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Four dimensional graphs of complex functions

Malcom Lee Murrill

FOUR - DIMENSIONAL GRAPHS OF COMPLEX FUNCTIONS

by

Malcolm Lee Murrill

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Approved, August 20, 1959

E. Sherman Grable
Wm. E. Trout, Jr.
W. Allan Powell
Francis B. Kay
H. Pearce Atkins
Nolan E. Rice
Jackson J. Taylor

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Chapter I. NATURE OF THE PROBLEM.

1. Functions of a Complex Variable.

Complex functions of a single complex variable involve four unknowns, two independent and two dependent variables, and thus cannot be adequately represented in two- or three-dimensional space. Various geometric constructions in both two and three dimensions have been devised in the past, however, in attempts to illuminate complex function theory. The standard, and most useful, of these representations is that developed by Gauss and Riemann employing two complex planes simultaneously[1*]. These show the correspondence between a particular curve or region in the object plane and its image, as mapped by a given transformation, in the image plane. Tables based on this system of representation have been compiled[2]. The chief disadvantage here is, of course, the fact that each pair of graphs shows only one facet of the particular complex function involved, i. e. its effect on some one region or set of curves. No overall graph of a complex function is presentable in this system.

A clearer idea of the effect of a particular transforma-

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* Numbers appearing as superscripts refer to notes and references at end of paper.

tion can be obtained under this system by constructing the images of a rectangular grid over the region[3] or of a family of concentric circles[4]. An important variation of this is the plotting of the images under a given transformation of the contour curves or level lines of the conjugate harmonic functions. In certain practical applications, these orthogonal sets of curves represent isotherms or equipotentials and their accompanying lines of flow, flux lines, or streamlines[5].

Other ingenious geometric configurations in two and three dimensions for representing (in part) the nature of complex functions are described in the following excerpts from articles, some not without a hint of frustration, which have appeared in the American Mathematical Monthly over the past forty years.

"The impossibility in three dimensions", writes Norman Miller[6], "of representing graphically a function of a complex variable makes it necessary for the student to call on his imagination in other ways in order to realize the properties of these functions. Two methods are common in the geometrical theory of functions. One is to represent in two different planes or in two Riemann surfaces the variables z and w and to study the correspondence between the points of the two planes or surfaces, which is determined by the relation $w = f(z)$. The second method, which does much to illuminate the subject for the beginner, is to represent in one plane both the independent and dependent variables and to interpret the transformation kinematically as a flow of the points in the plane[7].

"A complete graph of the function $w = f(z)$ or $u + iv = f(x + iy)$ consists of a two-dimensional manifold in space of four dimensions. Nevertheless the student, in his effort to visualize the function, thinks instinctively of a surface spread out over the plane of z. Such a surface is actually determined by taking for a third coordinate the absolute value of f(z)..."

In an earlier issue, A. F. Frumveller[8] had proposed two simultaneous three-dimensional graphs, reducing the four-dimensional problem to three dimensions by holding first one, then a second axis, equal to zero:

"Since [z] has been shuffled out of sight by projecting its field [xy] into the point (0,0) of the [w]-plane, two separate diagrams will be needed in plotting -- one, to show the path of [z] in its own plane -- the other, to show the position and length of the vector-ordinates in the plane of [w]."

A modification of the above, by E. L. Rees[9], suggests a single three-dimensional figure showing the surface $u(x,y)$ on which are drawn "the contours for $v = v_1, v_2$, etc., the consecutive v's differing by a constant. These contours enable us to visualize the variation of v, so that we have pictured the variation of both u and v and, therefore, of [w], all on one surface."

Twenty years later a refinement and extension of three-dimensional representation was given by Luise Lange[10] as follows:

"...Pairs of complex numbers interpreted as coordinates of a point would, indeed, require a four-dimensional space.

"The classical representation of functions of a complex variable, as developed by Gauss and Riemann, uses an altogether different idea, the functional equation $w = f(z)$ being interpreted as a transformation of the points of one two-dimensional continuum onto another. The Cartesian scheme, on the other hand, has also been adapted by plotting in rectangular space coordinates separately the two surfaces $u(x,y)$ and $v(x,y)$, or the surface of the modulus $R(x,y)$.

"In the following a somewhat different method is set forth to adapt the Cartesian scheme to the representation of functions of a complex variable. It consists in presenting on one coordinate axis linear fields of the complex independent variable, and on the other two axes the real and imaginary parts of the dependent variable. The function $w = f(z)$ thereby appears in the form of one-parameter families of space curves. These curves, which may be regarded as three-dimensional sections through the non-presentable four-dimensional loci, (or as one-dimensional sections of two-dimensional surfaces in four-space), are the complex generalizations of the familiar plane real curves.

"Various families of curves (or different sets of sections) for a given function are obtained by different choices of the parameter of the linear complex z-field. In the following have been treated some presentations using:

"(a) ϕ as parameter with r as independent variable ("radial sections"),

"(b) x as parameter with y as independent variable ("sections parallel to the imaginary axis"), and

"(c) y as parameter with x as independent variable ("sections parallel to the real axis")."

An extension of this idea to the general four-dimensional function of three independent variables and to complex functions of several complex variables, using families of successive three-dimensional sections, is suggested in a later article by Stefan Bergman[11].

However, in all of the geometric representations outlined above, the basic dilemma is still present: a four-dimensional quantity cannot be adequately and completely represented in two or three dimensions. Some characteristic feature of the complex function has to be sacrificed or suppressed to squeeze the four dimensions down to 2- or 3-space. A new approach is therefore needed before we can construct a full geometric analogue of a function of a complex variable.

2. Hyper-Analytic Geometry.

Under the impact of the purely analytical methods of modern mathematics, graphical representation has been relegated to the role of a picturesque but limited aid to man's mathematical comprehension. No doubt a contributing factor to this situation has been the traditional limitation of such representations to three dimensions. Man's "common-sense" experiences with the physical world were long a mental barrier to the extension of geometry into an "unreal" realm of four or more dimensions. Only since the early nineteenth century has four-dimensional geometry been seriously considered and developed[12].

It seems remarkable to the author, however, that this geometry of hyperspace was not founded on a simple extension of the cartesian coordinate system, to become a hyper-analytic geometry. Without a four, or more, dimensional frame of reference, the constructive features of hypergeometry are necessarily vague and inaccurate, and adequate graphical representation of functions of more than three variables thus remains impossible.

The value of an adequate geometrical analogue is, of course, undisputed. As Arnold Emch[13] points out:

"Even in more advanced fields and certain domains of mathematical research the establishment of the constructive features of a mathematical theory may sometimes be greatly aided and illuminated by appropriate graphs, diagrams and models.

"Not infrequently it happens that, after the actual construction of a figure or of a model, a close examination of the finished product reveals or suggests the existence of new properties of the form investigated which were not anticipated before the construction.

"Another important factor in the construction of figures and models lies in its strengthening of the geometrical imagination and of mathematical intuition in general...".

The road, however, has been paved with warning signs discouraging any attempt at a complete graphical representation of four or more dimensional functions:

"Of course," says E. T. Bell in Men of Mathematics[14], "if we take points as the elements out of which our space is to be constructed, nobody outside of a lunatic asylum has yet succeeded in visualizing a space of more than three dimensions."

In Mathematics and the Imagination[15], Kasner and Newman say: "Graphic representations of four-dimensional figures have been attempted: it cannot be said these efforts have been crowned with any great success."

And R. E. Gaskill in Engineering Mathematics[16] says: "[A graph of] the relationship w = f(z) ... would require four dimensions. Since we do not have a supply of four-dimensional graph paper, the best we can do is to provide the two complex planes ... and call attention to a few correspondences ...".

However, let us "rush in where angels fear to tread" and arbitrarily add a fourth "perpendicular" axis to the cartesian system, thus creating a hyper-analytic geometry. As we shall see, four-dimensional functions can be completely and accurately represented in such a system. And, in fact, by using n "perpendicular" coordinate axes, geometry is liberated from its three-dimensional prison and becomes once more a powerful friend and advisor in higher analysis in the n-dimensions of functions of n variables[17].

Chapter II. GRAPHS AND HYPERGRAPHS.

Geometric representation of complex functions, as was seen in the preceding chapter, has been confined in the past to the two- and three-dimensional space of which we have a first-hand knowledge. This ordinary space of our experience, however is basically inadequate for the representation of such functions.

The author presents in this paper what he believes to be an original method for the complete geometric representation of functions of a complex variable, using a hyper-analytic geometry of four dimensions.

1. The Four-Dimensional Coordinate System.

For our frame of reference in the hyper-analytic geometry of four dimensions, we postulate and construct four coordinate axes, x, y, u, and v, mutually perpendicular by definition at a common origin O. The number of mutually perpendicular p-dimensional coordinate manifolds in n-space is $C(n,p)$, the coefficient of x^p in the expansion of $(1 + x)^n$. In four dimensions, since $(1 + x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$, we thus have:

one "mutually perpendicular" coordinate point: the origin[18];
four mutually perpendicular coordinate axes: x, y, u, v;
six mutually perpendicular coordinate planes: xy, xu, xv,

yu, yv, uv;

four mutually perpendicular coordinate hyperplanes: xyu, xyv, xuv, yuv;

one "mutually perpendicular" coordinate hyperspace: xyuv. Also, n-dimensional space is partitioned by n mutually perpendicular (n - 1)-dimensional manifolds into 2^n distinct (i.e. disjoint and exhaustive) n-dimensional cells. Thus, in three dimensions, the three coordinate planes divide ordinary space into eight three-dimensional octants. Similarly, in four dimensions, the four coordinate hyperplanes divide hyperspace into sixteen four-dimensional cells which we will call "hexadekants".

To obtain a four-dimensional graph, or "picture" of an object in 4-space, we must project from the four dimensions of the figure to the two dimensions of the graph paper. Central projection (i.e. perspective) is too involved in construction and metrical determination for our purposes here; and of the parallel projections, orthogonal projection would require three views, to be unambiguous. We have available then, and will use, an oblique projection and, later, axonometric (isometric) and stereoscopic projections for our figures[19].

In three-dimensional oblique projection (that used for ordinary three-dimensional graphs), two axes are drawn in true length, while the third is foreshortened. This so-called "cabinet projection" gives a fairly natural appearance to three-dimensional drawings. An alternate system, called a "cavalier projection" extends the third axis to the same length as the other two, with a resulting distortion of the figures.

In our four-dimensional system, a "cabinet projection" would foreshorten two of the axes. Since, as we shall see later, this "double foreshortening" is a characteristic property of two-dimensional projections of four dimensions, it would be natural to use such a projection for our figures. However, for convenience in working in the z- and w-planes of complex variables, which is our primary objective here, we will use instead a "cavalier projection" with equal scales on all four axes, and preserve the right angle between the axes of the z-plane. In this way we will be seeing each of the two perpendicular planes, the xy and the uv coordinate planes, in true shape. Our hyper-figures then, while still perfectly representative, will be somewhat distorted from their "natural appearance" (whatever that is) in four-dimensions.

To avoid crowded figures and the confusion of coinciding lines[20], we will use the following particular asymmetrical arrangement of the axes. (See Figure 1). Starting with the horizontal half-line to the left of the origin and proceeding counterclockwise, the positive ends of the axes are located as follows:

the x-axis 30° below the left horizontal half-line,
the y-axis 90° to the right of this,
the u-axis 60° above this on the right horizontal half-line, and
the v-axis 90° from this and vertical.

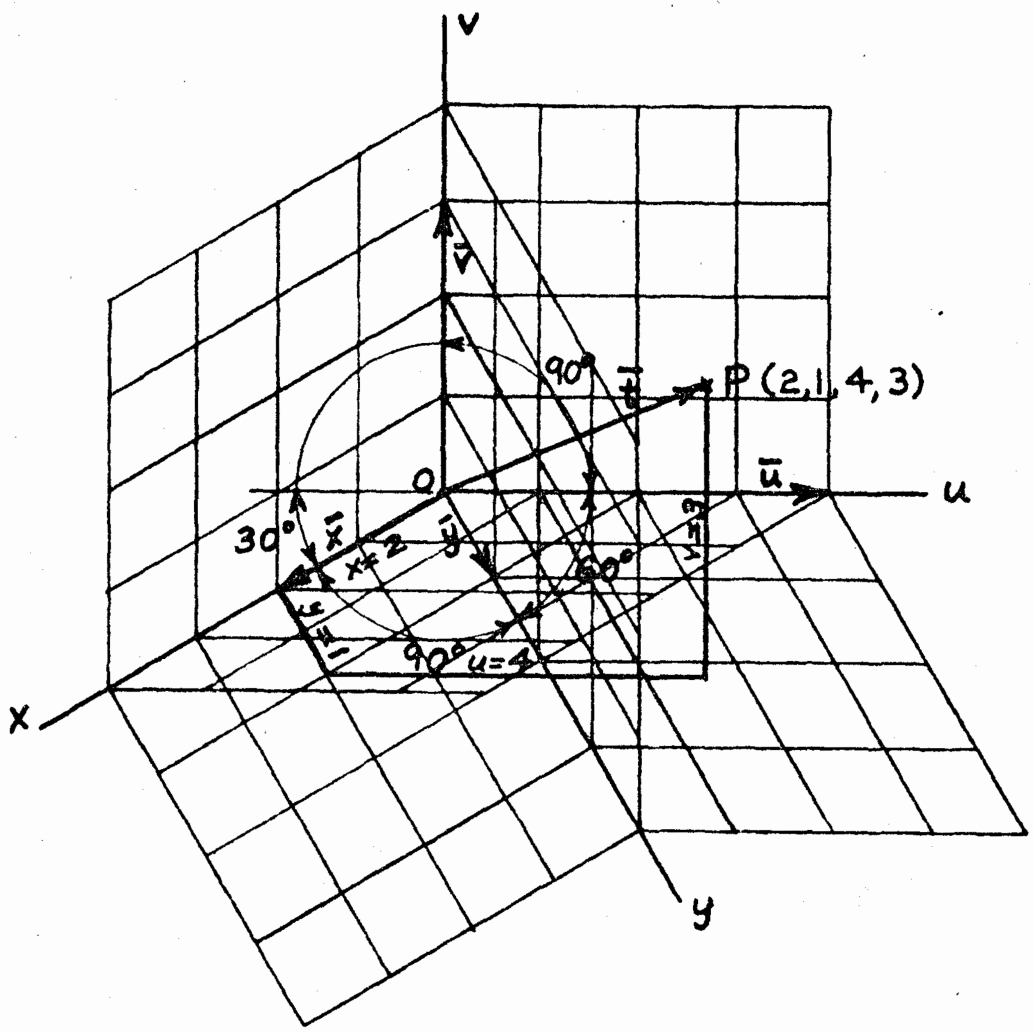


Fig. 1 The Four-Dimensional Coordinate System

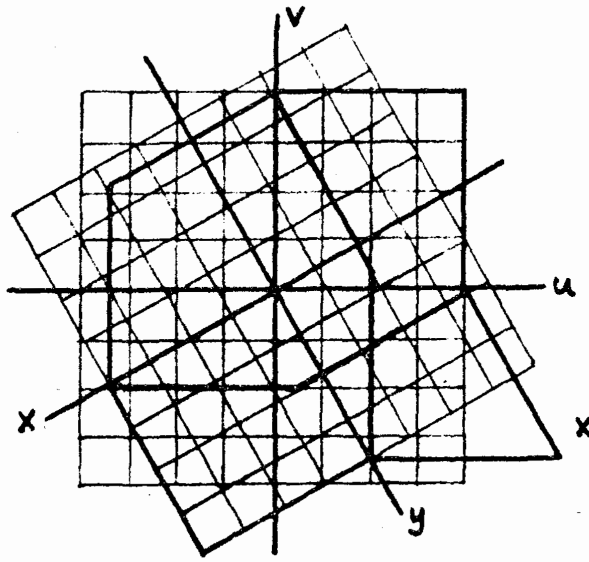


Fig. 2

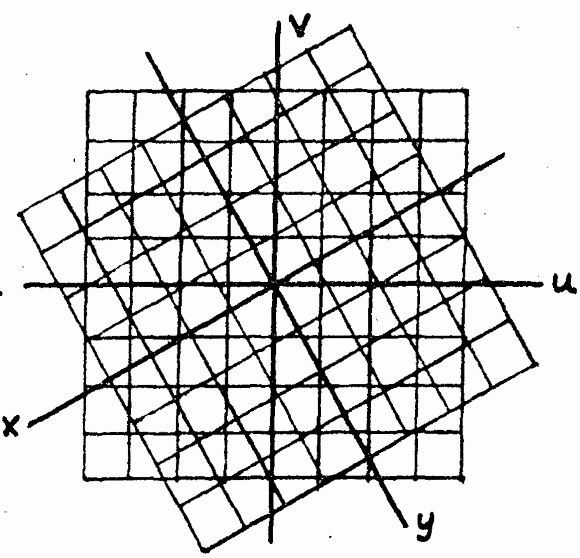


Fig. 3

Notice that in this arrangement the xy and the uv planes are not only in true shape but with the usual orientation of the axes, so that figures may be plotted directly in each of the two planes, it only being necessary to rotate the paper when working in the xy plane. Alternate designs for four-dimensional graph paper particularly suited for complex functions are shown in Figures 2 and 3. Other arrangements of the axes will be discussed in later chapters.

Points are plotted in this system, as indicated in Figure 1, by starting at the origin and moving successively, in any order, the given coordinate distance parallel to each axis. We define the vector $\bar{t}$, drawn from the origin to the point P(x,y,u,v), to be the vector sum of $\bar{x}$, $\bar{y}$, $\bar{u}$, $\bar{v}$. It may be represented by the notation $\bar{t} = (x,y,u,v)$, and will be called a "transformation vector".

2. Graphs, Manifolds, and Intersections.

DEFINITION: We shall define a graph in n-dimensions as the locus of the end-points of the transformation vector $\bar{t} = \sum_{i=1}^{n} \bar{x}_i$. Such a locus is a "transformation locus". When $n \leq 3$, this definition gives us the graphs obtained by the ordinary methods of plotting. For $n > 3$, we may call these loci "hypergraphs".

The kind of graph, whether it will be a point, curve, surface, solid, or hypersolid, is determined by the proportionate number of constraints on the degrees of freedom available. In three dimensions, for example, one equation defines a surface (a two-dimensional manifold), two consistent and independent equations define a skew curve (a one-dimensional manifold), while three such equations define isolated points (zero-dimensional manifolds). In general, if r constraints are to be imposed on the points of n-space, with their n variables or degrees of freedom, we will need r dependent relationships among the n variables, leaving n - r of the variables still independent. Given r consistent and independent equations expressing such dependence, r of the variables, usually those expressed or expressable as explicit functions of the others, are customarily called the dependent variables. The remaining n - r variables of the n-space are called the independent variables. If one or more of these do not occur explicitly in the equations then the manifold is some type of "cylindrical" manifold. Since an (n - r)-dimensional manifold is a set of points with n - r degrees of freedom or independent variables, we have the following theorem for determining the kind of graph we can expect:

<u>THEOREM:</u> "A set of r consistent and independent equations among n variables uniquely determines an (n - r)-dimensional manifold in n-space."[21]

We often speak of the manifold resulting from two or more simultaneous constraints as the <u>intersection</u> of the manifolds described by each of the separate constraints. Such intersections are uniquely determined by the given constraints; however, they are not uniquely described by them, since any number of combinations of sufficient constraints can be found to describe the same manifold. Thus, three planes, under suitable restrictions, determine a unique point in 3-space, but this point can be described in any number of other ways. The theorem tells us the <u>type of intersection</u> our manifold or graph will be.

Exact definitions of the usual names for the various manifolds in zero to five dimensions are given in the accompanying table. We have proposed two additional names. Just as a plane curve, a one-dimensional manifold in two dimensions, if given a new dimension in 3-space, will twist into a "skew curve" in that space while, however, still keeping its characteristic identity as a one-dimensional succession of points, so a surface or a solid in three dimensions, given an added fourth dimension of freedom, will twist and change into a new shape in four dimensions while still retaining the basic characteristics of a surface or a solid. For this reason we will sometimes call a two-dimensional manifold in 4- instead of 3-space a "skew surface", and a three-dimensional manifold in 4-space a "skew solid". Note that "skew solids", or more properly, hypersurfaces, include not only hyperplanes but also hypercubes, hyperspheres, etc. which suggest four dimensional hypersolids. Actually, of course, these are all hypersurfaces (three-dimensional solids), since their equations give us only the points on their hypersurfaces, not the points of the enclosed hypersolid. The same thing is true in three dimensions. A cube or a sphere, as defined by its equation, is, like a plane, a two-dimensional surface; if we

TABLE OF MANIFOLDS IN ZERO TO FIVE DIMENSIONS

n Dimension - of space	n - r Dimension of manifold	(Exact Definition)	"Common Designations"	[Proposed Name]

0-0 (Point Point) "Point"

1-0 (Line Point) "Point"

1-1 (Line Curve*) "Line"

2-0 (Plane Point) "Point"

2-1 (Plane Curve) "Plane curve" or "Curve"

2-2 (Plane Surface*) "Plane surface" or "Plane"

3-0 (Space Point) "Point"

3-1 (Space Curve) "Space curve" or "Skew curve"

3-2 (Space Surface) "Surface"

3-3 (Space Solid) "Solid"

4-0 (Hyperspace Point) "Point"

4-1 (Hyperspace Curve) "Hyperpoint"

4-2 (Hyperspace Surface) "Hypercurve" [Skew surface]

4-3 (Hyperspace Solid) "Hypersurface" [Skew solid]

4-4 (Hyperspace Hypersolid) "Hypersolid"

5-0 (Hyper-hyperspace Point) "Point" or "Point-System"

5-1 (Hyper-hyperspace Curve) "Curve"

5-2 (Hyper-hyperspace Surface) "Hyper-hyperpoint" or "Super-curve"

5-3 (Hyper-hyperspace Solid) "Hyper-hypercurve" or "Subsur-face"

5-4 (Hyper-hyperspace Hypersolid) "Hyper-hypersurface" or "Surface"

5-5 (Hyper-hyperspace Hyper-hypersolid) "Hyper-hypersolid" or "Space"

* If the manifold is linear, read "line" for "curve" and "plane" for "surface" throughout the table.

wish to describe a three-dimensional solid we have to use inequalities or parameters. The final names given in the table under five dimensions are the terms used by Cayley.

If, in plotting the graph of a set of equations, we let OA equal the vector sum of the $n - r$ vectors representing the independent variables, and OA' the vector sum of the r vectors representing the dependent variables, then by our definition, $\overline{t} = (OA, OA') = OA^*$, where A^*, the end of the $\overline{t}$ vector, is the corresponding transformation point on the graph of our function. Figures 4 through 9 illustrate the definition and theorem of this section for various manifolds when $n = 2, 3$, and 4, and show the nature of a graph as a transformation locus between the independent and dependent variables. Cylindrical manifolds, not shown here, are considered in the first section of Chapter III. (See Figures 10 and 11).

More generally, the process of plotting a graph is that of vector composition and consists of finding a set of points in n-space, given a set of coordinates. The general inverse problem, vector resolution, is to find the set of coordinates, given the set of points. From OA and its transformation point OA^*, we can readily find OA'. Whether, given OA^*, a method is available for finding OA (or vice versa), and whether we can further resolve OA and OA' into the vectors of the individual variables determines our ability to complete this inverse process. For a two-dimensional graph of a curve and a three-dimensional model of a skew curve or surface this is always possible by orthogonal projection. Under any other conditions,

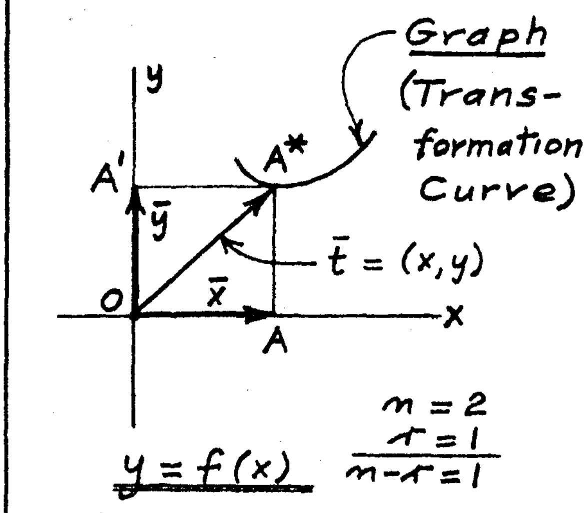


Fig. 4 - 2 dims. 1 dep. var.

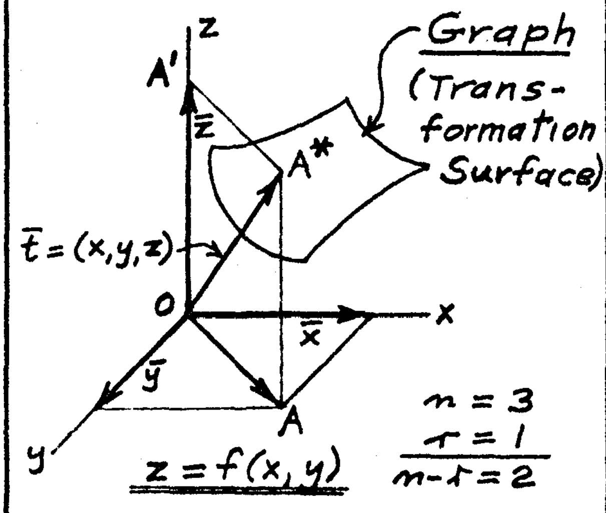


Fig. 5 - 3 dims. 1 dep. var.

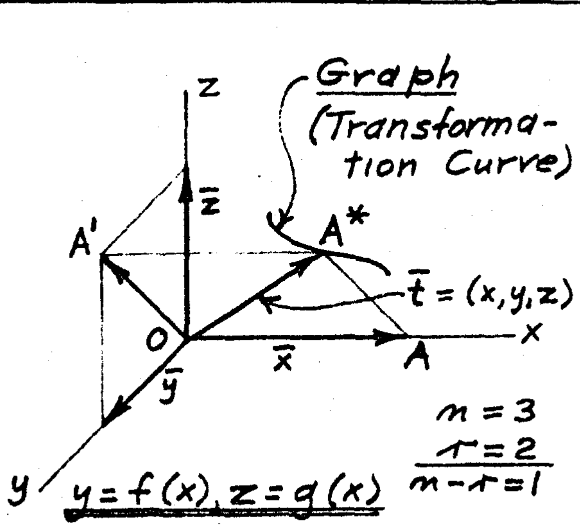


Fig. 6 - 3 dims. 2 dep. vars.

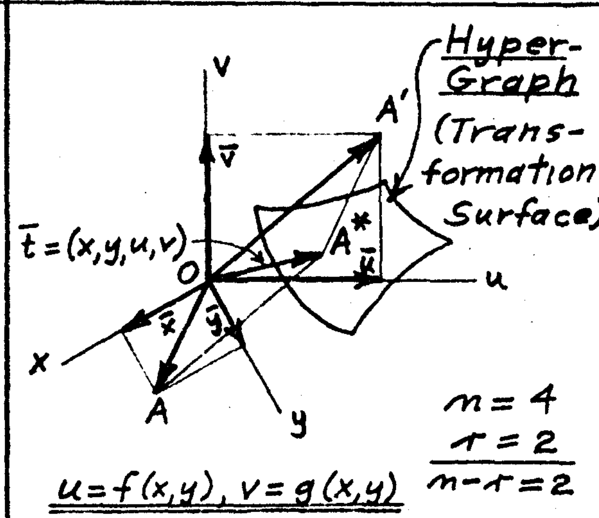


Fig 7 - 4 dims. 2 dep. vars.

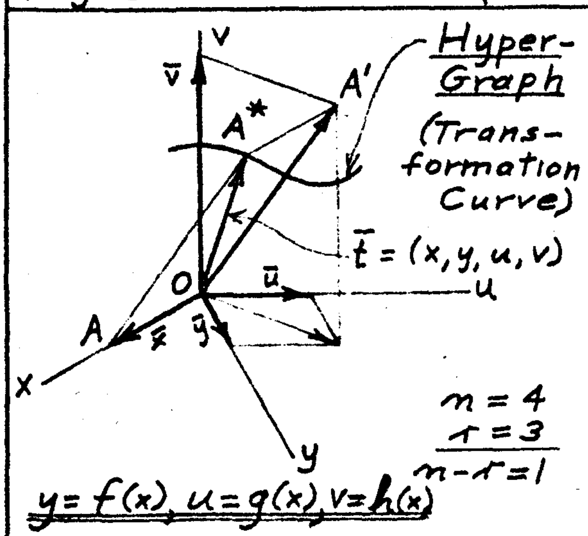


Fig. 8 - 4 dims. 3 dep. vars.

