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Communication games

Kimberly I. Noonan
University of Richmond

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Kimberly I. Noonan
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Department of Mathematics and Computer Science
University of Richmond

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[1]Under the direction of Dr. Kathy Hoke

Abstract

A communication game combines traditional n-person game theory with graph theory. The result is a model of a bargaining situation where communication is restricted. The game's multilinear extension (MLE), a polynomial that summarizes the solutions of the game, is well known for the case where the graph is a tree or simple cycle. This paper simplifies the computation of MLE of the communication game in the case when the graph is a series of simple cycles. The results are then applied to studying the power of each Canadian province in passing an amendment to the constitution, taking geographic location into account. Finally, we discuss simplifications of the computation of the MLE in the case of the complete graph and make conjectures about the coefficients in the MLE.

This paper is part of the requirements for honors in mathematics. The signatures below, by the advisor, a departmental reader, and a representative of the departmental honors committee, demonstrate that Kimberly Noonan has met all the requirements needed to receive honors in mathematics.

Kathy Hoke

(advisor)

Della D. Fenster

(reader)

[signature]

(honors committee representative)

1. Cooperative Game Theory Background

A communication game combines traditional n-person game theory with graph theory. The result is a model of a bargaining situation where communication is restricted. When studying cooperative games we are interested in all the possible coalitions that may form and how each coalition affects the power of a player in the entire game. [[5] and [6] include a more thorough treatment of cooperative game theory].

Definition: A **game**, (N, v), in characteristic form, is a finite set $N = \{1, 2, ..., n\}$ of players along with a real valued set function $v : 2^N \rightarrow \Re$, defined for all subsets $S \subseteq N$ with $v(\emptyset) = 0$. For convenience, we assume throughout this paper that the underlying game (N, v) is zero-normalized; i.e. $v(\{i\}) = 0$ for all $i \in N$.

The characteristic function, v, completely defines the game and so we may refer to any game by v. We will focus our study of cooperative games on the set of **simple** games in which some coalitions "win" while others "lose". Consider the following two examples of simple games:

Definition: Majority Rules Game:

$$v(S) = \begin{cases} 1 & \text{if } |S| \geq \frac{n}{2} \\ 0 & \text{otherwise} \end{cases}$$

The majority rules game defines the game where a coalition wins (value 1) if it has a majority of the players and it loses (value 0) otherwise.

Example: A committee has 7 members, denoted by the set $N = \{1, 2, 3, 4, 5, 6, 7\}$. A decision will be made if a majority of the committee members agree on an outcome. The value of the coalition $S = \{1, 3, 7\}$ under the majority rules game is $v(S) = v(1, 3, 7) = 0$, because the coalition does not have a majority of the players. The value of the coalitition $S' = \{2, 4, 5, 6\}$ is $v(S') = v(2, 4, 5, 6) = 1$.

Definition: Unanimity Game:

$$\mu_S(T) = \begin{cases} 1 & \text{if } S \subseteq T \\ 0 & \text{otherwise} \end{cases}$$

A unanimity game, denoted by μ_S, defines a special subset S such that a coalition wins if and only if it contains all of this special subset S.

Example: Recall the seven committee members in the previous example and suppose that in order to make a decision players 1,2,3, and 4 must be in agreement. So, we can define the special subset of players $S = \{1,2,3,4\}$. Then $\mu_S(\{1,3,5\}) = 0$, since $\{1,3,5\}$ does not contain S, where as $\mu_S(\{1,2,3,4,5\}) = 1$.

Unanimity games are very important in the study of cooperative games because these games form a basis for all other games. In other words, any cooperative game may be written as a linear combination of unanimity games.

The characteristic function v is a set function. The unique multilinear polynomial associated with the set function v provides one efficient method to study cooperative games.

Definition: A **characteristic vector**, $(x_1, x_2, ..., x_n)$, for a set S is defined as

$$x_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases}$$

Example: Let $S = \{1,3,4\}, n = 4$.
Then the characteristic vector is $(1,0,1,1)$.

Definition: The **Multilinear Extension (MLE)** of a game is a function $f(x_1, x_2, ..., x_n)$ such that if $(y_1, y_2, ..., y_n)$ is the characteristic vector for a set then $f(y_1, y_2, ..., y_n) = v(S)$.

