, then the domain of the map would be the following. An Euler path in a graph is a path which crosses each edge exactly once while beginning and ending at different vertices.

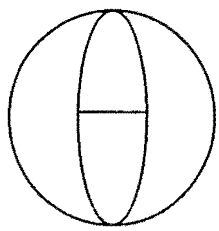
where f(X1) = f(X3) and f(X2) = f(X4)

The irreducibility can easily be seen because the inclusion of any one node in the map requires the inclusion of the rest of the nodes. In fact, we will show that whenever a graph has an Euler path and does not have a 2-to-1 cut set then this same construction will always produce an irreducible 2-to-1 map onto the graph.

There is no irreducible 2-to-1 map onto the following graph because it contains a 2-to-1 cut set, namely the two nodes.



The addition of just one edge to this graph creates a graph that admits an irreducible 2-to-1 map; The graph no longer contains a 2-to-1 cut set.

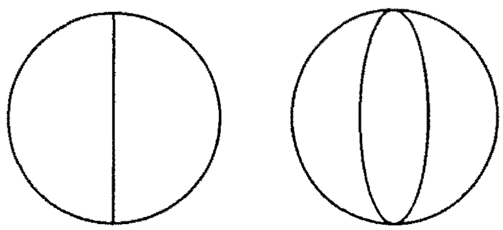


Note that this graph now has an Euler path as well.

Irreducible k-to-1 maps

A map $f : X \rightarrow Y$ where X and Y are continua is called an irreducible k-to-1 map if it is a k-to-1 map and for every proper subcontinuum A of X, $f|_A$ is not a k-to-1 map. Continua are compact, connected sets. This paper will focus on the specific case of irreducible 2-to-1 maps. O.G. Harrold did early work in this area when he proved that there is no 2-to-1 map onto an arc. If there was a 2-to-1 map onto an arc, then, by Zorn's Lemma, there must be a minimal set in which the map is still 2-to-1. However, this can not be because every subcontinua of an arc is an arc. So Harrold actually showed that there was no irreducible 2-to-1 map onto an arc. This was the original motivation for considering irreducible k-to-1 maps.

The comparisons between the two following potential ranges sparked many ideas concerning the conditions which guarantee the existence of an irreducible 2-to-1 map.



An irreducible 2-to-1 map can be quickly found onto the graph on the left. However, the graph on the right is a crucial example of a graph which does not admit an irreducible 2-to-1 map. The characteristics of these graphs lead to the idea of k-to-1 cut sets. Dr. Nall has shown the following theorem.

Theorem 1. *If f is an at most a k-to-1 map from a continuum X onto a continuum Y, and for no proper subcontinuum Y' of Y is $f^{-1}(Y')$ a continuum and B is a non-empty finite subset of Y, then the number of components of $Y \backslash B$ is less than $|B| \cdot k$.*

In other words, Y has no k-to-1 cut set. We have conjectured that the converse is also true, although no complete characterization of the image of irreducible 2-to-1 maps exists now.

Even if the characterization was fully developed, the process of generating an irreducible 2-to-1 map can be challenging. There are several different approaches examined in this paper to produce an irreducible 2-to-1 map onto a graph. We will look at Euler paths, edge-disjoint spanning trees and open ear decompositions as methods of furnishing these irreducible maps. Ear decompositions and spanning trees are both used in graph theory to ensure network reliability and we intend to suggest an application of irreducible 2-to-1 maps that is also related to reliability in distributed networks.

Irreducible maps and Euler paths

There are several methods of developing maps to ensure irreducibility. As mentioned earlier, Dr. Nall has shown the following theorem.

Theorem 1. *If $f: X \to Y$ is an irreducible k-to-1 map from a continuum X onto a continuum Y, then Y does not have a k-to-1 cut set.*

Although the converse has yet to be proven, we know that if a graph has an Euler path and no 2-to-1 cut set, then the Euler path on the graph produces an irreducible map. This theorem is presented in the following section.

Irreducible k-to-1 maps and spanning trees

The following theorem generalizes the situation from graphs having an Euler path to graphs having a map from a tree which is 1-to-1 on the edges and 2-to-1 on the nodes.