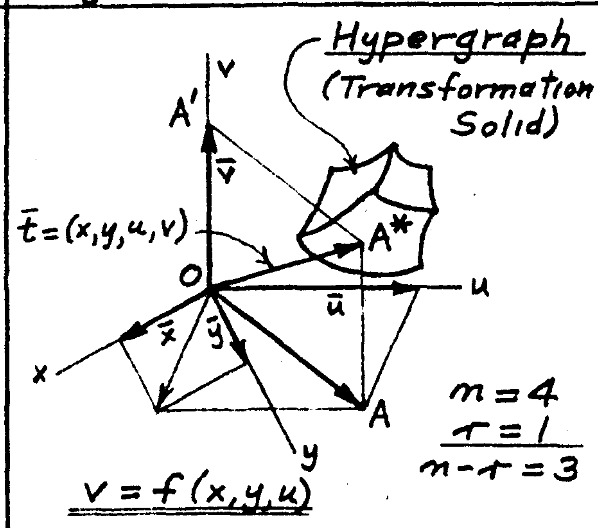


Fig. 9 - 4 dims. 1 dep. var.

additional information, including the correspondence between the vector sums of the independent or dependent variables and their transformation points, is needed to solve this inverse problem.

3. Graphic Transformation.

Given r equations in r dependent variables, expressable as explicit functions of n - r independent variables with, perhaps, some zero coefficients, we can assign values to the independent variables, determine the corresponding values of the dependent variables and find their vector sums $\bar{t}$. Thus we can plot a graph, or "transformation locus", which is an accurate geometric representation of our mapping function.

The use of the graph, on the other hand, to actually perform a transformation may not be so simple a problem, since it is one form of the inverse graphing process. In the projection of points between two manifolds, such as a p-dimensional manifold in n-space and the two-dimensional manifold of its graph on paper, ambiguous results occur if the dimensions of the two manifolds are different.[22] That is, the projection can then be accomplished in only one direction; the process has no inverse, or finite correspondence, in the reverse order. For example, we can project a point of a cube into only one point of its shadow on a plane, but the reverse process gives us an indeterminate location, actually an infinitude of possible locations, for our point projected from the shadow back into the cube. This means that, while our hyper-analytic geometry will enable us to graph the correspondences between the variables in n-space, the use of such graphs as actual transformation devices or nomographs is limited to a few cases in the lower dimensions, unless we employ some additional method for equalizing the difference in dimensions between the manifold and its projection. (Some methods for accomplishing this are given in Chapter IV.) Even though we may not desire to perform the actual graphic transformation, if a "read-out" is possible, the correspondences represented by the graph then become intelligible.

Fortunately, we can perform graphical transformations, at least in the usual direction from independent to dependent variables, in all cases through four dimensions, and in most of the five-dimensional cases, the particular case of the four-dimensional hyperpoint, and some of the five-dimensional cases, however, requiring additional features to make them amenable. (See Chapter IV).

In the particular case in which we are interested here, that of two independent and two dependent variables, our theorem tells us we are dealing with a two-dimensional manifold in 4-space; thus we can project from this to the two dimensions of our graph paper without ambiguity. The hypergraphs of complex functions, then, will not only give us pictures of the functions themselves but will also perform the actual transformations graphically. They therefore will be comprehensible and valid geometric representations of the actual correspondences expressed by the complex functions.

Chapter III. HYPERGRAPHS OF COMPLEX FUNCTIONS

1. The Special Four-Dimensional Case: Two Independent and Two Dependent Variables.

A function of a single complex variable has the form $w = f(z)$, where $w = u + iv$ and $z = x + iy$. From $u + iv = f(x + iy)$ we obtain, on separating the real and imaginary parts, $u = u(x,y)$ and $v = v(x,y)$, two equations in two independent and two dependent variables.

In three dimensions, two equations in one independent and two dependent variables give us a skew curve in 3-space, which is the intersection of two perpendicular cylindrical surfaces. (Figure 10). In a similar fashion, in four dimensions, two equations in two independent and two dependent variables give us a "skew surface" or hypercurve in 4-space, which is the intersection of two perpendicular cylindrical "skew solids" or hypercylindrical hypersurfaces[23]. (Figure 11). (The entire object in Figure 11 is a plano-cylindrical hypersurface and its interior.[24]) If w is an analytic function of z, then u and v are conjugate harmonic functions. Thus, the graph of an analytic function of a complex variable is the intersection of its conjugate harmonic hypercylindrical hypersurfaces.

In Figure 10, we see that a point $P_1(x_1,y_1,z_1)$ on the skew curve of intersection is the intersection of perpendicular

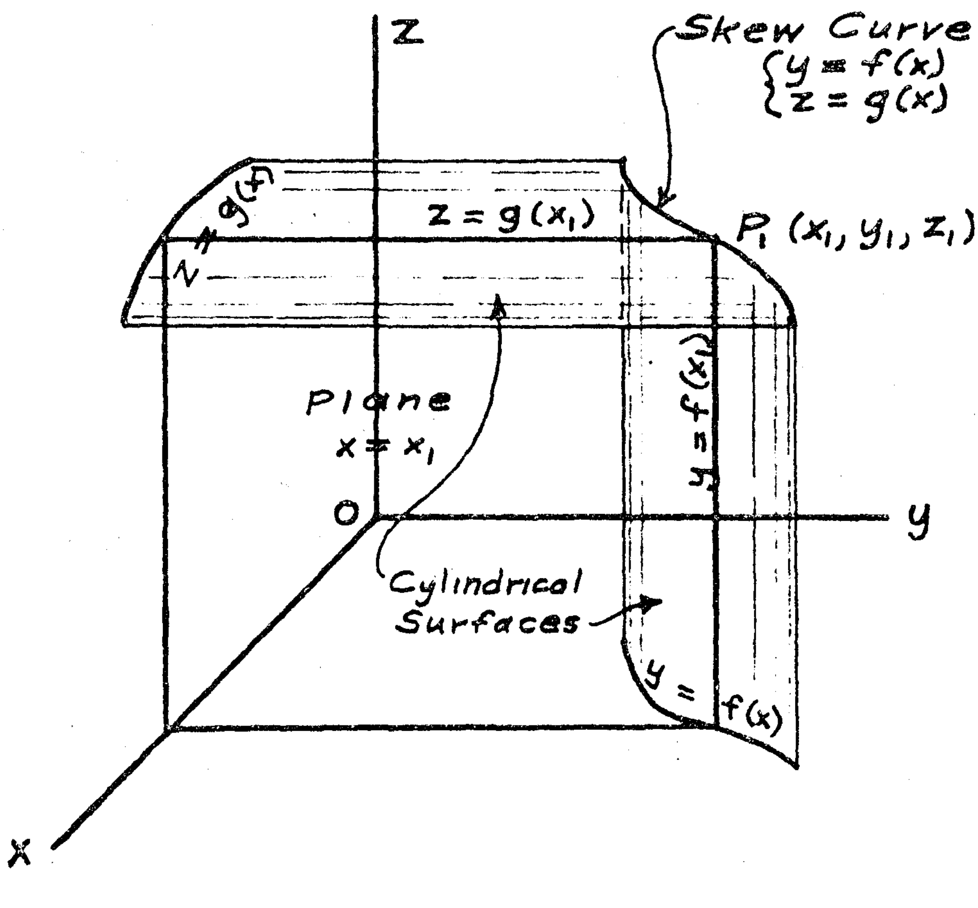


Fig. 10 Skew Curve in 3-Space

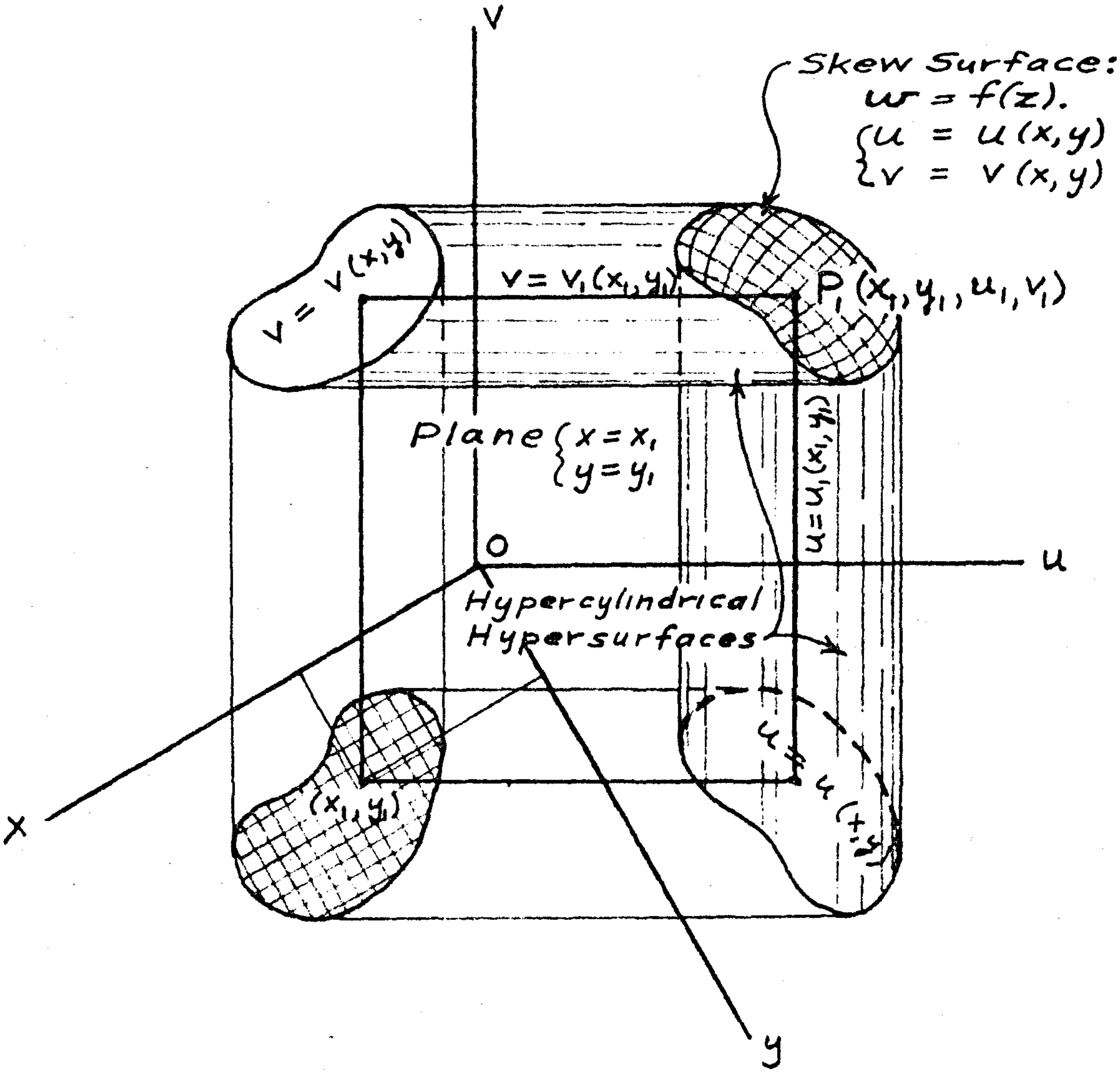


Fig. 11 Skew Surface in 4-Space

cylindrical elements cut from the two intersecting cylindrical surfaces by a plane through x_1 parallel to the plane of the dependent variables. That is, P_1 is the intersection of $y = f(x_1)$ and $z = g(x_1)$. Similarly, in Figure 11, a point $P_1(x_1, y_1, u_1, v_1)$ lying on the skew surface of intersection is the intersection of perpendicular cylindrical elements cut from the two cylindrical skew solids by a plane through the point (x_1, y_1) parallel to the plane of the dependent variables. Notice that, for each point (x_1, y_1), we have some particular pair of values $u = u_1(x_1, y_1)$, $v = v_1(x_1, y_1)$; i.e. each plane drawn parallel to the uv plane through some point in the xy plane cuts each of the cylindrical skew solids in only one element[25]. The intersections of these pairs of perpendicular elements determine the skew surface. We might call such a "skew surface" variously a "transformation surface", "t-surface", "hypergraph", or "hypercurve". Or, since it is interposed between and performs the transformation from an "object" in the z-plane to its "image" in the w-plane, we might call it a "mirror surface" or "m-surface"[26]. (See Figure 12).

2. Construction of Hypergraphs of Complex Functions.

In the geometry of real variables, there are some curves, such as circles and ellipses, which can be plotted in their entirety. Most functions, however, are unbounded in one or both variables, and we content ourselves with plotting and studying the salient features of the most interesting portion of the curve, often that near the origin. Similarly, there

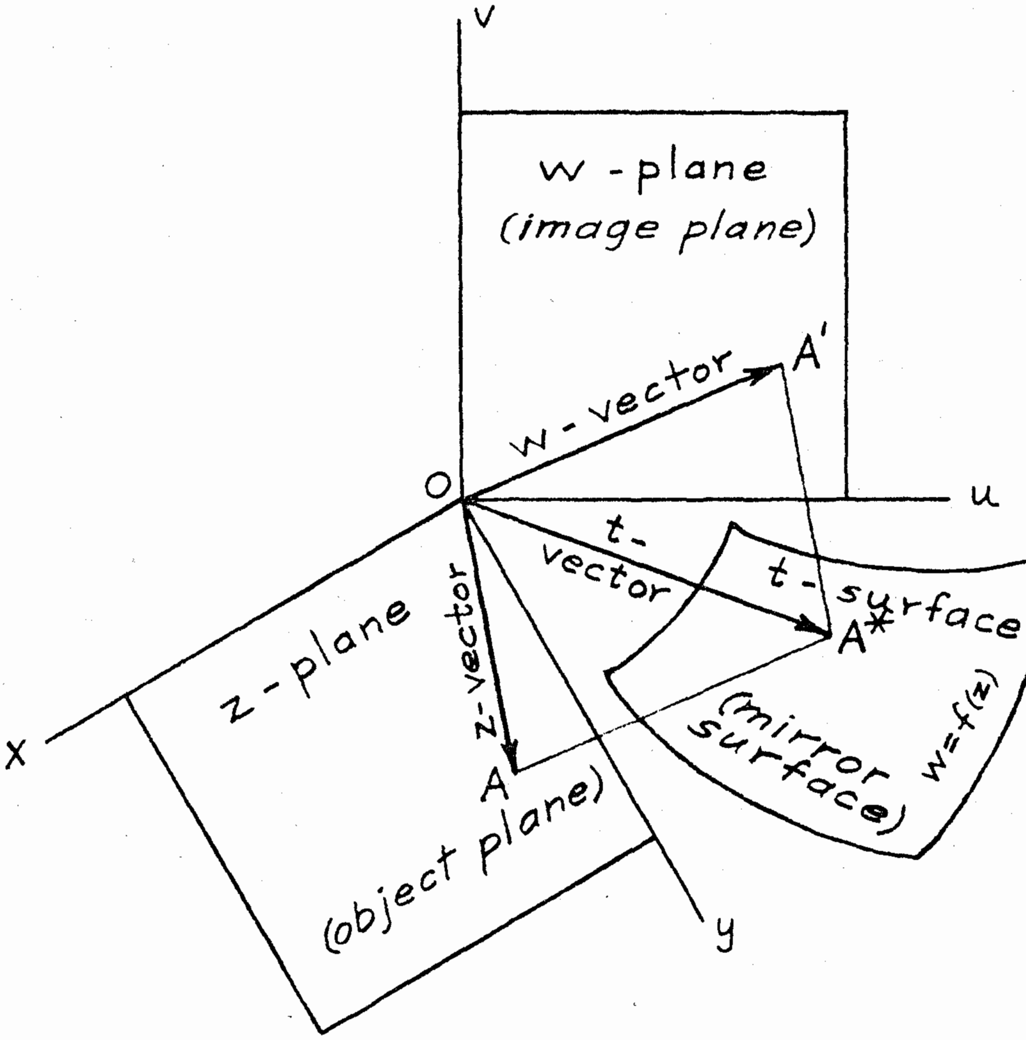


Fig. 12 Mirror Surface

are relatively few special analytic functions which have their regions of existence limited by natural boundaries; and even these functions are usually unbounded in absolute value on such boundaries. Most analytic functions (if we include their analytic continuations and exclude singular points) are defined over the entire complex plane. Furthermore, unless the function is a constant, by Liouville's Theorem, it is unbounded in absolute value for at least one point of this plane. In view of these considerations, we cannot expect to plot the entire graph of an analytic function, any more than we can plot an entire parabola. We can, however, construct and examine the transformation surface of a complex function for any portion of the complex plane we wish, such as a region near the origin or about a pole, and thus become familiar with its particular characteristic shape, just as in real variables.

We will use for our basic region in this paper a $3n \times 3n$ square grid in the xy plane, usually with $n = 1$, and in order to keep our figures here simple, we will confine ourselves for the most part to the first quadrant, starting at the origin. This will give us sixteen points on the transformation surface, from which we can gain some idea of the character of the function both individually and in comparison with other functions, as well as make use of the hypergraph to illustrate a few simple transformations.

As with ordinary graphs, we first compute a set of values for u and v corresponding to those assigned to x and y. Once we have these, we can use either of two methods for plotting the actual points of the transformation surface. First, as stated in Chapter II, we can plot successively, along lines parallel to the respective axes, each set of values of x, y, u, and v, taken in any order. A triangle with scales along the perpendicular sides is an aid in such plotting. A second method is to plot point (x,y) in the z-plane, and its corresponding point (u,v) in the w-plane, then draw vectors $\bar{z} = OA$ and $\bar{w} = OA'$ and from these, by completing the parallelogram and drawing its diagonal, find their vector sum $\bar{t} = \bar{z} + \bar{w} = OA^*$. A mechanical plotting aid for this method is described below. Each "mirror point" A* obtained is correlated with its "object point" A by some convenient notational system, as discussed in the next section. Using a French curve, smooth "mirror curves" of the grid lines are now drawn through the corresponding mirror points, giving us for the complex function being plotted a "mirror surface" in four dimensions of our basic grid.

To "read" the transformation surface thus obtained for the nature of the transformation it effects, it is only necessary to keep in mind that if the vector from each point in the z-plane to its mirror point on the transformation surface be moved to the origin, it maps the image of that point in the w-plane. Thus the "impending transfiguration" of the object plane can be visualized by reading, say, counterclockwise around the border of the grid in the z-plane, "carrying the origin with us" from each grid intersection to the next as we note the changing vectors to the transformation surface. See, for example, Plate 16.

Graphic transformation of points, curves, and regions from the z-plane to the w-plane can now be accomplished. From any object point A of the curve or region in the z-plane we draw the vector AA* to the corresponding mirror point A* on the transformation surface. The vector difference OA* − OA = OA', found by completing the parallelogram with OA* as a diagonal, gives us the vector OA' locating the image point A' in the w-plane. When point A does not lie on a grid intersection, we must either use a more finely divided grid system, compute a special point, or use judicious interpolation, in locating its mirror point A*.

Inverse graphic transformation from the w- to the z-plane can be accomplished by a "trial and error" search method. Better, of course, mirror curves of the grid lines $u = c_1$, $v = c_2$, can be plotted on the transformation surface also and inverse transformations performed directly.

Graphic transformation should be an aid in the solution, particularly where irregular paths or regions are involved, of certain boundary value problems connected with heat flow, electric potential, fluid flow, and air foils. For such practical applications, the transformation surfaces and the z- and w-grids would have to be drawn accurately to fine divisions, like fine graph paper, and perhaps printed in different colors; e.g. z-plane, light blue; t-surface, light red; w-plane, light green; reserving black for the curves and regions to be plotted and transformed.

In this connection, the work would be facilitated by some type of mechanical plotter constructed as a parallelogram, adjustable in size and shape, to duplicate the vector figure OAA*A'. A pair of surmounted parallel rulers or a system of cams or gears would maintain a true parallelogram for all positions. In use, the first or O vertex would be anchored at the origin, the second or $\bar{z}$ vertex with tracer point would be placed on a point in the z-plane, the fourth or $\bar{w}$ vertex with tracer point on the corresponding point in the w-plane. The third or $\bar{t}$ vertex with pencil would then mark the point on the transformation surface. Such a device would not only plot the transformation surface, but once this had been obtained, it could then be used, interchanging the pencil and tracer point between the $\bar{t}$ and $\bar{w}$ vertides, to plot the image in the w-plane of any figure in the z-plane, using its corresponding points on the transformation surface. Of course, such a mechanical plotter would not be limited to complex functions; since it is perfectly general in principle, it could be applied to any graph in any number of dimensions. (See Chapter II, Figures 4 through 9). Additional refinements could obviate the necessity of first summing the independent and the dependent variables.

3. <u>Point Notation.</u>

We noted, in Chapter II, Section 2, that one of the two problems connected with graphic transformation is the correlating of the vector sums of the independent variables and their transformation points. With a graph of a two-dimensional manifold in n-space, the best method for indicating this correspondence is by the use of mirror curves of the xy plane grid lines. Values of x and y can be written along the mirror axes; for added clarity, the y values can be underlined. (A memory aid here is that a number with a horizontal line under it, c, indicates the mirror of y = c, which is a horizontal line in the usual xy graph.) An alternate method of point notation for two-dimensional manifolds is given later in this section.

For a three-dimensional manifold in n-space we can substitute a series of successive two-dimensional layers or surfaces, joined by curves through their corners, each identified along one of these corner curves as to its position, u = c, in the hypersurface, and each marked with mirror grids. Additional curves can be drawn on two of the external faces to correlate the x and y values between the several surfaces. Clarity is increased by making the weight of the lines heavier on the border and visible faces. All actual intersections of mirror grids in the figure, as well as on two-dimensional surfaces which double back on themselves, should be marked with heavy dots or small x's to distinguish them from the many false or apparent intersections. Interpolation between grid lines becomes a major problem here.

Beyond a three-dimensional manifold, the projection on the plane of the graph paper becomes too cumbersome, in general, to use for graphic transformation, although some interesting hypergraphs can be obtained. We must then resort to projections on the various planes and hyperplanes, or successive intersections with these, or to purely nomographic devices. This is our difficulty in the general five-dimensional case with one dependent and four independent variables.

If it is desired instead to assign letters to the grid intersections of a two-dimensional manifold, the following system for a 6n x 6n square array of points, with any scale and any center, is convenient. Since most of the interesting features of the common functions occur within such a region, with n = 1, centered at the origin, much of our work will be done there, and the letter notation indicated below will be adopted.

Regular point notation is shown in Figure 13, which gives the mirror point letters for the corresponding grid points of the z-plane. The notation k^- can be read "k minus". Plotting can be facilitated by writing these symbols in lightly on the z-grid. It is then only necessary to plot the u and v values of each point from its symbol, which is then erased and written in clearly beside the final point obtained.

Singular point notation is shown in Figure 14 for a pole at point k. About a singular point, a "spider-web" pattern of nested regions is used, and the letters of the original square are repeated in the smaller squares, with dots over the letters indicating the successive subdivisions of the region. If quarter points are desirable, they can be lettered as shown with the unused symbols left over from the regular point nota-

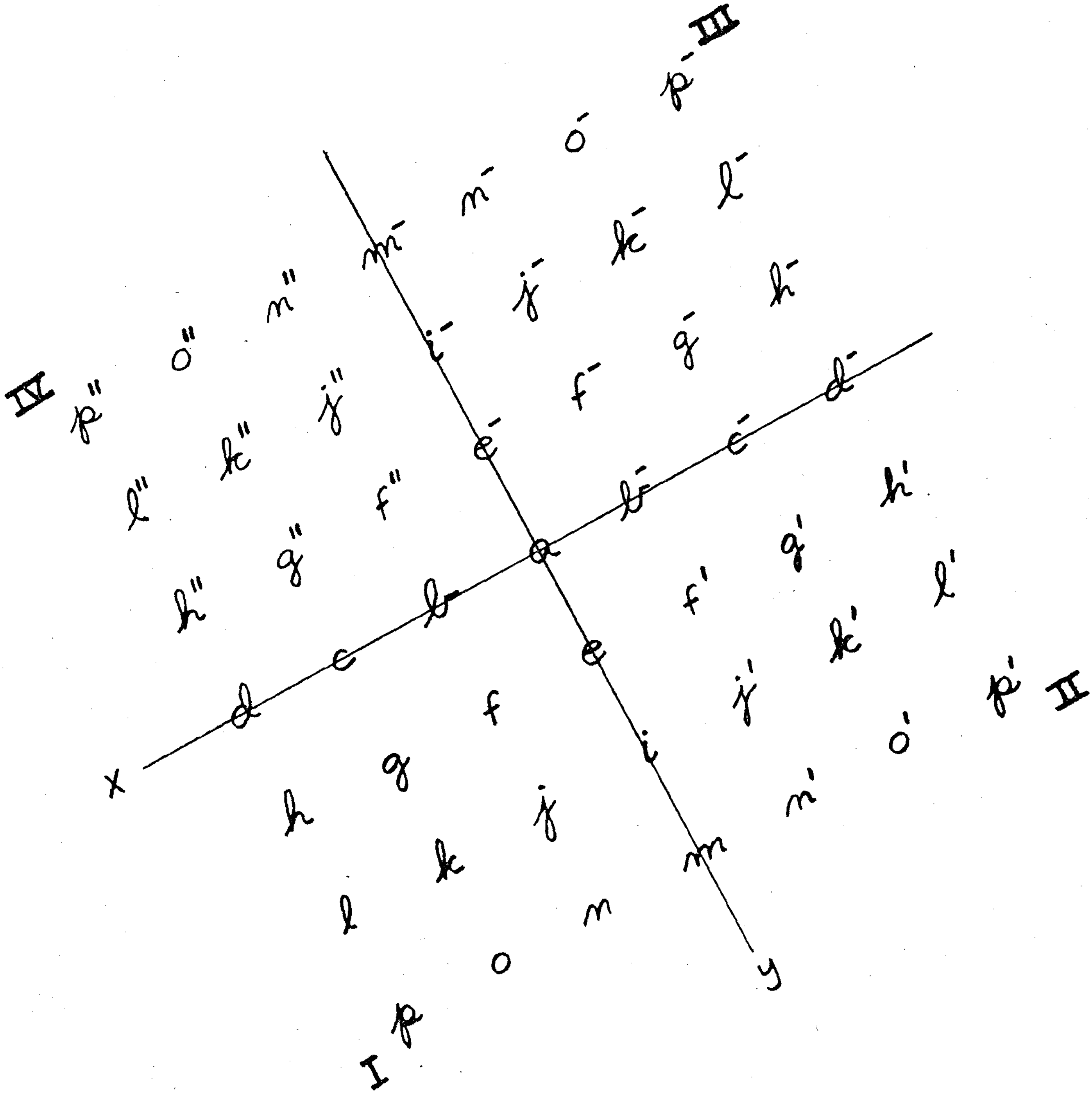


Fig. 13 Regular Point Notation

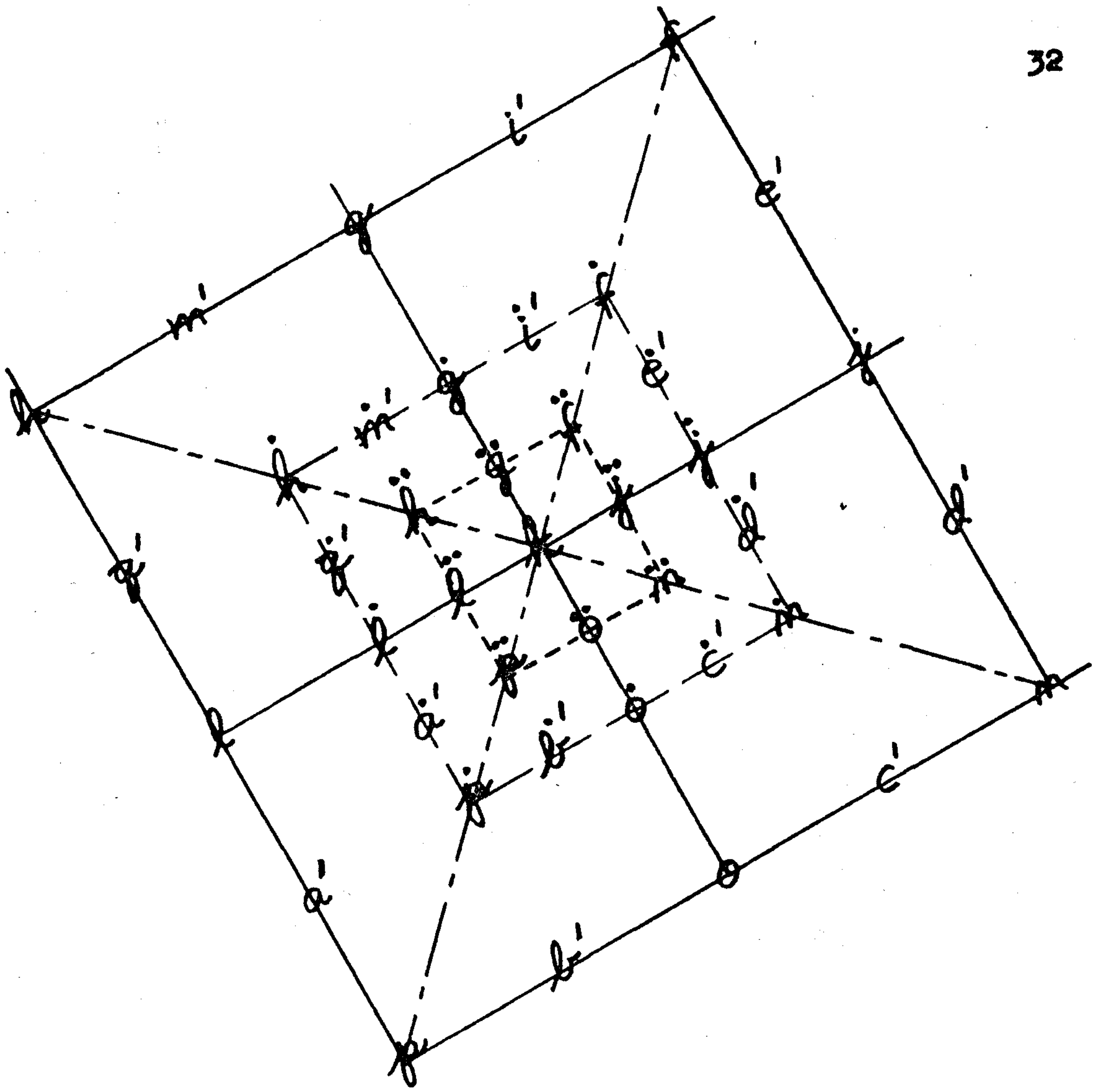


Fig. 14 Singular Point Notation

	a	b	c	d	e	i	m	q	r...	a_1...	a_2...	...
I	{ Additional		Points,			or		}		{ ~~Riemann~~		}
II	{ Quarter		Points		at	Poles		}		{ ~~Surface~~		}
–										{ ~~Points~~		}
III	a'''...		Additional					Points				

Fig. 15 Special Point Notation

tion scheme.