To simplify the terms in the following theorems and examples we will use the following notation:

Notation:

$$X_S = \prod_{i \in S \subseteq N} x_i$$

Example: $X_{124} = x_1 x_2 x_4$

The MLE has the form

$$\sum_{S \subseteq N} \Delta_S X_S$$

where Δ_S represents the coefficient of the term X_S in the MLE. The following theorem shows how to compute the coefficients, Δ_S.

Theorem 1.1 (Owen, 1992): Given a game, v, the coefficients of the MLE(v) can be computed as follows:

$$\Delta_S = \sum_{T \subset S} (-1)^{|S|-|T|} v(T)$$

Example: Consider the majority rules game on 4 players, ($n = 4$). The coefficients can be computed in the following way,

$$\Delta_{134} = v(134) - v(13) - v(14) - v(34) + v(1) + v(3) + v(4) = 1$$

$$\Delta_{12} = v(12) - v(1) - v(2) = 0$$

similarly

$$\Delta_{1234} = -3$$

In general, for the majority rules game on 4 players,

$$MLE(v) = x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 - 3x_1x_2x_3x_4$$

Definition: A player i is called a **dummy player** for a game v if $v(T \cup \{i\}) = v(T)$ for all T.

Theorem 1.2: If player i is a **dummy player**, then $\Delta_S = 0, \forall i \in S$. That is, the dummy player contributes nothing to the game.

Proof: Let S be a set of arcs containing arc i.

$$\Delta_S = \sum_{T \subseteq S} (-1)^{|S|-|T|} v(T), \text{ by definition.}$$

$$\Delta_S = \sum_{T \subseteq S, i \in T} (-1)^{|S|-|T|} v(T) + \sum_{T \subseteq S, i \notin T} (-1)^{|S|-|T|}$$

$$\Delta_S = \sum_{T \subseteq (S-i)} (-1)^{|S|-|T \cup i|} v(T \cup i) + (-1)^{|S|-|T|} v(T).$$

Since i is a dummy arc, $v(T \cup i) - v(T) = 0$ for all T.
Note if $T = \emptyset, v(\emptyset \cup i) - v(\emptyset) = 0$, since $v(i) = 0$.
So $\Delta_S = 0.\square$

We have seen how to derive the MLE from the characteristic function. Note that we can also use the MLE, $f(x_1, x_2, \ldots, x_n)$, to determine the value of any coalition S, by simply plugging the characteristic vector of S into f.

As shown above, the MLE contains all of the information that is given by the characteristic function of a game and allows us to manipulate this information in many ways. We are primarily interested in solutions to the game. In particular, we want to assess the power of the individual players. The Shapley value and the Banzaaf values represent two of several examples of such power indices [5]. Throughout this paper, we consider the Shapley value.

Definition: The **Shapley value**, $\Phi(v)$, of a game v is defined by

$$\phi_i(v) = \sum_{S|i\in S} \gamma(s)\,[v(S) - v(S - \{i\})]$$

$$\text{where } s = |S| \text{ and } \gamma(s) = \frac{(s-1)!\,(n-s)!}{n!}$$

Example: In a majority rules game of three players, the winning coalitions are {1,2},{1,3},{2,3},{1,2,3}. Each player is contained in 3 winning coalitions, 2 of size 2 and one of size 3, so each player has the same Shapley value, namely $\frac{1}{3}$.

The following theorem from [4] describes how to compute the Shapley value from the MLE of a game.

Theorem 1.3: The Shapley value for player i in game v can be computed using the MLE as follows:

$$\Phi_i(v) = \sum_{T\subset N: i\in T} \frac{\Delta_T}{|T|}$$

for all $i \in N$.

Example: Consider again the majority rules game on three players. First we determine the MLE of the game to be as follows:

$$MLE(v) = -2x_1x_2x_3 + x_1x_2 + x_1x_3 + x_2x_3$$

Then we find the Shapley values:

$$\Phi_i = \frac{-2}{3} + \frac{1}{2} + \frac{1}{2} + 0 = \frac{1}{3},\ i = 1,2,3$$

2. Graph Theory Background

In this section, we discuss some graph theory concepts necessary for our analysis later. For more details, see Roberts [5].

Definition: A **graph** is a pair (N, A), where N is a set of vertices and A is a set of arcs joining pairs of vertices. An arc is said to be **incident** with each of the vertices it joins.