Theorem 2. *Suppose G is a graph that does not have a 2-to-1 cut set. Let T be a tree with $f\colon T \to G$ such that $f(V_T) = V_G$, f/V_T is 2-to-1 and f is 1-to-1 on $T \backslash V_T$. Then there is no proper connected subtree T' of T such that f maps the nodes of T' onto the nodes of f(T').*

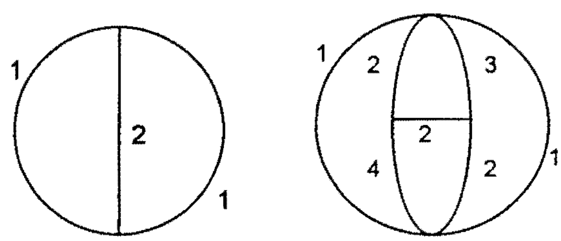
Proof. Assume there is a proper connected subtree T' of T such that $f|V_{T'}$ is 2-to-1 onto the nodes of f(T'). Let $|V_T| = 2n$ and $|V_{T'}| = 2m$. The number of components of $T' \backslash V_{T'}$ must be 2m-1 because T' is a tree having 2m vertices. $T \backslash V_{T'}$ must have at least 2m-1 components since $T \backslash T' \neq \emptyset$. So $G \backslash f(V_{T'})$ must also have at least 2m components since f is 1-to-1 on $T \backslash V_T$. Because $f|V_{T'}$ is 2-to-1, $|V_{f(T')}| = m$. Therefore, $f(V_{T'})$ is a 2-to-1 cut set of G ■

Theorem 3. *If a graph G has an Euler Path and no 2-to-1 cut set then there exists an irreducible 2-to-1 map onto G.*

Proof. In order to have an Euler Path, a graph must have two odd ordered vertices and the rest even. Moreover, as mentioned earlier, the only points that need to be considered as part of a cut set are the branch points. In a graph, $|E| = (\Sigma o(V))/2$. Moreover, since there is no 2-to-1 cut set $|E| < 2|V|$ or else the vertices would be a 2-to-1 cut set. We now have $|E| = \Sigma o(V)/2 < 2|V|$. So, $\Sigma o(V) < 4|V|$. If there exists $v \ni o(v) > 5$ then there must be at least three vertices of order 3. Furthermore, if there exists $v \ni o(v) = 5$, then there must be at least two vertices of order 3. However, there can only be two odd ordered vertices in G. So, there does not exist $v \ni o(v) \geq 5$. The Euler path is the map of an arc onto the graph and since all the nodes are of order 3 or 4, the path passes through each node exactly twice. So, an arc that follows the Euler path would be 2-to-1 on the nodes of G and 1-to-1 on the edges. By Theorem 2, doubling the sections of the arc that map onto edges of G yields an irreducible 2-to-1 map∎

Building Ranges and Ear Decompositions

An ear decomposition is a sequence of graphs $\{G_1, G_2, \ldots G_n\}$ where G_1 is a simple loop with one node and $G_n = G$. G_i can be obtained by subdividing at most two edges of G_{i-1} and adding at most one new edge. An open ear decomposition is one in which none of the edges added form loops. An increasing open ear decomposition is one that is open and requires the addition of at least one new node at each stage. Dr. Nall has shown that a graph with an increasing open ear decomposition is the image of an irreducible 2-to-1 map. Sometimes it is easy to find the increasing open ear decomposition as shown in the following familiar graphs. The edges numbered 1 are those that make up the first stage, those numbered 2, the second stage, and so on.



If we add to the increasing open ear decomposition the condition that exactly one new node is added at each stage, then the resulting graph will have two edge-disjoint spanning trees each having $|V| - 1$ edges. In addition to having two edge-disjoint spanning trees, it is also easy to see that a graph with this sort of ear decomposition has $2|V| - 1$ edges since at each stage the number of nodes increases by one and the number of edges increases by two.

We will use this technique to build an irreducible 2-to-1 map onto a $n \times m$ grid by starting with a loop composed of all of the outer edges and then adding the middle edges one by one, each time creating a new node. Since distributed networks are most often in the form of $n \times m$ grids, a potential application exists here.