Special point notation is shown in Figure 15.

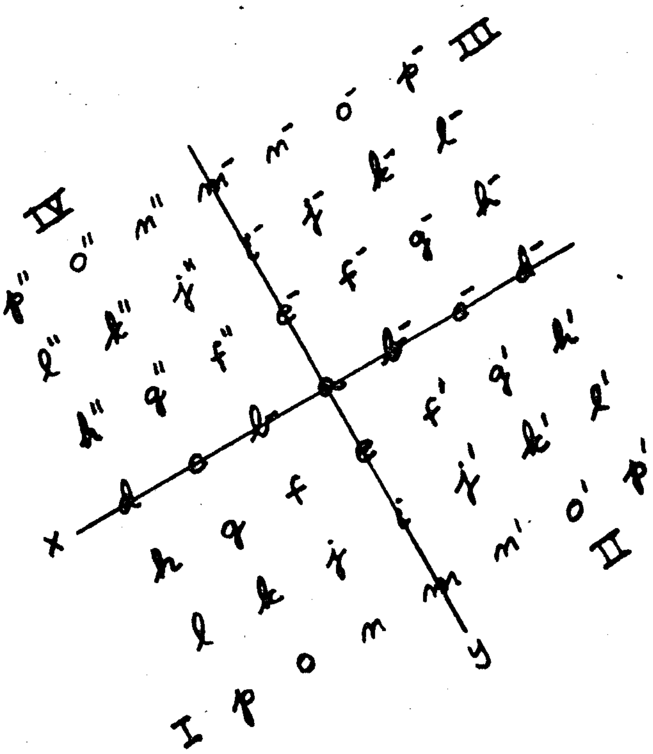
This system of notation accomplishes an orderly yet flexible method of identification of points in the region with an efficient use of a minimum of basic symbols. Two worksheet forms based on the system are included. (Figures 16 and 17). If desired, any point or the entire system can be extended to the corresponding points in the z- and w-planes by using the subscripts: k_z, k_t, k_w.

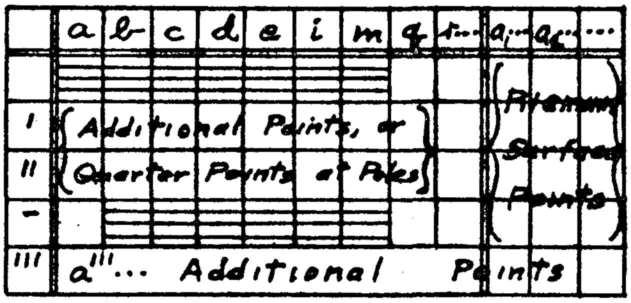
4. Transformation Surfaces for Various Complex Functions.

In the following pages we present a number of transformation surfaces obtained with the plotting methods we have described as applied to various elementary functions of a complex variable. Some transformations with these surfaces are also shown, in red.

Many of these surfaces are the "complex generalizations" of real plane curves. In these instances, the function $w = f(z)$ is real when z is real, i.e. $v = 0$ when $y = 0$, and the corresponding real variables curve is then $u = f(x)$, seen in the hypergraph as the trace "abcd" of the surface on the horizontal real xu plane. In other instances, the complex surface has no real counterpart, (i.e. no xu trace), or sometimes only one real point or two. Traces on various planes and three-dimensional sections in various hyperplanes will help us to better understand the nature of the surfaces. Note that the mirror point of the origin in these graphs is "a", the mirror

Regular and Special Points



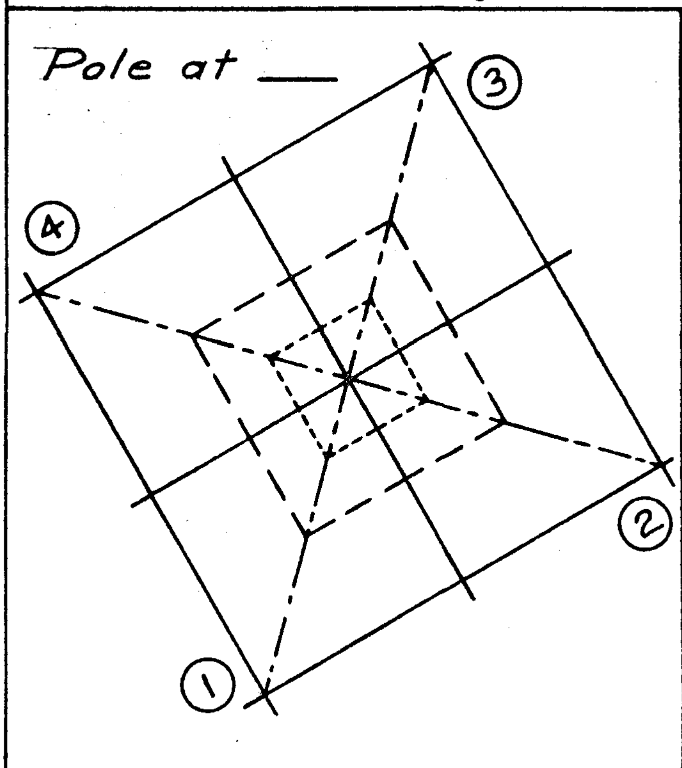


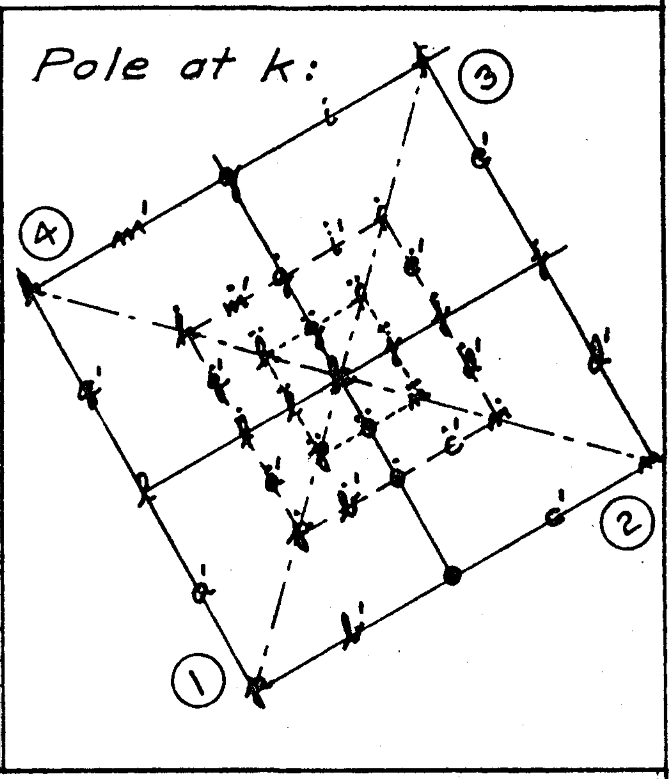
◯ w = → u = , v =

		x	y	u	v		x	y	u	v		x	y	u	v		x	y	u	v
I	a					b					c					d				
	e					f					g					h				
	i					j					k					l				
	m					n					o					p				
II						f'					g'					h'				
						j'					k'					l'				
						m'					o'					p'				
III						b^{-}					c^{-}					d^{-}				
	e^{-}					f^{-}					g^{-}					h^{-}				
	i^{-}					j^{-}					k^{-}					l^{-}				
	m^{-}					n^{-}					o^{-}					p^{-}				
IV						f''					g''					h''				
						j''					k''					l''				
						m''					o''					p''				

Fig. 16 Worksheet Form for Regular and Special Points

Singular Point





○ w = ⟶ u = , V=

		x	y	u	v		x	y	u	v		x	y	u	v		x	y	u	v
①	•					••					a′					ȧ′				
	•					••					b′					ḃ′				
②	•					••					c′					ċ′				
	•					••					d′					ḋ′				
③	•					••					e′					ė′				
	•					••					i′					i̇′				
④	•					••					m′					ṁ′				
	•					••					q′					q̇′				
○																				
○																				
○																				
○																				

Fig. 17 Worksheet Form for Singular Points

curve of the x-axis the line "abcd", and of the y-axis "aeim". The hypergraphs shown are constructed, for most of the functions, over a 3 x 3 region in the first quadrant of the xy plane. A large unit scale has been used to magnify the characteristic features of the surfaces at the origin. For $w = z^{\frac{1}{2}}$, however, we have used a 12 x 12 region centered about the origin to show the continuous mirror surface corresponding to the Riemann surface of two sheets in the z-plane.

In "visualizing" the surfaces, without the aid of stereoscopic drawings as described in the next section, it should be kept in mind that all u and v values are plotted in planes parallel to the uv plane, which is the plane of the paper, from points in the xy plane, which extends from the uv plane toward the observer. Therefore the relative "nearness" of any point of the transformation surface corresponds to that of the object point in the xy plane treated as a "plan view", showing the xy plane from above with the observer at the lower edge of the paper. Thus, for example, in the graph of $w = z$, the surface does not lie in the plane of the paper, but extends from the origin toward the observer, point p, as in all of these graphs, being the closest. This is readily seen in the stereoscopic view of $w = z$. Using this proximity principle, we can draw a series of horizontal lines through the grid points of the xy plane and obtain the following "order of proximity" of the points to the observer, <u>for our particular arrangement of the axes</u>, starting with the closest point p:

p o l n k m h j n' g i o' d f j' p' c e k' h" b f' l' g" a

g' l'' f'' b' h' k'' e^- c^- p'' j'' f^- d^- o'' i^- g^- n'' j^- h^- m^- k^- n^- l^- o^- p^-.

For the first quadrant points only, we have:

p o l n k m h j g i d f c e b a.

In determining the relative positions of the mirror surface and the xy-plane, it should be kept in mind that if v is negative, the mirror points are below both the xu and the xy plane. From the graph itself, we can reason as follows. The object point and its mirror point lie in the same vertical plane parallel to the paper, so they are each the same distance from the observer. If the mirror point is farther from the bottom edge of the paper than its object point, the "slope" of a line from the origin to the mirror point is less than that of one to the object point for points this side of the origin, greater beyond. In either case this point of the mirror surface lies above the z-plane. Conversely, if the mirror point is nearer to the bottom edge than its object point, it lies below the z-plane.

Commentaries on the individual plates are given below. The graphs, as mentioned previously, are only small portions of the entire (usually unlimited and unbounded) surfaces.

(1) $\underline{w = 0}$: The hypergraph coincides with the xy plane, analogous to the real variables case, u = 0, which coincides with the x-axis. The latter is seen here as the trace abcd on the real xu plane. Since the object and mirror points coincide throughout, all vector differences are zero, i.e., every

① w = 0 → u = 0 , v = 0

	x	y	u	v		x	y	u	v		x	y	u	v		x	y	u	v
a	0	0	0	0	e	0	1	0	0	i	0	2	0	0	m	0	3	0	0
b	1	0	0	0	f	1	1	0	0	j	1	2	0	0	n	1	3	0	0
c	2	0	0	0	g	2	1	0	0	k	2	2	0	0	o	2	3	0	0
d	3	0	0	0	h	3	1	0	0	l	3	2	0	0	p	3	3	0	0

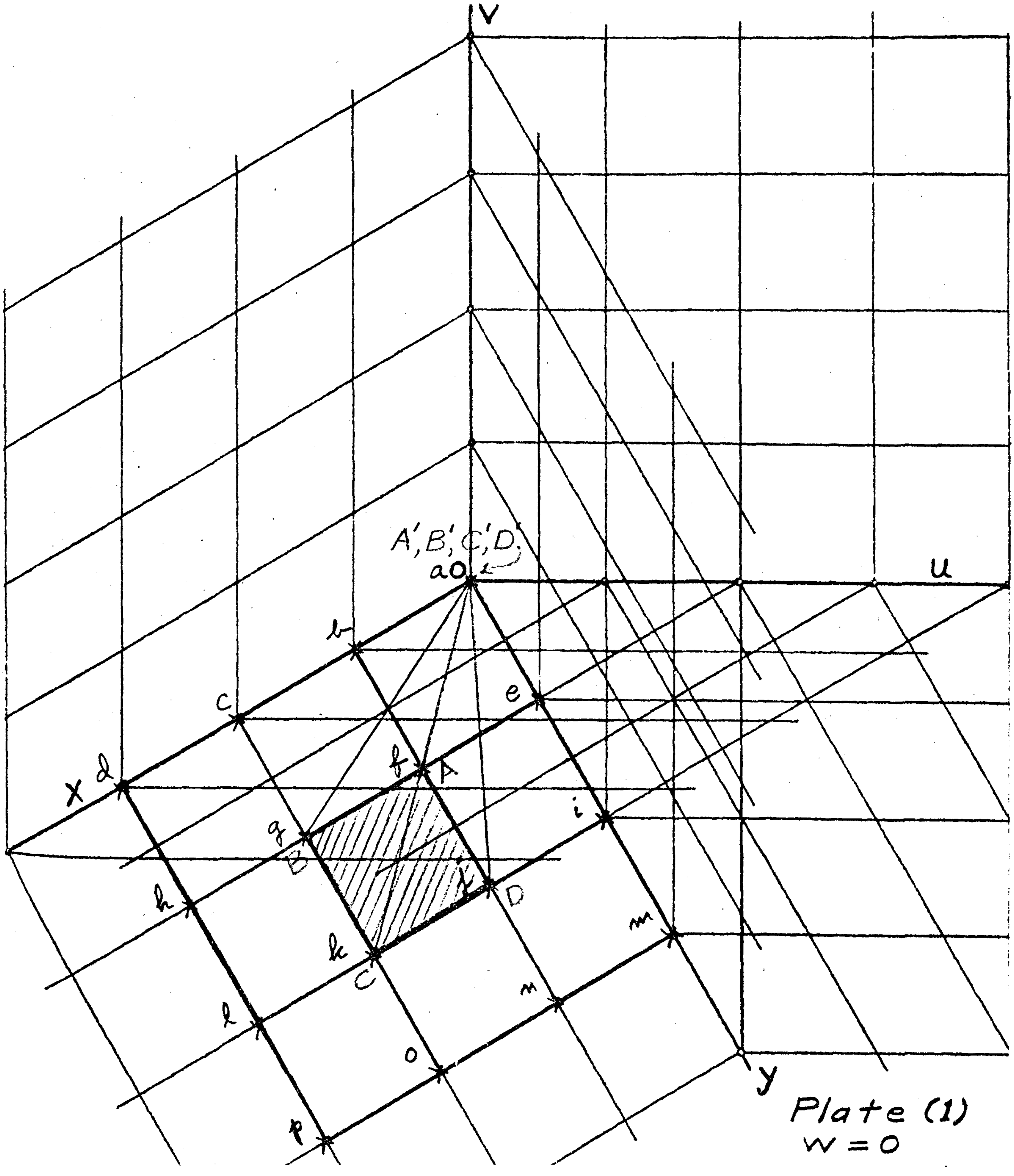


Plate (1)
W = 0

point of the xy plane is mapped into the origin by the surface $w = 0$.

(2) $\underline{w = 2 - i}$: Here our xy plane is represented by a parallel plane through the point $2 - i$ to the right and below the real xu plane. Since the mirror and object planes are parallel, they do not intersect; thus there is no real counterpart to this complex function. By completing parallelograms between the z-, t-, and w-points, we see that all points of the xy plane are mapped by the surface into the complex point $2 - i$.

(3) $\underline{w = z}$: As pointed out above, we have here a surface extending out from point a at the origin to point p nearest the observer. The fact that the mirror grids, which are actually orthogonal on the mirror surface, also plot out on our projection of this surface as perpendiculars although we are obviously seeing the surface "at an angle", is due to the use of equal scales and perpendicular axes in our oblique projection of the z-plane. If we had used a cabinet projection with equal foreshortening on the two axes, the surface would still appear square, though smaller. (See, for example, the graph of $w = 2z$). Unequal foreshortening or nonperpendicular axes will, in general, give us an inclined mirror of the plane as a parallelogram.

If we "shift our viewpoint" here by rotating the z-plane in the plane of the paper about the origin, we find that the square projection of the mirror surface remains square, illus-

② $w = 2 - i \rightarrow u = 2, v = -1$

	x	y	u	v		x	y	u	v		x	y	u	v		x	y	u	v
a	0	0	2	-1	e	0	1	2	-1	i	0	2	2	-1	m	0	3	2	-1
b	1	0	2	-1	f	1	1	2	-1	j	1	2	2	-1	n	1	3	2	-1
c	2	0	2	-1	g	2	1	2	-1	k	2	2	2	-1	o	2	3	2	-1
d	3	0	2	-1	h	3	1	2	-1	l	3	2	2	-1	p	3	3	2	-1

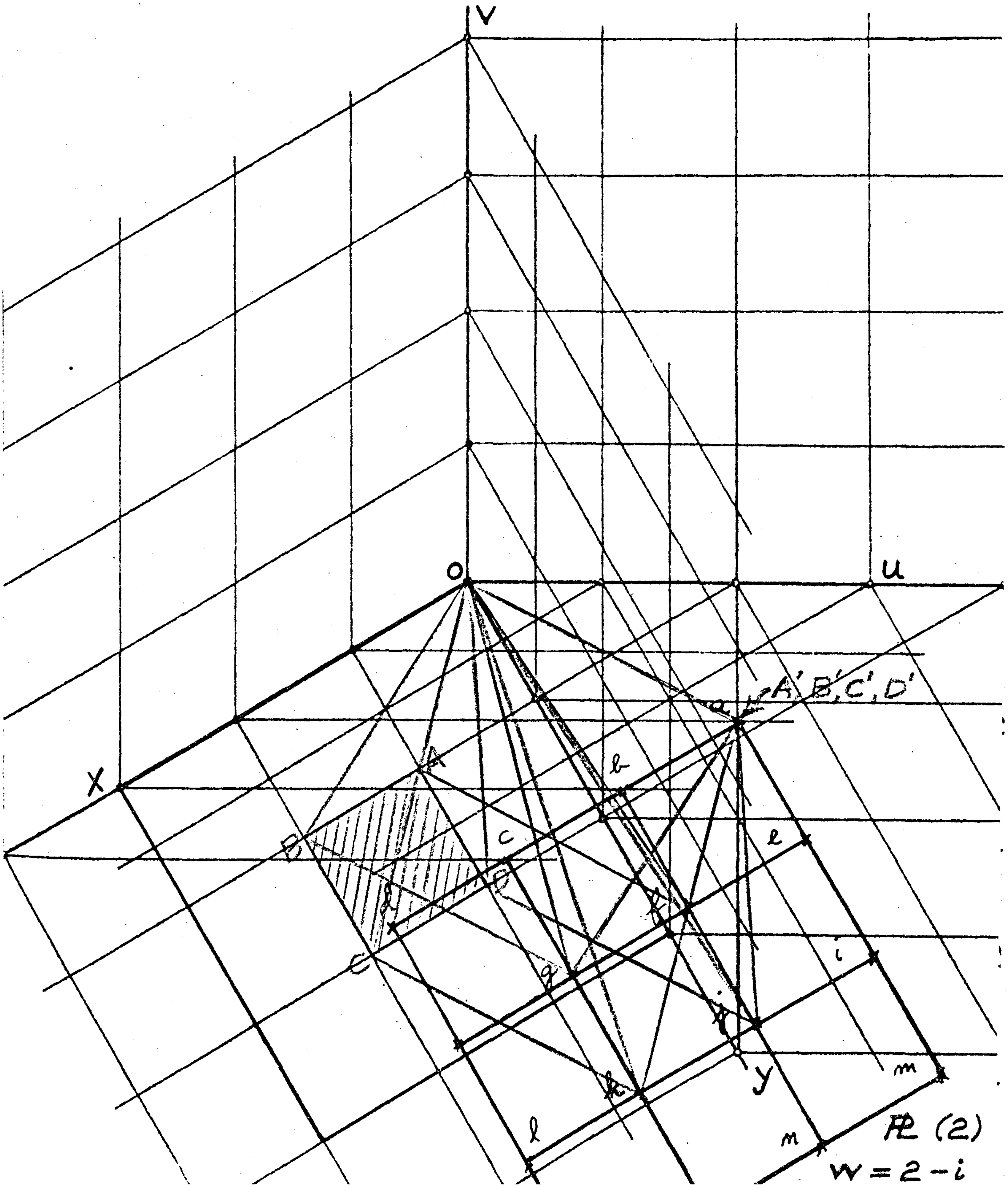


Fig (2) $w = 2 - i$

③ $w = z \rightarrow u = x, \; v = y$

	x	y	u	v		x	y	u	v		x	y	u	v		x	y	u	v
a	0	0	0	0	e	0	1	0	1	i	0	2	0	2	m	0	3	0	3
b	1	0	1	0	f	1	1	1	1	j	1	2	1	2	n	1	3	1	3
c	2	0	2	0	g	2	1	2	1	k	2	2	2	2	o	2	3	2	3
d	3	0	3	0	h	3	1	3	1	l	3	2	3	2	p	3	3	3	3

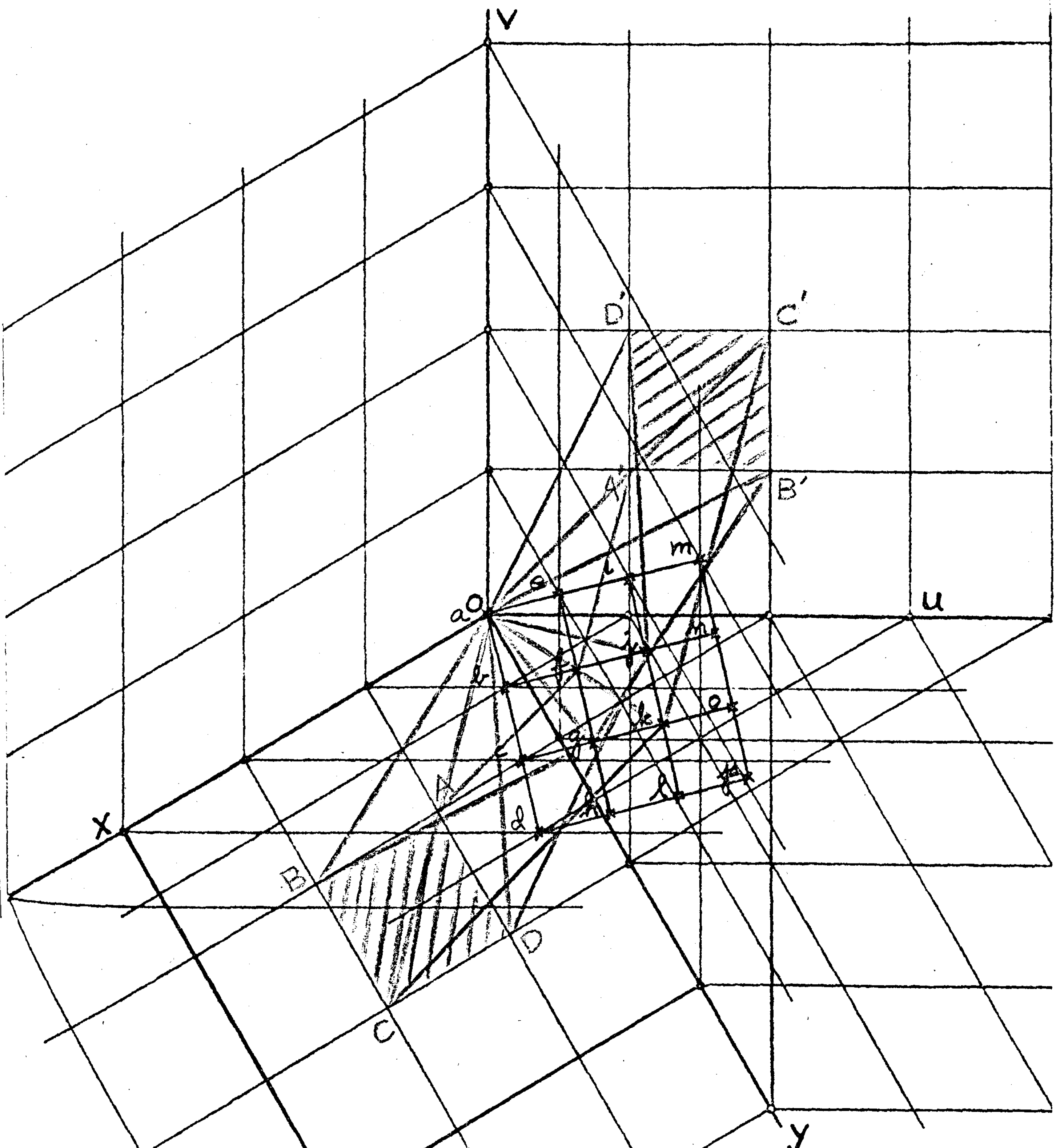


Plate (3)
w = z

trating a four-dimensional extension of the fact that we can see the true length of a line from more than one position, if we are in a third-dimensional plane perpendicular to the line. Here we are in a fourth-dimensional plane perpendicular to the plane in which the right angles occur, so we can "walk around" it in this perpendicular plane and still always see the mirror plane as a true square. A "side effect", however, is a change in the apparent size of the plane (compare the graphs of w = z and w = -z) caused by a "double foreshortening" in four dimensions, which will be discussed in a later chapter.

Notice the complex meaning of "linearity" and "slope" here. Since w = z is linear, our hyperspace surface is "flat", i.e., a hyperspace plane or "hyperline". Since this function has a "slope" of one, its mirror plane makes equal "angles" with the z- and w-planes. Without defining "flatness", "slope", or "angles" in four dimensions, our hypergraph of w = z still gives us a geometrical feeling for these qualities. It is interesting to see the interpretation of the Cauchy-Riemann equations in this graph as the slope $\frac{\partial u}{\partial x}$ of line abcd equal to the slope $\frac{\partial v}{\partial y}$ of line aeim, with $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0$ at the origin. Mirror point g appearing near the object point (0,1) is the mirror of object point (2,1), and the latter is "closer to the observer" (at the bottom of the page) than is point (0,1). From this we know that g, and thus the entire mirror surface shown, is "nearer" to the observer than is the object plane, i.e., it will appear to be "above" the xy plane in a stereoscopic view, a fact which is confirmed in the next section. The real counterpart, u = x, is the line abcd in the xu plane. A pure imaginary counterpart is also seen as the trace aeim in the vertical "pure imaginary" yv plane. The four-dimensional line x = y = u = v, which is equidistant from all four axes, lies in the mirror plane and is seen there as the diagonal afkp. Points of the z-plane are "reflected" by the mirror as shown into the same relative positions in the w-plane.