Definition: A **path** between 2 vertices p and q is a sequence of vertices $p = p_1, p_2, \ldots, p_K = q$ such that there is an arc between p_i and p_{i+1} for $i = 1, 2, \ldots, K-1$.

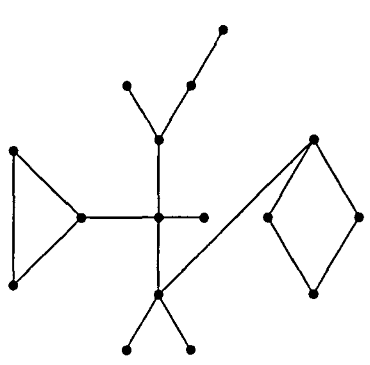
Definition: Two vertices are said to be **connected** if there is a path between them. A set of vertices is **connected** if every pair of vertices in the set is connected. A graph is said to be **connected** if the set of all vertices, the components of the graph, are maximal connected sets of vertices, i.e. S is a **component** if S is connected and if $S \subseteq T$ and T is connected, then $S = T$.

Definition: A **simple cycle** is a connected graph where exactly two arcs are incident with each vertex.

Definition: A **tree** is a connected graph with no cycles.

Definition: A **series of simple cycles** is a graph consisting of simple cycle(s) and tree(s) where at most one vertex in any simple cycle can be connected to a vertex not in the cycle.

Example: Series of simple cycles graph:



Defintion: The **extreme points** of a tree P, $E(P)$, is the set of all vertices in P that are incident with exactly one arc.

Notation: Let G be a graph. The set of vertices and arcs of G are denoted $vert(G)$ and $arc(G)$ respectively.

3. The Arc Game

Thus far, we have studied games defined by a characteristic function and represented by a unique multilinear extension. Now suppose we impose a graph on the situation, where the vertices of the graph represent the players and the lines between the players represent arcs or lines of communication.

This new way of studying the cooperative situation defined by the original game allows us to restrict communication between any two players. Now we can define a new game, the arc game, which incorporates the communication restrictions imposed by the graph on the game.

Definition: Given an original game (N, v) on n players and a graph $G = (N, A)$ with the same n vertices and let $T \subseteq A = arc(G)$, then the **Arc Game** (A, V^A) is a game with player set $A = arc(G)$ defined in coalition form as follows:

$$v^A(T) = \sum_{C_i \in C(T)} v(C_i).$$

where $C(T)$ is the set of components of T.

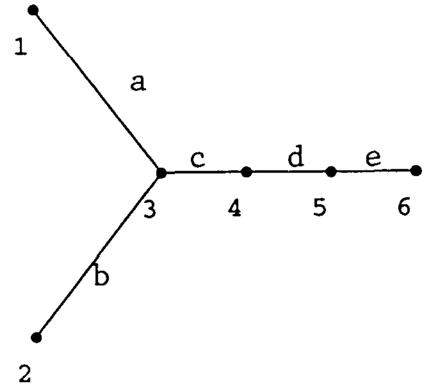
We are now interested in studying solutions to the arc game. In the previous examples, we have been interested in the power indices of the players, determined by the Shapley value. Now we will compute the multilinear extension in terms of the arcs of communication as opposed to the players of the game. From this new multilinear extension we can compute the Shapley value of the arc game, which will allow us to compare the relative power of the arcs. We can also determine a new index of power for the players, called the position value, by using the Shapley value of the arc game. This value was defined in [1].

Definition: The **position value**, $\pi(N, v, A) \in \Re^N$, is given by

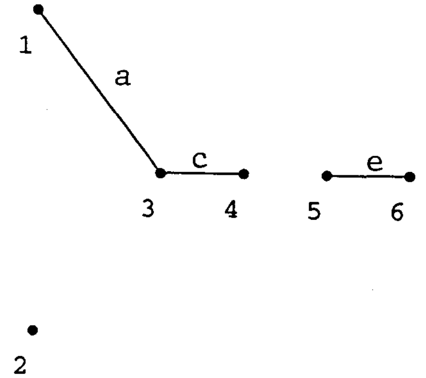
$$\pi_i(N, v, A) = \sum_{a \in A_i} \frac{1}{2} \Phi_a(A, r_N^v) \; \forall i \in N$$

where A_i is the set of arcs in A incident with player i and Φ_a is the Shapley value for arc a in the arc game.