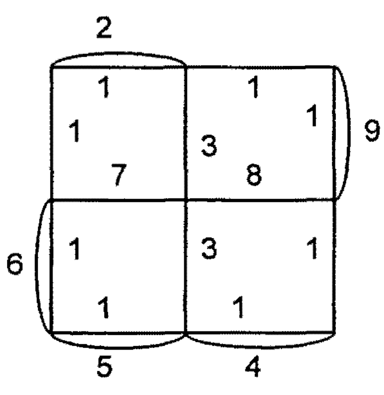
Irreducible maps onto grids

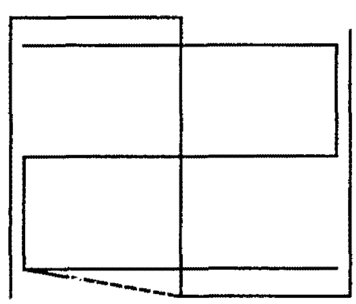
It is an easy exercise to construct an increasing open ear decomposition on an $n \times m$ grid and there is a general construction of an irreducible 2-to-1 map that coincides with each increasing open ear decomposition. However, the most obvious ear decompositions for an $n \times m$ grid do not produce very natural irreducible 2-to-1 maps. On the other hand, there is a natural irreducible 2-to-1 map onto a graph which is obtained by adding a few extra edges to a grid. The additional edges guarantee that the new graph has two edge-disjoint spanning trees with one edge leftover which is not covered by the trees. Now we may think of this augmented graph G as the image of a map which is 2-to-1 onto the nodes of G and whose whole domain consists of two disjoint copies of the spanning trees joined together by an edge that maps onto the leftover edge of G. By Theorem 2, this will be irreducibly 2-to-1 with respect to nodes onto the augmented graph. These spanning trees along with the final edge can be created in a systematic manner by using increasing open ear decompositions which force the graph G to have no 2-to-1 cut set. Finally, collapsing the additional edges down on the augmented graph will not affect the irreducibility of the map.

Generating Spanning Trees

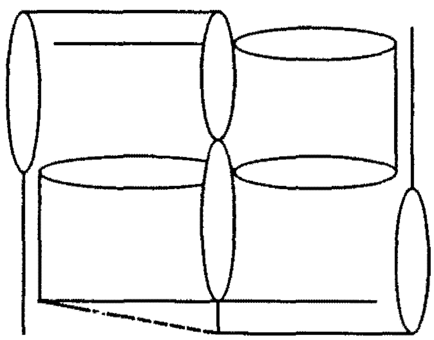
Odd by Odd grids

Below, on the left, is the augmented graph for a 3 X 3 grid. Clearly, the numbering on the augmented graph gives an increasing open ear decomposition and produces a graph with 2 | V | - 1 edges. So, there exist two edge-disjoint spanning trees, of which two natural ones are shown on the right. In fact, the addition of edges to the 3 X 3 grid was done with these spanning trees in mind. The dotted line indicates the extra edge.



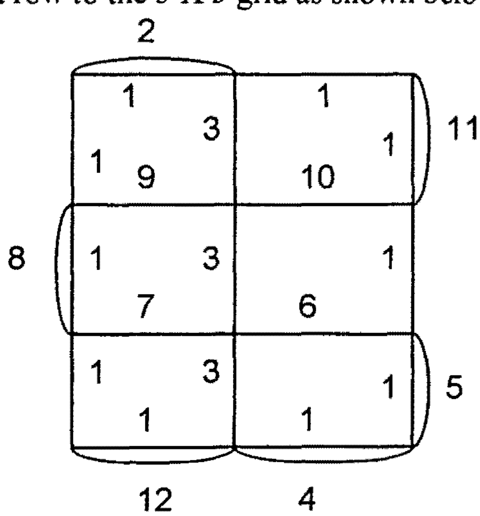


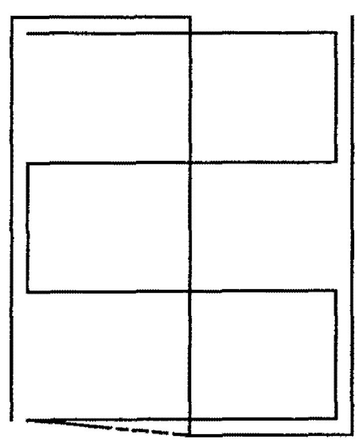
If the double edges in the range are collapsed into single edges and the edges in the trees which map 1-to-1 into the range are doubled, then the spanning trees and the augmented graph form the domain and range, respectively, of an irreducible 2-to-1 map. Neither of these operations affect the irreducibility of the map. The domain for an irreducible 2-to-1 map onto the 3 X 3 grid is shown below.