(4) w = z + 2 : The effect of an additive constant, as seen in Plate (2) above, is to shift the entire surface in translation to a parallel position through a new point. This is the complex counterpart of translation of a curve to a parallel position by an additive constant, illustrated here by the real line abcd with slope one through the point (0,0,2,0). The mapping is correspondingly translated by this shift.[27]

(5) w = -z : Notice here that we have been forced to reduce the scale of the xy plane in order to accomodate the change in "apparent area" between this surface and that for w = z. Actually the two have the same area, as will be easily verified in the next chapter. The change in "apparent area" results from our change of viewpoint for the two surfaces. Since mirror point j, in the same vertical plane as its object point (1,2), appears near point (3,3) which is closer to the observer, we know that j and thus the entire surface lies "below" the z-plane, but "in front of" the uv plane through (0,0). The stereoscopic view will confirm this. The real counterpart

(4) $w = z+2 \rightarrow u = x+2, v = y$

	x	y	u	v		x	y	u	v		x	y	u	v		x	y	u	v
a	0	0	2	0	e	0	1	2	1	i	0	2	2	2	m	0	3	2	3
b	1	0	3	0	f	1	1	3	1	j	1	2	3	2	n	1	3	3	3
c	2	0	4	0	g	2	1	4	1	k	2	2	4	2	o	2	3	4	3
d	3	0	5	0	h	3	1	5	1	l	3	2	5	2	p	3	3	5	3

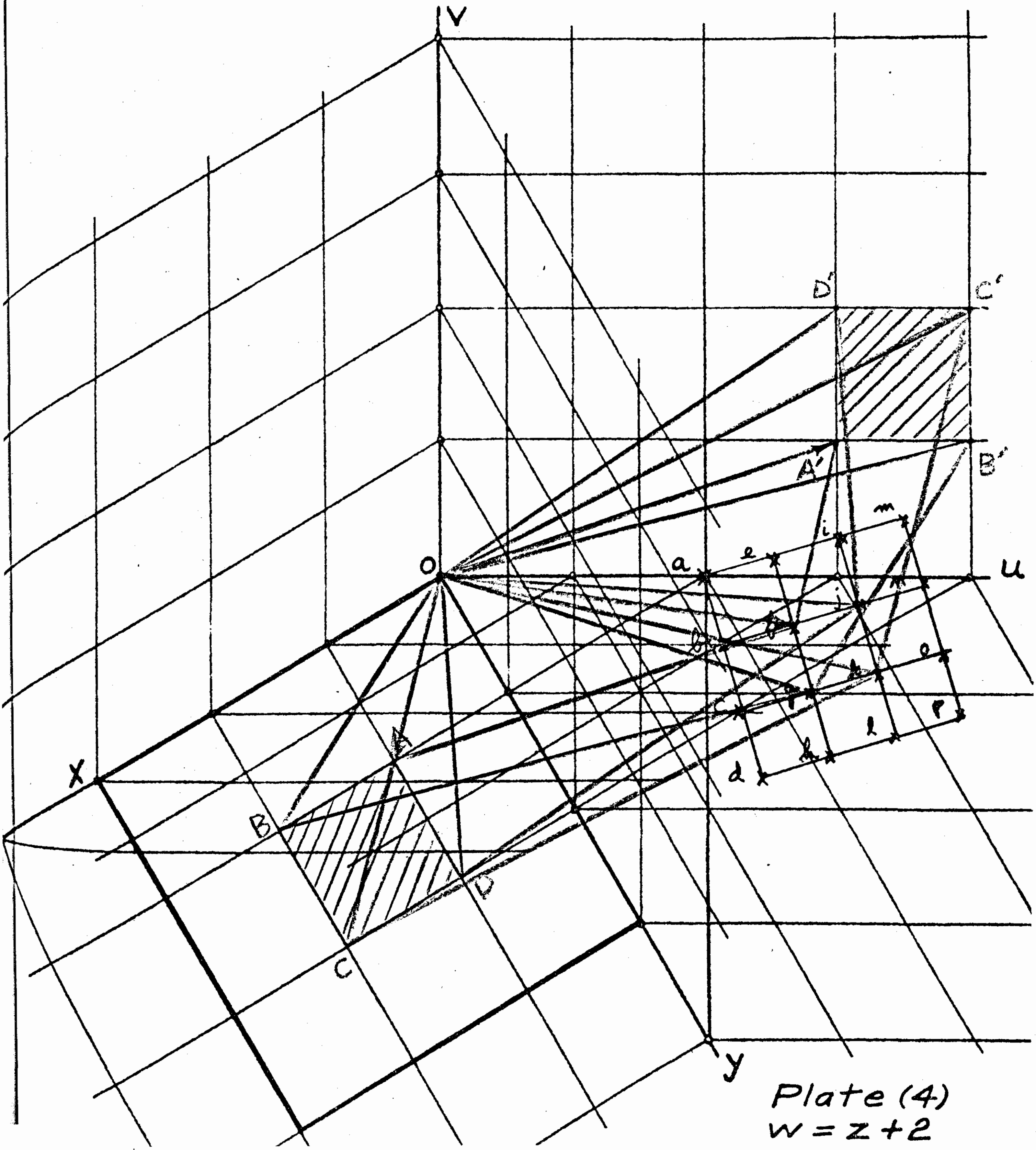


Plate (4)
$w = z+2$

⑤ $w = -z \rightarrow u = -x, \; v = -y$

	x	y	u	v		x	y	u	v		x	y	u	v		x	y	u	v
a	0	0	0	0	e	0	1	0	-1	i	0	2	0	-2	m	0	3	0	-3
b	1	0	-1	0	f	1	1	-1	-1	j	1	2	-1	-2	n	1	3	-1	-3
c	2	0	-2	0	g	2	1	-2	-1	k	2	2	-2	-2	o	2	3	-2	-3
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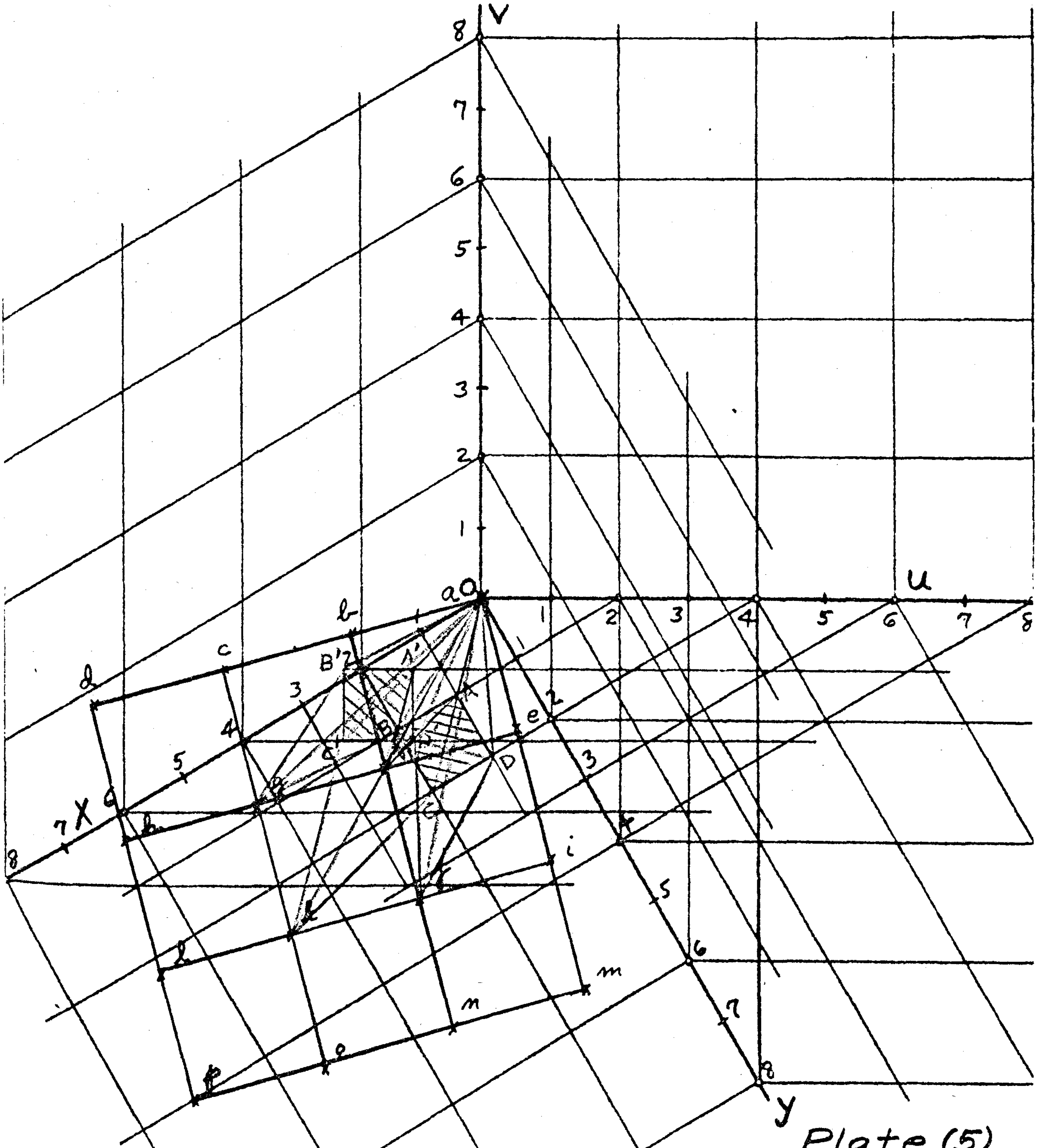


Plate (5)
W = -Z

$u = -x$ is the line abcd in the xu plane. The surface maps regions from the first quadrant of the z-plane to the third quadrant of the w-plane.

(6) $w = -z - 1 + i$: Another illustration of the effect of an additive constant, this time shifting the mirror plane up and to the left, so that it is "hinged" at point $-1 + i$. Since the constant is complex, there is no real counterpart. The line abcd lies in the plane $v = 1$ parallel to the real plane. The image is correspondingly shifted.

(7) $w = 2z$: With the multiplicative factor, the transformation plane has increased in actual as well as apparent area, being two and a half times larger in actual (computed) area than the mirror surface of $w = z$ for the same region. The map of course, has doubled in size. It is interesting to note that the transformation surface is rotated by this real factor, although the map of the region is not. With a complex factor the map would also be rotated[27]. The line abcd is the line $u = 2x$ in the real plane.

(8) $w = R(z)$ and (9) $w = I(z)$: These two mirror planes, as one might have expected, are orthogonal in 4-space. Like the z- and w-planes, they have only one point (the origin) in common; thus they are "absolutely perpendicular" planes.[28] Their complementary nature is evident from a comparison of the two surfaces, shown here in the same plate. Together they resolve any vector in the z-plane into its real and

⑥ $w = -z - 1 + i \longrightarrow u = -x - 1, \; v = -y + 1$

	x	y	u	v		x	y	u	v		x	y	u	v		x	y	u	v
a	0	0	-1	1	e	0	1	-1	0	i	0	2	-1	-1	m	0	3	-1	-2
b	1	0	-2	1	f	1	1	-2	0	j	1	2	-2	-1	n	1	3	-2	-2
c	2	0	-3	1	g	2	1	-3	0	k	2	2	-3	-1	o	2	3	-3	-2
d	3	0	-4	1	h	3	1	-4	0	l	3	2	-4	-1	p	3	3	-4	-2

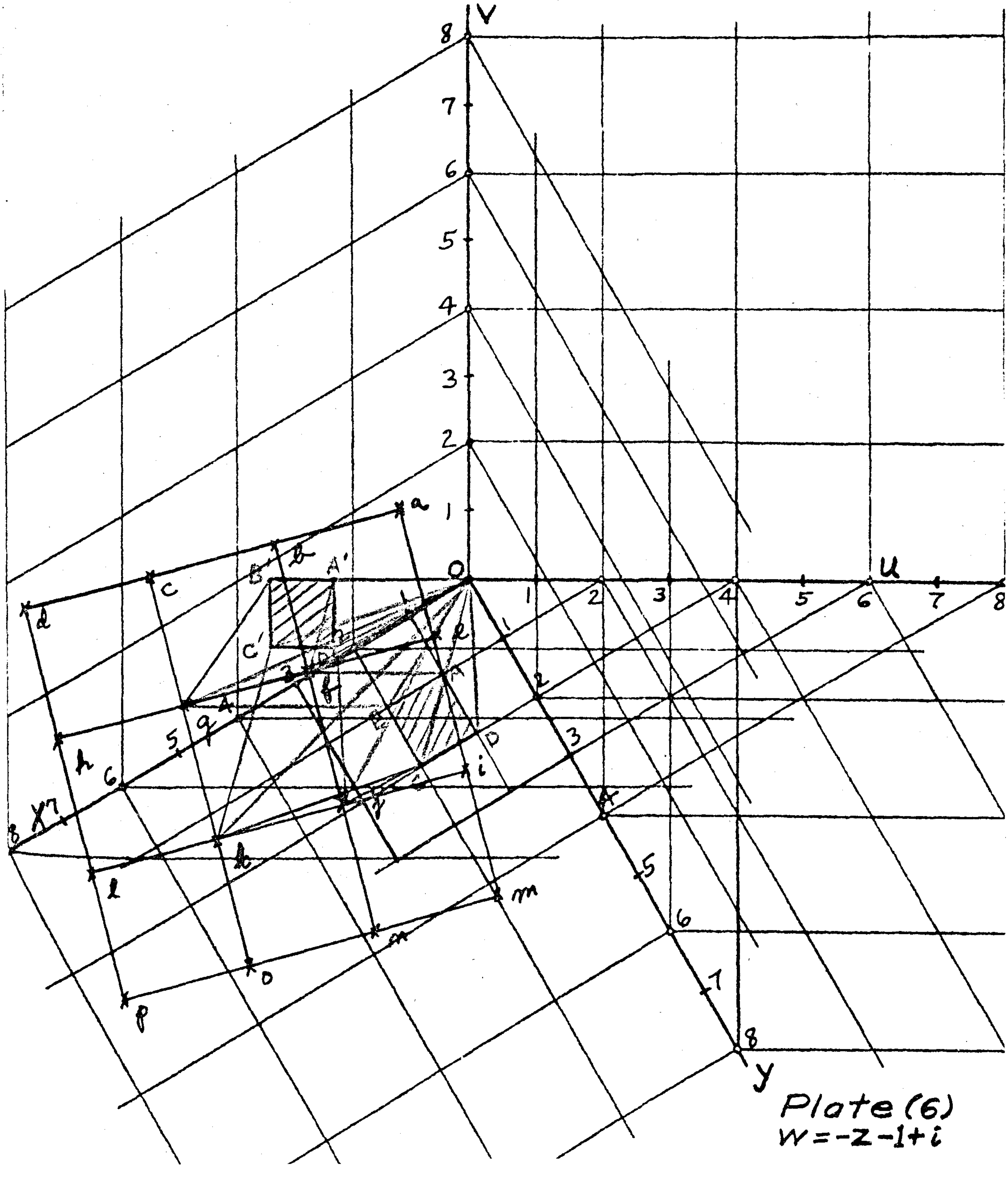


Plate (6)
$w = -z - 1 + i$

⑦ $w = 2z \rightarrow u = 2x, \; v = 2y$

	x	y	u	v		x	y	u	v		x	y	u	v		x	y	u	v
a	0	0	0	0	e	0	1	0	2	i	0	2	0	4	m	0	3	0	6
b	1	0	2	0	f	1	1	2	2	j	1	2	2	4	n	1	3	2	6
c	2	0	4	0	g	2	1	4	2	k	2	2	4	4	o	2	3	4	6
d	3	0	6	0	h	3	1	6	2	l	3	2	6	4	p	3	3	6	6

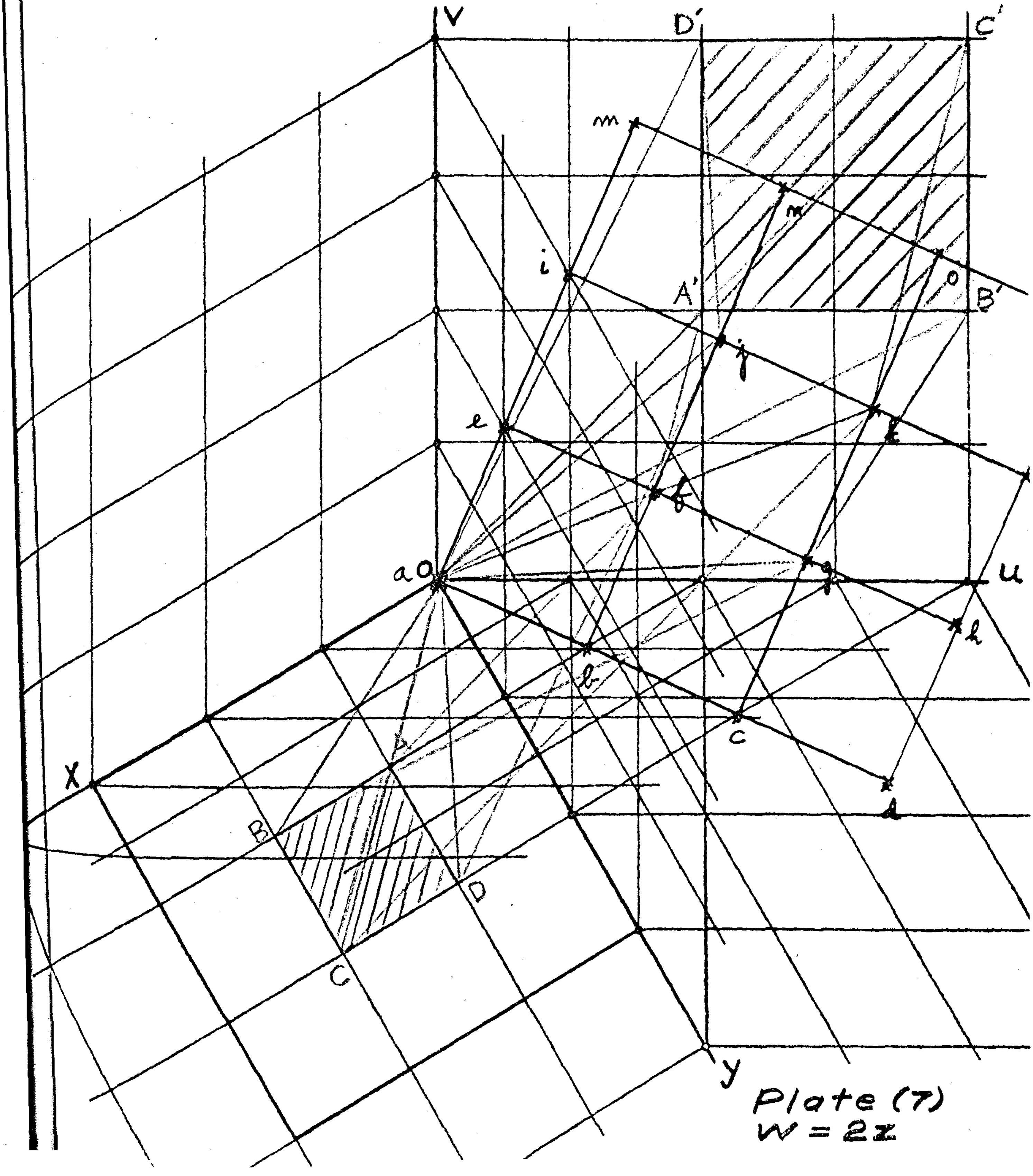
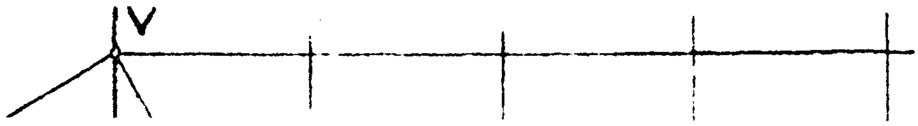


Plate (7)
$w = 2z$

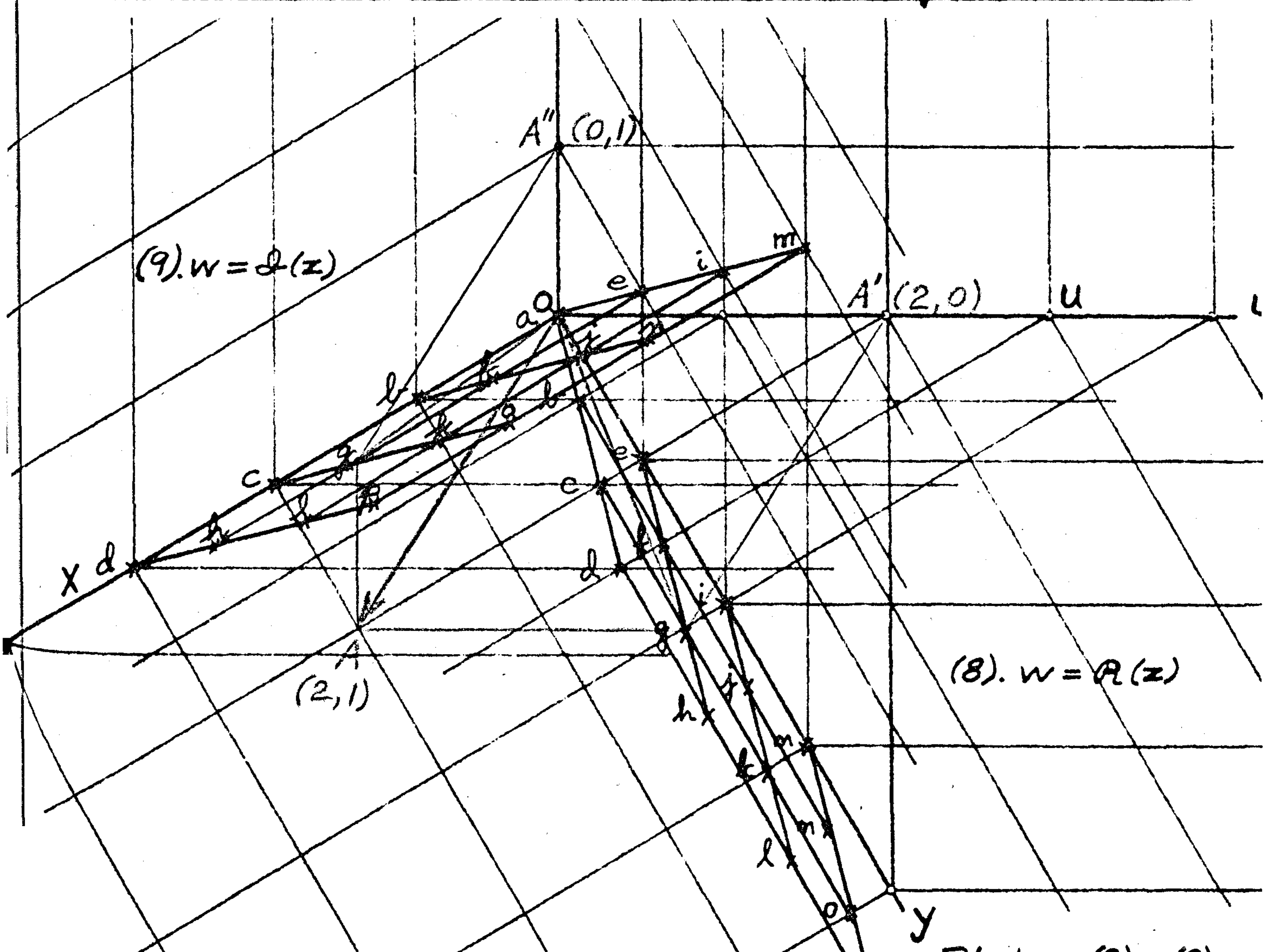
(8) $w = \mathcal{R}(z) \rightarrow u = x, \ v = 0$

	x	y	u	v		x	y	u	v		x	y	u	v		x	y	u	v
a	0	0	0	0	e	0	1	0	0	i	0	2	0	0	m	0	3	0	0
b	1	0	1	0	f	1	1	1	0	j	1	2	1	0	n	1	3	1	0
c	2	0	2	0	g	2	1	2	0	k	2	2	2	0	o	2	3	2	0
d	3	0	3	0	h	3	1	3	0	l	3	2	3	0	p	3	3	3	0



(9) $w = \mathcal{I}(z) \rightarrow u = 0, \ v = y$

	x	y	u	v		x	y	u	v		x	y	u	v		x	y	u	v
a	0	0	0	0	e	0	1	0	1	i	0	2	0	2	m	0	3	0	3
b	1	0	0	0	f	1	1	0	1	j	1	2	0	2	n	1	3	0	3
c	2	0	0	0	g	2	1	0	1	k	2	2	0	2	o	2	3	0	3
d	3	0	0	0	h	3	1	0	1	l	3	2	0	2	p	3	3	0	3



Plates (8) & (9)
$w = \mathcal{R}(z)$; $w = \mathcal{I}(z)$

imaginary components in the w-plane.

(10) <u>$w = \bar{z}$</u> : The transformation surface of this non-analytic function is an "edge view" of the mirror plane, which maps the region into a reflection in the real axis. The consecutive alphabetical order is probably meaningless, since a plane lettered by the system used here can be viewed from a slightly different position and give a non-consecutive reading of its points. The real part of this surface is its intersection with the real plane, the line $u = x$.

(11) <u>$w = |z|$</u> : Since this is a real function, $v = 0$, and we are confined to the xyu hyperplane. The equation is that of a cone, two elements of which are the straight lines abcd in the xu plane and aeim in the yu plane. Though not appearing so, in our projection, these elements are actually of equal length.

Despite the fact that this is a real function, its mirror surface lies in a complex hyperplane, since its argument is complex. Only the mirror loci of real functions of real variables can lie totally in the real xu plane. The conical surface here maps every point of the xy plane into its "absolute value" on the u-axis. A three-dimensional model of this mapping surface can be constructed.

(12) <u>$w = \arg z$</u> : Like its companion, $w = |z|$, this is a real function of a complex variable. The value of this function as we near the origin depends upon the direction of approach; the surface is thus discontinuous there. The mirror surface is

(10) $w = \bar{z} \rightarrow u = x, v = -y$

	x	y	u	v		x	y	u	v		x	y	u	v		x	y	u	v
a	0	0	0	0	e	0	1	0	-1	i	0	2	0	-2	m	0	3	0	-3
b	1	0	1	0	f	1	1	1	-1	j	1	2	1	-2	n	1	3	1	-3
c	2	0	2	0	g	2	1	2	-1	k	2	2	2	-2	o	2	3	2	-3
d	3	0	3	0	h	3	1	3	-1	l	3	2	3	-2	p	3	3	3	-3

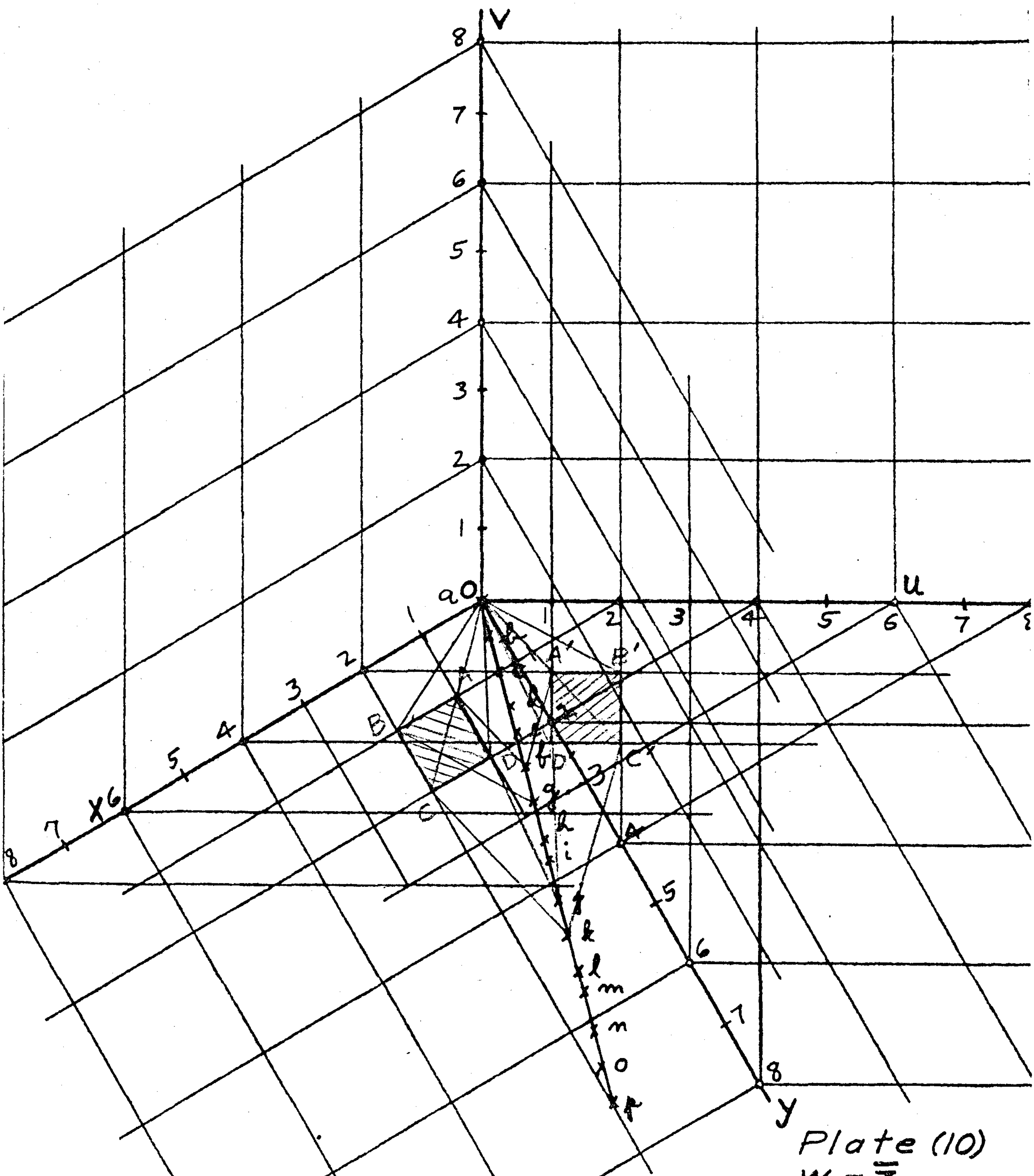


Plate (10)
$W = \bar{Z}$

(11) $w = |z| \quad \rightarrow \quad u = \sqrt{x^2+y^2}, \; v = 0$

	x	y	u	v		x	y	u	v		x	y	u	v		x	y	u	v
a	0	0	0	0	e	0	1	1	0	i	0	2	2	0	m	0	3	3	0
b	1	0	1	0	f	1	1	1.4	0	j	1	2	2.2	0	n	1	3	3.2	0
c	2	0	2	0	g	2	1	2.2	0	k	2	2	2.8	0	o	2	3	3.6	0
d	3	0	3	0	h	3	1	3.2	0	l	3	2	3.6	0	p	3	3	4.2	0