Example: Let G be the graph shown below. Let μ_S^A be the arc game on the graph G, with underlying unanimity game μ_{35}.



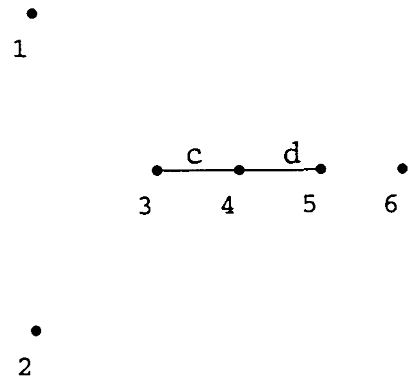
Let $L = \{a, c, e\}$. Note this coalition divides the graph into three subcomponents.



Recall that for the original game, μ_s, that a coalition has value 1 if the coalition completely contains S and 0 otherwise. Now we can compute the value of this coalition in the game from the following computation:

$$\begin{aligned}\mu_{35}^{A}(ace) &= v(134) + v(2) + v(56) \\ &= 0 + 0 + 0 \\ &= 0\end{aligned}$$

Next let $L = \{cd\}$ and notice how this coalition divides the graph into four components as shown below.



We now compute the value of this coalition and compare it to the value of the first coalition.

$$\begin{aligned}\mu_{35}^{A}(cd) &= v(1) + v(2) + v(345) + v(6) \\ &= 0 + 0 + 1 + 0 \\ &= 1\end{aligned}$$

Above, we calculated the value of two different coalitions for the unanimity game μ_{35}^{A} on the graph, G. In order to find the multilinear extension for the arc game on this graph we need to calculate the value for every possible coalition and then use Theorem 1.1 to determine the coefficient of each term of the MLE. Since there were six players ($n = 6$) there are 63 possible coalitions. The following theorem not only defines the terms in the MLE with a non-zero coefficient, but also shows how to compute the coefficient in another way.

Theorem 3.1 (Hoke): Let $S \subseteq N = vert(G)$ with the size of $S \geq 2$. Define

$$\sum(S) = \{T_i \subseteq G \,|T_i \text{ is a tree, } S \subseteq \text{vert}(T_i), \text{ and } E(T_i) \in S\}$$

We call the members of $\sum(S)$ *minimal trees.* Then the MLE for μ_S^A is given by

$$\begin{aligned}\text{MLE}(\mu_S^A) &= \sum_{T_i \in \sum(s)} X_{arc(T_i)} - \sum_{T_i, T_j \in \sum(s)} X_{arc(T_i) \cup arc(T_j)} \\ &\pm \ldots \pm (-1)^{K+1} \sum_{T_{i_1}, T_{i_2}, \ldots, T_{i_k}} X_{arc(T_{i_1}) \cup arc(T_{i_2}) \cup \ldots \cup arc(T_{i_k})} \\ &+ (-1)^m X_{arc(T_1) \cup arc(T_2) \cup \ldots \cup arc(T_m)}\end{aligned}$$

Example: Let G be the graph in the last example. First we must determine the minimal trees of our game on the graph G above. By the definition above, a minimal tree must contain all of S and the endpoints of the tree must be contained in S. Since $S = \{3, 5\}$, we can see from the graph that X_{cd} is the minimal tree. The $\text{MLE}(\mu_{35}^A) = X_c X_d$

The MLE for any unanimity game on a tree is actually very simple. Since a tree is connected, there is only one minimal tree on the graph that contains all the elements of S and whose endpoints are contained in S. Borm, Owen and Tijs stated this result in [1].

Notation: We will denote the coefficent of the term X_S in the arc MLE as Γ_S.

Theorem 3.2 (Owen, Borm,Tijs): Let G be a tree with the arc set A. Let $\sum \Delta_T x_T$ be the MLE for μ_S.
The coefficient of the term X_L of the MLE (μ_S^A) is:

$$\Gamma_L = \begin{cases} \sum \Delta\mu_T, \text{ where } E(L') \subseteq T \subseteq L' & \text{if } L' \text{ is connected} \\ 0 & \text{otherwise} \end{cases}$$

where L is the set of arcs and L' is the set of vertices incident with at least one arc in L.

Next, we look at the case when the graph is a simple cycle.