Even by Odd grids

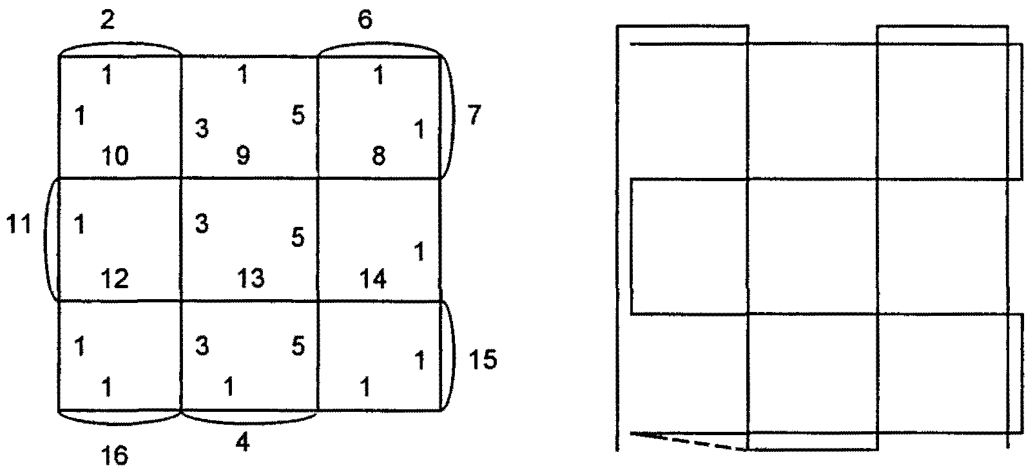
Irreducible 2-to-1 maps onto even by odd grids can be obtained by adding a row to the odd by odd map in a particular way. The general method for doing this is the same as adding a row to the 3 X 3 grid as shown below.





Even by Even grids

Similarly, a row and a column may be added at the same time to create the general pattern for an even by even grid.



Therefore, there exists an irreducible map onto all grids. This result leads to another question. Can we find an irreducible 2-to-1 map onto a 3 dimensional grid? The formula for the number of edges on an m x n x k grid is $[(2\ m\ n\ k) + (k - 1)(m\ n) - k(m + n)]$ and the number of nodes m n k. Note that $(m\ n) \geq (m + n)$. Since $[(k - 1)(m\ n) - k(m + n)] \geq [(k - 1)(m\ n) - (k - 1)(m + n)] = [(k - 1)((m\ n) - (m + n))] \geq 0$, we have $[(2\ m\ n\ k) + (k - 1)(m\ n) - k(m + n)] \geq 2\ m\ n\ k$. So, a 3 dimensional cube has a 2-to-1 cut set, namely the nodes, and therefore, there does not exist an irreducible 2-to-1 map onto the cube. However, we can generate an irreducible 2-to-1 map which maps onto the nodes of a cube. A m x (n k) grid may be folded over back and forth to form an m x n x k cube.

Since there is an irreducible 2-to-1 map onto the grid, this map forms a nodally irreducible 2-to-1 map into the cube.

Applications

The reason we are concerned with irreducible 2-to-1 maps onto grids is that for the most part, parallel computers are arranged by grids and the maps may be used to ensure network reliability. However, our technique for developing irreducible 2-to-1 maps onto grids can be used in another similar case. The application we have in mind involves thinking of the trees that form the domain as information routes. Suppose that a commanding officer is at a root node in the domain tree and needs to send ready, aim, fire signals to all supporting nodes. The officer must ensure that no node fires until all nodes are ready and have aimed. First, the officer will send a ready signal down the tree. The signals are passed along the tree by each node as soon as they are in fact ready. The commanding officer will receive her own ready signal back as a sort of echo because the map is 2-to-1. After receiving the ready signal echo, she then sends on the aim signal. No one is allowed to send on the aim signal until they have received two ready signals. When the officer receives the aim echo, she passes on a fire signal which, similarly, can not be passed on by a node until it has received two aim signals. Because the map is irreducibly

2-to-1, no one will receive a fire signal unless everyone has responded to a ready. The irreducible 2-to-1 map acts as a fail safe mechanism to ensure a rough synchronization of the nodes.

References

1. O.G. Harrold, *Exactly (k, l) transformations on connected linear graphs,* Amer. J. Math. 62 (1940), 823-834.