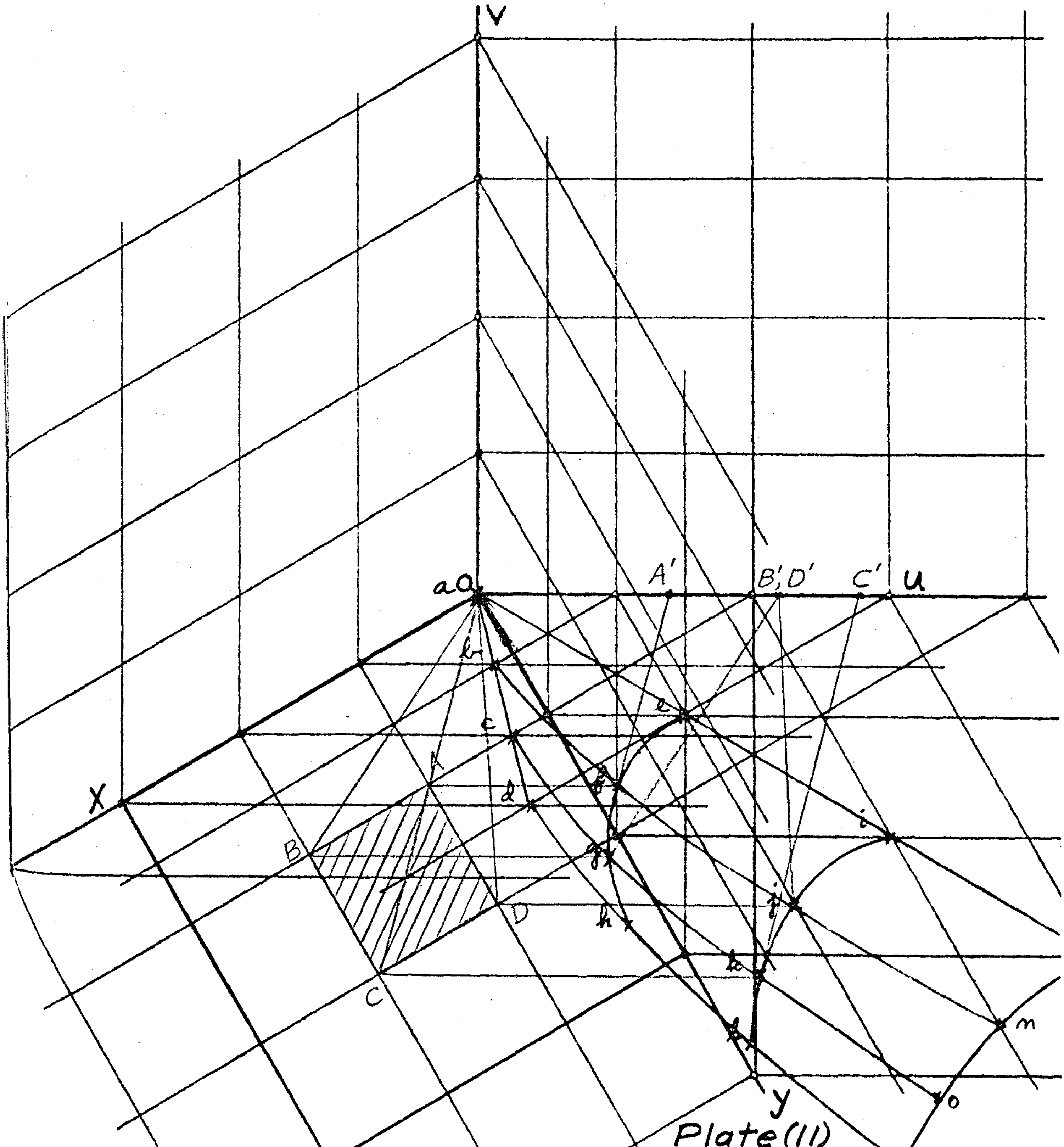


Plate (11)
$w = |z|$

(12) $w = \arg z \longrightarrow u = \arctan \frac{y}{x}, \quad v = 0$

		x	y	u	v		x	y	u	v		x	y	u	v		x	y	u	v
I	a	0	0	Indet.	0	b	1	0	0	0	c	2	0	0	0	d	3	0	0	0
	e	0	1	1.57	0	f	1	1	0.79	0	g	2	1	0.46	0	h	3	1	0.32	0
	i	0	2	1.57	0	j	1	2	1.11	0	k	2	2	0.79	0	l	3	2	0.59	0
	m	0	3	1.57	0	n	1	3	1.25	0	o	2	3	0.98	0	p	3	3	0.79	0
II						f'	−1	1	−0.79	0	g'	−2	1	−0.46	0	h'	−3	1	−0.32	0
						j'	−1	2	−1.11	0	k'	−2	2	−0.79	0	l'	−3	2	−0.59	0
						n'	−1	3	−1.25	0	o'	−2	3	−0.98	0	p'	−3	3	−0.79	0
III						b^{-}	−1	0	0	0	c^{-}	−2	0	0	0	d^{-}	−3	0	0	0
	e^{-}	0	−1	−1.57	0	f^{-}	−1	−1	0.79	0	g^{-}	−2	−1	0.46	0	h^{-}	−3	−1	0.32	0
	i^{-}	0	−2	−1.57	0	j^{-}	−1	−2	1.11	0	k^{-}	−2	−2	0.79	0	l^{-}	−3	−2	0.59	0
	m^{-}	0	−3	−1.57	0	n^{-}	−1	−3	1.25	0	o^{-}	−2	−3	0.98	0	p^{-}	−3	−3	0.79	0
IV						f''	1	−1	−0.79	0	g''	2	−1	−0.46	0	h''	3	−1	−0.32	0
						j''	1	−2	−1.11	0	k''	2	−2	−0.79	0	l''	3	−2	−0.59	0
						n''	1	−3	−1.25	0	o''	2	−3	−0.98	0	p''	3	−3	−0.79	0

(12) $w = \arg z \rightarrow u = \arctan \frac{y}{x},\ v = 0$

	x	y	u	v		x	y	u	v		x	y	u	v		x	y	u	v
a	0	0	Indet	0	e	0	1	1.57	0	i	0	2	1.57	0	m	0	3	1.57	0
b	1	0	0	0	f	1	1	0.79	0	j	1	2	1.11	0	n	1	3	1.25	0
c	2	0	0	0	g	2	1	0.46	0	k	2	2	0.79	0	o	2	3	0.98	0
d	3	0	0	0	h	3	1	0.32	0	l	3	2	0.59	0	p	3	3	0.79	0

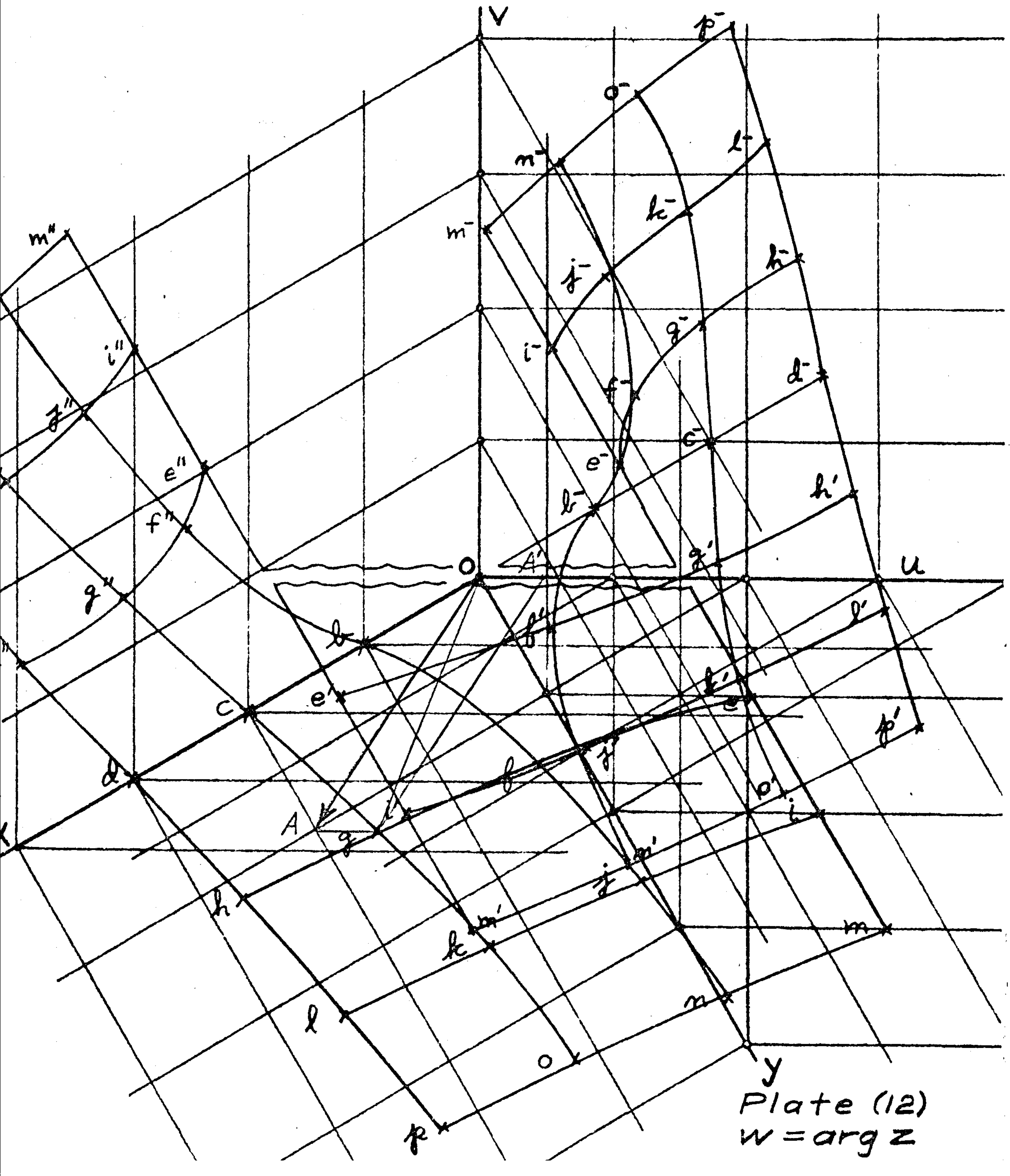


Plate (12)
w = arg z

shown for all four quadrants; it is actually a single-valued branch (obtained by restricting the arc tangent function to its principal values) of an infinitely repeating periodic surface. Each point in the z-plane is mapped by this surface into its vector angle value on the u-axis.

(13) $\underline{w = z^2}$ and (14) $\underline{w = z^{\frac{1}{2}}}$: These two functions represent the same characteristic surface, since if we reverse the variables in one of them, they become inverses. Thus it will be instructive to examine their transformation surfaces together, as two different views of the same surface. This surface is seen in full in Plate (14) for a 12 x 12 square region about the origin. For work with the multiple-valued function $w = z^{\frac{1}{2}}$, the z-plane is customarily replaced by a Riemann surface of two sheets, connected across a branch cut to form one continuous surface on which the function is single-valued. The corresponding continuous single-surface character of the transformation surface is seen in Plate (14).

By comparing pairs of neighboring points on the two mirror sheets, using the "order of proximity" described previously, we see that the mirror surface of sheet one ($d_1p_1p_1'p_2''p_2''d_2$) is above that of sheet two ($d_2p_3p_3'p_4''p_4''d_1$). The entire surface is sloping away from us, with the lower corners p_1 and p_3 nearest, and the upper corners p_2'' and p_4'' the farthest.

The effect of "double foreshortening" is seen in the apparent difference in areas of the equal quadrants of the two mirror sheets, the direction of viewpoint and not the proxim-

(13) $w = z^2 \rightarrow u = x^2 - y^2,\ v = 2xy$

	x	y	u	v		x	y	u	v		x	y	u	v		x	y	u	v
a	0	0	0	0	e	0	1	−1	0	i	0	2	−4	0	m	0	3	−9	0
b	1	0	1	0	f	1	1	0	2	j	1	2	−3	4	n	1	3	−8	6
c	2	0	4	0	g	2	1	3	4	k	2	2	0	8	o	2	3	−5	12
d	3	0	9	0	h	3	1	8	6	l	3	2	5	12	p	3	3	0	18
q	√2	√2	0	4	r	8/5	6/5	1.1	3.8										

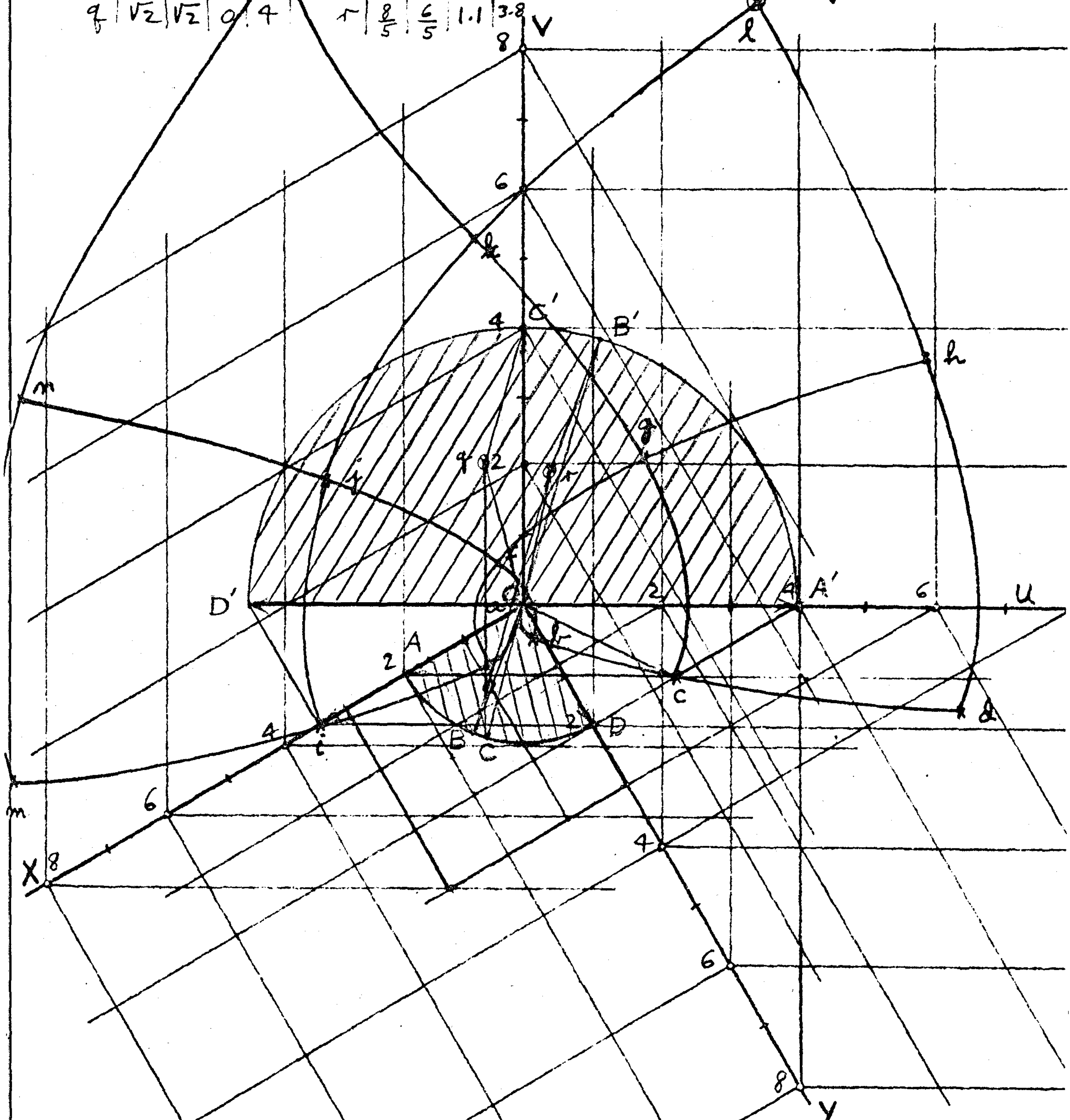


Plate (13)
$W = Z^2$

(14) $w = z^{\frac{1}{2}} \longrightarrow u = \pm\sqrt{\frac{1}{2}\left(\sqrt{x^2+y^2}+x\right)},\ v = \pm\sqrt{\frac{1}{2}\left(\sqrt{x^2+y^2}-x\right)}$

For $y \geqq 0$: $u_1 \geqq 0$, $v_1 \geqq 0$ and $u_3 \leqq 0$, $v_3 \leqq 0$.
For $y \leqq 0$: $u_2 \leqq 0$, $v_2 \geqq 0$ and $u_4 \geqq 0$, $v_4 \leqq 0$.

Points for 1st sheet of surface listed below.
Points on 2nd sheet have opposite u, v signs from these.

		x	y	u	v		x	y	u	v		x	y	u	v		x	y	u	v
I	a	0	0	0	0	b	2	0	1.4	0	c	4	0	2	0	d	6	0	2.4^+	0
	e	0	2	1	1	f	2	2	1.6^-	0.6^+	g	4	2	2.0^+	0.4^+	h	6	2	2.5^-	0.4^-
	i	0	4	1.4	1.4	j	2	4	1.8	1.1	k	4	4	2.2	0.9	l	6	4	2.6^-	0.8^-
	m	0	6	1.7^+	1.7^+	n	2	6	2.0^+	1.5^-	o	4	6	2.4^-	1.3^-	p	6	6	2.7	1.1
II	a'''	0	-0.5	-0.5	0.5	f'	-2	2	0.6^+	1.6^-	g'	-4	2	0.4^+	2.0^-	h'	-6	2	0.4^-	2.5^-
	b'''	2	-0.5	-1.4	0.2	j'	-2	4	1.1	1.8	k'	-4	4	0.9	2.2	l'	-6	4	0.8^-	2.6^-
	c'''	4	-0.5	-2.0^+	0.0^+	n'	-2	6	1.5^-	2.0^+	o'	-4	6	1.3^-	2.4^-	p'	-6	6	1.1	2.7
III	d'''	6	-0.5	-2.5^-	0.0^+	b^-	-2	0	0	1.4	c^-	-4	0	0	2	d^-	-6	0	0	2.4^+
	e^-	0	-2	-1	1	f^-	-2	-2	-0.6^+	1.6^-	g^-	-4	-2	-0.4^+	2.0^-	h^-	-6	-2	-0.4^-	2.5^-
	i^-	0	-4	-1.4	1.4	j^-	-2	-4	-1.1	1.8	k^-	-4	-4	-0.9	2.2	l^-	-6	-4	-0.8^-	2.6^-
	m^-	0	-6	-1.7^+	1.7^+	n^-	-2	-6	-1.5^-	2.0^+	o^-	-4	-6	-1.3^-	2.4	p^-	-6	-6	-1.1	2.7
IV	e'''	0	-1	-0.7	0.7	f''	2	-2	-1.6^-	0.6^+	g''	4	-2	-2.0^+	0.4^+	h''	6	-2	-2.5^-	0.4^-
	f'''	2	-1	-1.5^-	0.3^+	j''	2	-4	-1.8	1.1	k''	4	-4	-2.2	0.9	l''	6	-4	-2.6^-	0.8^-
	g'''	4	-1	-2.0^+	0.2^+	n''	2	-6	-2.0^+	1.5^-	o''	4	-6	-2.4^-	1.3^-	p''	6	-6	-2.7	1.1

(14) $w = z^{\frac{1}{2}}$ → $u = \pm\sqrt{\frac{1}{2}\left(\sqrt{x^2+y^2}+x\right)}$, $v = \pm\sqrt{\frac{1}{2}\left(\sqrt{x^2+y^2}-x\right)}$

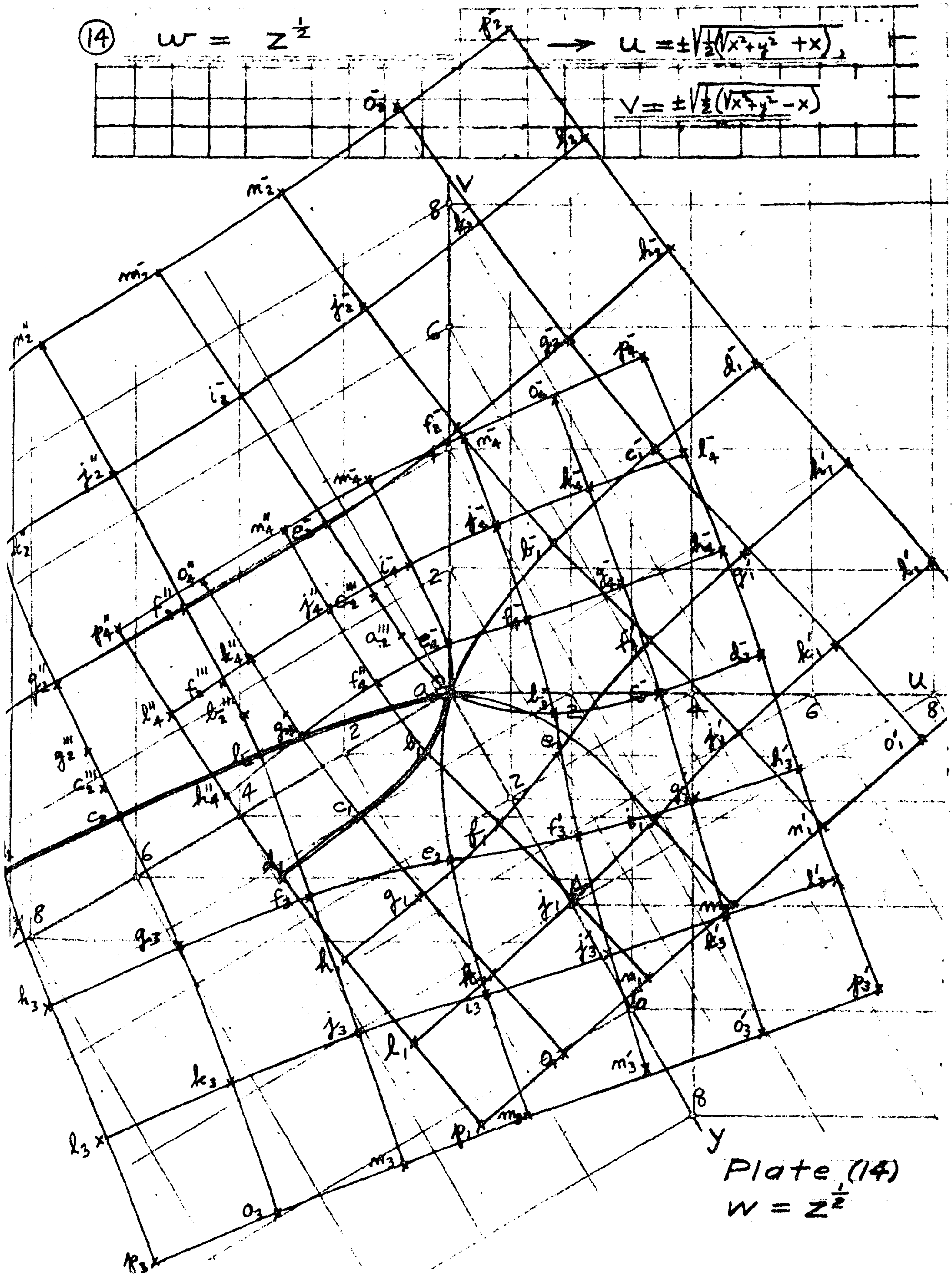


Plate (14)

$w = z^{\frac{1}{2}}$

ity (as in three-dimensional perspective) determining the apparent size. Compare, for example, the equal lower corner areas, $o_1k_1l_1p_1$ and $o_3k_3l_3p_3$, of which the first appears smaller, even though it is nearer, because we are viewing it at more of an "angle". Visualization of the cross-over in four dimensions between the two sheets, along the real parabola branch-cut $x = u^2$ (curve $d_1a_1d_2$), is difficult; especially when we find that the order of proximity of the following points is d_1 f_3 c_1 h_4" b_2 g_4".

In comparing the two plates, we must keep in mind that our grid systems and hence outlines are diagonally opposed, so that point p of Plate (13), for example, lies on the total surface somewhere beyond and between p_1 and p_1' of Plate (14). Our first quadrant surface shown for $w = z^2$ thus is a portion of the surface in the vicinity of the first half of mirror sheet one of $w = z^{\frac{1}{2}}$, so that in Plate (13) we are viewing Plate (14) roughly from its positive v-axis. The surface here extends out toward us from the origin, point p (off the top of the paper) being the peak of the roof above us.

A look at the mirror curves of the x- and y-axes in Plate (14) proves interesting. The positive x-axis, on substitution into the u and v functions for $w = z^{\frac{1}{2}}$, yields the parabola $u^2 = x$, $v = 0$ in the real xu plane, seen here as curve $d_1a_1d_2$. The negative x-axis gives the parabola $v^2 = x$, $u = 0$ in the complex xv plane, curve $d_1^-a_1d_3^-$. (The corresponding curves are seen partially in Plate (13) as abcd and aeim respectively.) Thus we have two parabolas joined at the origin and each curled about the x-axis, but extending in opposite directions and lying in perpendicular planes. These parabolas represent the corresponding real and complex graphs of the real variable function, $u = x^{\frac{1}{2}}$, for both positive and negative values of x. We have therefore arrived, through one facet of our complex surface, at the same method for displaying the graph of a complex function of real variables as described by Ward[29], Kempner[30], Frumveller[8], Lange[10], and others. Our graphs of functions of a complex variable thus include not only the real functions of real variables as a special case, but also the complex functions of real variables, and, as seen in Plates (11) and (12) for $w = |z|$ and $w = \arg z$, the real functions of complex variables. A four-dimensional hypergraph, then, is a complete graphical representation of functions of a single variable in the field of real and complex numbers.

The mirror curves of the y-axis are the curves $m_1a_1m_3$ and $m_2^{-}a_1m_4^{-}$. By substituting $x = 0$ into the u and v functions of $w = z^{\frac{1}{2}}$, we obtain $y = 2u^2$ and $y = 2v^2$, from which we have $u = \pm v$. Thus these mirror curves are the intersections, lying in the planes $u = \pm v$ of the yuv hyperplane, of the parabolic cylinders $y = 2u^2$ and $y = 2v^2$. But when $y > 0$, $u = v$, and the cylinders thus intersect in the $u = v$ plane. When $y < 0$, $u = -v$ and the cylinders intersect in this perpendicular plane, as can be seen in the graph.

We see then that the mirror curves of the two axes are both plane curves. It is interesting to note that, although these curves lie in different hyperplanes, in the graph the branches of the mirror x-axis appear to flow smoothly along the surface into branches of the mirror y-axis. When we remember that one transformation of this surface, in the form $w = z^2$, doubles the angle at the origin, this does not seem unreasonable.

If we plot $w - K = (z - H)^2$, H, K complex, the "vertex" of the four-dimensional complex parabola is moved to the complex four-dimensional point (H,K). For the particular case when $H = h$, $K = k$, h and k real and positive, this is the point (h,0,k,0) and the real trace abcd... in the xu plane no longer intersects the x-axis, for the real variables equation $u - k = (x - h)^2$ yields, when $u = 0$, the quadratic equation $x^2 - 2hx + h^2 + k = 0$ which has no real roots. However, the imaginary roots of this equation are now seen in the hypergraph, for $u = v = 0$, as the intersection of the vertical parabola aeim... with the xy plane in the points $x = h \pm i\sqrt{k}$. Similar figures are obtained for complex values of H and K, i.e., for roots of quadratic equations with complex coefficients.

Just as the parabola $u = x^2$ occupies two of four quadrants in a plane, we find, by checking the possible sign combinations of the complex function, that the surface $w = z^2$ extends through eight of the sixteen hexadekants in 4-space. Also, $u = x^2$ and $u = x^{\frac{1}{2}}$ share one quadrant out of four. The two complex analogues similarly share one-fourth, i.e., four, of the hexadekants. The surface $w = z^2$ transforms a quarter circle in the first quadrant into a semi-circle in the upper half-plane as shown. Each mirror sheet of the function $w = z^{\frac{1}{2}}$

maps a circle into a semi-circle (not shown here), the upper sheet into the upper half-plane and the lower sheet into the lower half-plane.

So much for the complex generalization of the parabola, the four-dimensional complex surface of which we might call a "parabolex". Next we look at the complex hyperbola, or "hyperbolex".