Definition: Let $P_1, P_2, ..., P_K$ be the set of all paths containing S with both extreme points in S. $Arc(P_i)$ is the set of all arcs that connect 2 vertices in P_i.
Note: $K =$ the size of S.

Theorem 3.3 (Mitchell): Let G be a simple cycle.
The MLE for μ_S^A can be expressed as

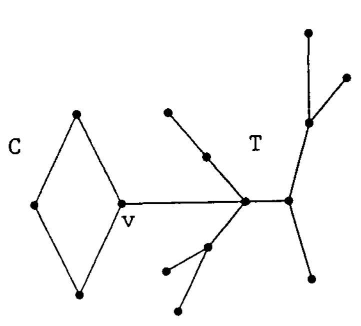
$$X_{arc(P_1)} + X_{arc(P_2)} + ... + X_{arc(P_K)} - (K-1)X_{arc(G)}$$

Example: Let G be a simple cycle.
Let $S = \{1, 5\}$ with the player set $N = \{1, 2, ..., 6\}$
Note: $K = 2, n = 6$

So $arc(P_1) = abcd, arc(P_2) = ef$
By theorem 3.3

$$\begin{aligned} MLE &= X_{arc(P_1)} + X_{arc(P_2)} - (2-1)X_{arc(G)} \\ &= x_a x_b x_c x_d + x_e x_f - x_a x_b x_c x_d x_e x_f \end{aligned}$$

From the two previous theorems, we can easily compute the MLE for the arc game on graphs which are either trees or simple cycles. Next we will consider what happens to the MLE when a tree (T) and one simple cycle (C) are connected at one vertex (v), as in the graph below:



The MLE for this type of graph depends on the location of the set of vertices, S. In particular, the set of vertices lies either (1) entirely on the tree, (2) entirely on the simple cycle, or (3) on both the cycle and the tree. We compute the MLE for each of the three cases as follows:

Case 1: The vertices of S lie entirely on the tree.
Notice that in this case, the vertices and arcs on the cycle are dummy players and arcs, respectively. By Theorem 1.2, the coefficient in the MLE for any term containing an arc on the cycle will be zero. Thus, we need to consider only the coalitions of arcs on the tree. This reduces the graph to a tree, so we can apply Theorem 3.2 to compute the MLE for the arc game.

Case 2: The vertices of S lie entirely on the simple cycle.
In this case, the vertices and arcs on the cycle are dummy players and arcs respectively. This reduces the graph to a simple cycle, for which Theorem 1.2 can be applied to compute the arc MLE.

Case 3: The vertices of S lie both on the tree and the simple cycle.
We will define v to be the vertex that lies both on the cycle, i.e. $v = vert(T) \cap vert(C)$. Recall from Theorem 3.1, that the terms in the MLE are unions of the minimal trees on the graph which contain all of S and whose extreme points are in S. In this case, S lies on the tree and on the simple cycle, therefore each minimal tree must contain v. The following theorem simplifies the minimal trees and the MLE for Case 3.

Theorem 3.4: Let G be a graph with a series of simple cycles which consists of exactly one simple cycle, C, and one tree, T, with $v = vert(C) \cap vert(T)$.

Let S be a set of vertices such that $S \cap \{vert(C) - \{v\}\} \neq \emptyset$ and $S \cap \{vert(T) - \{v\}\} \neq \emptyset$.

Then the MLE for the arc game on a set S is

$$\text{MLE}(\mu_S^A) = \sum_{T_i \in \sum(S)} X_{arc(T_i)} - (K-1)X_{arc(T_1)\cup arc(T_2)\cup \ldots \cup arc(T_k)}$$

Where $\sum(S) = \{T_i \mid T_i \text{ is a tree, } S \subseteq T_i \text{ and } E(T_i) \in S\}$

Note that the size of $\sum(S) = K$, where $K = |S \cap \{vert(C) - \{v\}\}| + 1$.

Next, we will consider a graph with exactly two simple cycles connected by a tree. Again, the MLE depends on the vertices in S. There are four distinct cases concerning the vertices of S for this graph, a series of simple cycles. The arc MLE for the cases I, II, and III are described above. Case IV is when the vertices of S lie on the tree and both cycles. The following theorem simplifies the MLE for this case.

Theorem 3.5: Let G be a series of simple cycles with exactly two cycles, C_1 and C_2, connected by one tree T.