(15) $\underline{w = \frac{1}{z}}$: Here is our first surface with a singular point, the origin. Only the first quadrant surface is shown here; the reader will find it instructive to plot the complete surface, and to note the intersection with the 45° vertical plane, $x = u$, $y = 0$, through the v-axis, which gives a circle about the origin. This vertical complex circle joins the two horizontal branches of the real equilateral hyperbola, $u = \frac{1}{x}$.[31] The positive branch of this hyperbola is seen here in the xu plane as $\ddot{b}\,\dot{b}$ bcd. Since v is negative throughout the first quadrant, the rest of the surface shown here lies below both the xu and the xy planes. As the z values approach the origin from the first quadrant, the surface twists into a vortex away from the observer, then spreads out indefinitely downward and to the right, approaching the fourth quadrant of the w-plane. In the complex analogy to the real hyperbola, the surface is asymptotic to the z and w coordinate planes. Although it is symmetrical with respect to the origin, the "hyperbolex" may be a little difficult to visualize near this singular point, since it changes here from a surface which

(15) $w = \frac{1}{z} \rightarrow u = \frac{x}{x^2+y^2}, \; v = \frac{-y}{x^2+y^2}$

	x	y	u	v		x	y	u	v		x	y	u	v		x	y	u	v
a	0	0	Undef	Undef	e	0	1	0	-1	i	0	2	0	-0.50	m	0	3	0	-0.33
b	1	0	1.0	0	f	1	1	0.50	-0.50	j	1	2	0.20	-0.40	n	1	3	0.10	-0.30
c	2	0	0.50	0	g	2	1	0.40	-0.20	k	2	2	0.25	-0.25	o	2	3	0.15	-0.23
d	3	0	0.33	0	h	3	1	0.30	-0.10	l	3	2	0.23	-0.15	p	3	3	0.17	-0.17
ḃ	½	0	2	0	ė	0	½	0	-2	ḟ	½	½	1	-1	a'	1	½	0.80	-0.40
b'	½	1	0.40	-0.80	b̈	¼	0	4	0	ë	0	¼	0	-4	f̈	¼	¼	2	-2
ȧ'	½	¼	1.60	-0.80	ḃ'	¼	½	0.80	-1.6										

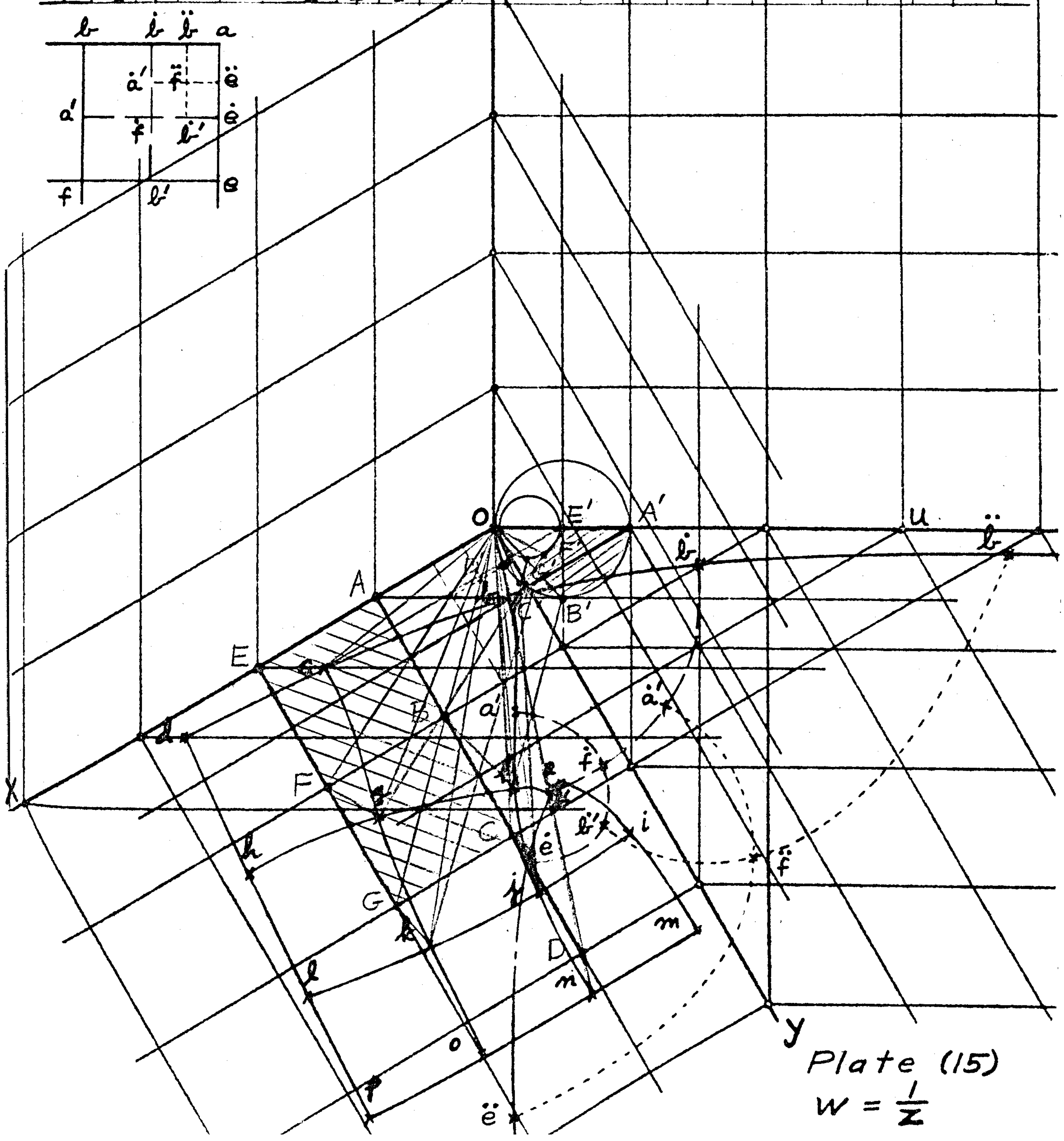


Plate (15)
$w = \frac{1}{z}$

was asymptotic to the 1st, 2nd, 3rd, and 4th quadrants of the z-plane into a surface asymptotic respectively to the 4th, 3rd, 2nd, and 1st quadrants of the w-plane.

A rectangular strip in the first quadrant of the z-plane parallel to the y-axis is mapped by the surface, as shown, into a fourth quadrant portion of the area between two circles in the w-plane passing through the origin and with centers on the u axis.

(16) $w = \frac{-iz + i}{z + 1}$: The first quadrant surface, shown here, of this linear fractional transformation presents a pleasantly uncomplicated appearance. However, this appearance is deceptive, as the surface has a singular point at $z = -1$, toward which the mirror curve of the x-axis dcba is heading. This trace on the xv plane is an equilateral hyperbola with center at point (-1,-1), so the surface "comes in again" from the minus v direction past this point. The xu trace yields only one real point, b(1,0). The mirror curve of the y-axis is the three-dimensional skew curve aeim in the yuv hyperplane. In visualizing the portion of the surface shown, we note that point e lies in the yu plane, and that all other mirror grid points except a, b, and e lie below the xy plane.

The word "linear" for these fractional transformations is misleading, as the mirror surface here indicates. Actually, the most interesting part of this surface, that around the singular point, is not shown here. Since the complex hyperbola, $w = \frac{1}{z}$, is one form of a linear fractional transformation,

(16) $w = \dfrac{-iz+i}{z+1}$ (A Linear Fractional Transformation) $\rightarrow$ $u = \dfrac{2y}{(x+1)^2+y^2}$, $V = \dfrac{-x^2-y^2+1}{(x+1)^2+y^2}$

	X	Y	u	V			X	Y	u	V			X	Y	u	V			X	Y	u	V	
a	0	0	0	1	✓	e	0	1	1	0	✓	i	0	2	$\frac{4}{5}$	$-\frac{3}{5}$	✓	m	0	3	$\frac{3}{5}$	$-\frac{4}{5}$	✓
b	1	0	0	0	✓	f	1	1	$\frac{2}{5}$	$-\frac{1}{5}$	✓	j	1	2	$\frac{1}{2}$	$-\frac{1}{2}$	✓	n	1	3	$\frac{6}{13}$	$-\frac{9}{13}$	✓
c	2	0	0	$-\frac{1}{3}$	✓	g	2	1	$\frac{1}{5}$	$-\frac{2}{5}$	✓	k	2	2	$\frac{4}{13}$	$-\frac{7}{13}$	✓	o	2	3	$\frac{1}{3}$	$-\frac{2}{3}$	✓
d	3	0	0	$-\frac{1}{2}$	✓	h	3	1	$\frac{2}{17}$	$-\frac{9}{17}$	✓	l	3	2	$\frac{1}{5}$	$-\frac{3}{5}$	✓	p	3	3	$\frac{6}{25}$	$-\frac{17}{25}$	✓
d'''	0	$\frac{1}{2}$	$\frac{4}{5}$	$\frac{3}{5}$		b⁻	−1	0	Indet	Indet		c⁻	−2	0	0	−3		d⁻	−3	0	0	−2	

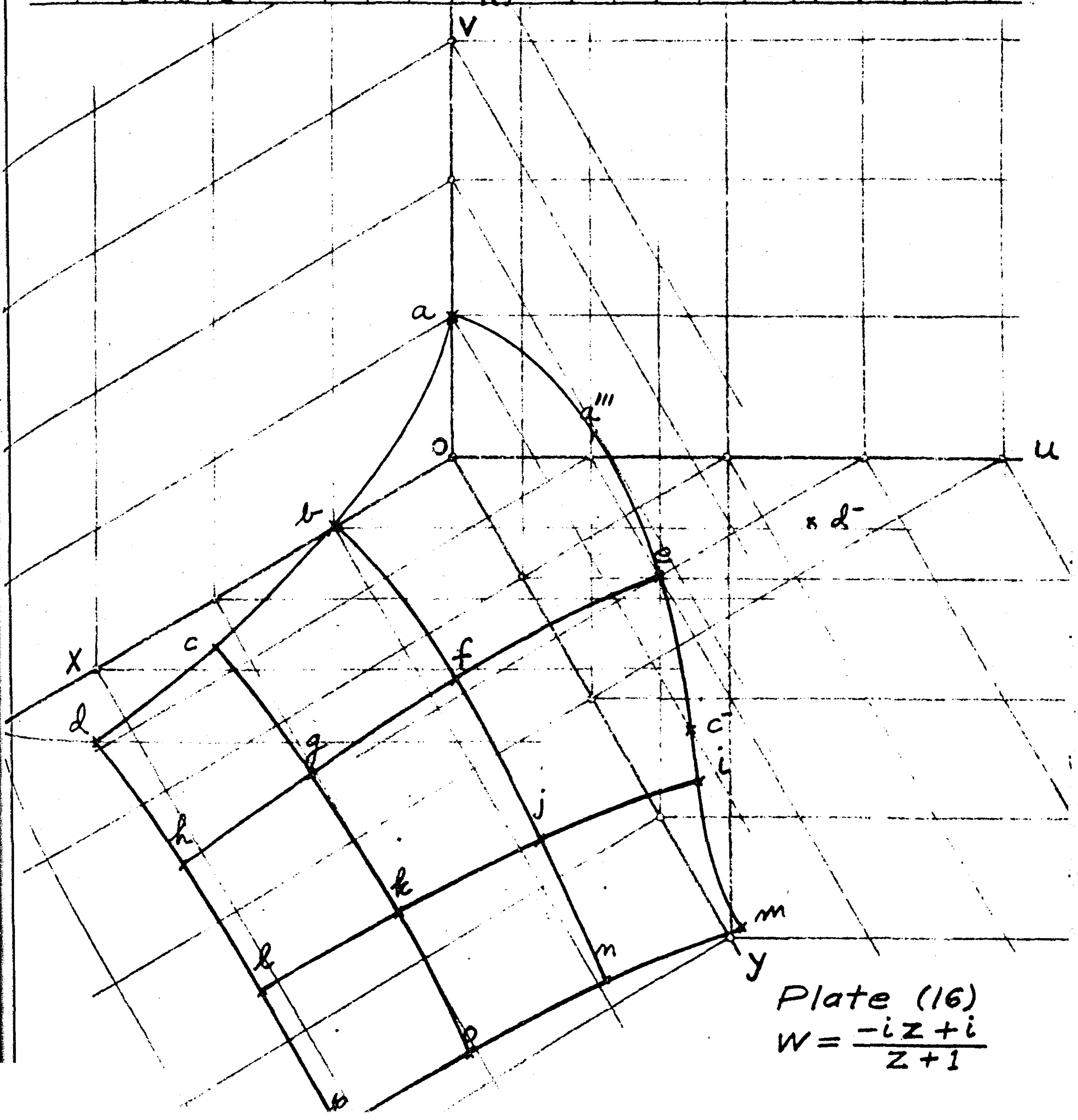


Plate (16)
$W = \dfrac{-iz+i}{z+1}$

we can expect many similarities between these two surfaces, such as the plane equilateral hyperbolas in each, asymptotic planes, singular points, the type of transformation effected, etc.

Looking more closely at the general linear fractional transformation, $w = \frac{\alpha z + \beta}{\gamma z + \delta}$, $\alpha\delta - \beta\gamma \neq 0$, we see that it can be written in the form of the successive transformations: (1): $w' = \frac{1}{z'}$; (2): $w' = w - \frac{\alpha}{\gamma}$; (3): $z' = \frac{\gamma(\gamma z + \delta)}{\beta\gamma - \alpha\delta}$, $\alpha\delta - \beta\gamma \neq 0$.

As in Plates (4), (6), and (7), the transformation (2) above is simply a translation of the w-plane, while the transformation (3) translates, rotates, and expands the z-plane. Since (1) above is the complex hyperbola, this means that the mirror surface of any linear fractional transformation is simply that of the complex equilateral hyperbola, distorted somewhat by being plotted from coordinate planes which have been translated, rotated, and expanded by the amounts determined by the constants of the linear fraction.

(17) <u>$w = e^z$</u> : In this first quadrant portion of the exponential surface we see the real exponential curve represented by the trace abcd in the xu plane, while the pure imaginary counterpart is the skew curve aeim in the yuv hyperplane. This surface is periodic along the y-axis; approximately half a period is shown here. To plot the full period of $2\pi i$, we would have to use a considerably reduced scale on the u and v axes. This is the same surface, of course, viewed from a different direction, for the inverse function $w = \log z$.

(17) $w = e^z \rightarrow u = e^x \cos y,\ v = e^x \sin y$

	x	y	u	v		x	y	u	v		x	y	u	v		x	y	u	v
a	0	0	1	0	e	0	1	0.54	0.84	i	0	2	-0.42	0.91	m	0	3	-0.99	0.14
b	1	0	2.72	0	f	1	1	1.47	2.29	j	1	2	-1.13	2.47	n	1	3	-2.7	0.38
c	2	0	7.4	0	g	2	1	4.0	6.2	k	2	2	-3.08	6.7	o	2	3	-7.3	1.04
d	3	0	20	0	h	3	1	10.8	16.8	l	3	2	-8.3	18.2	p	3	3	-19.8	2.8

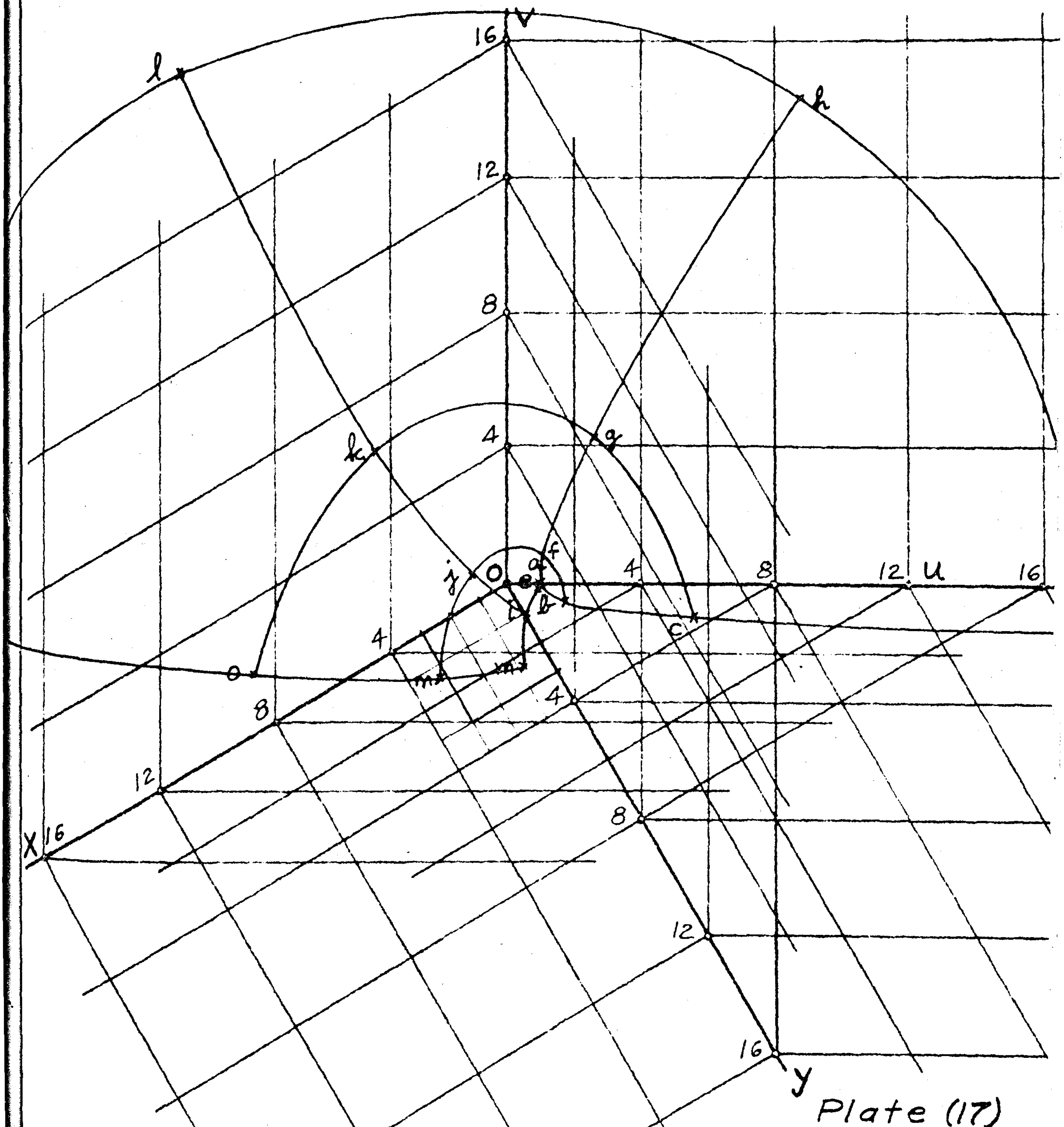


Plate (17)
$w = e^z$

The impending transformation of the z-plane grid lines into w-plane circles and rays is seen here foreshadowed by the pattern of the successive mirror grids.

(18) and (19) $w = \sin z$: We include Plate (18) for its shock value. At the center of this mathematical whirlwind, the discerning reader will find the sine curve dcbab‴c‴d‴ in the xu plane, and the hyperbolic sine surve mieae‴i‴m‴ in the yv plane. Not much else can be gathered from this asymmetrical projection, plotted here for all four quadrants, except to note that the surface twists back on itself doubly in some complicated fashion.

Our serious purpose here is to illustrate the advantages of the symmetrical projection, discussed later in this chapter, for these more complicated surfaces. This same function is plotted again in Plate (19) using these symmetric axes, and now the surface presents a much more orderly appearance. The sine curve dd‴ and the hyperbolic sine curve mm‴ are again seen in this plate, this time both in true shape. Other advantages of the symmetric system will be given in Section 6 of this chapter.

Since $w = \sinh z$ can be written in the form of the successive transformations

(1) $w' = \sin z'$, (2) $z' = iz$, (3) $w' = iw$,

we see that if we rotate the surface for $w = \sin z$ counterclockwise through 90° and reletter the axes, we have the surface for $w = \sinh z$. Similarly, $w = \cos z$ and $w = \cosh z$ are essentially the same. Thus these corresponding circular and

(18) $w = \sin z \rightarrow u = \sin x \cosh y,\ v = \cos x \sinh y$

		x	y	u	v		x	y	u	v		x	y	u	v		x	y	u	v
I	a	0	0	0	0	b	1	0	.84	0	c	2	0	.91	0	d	3	0	.14	0
	e	0	1	0	1.18	f	1	1	1.30	.63	g	2	1	1.40	-.49	h	3	1	.22	-1.16
	i	0	2	0	3.63	j	1	2	3.17	1.96	k	2	2	3.42	-1.51	l	3	2	.53	-3.59
	m	0	3	0	10.02	n	1	3	8.43	5.41	o	2	3	9.10	-4.17	p	3	3	1.42	-9.91
II						f′	-1	1	-1.30	.63	g′	-2	1	-1.40	-.49	h′	-3	1	-.22	-1.16
						j′	-1	2	-3.17	1.96	k′	-2	2	-3.42	-1.51	l′	-3	2	-.53	-3.59
						n′	-1	3	-8.43	5.41	o′	-2	3	-9.10	-4.17	p′	-3	3	-1.42	-9.91
III						b^-	-1	0	-.84	0	c^-	-2	0	-.91	0	d^-	-3	0	-.14	0
	e^-	0	-1	0	-1.18	f^-	-1	-1	-1.30	-.63	g^-	-2	-1	-1.40	.49	h^-	-3	-1	-.22	1.16
	i^-	0	-2	0	-3.63	j^-	-1	-2	-3.17	-1.96	k^-	-2	-2	-3.42	1.51	l^-	-3	-2	-.53	3.59
	m^-	0	-3	0	-10.02	n^-	-1	-3	-8.43	-5.41	o^-	-2	-3	-9.10	4.17	p^-	-3	-3	-1.42	9.91
IV						f″	1	-1	1.30	-.63	g″	2	-1	1.40	.49	h″	3	-1	.22	1.16
						j″	1	-2	3.17	-1.96	k″	2	-2	3.42	1.51	l″	3	-2	.53	3.59
						n″	1	-3	8.43	-5.41	o″	2	-3	9.10	4.17	p″	3	-3	1.42	9.91

(19) $w = \sin z$ (Symmetric Axes) $X = x - v;\ Y = y + u$

		X	Y		X	Y		X	Y		X	Y
I	a	0	0	b	1	.84	c	2	.91	d	3	.14
	e	-1.18	1	f	.37	2.30	g	2.49	2.40	h	4.16	1.22
	i	-3.63	2	j	-.96	5.17	k	3.51	5.42	l	6.59	2.53
	m	-10.02	3	n	-4.41	11.43	o	6.17	12.10	p	12.91	4.42
II				f′	-1.63	-.30	g′	-1.51	-.40	h′	-1.84	0.78
				j′	-2.96	-1.17	k′	-.49	-1.42	l′	.59	1.47
				n′	-6.41	-5.43	o′	2.17	-6.10	p′	6.91	1.58
III				b^-	-1	-.84	c^-	-2	-.91	d^-	-3	-.14
	e^-	1.18	-1	f^-	-.37	-2.30	g^-	-2.49	-2.40	h^-	-4.16	-1.22
	i^-	3.63	-2	j^-	.96	-5.17	k^-	-3.51	-5.42	l^-	-6.59	-2.53
	m^-	10.02	-3	n^-	4.41	-11.43	o^-	-6.17	-12.10	p^-	-12.91	-4.42
IV				f″	1.63	.30	g″	1.51	.40	h″	1.84	-.78
				j″	2.96	1.17	k″	.49	1.42	l″	-.59	-1.47
				n″	6.41	5.43	o″	-2.17	6.10	p″	-6.91	-1.58

(18) $w = \sin z \rightarrow u = \sin x \cosh y,\ v = \cos x \sinh y$

	X	Y	U	V		X	Y	U	V		X	Y	U	V		X	Y	U	V
a					e					i					m				
b					f					j					n				
c					g					k					o				
d					h					l					p				

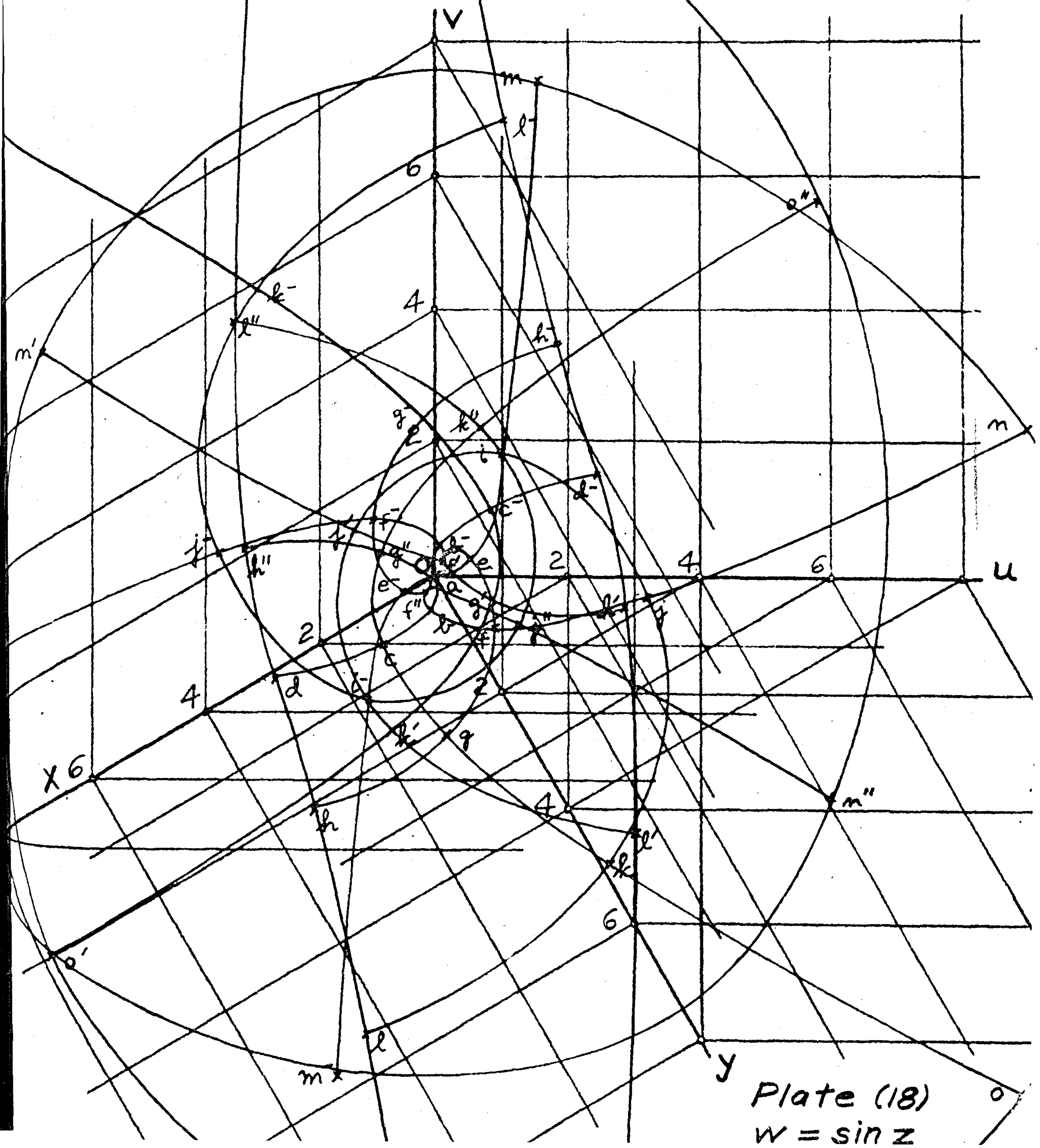


Plate (18)
$w = \sin z$

(19) w = sin z. (Symmetric axes)

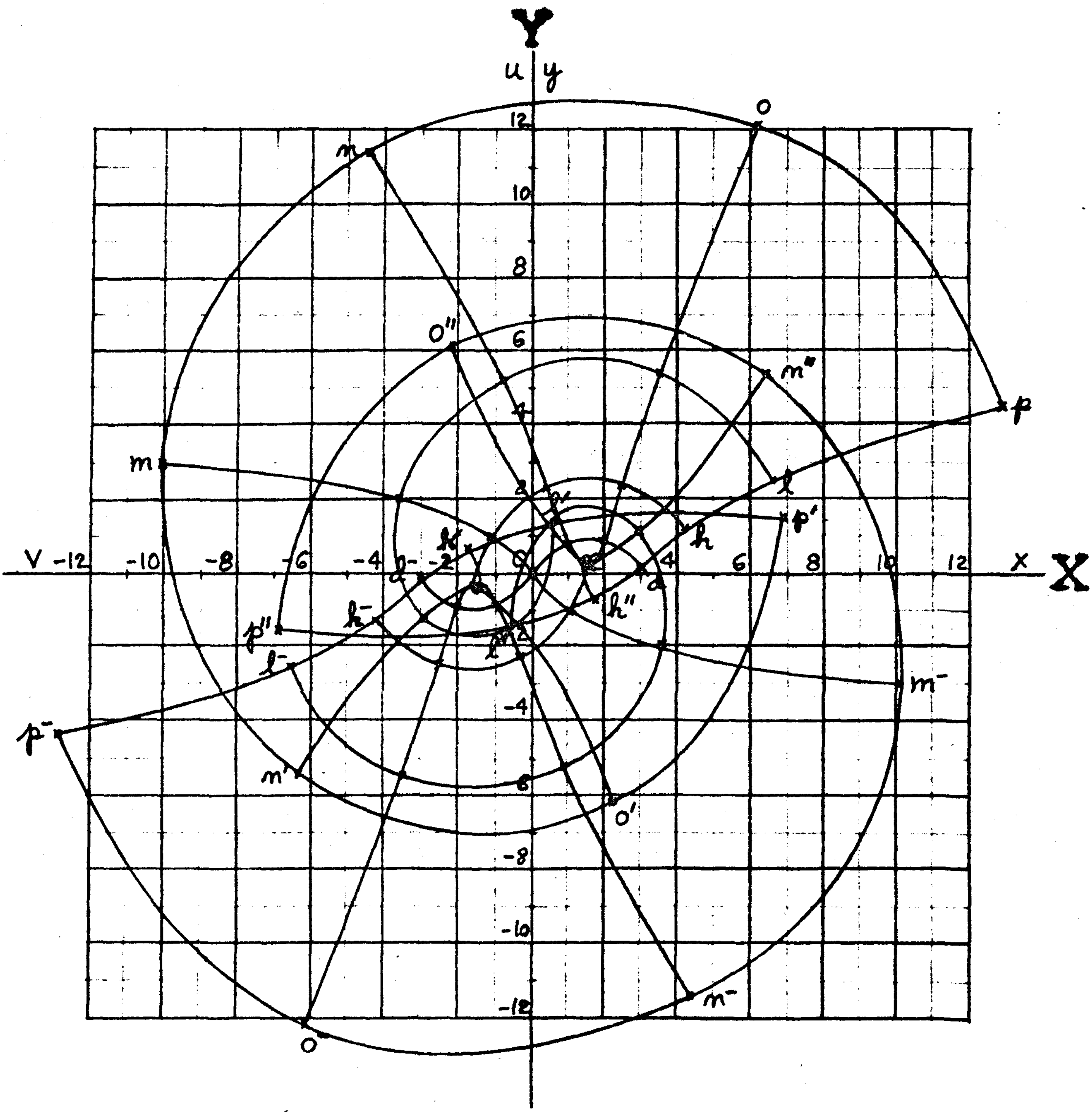


Plate (19)
w = sin z

hyperbolic functions are but two aspects of the same complex function: in their complex form they have the same transformation surface; in their real variables form, they are seen as the traces of this surface on the two absolutely perpendicular xu and yv planes. These same surfaces also represent, from different viewpoints, the inverse trigonometric and hyperbolic functions, $w = \sin^{-1}z$, $w = \sinh^{-1}z$, etc.

5. Stereoscopic Hypergraphs.

Visualization of the four-dimensional transformation surfaces of complex functions is greatly aided by viewing them in three dimensions instead of two. This is accomplished by the usual process of making two stereoscopic drawings from slightly different angles, and viewing them in such a manner as to see only one image with each eye. The figure will then be seen in depth, and the spatial relationships of the various surfaces and coordinate planes will be readily apparent.

The main difficulty in the use of stereoscopic views is in acquiring a method for separating the images to each eye; the actual plotting of the two images is basically simple. Any available stereoscopic device will be suitable: prisms, mirrors, red and blue filters, etc.; or the images can be superimposed directly by use of the eye muscles. We have included a few stereoscopic drawings in this section which can be seen in three dimensions by this last device. A little practice in moving one image toward the other with the eyes will enable most individuals to eventually see these figures in full three-dimensional depth. (Figures 18, 19, 20, and 21).

The relationship of the four coordinate hyperplanes to each other in four-dimensional space is seen in Figure 18. Only the first octants of the three-dimensional hyperplanes and the first hexadekant of 4-space are shown here. Notice the "apparent intersection" of the real xu plane and the pure imaginary yv plane along the four-dimensional line $x = y = u = v$. In Figure 19, the plane $w = c$ is seen parallel to and above the z-plane. In Figures 20 and 21, the relative positions of the transformation surfaces for $w = z$ and $w = -z$ are clearly apparent, as well as the mappings of the z-plane onto the w-plane by these sloping mirror surfaces.

To draw a pair of stereoscopic views, we first plot the right hand image by the usual methods. Then, on a horizontal line to the left at a distance determined by the particular viewing device we are using (two inches for the unaided eyes), we draw a set of axes with the z-plane rotated counterclockwise five more degrees, so that it is tilted down at 35° instead of the usual 30°. We have thus shifted our viewpoint slightly for the second image. The scales along the x and y axes in this left view are now adjusted by projecting horizontally from the normal x and y scales in the right hand view. Or, we can use the formulas:

$$x_L = \frac{\sin 30^\circ}{\sin 35^\circ} x_R = 0.87x_R; \quad y_L = \frac{\cos 30^\circ}{\cos 35^\circ} y_R = 1.06y_R.$$

Every point of the left z-plane has now been displaced in a horizontal direction only, and by an amount proportional to the distance of the point from the origin, our two require-

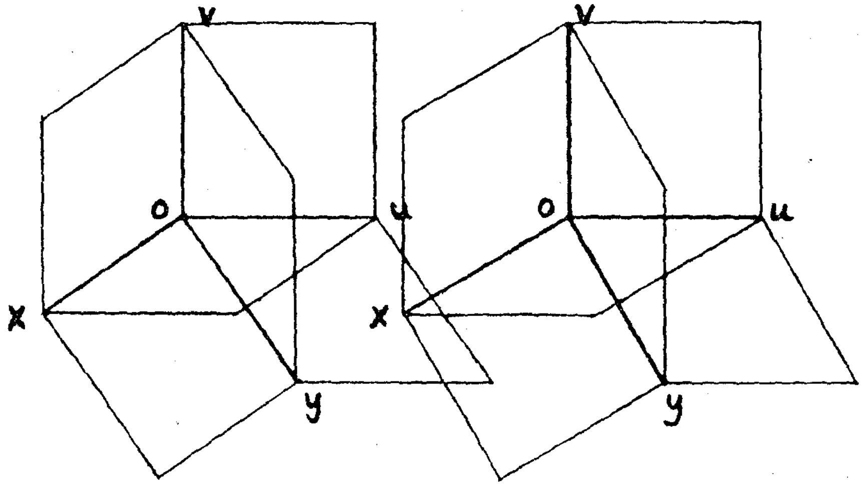


Fig. 18 First Hexadekant

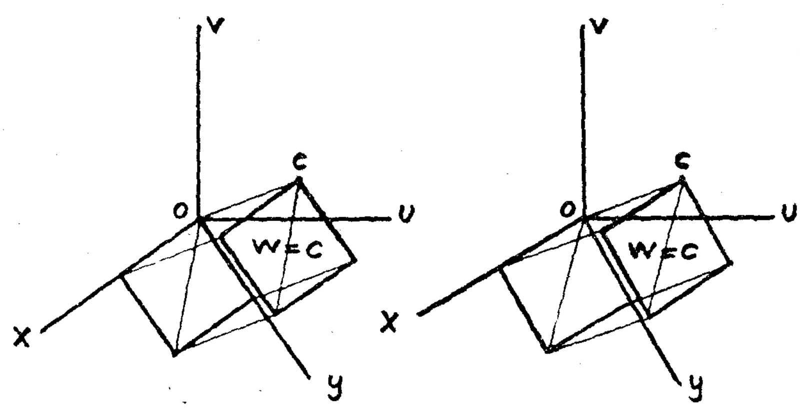


Fig. 19 w = c

Stereoscopic Projection

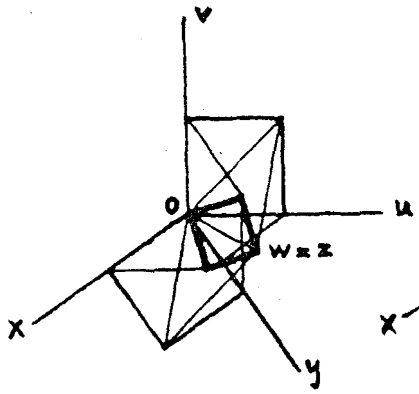


Fig. 20   w = z

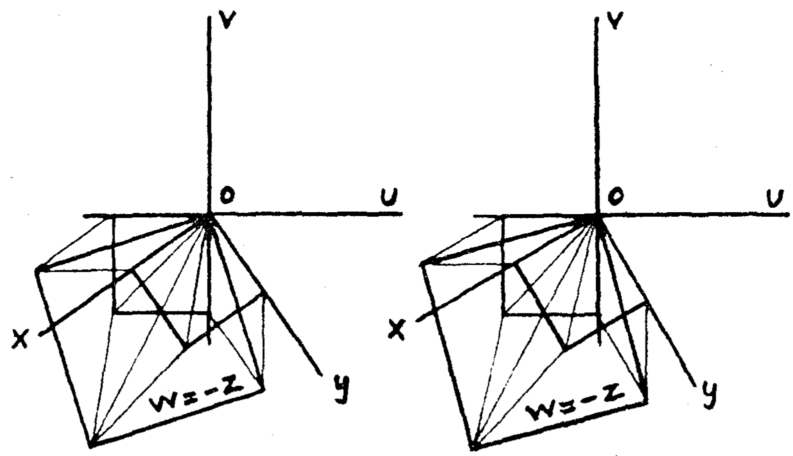


Fig. 21   w = -z

Stereoscopic Projection

ments for a stereoscopic conjugate.

It remains simply to replot the points of the hypergraph in this left view from our new axes by the usual methods. Since we are using the same u and v values for each point in both views, all of our points will be paired horizontally, and proportionally displaced, as required.

For a more formal approach, we introduce the following notation:

@: "appears closer (to the observer) than", or "is ahead of"

$\overline{@}$: "appears equidistant with", or "is the same distance as"

$\not@$: "appears farther away than", or "is behind".

Given the stereoscopic pair of views, t_L and t_R, of a point t, let h_L and h_R be the horizontal <u>directed</u> distances (positive if t is to the right of the origin, negative to the left) respectively of t_L and t_R from the vertical axis v through the origin. Then if we disregard the complicating secondary effects of perspective, the relative proximity, p, of point t to the observer is the horizontal displacement of t between the two views, and is given by $p = h_L - h_R$. If $p > 0$, t @ uv plane through the origin; if $p = 0$, $t\ \overline{@}$ uv plane; if $p < 0$, $t \not@$ uv plane. In general, for any two points, t_1 and t_2, if $p_1 > p_2$, then t_1 @ t_2; if $p_1 = p_2$, $t_1\ \overline{@}\ t_2$; and if $p_1 < p_2$, $t_1 \not@ t_2$.

A formula for p in terms of the coordinates x and y and the angles of the two stereoscopic planes can be obtained, with which we can then determine the relative apparent proximity of any point to the observer without actually constructing a stereoscopic view. We must be careful, however, to construct the figures and derive the formula in such a manner as not to shift the observer's position to that of the added view. By setting p equal to zero in this formula, we find the line of "apparent intersection" of the z- and w-planes from the particular viewpoint being used, as determined by the arrangement of the axes. This line is not actually in either of the planes, of course, as the z- and w-planes intersect in only one point, the origin. It is the four-dimensional line: $x = y = u = v$, which is the apparent intersection of all three pairs of coordinate planes intersecting in only a point, and is the actual intersection of the three planes each equidistant from a pair of these coordinate planes: (1) $x = y$, $u = v$; (2) $x = u$, $y = v$, (the $w = z$ plane); and (3) $x = v$, $y = u$. The analogy to the three-dimensional line, $x = y = z$, is close. The line perpendicular to this apparent intersection is the line of "apparent maximum slope" of the z-plane, from our viewpoint.

Stereoscopic views and equations are a considerable help with the more complicated figures. For many of the simpler surfaces, however, the "order of proximity" principle given in the preceding section is sufficient to enable us to visualize the surface in three dimensions, if not in four.

6. Other Representations of Complex Functions.

Flow Lines: Instead of showing a mirror surface, we can, if we wish, represent the transformation of a complex function kinematically as a field of vectors or flow lines between the two planes. Various versions are possible. We can connect each point of the z-plane directly by a vector to its image point in the w-plane (see Chapter I, page 2). Or we can construct one field of vectors from each object point to its mirror point, then add a second field from these to the corresponding image points. (See Figure 22). Finally, we can also include the intermediate three-dimensional harmonic surfaces by running the first set of vectors from the object plane to its harmonic surface, from this a second set to the mirror surface, another set from the mirror to the conjugate harmonic surface, and a final set from this to the image plane.

Models: Our ability to obtain three-dimensional "pictures", as seen in the preceding section, of the four-dimensional transformation surfaces leads us to further wonder if three-dimensional models of these surfaces are not possible. Such models would represent the surface from only one particular viewpoint, of course, since the apparent area and configuration of the four-dimensional surface are changed by different angles of projection, just as with the projections of a skew curve in three dimensions. Right angles and distances in the four-dimensional figure would not necessarily be preserved in this three dimensional projection. Strings running from object to mirror to image points would indicate the "flow lines" of the transformation in these models. Even "pop-up" models, which could be included in a text book, might be possible for

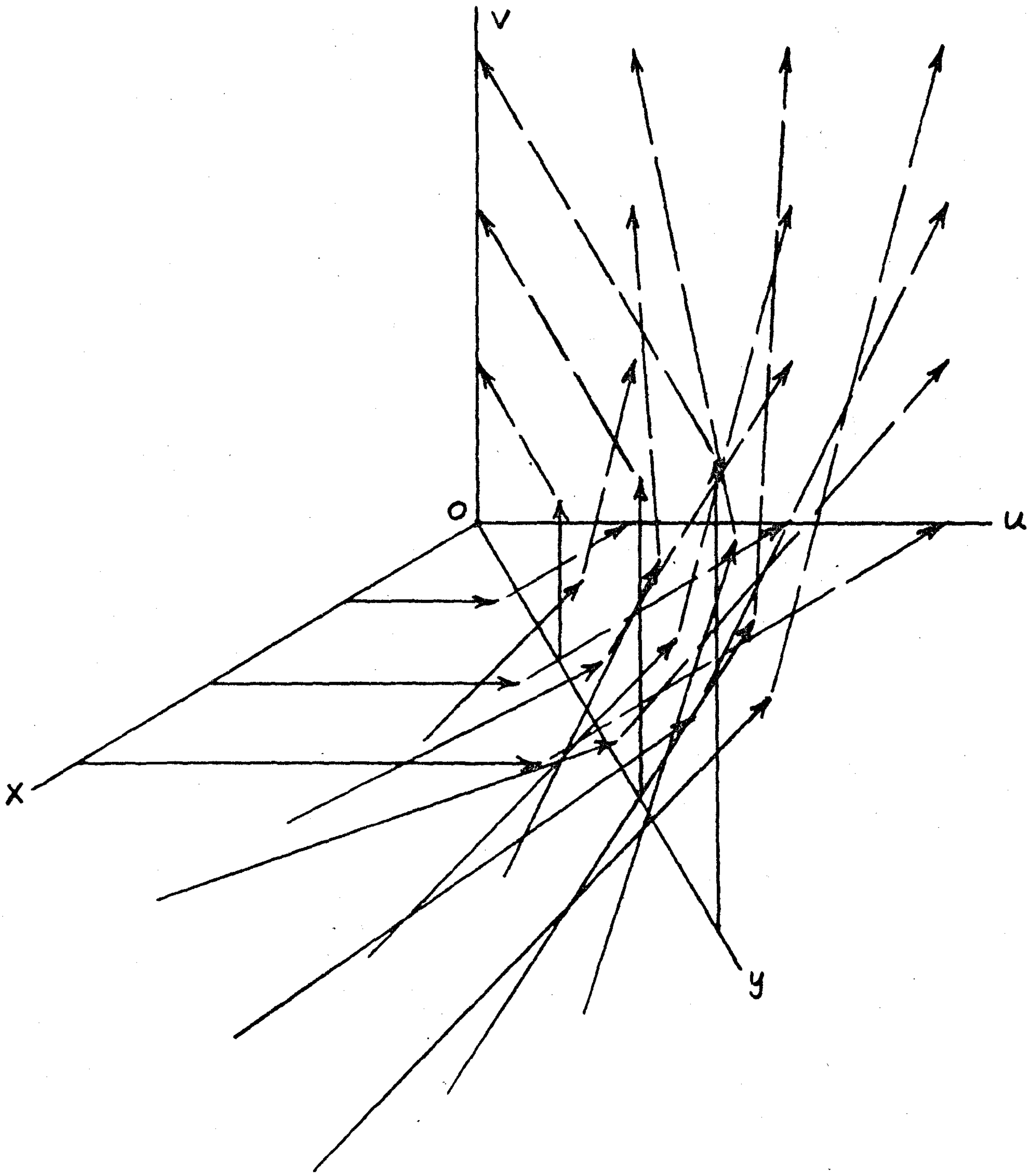


Fig. 22 Flow Lines: w = z

a surface such as $w = z$.

Besides representative visual models, two and three-dimensional dynamic models or analogues of the transformation surfaces might be constructed along mechanical, electrical, or optical lines for a wide variety of applications in such fields as aerodynamics, thermodynamics, fluid mechanics, electrical theory, optics, cartography, analogue computers, etc., that is, in any application where automatic or continuous conformal mapping is desired.

Symmetric axes: We have chosen in this paper to construct the hypergraphs with an asymmetrical arrangement of the axes, because this gives more of a "four-dimensional feel" to the figures. However, often the best and most practical arrangement of the axes to use in four-dimensional plotting is the "symmetrical" arrangement introduced in Plate (19) with $w = \sin z$. In this symmetrical projection, we use a viewpoint for our "cavalier projection" which rotates the z-plane another 60° counterclockwise until the y and u axes coincide. Then, for convenience, we rotate the entire figure another 90°, so that the x and y axes are given their customary "real variables" position in the first quadrant, with the u and v axes falling in the second. This is, in effect, a four-dimensional "isometric projection", obtained by sighting along the line $x = y = u = v$, with the axes renamed for convenience. (Or we can obtain this projection by sighting directly along a particular one of the sixteen directed lines: $\pm x = \pm y = \pm u = \pm v$.) In such a projection the four axes, and consequently the four full planes, will be seen foreshortened. More properly, the "symmetric projection" here is an "isometric drawing", with the foreshortened scales restored to full size. We are thus viewing four of the six planes in true shape and the remaining two edge on, analogous to the three-dimensional viewing of two of three perpendicular lines in true length with the third end on.

There are many advantages in the use of the symmetric form of the axes for hypergraphs of complex functions. First, of course, we no longer need a special four-dimensional graph paper. We can plot our four-dimensional figures directly on ordinary graph paper, and by rotating it but a quarter of a turn, we can work in either of the complex planes. Second, the plotting is considerably simplified. Since the horizontal coordinate (X) of any point is the difference between the x and v values, and the vertical coordinate (Y) is the sum of the y and u values, we can first find the values for $X = x - v$ and $Y = y + u$, as in Table (19), then plot these pairs as ordinary points in the XY plane.[17] Third, this arrangement discloses many of the symmetries of the surface and of its traces with respect to the coordinate planes, the axes, and the origin, as Plate (19) illustrates, which helps in both the plotting and the understanding of the figure. This is our reason for calling these "symmetric axes".

Fourth, visualization of the mirror surface is greatly simplified. Since we are still considering the uv plane as the plane of the paper and the xy plane as extending toward us (though lengthened by a cavalier projection), then for any two points P_1 and P_2 on the mirror surface, we have simply that

$$P_1\ (@,\ \overline{@},\ @\!\!\!/)\ P_2 \text{ as } x_1\ (>,\ =,\ <)\ x_2.$$

This means then that the mirror curves of the parallel lines $x = c$ are the "contours" of the surface; i.e., they are the intersections of the surface with a series of planes parallel to the paper in front of, in, and behind it. Thus, in Plate (19), curve pp" is a plane curve and the closest part of the surface to us. One unit behind this, as measured on the sloping x-axis scale, is a plane containing the curve oo" of the surface, and so on to the curve in the farthest plane from us, p'p‴. Contrarily, the mirror curves of the parallel lines $y = c$, such as pp', p"p‴, etc. are the "receding curves" or "ebb curves", i.e., the curves of maximum recession of the surface from the observer.

Fifth, the graph of the special case for real functions of real variables now appears in true size and shape in the xu plane, and similarly for pure imaginaries in the yv plane. (See, for example, the sine and hyperbolic sine curves in Plate (19).) And sixth, we will find in the next chapter that by viewing four of the six coordinate planes in true shape, we also, as a bonus, see the transformation surface of the linear function, $w = az + \beta$, where a is real, in true size. (The analogy to three dimensions is obvious.) Thus the effects of double foreshortening are obviated for the linear function with a real coefficient, so that the mirror surfaces of $w = z$ and $w = -z$, for example, will now both appear in the same (true) size.

The price we pay for these advantages is the loss of the traces in the yu and xv planes (See, for example, trace aeim in Plate (13), and trace abcd in Plate (16).) Other "symmetric" arrangements can be used showing these planes and suppressing another pair, if our primary objective is thus better served. Of course, when desired, we can always abandon the symmetric projection and its advantages and view all six planes simultaneously in some asymmetrical arrangement such as that used for most of the plates in this chapter.

Functions of Several Complex Variables: The hyper-analytic geometry we have used to represent functions of a single complex variable can be extended to the study of functions of several complex variables, by adding extra "perpendicular" axes to the system. For functions of two complex variables, $q = f(z,w)$, where $q = r + is$, $z = x + iy$, and $w = u + iv$, two axes are added to the asymmetrical system, 15° to the left of the previous v-axis and 15° above the previous x-axis. The axes are then renamed, starting with the upper left, in the counterclockwise order x, y, u, v, r, s. (Figure 23). Alternatively, three 90° pairs of axes can be equally spaced 30° apart around the origin (Figure 24). For symmetric axes, we add the two new variables, r and s, to the left horizontal (v) axis and the lower vertical axis respectively (Figure 25). For functions of more than two complex variables, an extension of this symmetric system is probably the most suitable.

We cannot hope to obtain usable hypergraphs for a complete region of these functions, since the transformation locus of

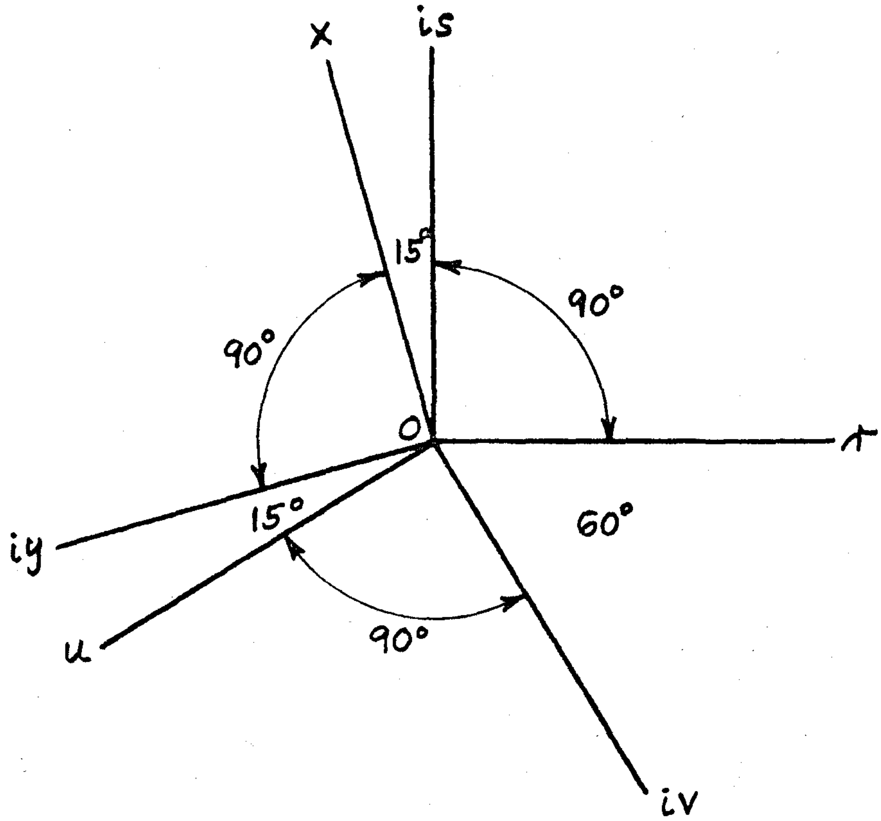


Fig. 23

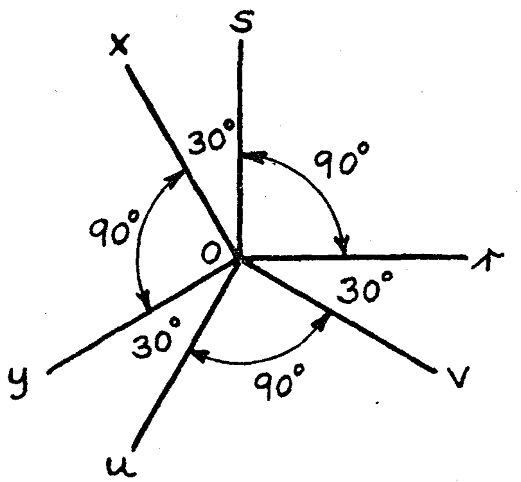


Fig. 24

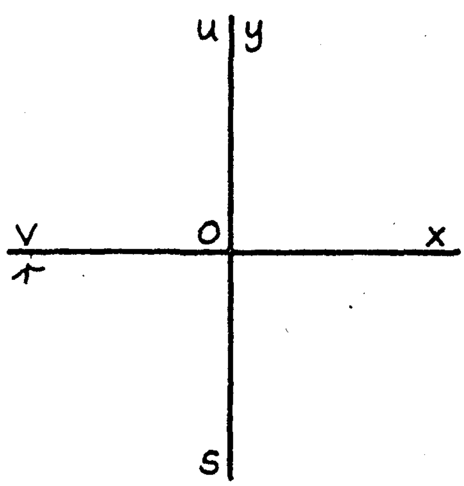


Fig. 25

a function of just two complex variables is a six-dimensional hypersolid, while the argument (z,w) itself ranges over a four-dimensional hyperspace. We can, however, use one of the above systems to investigate certain subregions of the locus, consisting of mirror points of given curves and surfaces in the hyperspace of the argument. By our definition in Chapter II, these transformation locus points, as with all functions, are the vector sums of the n variables involved.

Chapter IV. HYPER - ANALYTIC GEOMETRY

1. Résumé.

With the construction of the four-dimensional graphs of functions of a complex variable, our main objective in this paper has been accomplished. However, there are several additional features of the system we have used which are worth presenting here.

We have introduced, without developing systematically, a hyper-analytic geometry of four (or more) dimensions, based on a direct graphical extension of the cartesian coordinate system. Following the pattern established in classical plane and solid analytic geometry, we may define a four-dimensional space as a space which can be put into one-to-one correspondence with the quadruple of real numbers (x, y, u, v). We can then proceed to develop the geometry of this space along parallel algebraic and graphical lines in a similar fashion to the plane and solid analytic geometries. Since functions of a complex variable are representable as sets of quadruples of real numbers, this hyper-analytic geometry can represent such functions, as we have seen in this paper.

The usual extensions to four dimensions of the distance formula, direction cosines, etc. are adopted; thus, for example,

$$d^2 = x^2 + y^2 + u^2 + v^2,$$

and

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = 1.$$

Certain aspects of hyper-analytic geometry have already been presented in the preceding chapters in connection with the exposition of our main subject. We have given, for example, some general results for the hypergraphs of n-dimensional hyper-analytic geometry in our definition of n-dimensional graphs, the number of p-dimensional coordinate manifolds in n-space, the number of partitioned cells, and the theorem on manifolds in n-space, as well as some fundamentals of graphic transformation, stereoscopic methods, etc.

The various cases of the general four- and five-dimensional graphs have been mentioned in passing; to these we will return later for a more detailed treatment.

In the particular application to complex functions, enough of the hyper-analytic geometry background has been given to show that here we are dealing basically with a "geometry of surfaces". Thus, whereas the analytic geometry of functions of a single real variable is a study of the relationships between two-dimensional curves and the correspondences between sets of points on a pair of perpendicular lines, the coordinate x and y axes, the hyper-analytic geometry of functions of a single complex variable is concerned with the relationships between four-dimensional surfaces and the correspondences between sets of points in a pair of perpendicular coordinate z- and w-planes. The basic element in this "surface geometry" is no longer a point moving on a coordinate axis, but a line (vector) moving in its coordinate plane. With this analogy, features and theorems of plane analytic geometry can often be extended to hyper-analytic geometry with the simple substitution of the words, "line" for "point", "plane" for "line", and "surface" for "curve". For example, geometrically, y = x is a line which is the locus of all points equidistant from the two perpendicular coordinate lines. Similarly, w = z, hyper-geometrically, is a plane which is the locus of all lines equidistant from the two perpendicular coordinate planes.

The analogy brings up further interesting questions. Since the formal manipulations with x, y, and z are unchanged by replacing them with z, w, and q, are the hypersphere, the parabolex, the hyperbolex, etc., "hyper-conic sections"? That is, are they the intersections of a hyper-cubed-plane with a hyper-cubed-cone in the six-dimensional space of functions of two complex variables?

Is the limiting form of the parabolex (Plate (14)) two parallel planes, just as a parabola degenerates into two parallel lines?

Is the limiting form of the hyperbolex (Plate (15)) the "rectangular system"[32], i.e., the four coordinate planes, of four-dimensional space, just as an equilateral hyperbola degenerates into the asymptotic coordinate axes?

Is the circular trace about the origin in the hyperbolex cut from a tangent and perpendicular (in six dimensions) hypersphere whose limit is the hyperpoint at the origin?

Is there an imaginary skew torus in four dimensions wrapped about a regular three-dimensional cone such that a hyperline cuts the imaginary circle from it, as well as the real hyperbola from the cone?

Returning to more tangible results, other aspects of hyper-analytic geometry which have been presented in the preceding chapters are the graphical analysis of complex equations, including a hyper-geometrical interpretation of intercepts, traces, and three-dimensional sections of the surfaces representing these equations, the special cases of real and pure imaginary variables, complex roots of quadratic equations with real and complex coefficients, etc. Also we have seen the geometrical results of additive and multiplicative complex constants.

Transformation of the coordinate planes (analogous to the transformation of axes in plane analytic geometry) was illustrated by a translation of the parabolex, and by the successive transformations of the general linear fractional transformation which reduces to translations, a rotation, and an expansion of the coordinate planes of a hyperbolex.

A formal presentation of hyper-analytic geometry, however, would require many aspects we have not touched on here. Some of these are already defined and available from the usual treatments of n-dimensional geometry. We would need, for example, definitions of "slope", "angle", "parallelism", "perpendicularity", etc., as well as a general treatment of such things as rotation of axes, length of arc, area of surfaces, volume of hypersurfaces, hypervolume of hypersolids, and, for the two dependent variables case, the exact geometrical meaning of the complex derivitive, the complex integral, residues, the Cauchy-Riemann conditions, the Cauchy-Goursat theorem, etc.

However, we will leave this geometry in the embryo stage, and turn instead to some final interesting aspects of four-dimensions and of the particular graphical system we have employed.

2. Some Peculiarities of Four Dimensions.

Most of the unusual features of four-dimensional geometry are presented in the standard literature on the subject, and the reader is referred to this for a basic orientation in this field.[33]

Some of these oddities are apparent from the stereoscopic view (Figure 18) of the first hexadekant. By adding "against nature" a fourth perpendicular axis, we are rewarded with a system in which there are four unlimited but non-overlapping three-dimensional spaces or hyperplanes. Also we now have six mutually perpendicular planes, two pairs of which intersect in lines while the remaining pair intersects in only a point.[28] Further, there are three-dimensional spaces (hyperplanes) perpendicular to lines, planes and other three-dimensional spaces, etc. Of course, our basic assumption of a fourth perpendicular axis means that we can have any number (a whole plane) of lines perpendicular to a plane at a point. And, as we shall see below, just as a point in a graph of three dimensions may represent the end view of a line, in a hypergraph of four dimensions it may represent the "endview" of a plane. (Figure 26, (1)).

One of the more startling effects is the apparent change in size of an object as it is rotated or translated in a fourth dimension. In three-dimensional perspective we expect the apparent size of an object to change with distance. In four dimensions (neglecting perspective), the object changes in apparent size with our direction of viewing. This results from the fact, stated in an earlier chapter, that a two-dimensional projection of a four-dimensional object introduces a double foreshortening effect, or change of projected distance in two perpendicular directions, which thus produces a change in projected apparent area, i.e., our impression of the "size" of the object. (Compare, for example, the equal mirror surfaces of $w = z$, Plate (3), and $w = -z$, Plate (5).) We can experience the same thing in three dimensions. If we look edgewise along a table at the shadow of an object on it, we see a one-dimensional line which is a projection of a three-dimensional object doubly foreshortened. Similarly, the shadow of a plane "edge on" to the sun is a line, which if we look along it will be a point. The plane has thus been doubly foreshortened from two to zero dimensions.

As we either "move around" or "move past" a four-dimensional object, we will see this object apparently change in size (even diminishing to zero), since a four-dimensional

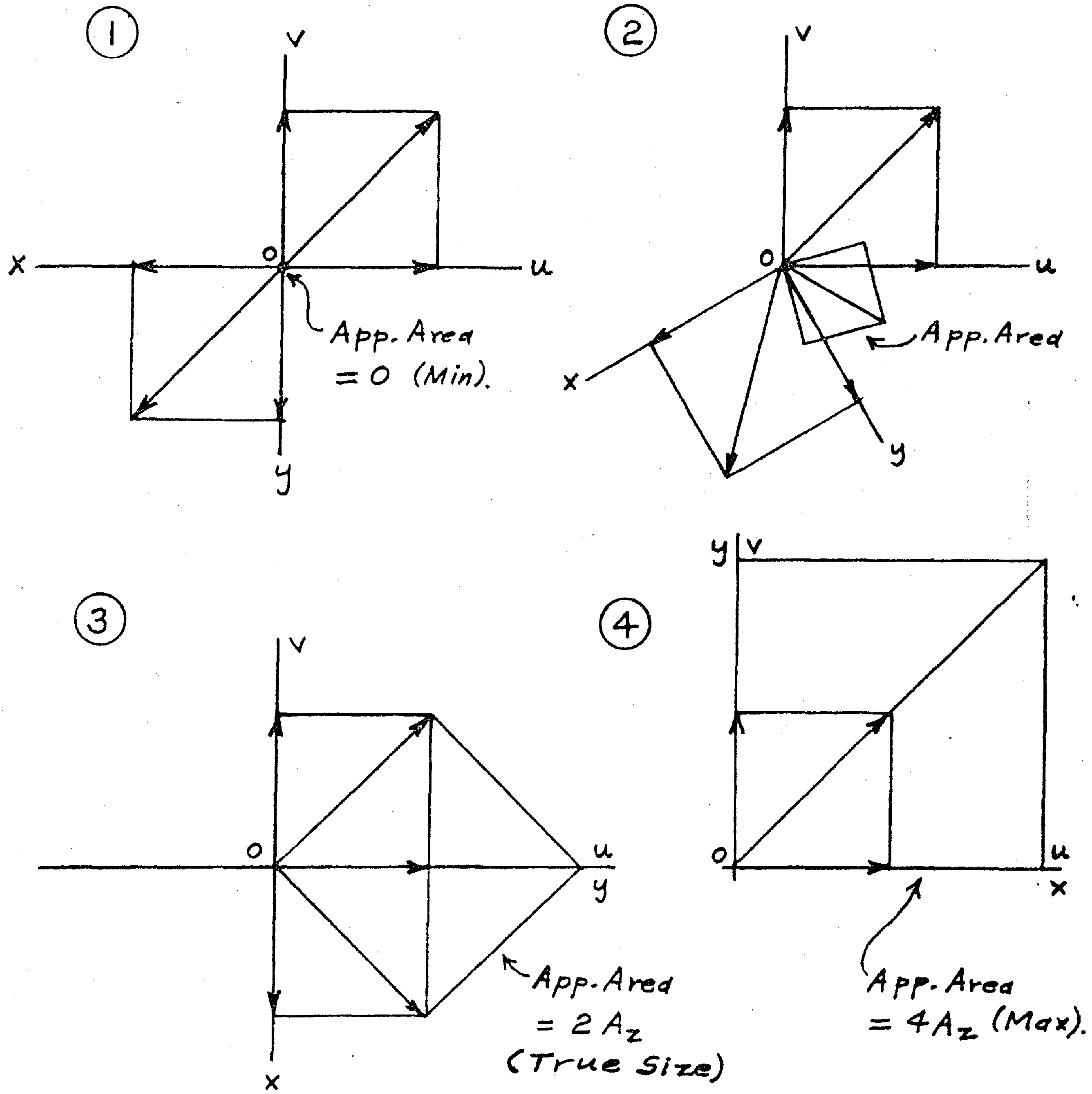


Fig. 26 Variation in Apparent Area
(w = z)

solid, surface, or curve is a <u>changing</u> three dimensional solid, surface, or curve in this fourth direction.[34] Figure 26 shows this variation in apparent area for a moving viewpoint for the identity surface $w = z$. In the distortion of our cavalier projection, the variation here ranges from a minimum of zero to a maximum area twice the actual size.

3. <u>A Few Theorems</u>.

We proceed to prove a few simple theorems in the hyper-analytic geometry of complex functions.

(1) For the general linear function, $w = Az + B$, A and B complex, each square region of the z-plane has a corresponding square mirror in the t-plane and a square image in the w-plane. Since the z- and w-planes are absolutely perpendicular, any line in one plane through their common point O is perpendicular to any line in the other through O.[28] Therefore z is perpendicular to w for all z's and all w's. But $\bar{t} = \bar{z} + \bar{w}$, so $|\bar{t}|^2 = |\bar{z}|^2 + |\bar{w}|^2$. Thus each square in the t-plane is equal to the sum of the corresponding squares in the z- and w-planes. We thus have the following

<u>THEOREM (1)</u>: Under the general linear transformation, $w = Az + B$, A and B complex, the area of any region of the t-plane is the sum of the areas of the corresponding regions of the z- and w-planes.

(2) For the linear function $w = \alpha z$, where $\alpha = c_1 + c_2 i$, we have $u = c_1 x - c_2 y$, $v = c_1 y + c_2 x$. If we compare the areas of

the corresponding square regions of the object, mirror and image planes:

$$(abfe)_z = 1, \; (abfe)_t, \text{ and } (abfe)_w$$

under this transformation plotted for symmetric axes, we find, by computing the lengths $(ab)_t$ and $(ab)_w$, and using the relationships for these axes: $X = x - v$, $Y = y + u$, and $u = y$, $v = -x$, that these areas are related as follows:

$$A_t = A_z + A_w - 2c_2$$

When α is real, $c_2 = 0$, and we have $A_t = A_z + A_w$, which is the true size of the mirror plane, by Theorem (1). We can thus consider this particular viewpoint as "perpendicular" to the mirror plane of the function $w = az$, where a is real. Since the effect of an additive constant, real or complex, is simply a translation of the mirror plane, we have the result mentioned in the last chapter:

THEOREM (2): In symmetric axes, the mirror plane of the linear function, $w = az + \beta$, where a is real, is seen in true size.

(3) In three dimensions we have paired words to indicate our three directions, for example, "up, down", "right, left", "toward, away". The human race has no experience with a fourth dimension, however; consequently we have no corresponding pair of words in our language for such a direction. Since we need for this two words denoting opposite directions, but unattached to any of the three ordinary dimensions, we will call this fourth direction "zig" and "zag".

If the directions indicated by the x, y, u, and v vectors in (1) of Figure 26 are considered the four positive directions: "zig", "toward", "right", and "up" respectively, then using Theorem (2), we have the following

THEOREM (3): For the function $w = az + \beta$, where a is real, when our viewpoint is such that the product of the signs of direction of the four axes is negative the apparent area is the true area; when their product is positive, the apparent area is at an extreme of convergence or divergence.

(4) Analytic functions have the important property of determining a conformal mapping between the object and image planes, i.e., angles are preserved under such a transformation. The following conjecture (proved here in a limited form) states that the projection on the mirror surface is also conformal. Taking the perpendicular segments of the x and y axes through points (1,0) and (0,1), we have for the direction numbers of their mirror segments, respectively:

$$1, 0, \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \text{ and}$$
$$0, 1, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}.$$

Under the Cauchy-Riemann conditions, the cross-product here is zero, therefore the mirror lines are perpendicular. Hence we have a partial proof of the following

CONJECTURE (1): Projections from the coordinate planes onto mirror surfaces of analytic functions are conformal.

4. Hypergraphs of Four and Five Dimensions.

We have been working with hypergraphs of two independent and two dependent variables. In this section we will discuss the other four-dimensional cases, as well as five-dimensional hypergraphs.

In Figure 27 is shown the graph of a curve in hyperspace, determined by three equations in one independent variable, x, and three dependent variables, y, u, and v. We can plot this graph without difficulty. To make it "readable", i.e., amenable to graphic transformation, however, we need an additional feature, since the vector sum of the dependent variables is in a higher dimensional space (three) than the two dimensions of the graph paper. A logical solution here is to add the projection of the curve on the uv plane to the graph, and correlate the points between the curve (whose x values are labelled) and its projection, by a series of straight line elements of the projecting surface, as shown. We can now perform graphic transformation with the curve, or read the correspondences represented by it, by following the steps given in the figure.

In Figure 28 we have the graph of a hypersurface, or three-dimensional transformation solid in four dimensions, whose equation is in the form of one dependent variable, v, and three independent variables, x, y, and u. The hypersurface graph can be plotted for any three-dimensional region of the argument desired, but it must be drawn and labelled in such a fashion as to make it possible to find the corresponding three-dimensional points within it. (See the instructions in Chapter III, Section 3.) Once this is done, it is a simple

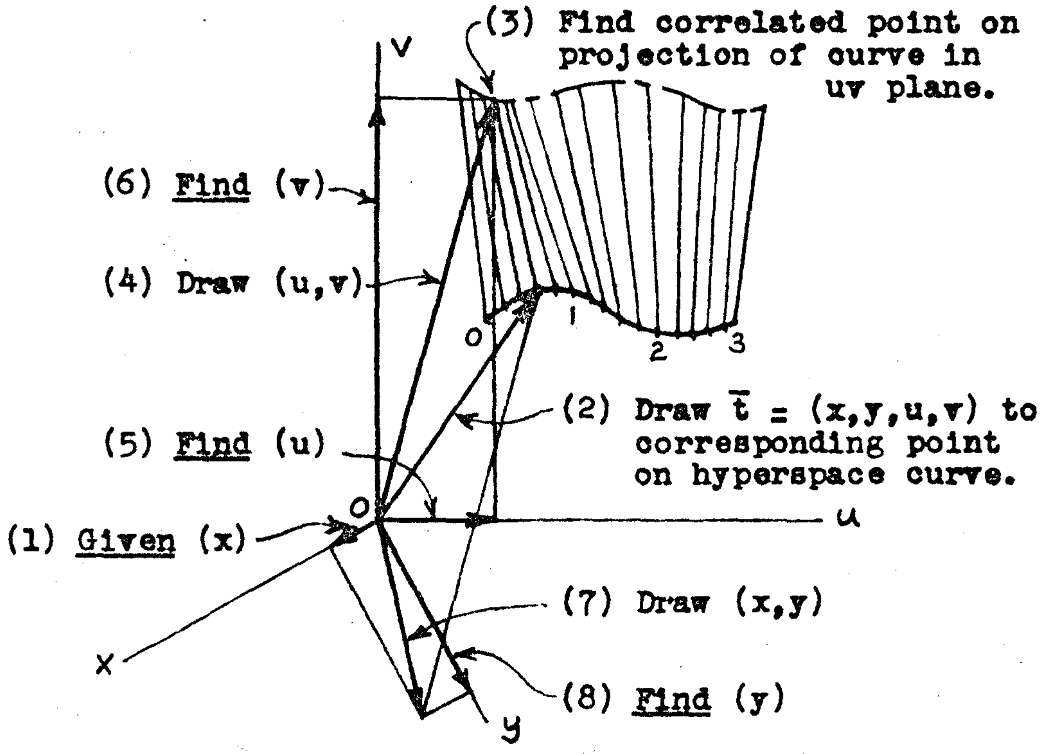


Fig. 27 4 dims. 3 dep. vars.

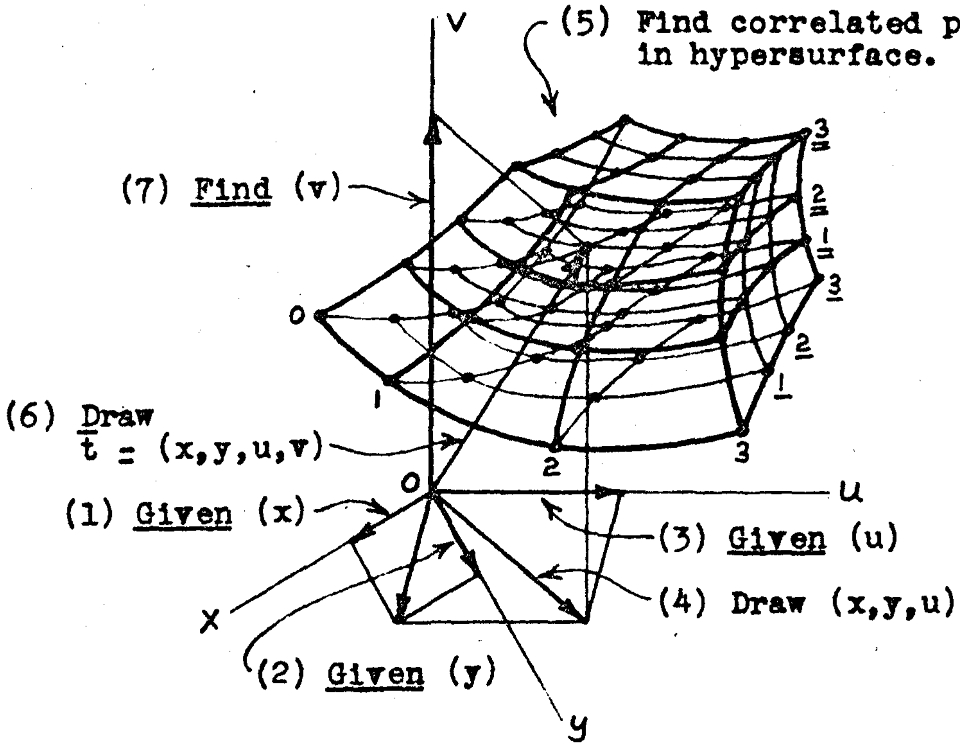


Fig. 28 4 dims. 1 dep. var.

matter to find the value of the dependent variable, v, following the steps given in the figure. The task of finding the correlated point in the hypersurface is made simpler by the fact that we project directly up from the (x,y,u) point to that particular mirror surface within the mirror solid which corresponds to the value given for u.

In five dimensions we add a fifth axis, s, 60° counterclockwise from the vertical axis, v. With four dependent variables we will have a curve, which can be made amenable to graphic transformation in the same fashion as that shown in Figure 27 for four dimensions. For the case of three dependent variables, we will have a transformation surface, which we can handle by including its correlated projection on the uv plane, in a similar manner to our treatment of a curve. For two dependent variables we have a transformation solid, which can be made tractable in the same way shown in Figure 28. The case of one dependent and four independent variables, however, we will leave for the reader to attempt for himself. (See Chapter III, Section 3).

We can perform graphic transformations, in fact, from any graph of a one- or two-dimensional manifold in n-space. And, as we have seen, even a three-dimensional manifold can be handled in 4- or 5-space. Beyond this, however, it becomes difficult to actually use the hypergraph itself to perform a transformation between the variables represented.

5. <u>Hexadekant Patterns</u>:

If we arrange the 16 hexadekants of 4-space according to some orderly cycle of sign changes such as that shown in the table of Figure 29, and number the approximate locations of these hexadekants in our four-dimensional coordinate system, we can investigate the pattern of the particular hexadekants occupied by a hypergraph, and thus increase our conception of the geometrical figure involved. By determining the possible sign combinations of the following functions, for example, we can obtain from the table the numbers of the hexadekants occupied and note the corresponding patterns in Figure 29.

$w = 2 - i$: 2, 7, 10, 15

$w = z$: 1, 7, 11, 13

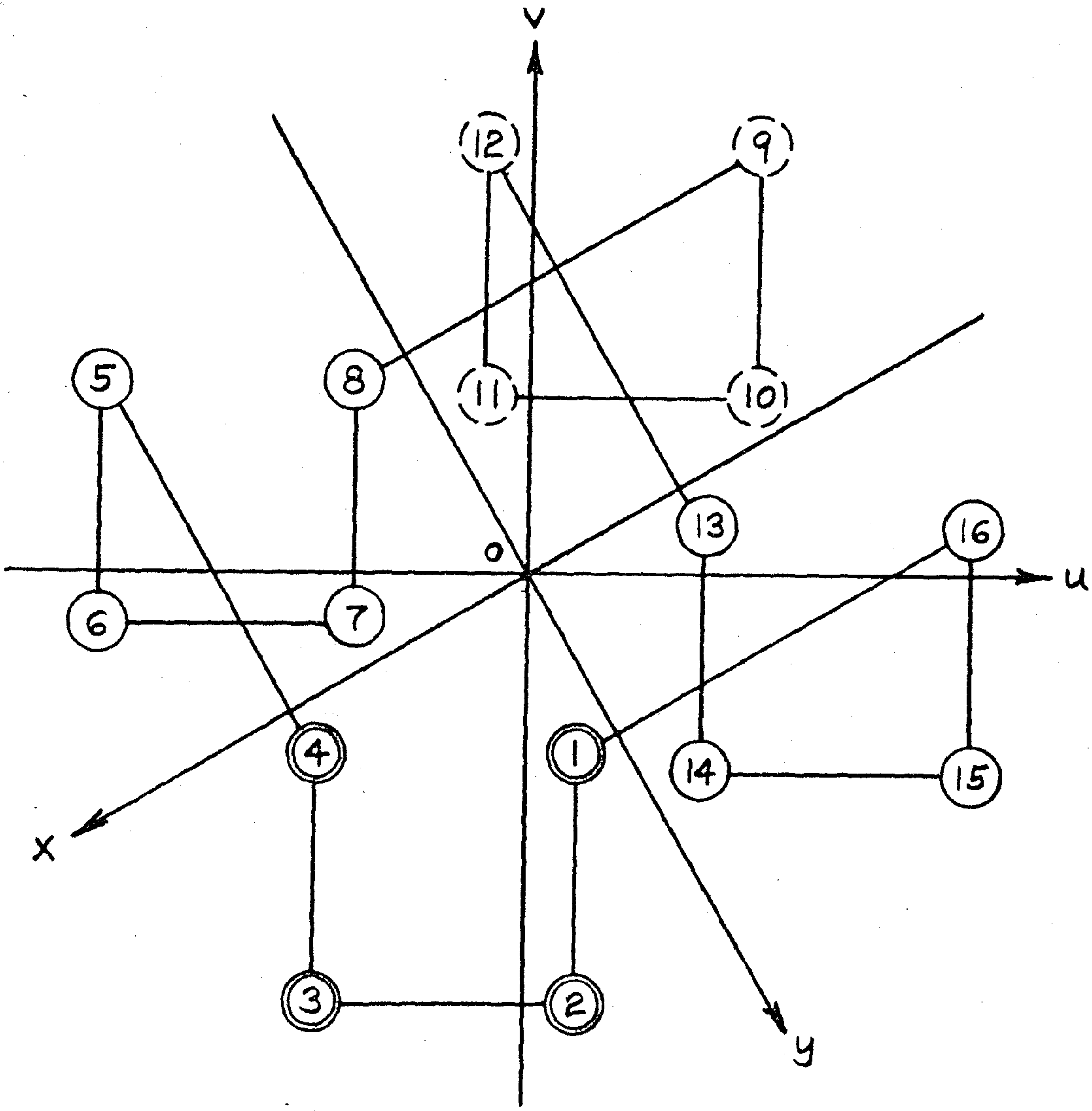
$w = z^2$: 1, 4, 7, 6, 15, 14, 9, 12

$w = z^{\frac{1}{2}}$: 1, 16, 12, 5; 3, 14, 10, 7

We can now "play" with these numbers, noting the common hexadekants of two functions, or that their sums paired consecutively or from each end total 17 fairly consistently, sometimes 13 or 21, etc. Actually, of course, we are dealing here with cyclic group transformations.

6. <u>Extended Analytic Geometry</u>.

The method of plotting four-dimensional graphs of complex functions presented in this paper was developed independently by the present author. In making a search to ascertain if anything of a similar nature had been done previously, the



No.	x	y	u	v	No.	x	y	u	v
1	+	+	+	+	9	−	−	+	+
2	+	+	+	−	10	−	−	+	−
3	+	+	−	−	11	−	−	−	−
4	+	+	−	+	12	−	−	−	+
5	+	−	−	+	13	−	+	−	+
6	+	−	−	−	14	−	+	−	−
7	+	−	+	−	15	−	+	+	−
8	+	−	+	+	16	−	+	+	+

Fig. 29. The 16 Hexadekants

author discovered the various two- and three-dimensional representations of complex functions discussed in Chapter I, and also a series of articles by R. S. Underwood[35] on an "Extended Analytic Geometry for n Variables". Though this geometry was not applied to complex functions in the above articles, the methods given for plotting points in this extended analytic geometry were basically the same as those developed for the hyper-analytic geometry of the present paper.

The two geometries themselves, however, are basically different. Professor Underwood postulates an "n-axes plane", and proceeds to develop the equations for the <u>traces</u> of the higher-dimensional functions on this plane. In hyper-analytic geometry, on the other hand, an n-dimensional space with n mutually perpendicular axes is postulated and constructed. As we have seen, projected lengths <u>and areas</u> are a function of the angle of projection choosen. Thus different results will be obtained in a geometry of an n-dimensional manifold and a geometry of its traces. In particular, distance, which is given by the usual extension of the distance formula and is invariant in hyper-analytic geometry, becomes a function of the angle of projection and is thus no longer invariant, in extended analytic geometry.

These basic differences between the two systems, of course, do not detract from the importance of Professor Underwood's work; Minkowski developed a very valuable geometry of relativity in which the usual distance formula is no longer valid either.

7. Conclusion.

Hypergeometrical interpretations of the basic theorems and operations in complex function theory still remain to be made. It is hoped that the graphical representations of complex functions presented here will not only help in a better understanding of these functions, but will also inspire further investigation into the four-dimensional meaning of such things as the complex derivative and the Cauchy-Riemann conditions, the complex integral and the Cauchy-Goursat theorem, and the value of residues at poles.

The author conjectures that the derivative represents the "slope" of a four-dimensional tangent plane to the mirror surface, that the Cauchy-Riemann conditions require that the normal to this plane make equal "angles" with the u and v axes (or perhaps the x and y axes) and that an extension of this is the general requirement for analyticity in n-space, that the value of the definite integral is connected with the projected areas of the harmonic hypercylindrical hypersurfaces of the function, in an analogous fashion to the three-dimensional interpretation of line integrals, that this value is zero around a closed curve if the mirror surface is continuous within this region, and that the value of the residue at a pole is a property of the hypersphere about the pole tangent to a hyperbolex of first order approximation to the mirror surface of the function.

But if the reader is not inclined to look into these subtleties, or apply hyper-analytic geometry to, say, relativity theory, he can still pass an amusing and profitable hour using the methods of this paper to plot a few complex functions, or even the familiar hypersphere or hypercube.

THE END

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17. And see Chapter IV, Section 6.

18. Birkhoff and MacLane: "A Survey of Modern Algebra", The Macmillan Company, New York, 1953, p.191.

19. For descriptions of the various types of projections, see W. H. Roever: "Some Frequently Overlooked Mathematical Principles of Descriptive Geometry", The American Mathematical Monthly, vol. 41, March 1934, p.142; or see any standard engineering drawing or descriptive geometry text.

20. Our "cavalier projection" does give us certain coinciding curves in the plotting of some four-dimensional figures, such as the hypersphere; and our choice of a 30° asymmetrical system gives us an edge view of the plane, $x = v$, $y = u$.

21. This theorem is similar to one given by Cayley in his "Mathematical Papers", vol. VI, p.458.

22. W. H. Roever: "Meaning and Function of a Picture", The American Mathematical Monthly, vol. 44, Oct. 1937, p.521.

23. Manning. (book) op. cit., p.253, 262 footnote.

24. Manning, (book) <u>op. cit.</u>, p.255-7.

25. Just as in three dimensions a line perpendicular to a surface intersects that surface in only one point, so in four dimensions a hyperline (a hyperspace plane) perpendicular to a hypersurface (a hyperspace solid) intersects that hypersurface in only one hyperpoint (a hyperspace line). See the book by Manning, <u>op. cit.</u>, p.60, Theorem 1.

26. Frumveller, <u>op.cit.</u>, On the transformation from z to w, Professor Frumveller says: "Its action is analogous to that of a mirror with a bent surface". This article also mentions four perpendicular axes and the use of the fourth dimension for complex functions, although only three axes are actually used in the plotting method presented. See Chapter I, p.3 of this paper.

27. Churchill, <u>op. cit.</u>, p.51.

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VITA

Malcolm Lee Murrill was born in Bedford, Virginia, February 27, 1916; since 1920 his home has been in Richmond. He has three brothers and one sister. His father, Pitt Samuel Murrill, received a B.S. degree at Virginia Polytechnic Institute in 1902 and an M.S. in 1904 and took an M.A. at Columbia University in 1912. His mother, whose maiden name was Edith Branson Simmons, attended Littleton College in North Carolina, Columbia University, the American Institute of Normal Methods in Boston, and Northwestern University.

Mr. Murrill was educated in the Richmond Public Schools, graduating from John Marshall High School in 1931. He spent two years with the Class of 1935 at the Richmond Branch of the Virginia Polytechnic Institute. In 1933 he entered Yale University where he was awarded the second Benjamin F. Barge Mathematical Prize and had a second rank stand in his Freshman year. He received a Junior high oration appointment, and held a Philip Marett Scholarship for the last three years of his course. In 1937 he was graduated with the degree of Bachelor of Arts in Mathematics.

After graduation, he was employed as a high school teacher in Fries, Virginia. Later he joined the Virginia State Highway Department as a draftsman and subsequently worked for the Newport News Shipbuilding and Drydock Company.

In 1942 he joined the United States Navy, in which he served as an Armed Guard gunner and later as an aerial navigator, being discharged in April 1946 with the rank of Lieutenant.

In February 1948, he joined the faculty of Virginia Polytechnic Institute, Richmond Branch, where at present he is Associate Professor of Mathematics and Engineering.

He attended one summer session at Columbia University before enrolling in the University of Richmond Graduate School.

He and his wife, the former Betty Stuart Hasher, have one son, Branson.