Let $v_1 = vert(C_1) \cap Vert(T)$ and $v_2 = vert(C_2) \cap vert(T)$. Let $S \subseteq vert(G)$ be a set of vertices such that $S \cap \{vert(C_1) - v_1\} \neq \emptyset$ and $S \cap \{vert(C_2) - v_2\} \neq \emptyset$.

Let $K_i = |S \cap \{vert(C_i) - v_i\}| + 1$ for $i = 1, 2$.

To compute the coefficients for the MLE(μ_S^A), we only need to consider the following sets of vertices.

1) $P_1, P_2, \ldots, P_{K_1}$, the simple paths on cycle C_1 such that $\{S \cup \{v_1\}\} \cap vert(C_1) \subseteq P_i$ and $E(P_i) \subseteq S \cup \{v_1\}\ \forall\ i$.

2) $Q_1, Q_2, \ldots, Q_{K_2}$, the simple paths on cycle C_2 such that $\{S \cup \{v_2\}\} \cap vert(C_2) \subseteq Q_j$ and $E(Q_j) \subseteq S \cup \{v_2\}\ \forall\ j$.

3) R, the subgraph of T such that $S \cap vert(T) \subseteq R$ and $E(R) \subseteq \{S \cup \{v_1, v_2\}\}$.

Thus the MLE for the arc game can be expressed as follows:

$$\begin{aligned}\mathrm{MLE}(\mu_S^A) &= \sum_{i=1}^{K_1}\sum_{j=1}^{K_2} X_{arc(P_i\cup R\cup Q_j)} \\ &\quad -(K_2-1)\sum_{i=1}^{K_1} X_{arc(P_i\cup R\cup C_2)} \\ &\quad -(K_1-1)\sum_{i=1}^{k_2} X_{arc(C_1\cup R\cup Q_j)} \\ &\quad -(1-K_1-K_2+K_1K_2)X_{arc(C_1\cup R\cup C_2)}\end{aligned}$$

Proof of Theorem 4.4:
$\sum(S) = \{P_i \cup R \cup Q_j\}\ \forall i = \{1, 2, \ldots, K_1\}, j = \{1, 2, \ldots, K_2\}$,
because $S \cap vert(C_1) \neq \emptyset$ and $S \cap vert(C_2) \neq \emptyset$.
So $|\sum(S)| = K_1K_2$.
Let $T_i \in \sum(S),\ I = \{1, 2, \ldots, K_1K_2\}$.
Note that the union of any two $P_{i's}$ is C_1 and the union of any two $Q_{j's}$ is C_2.

So $T_{i_1} \cup T_{i_2} \cup \ldots \cup T_{i_K}$ for $K \geq 2$ will be of one of the following forms:

(i) $P_i \cup R \cup C_2$
(ii) $C_1 \cup R \cup Q_j$
(iii) $C_1 \cup R \cup C_2$

From Theorem 4.1 we can write the MLE

$$\begin{aligned}\mathrm{MLE}(\mu_S^A) &= \sum_{i=1}^{K_1K_2} X_{arc(T_i)} \\ &\quad +\sum_{i=1}^{K_1} A_i X_{arc(P_i\cup r\cup C_2)} \\ &\quad +\sum_{i=1}^{K_2} B_j X_{arc(C_1\cup R\cup Q_j)} \\ &\quad +D X_{arc(C_1\cup R\cup C_2)}.\end{aligned}$$

To compute A_i, we consider unions of minimal trees in the form $P_i \cup R \cup Q_j$, with i fixed. There are $\binom{K_2}{2}$ double unions of this form, $\binom{K_2}{3}$ triple unions, etc.

So $A_i = (-1)(\binom{K_2}{2}) + (-1)^2(\binom{K_2}{3}) \pm \cdots (-1)^{K_2+1}(\binom{K_2}{K_2}) = -(K_2 - 1)$.

Similarly, $B_j = -(K_1 - 1)$.

To find D, note that $\mu_S^A(C_1 \cup R \cup C_2) = 1$. So by the characteristic vector and the definition of MLE, the coefficients of each term of the MLE must sum to 1. Thus $D = 1 - K_1 - K_2 + K_1 K_2$. □

4. An Application: The Canadian Government and Constitution:

Canada's history is full of disputes over how to amend the constitution. For example, one way to amend the constitution would be to gain the approval of a majority of the ten provinces. Another example is the Victorian amendment which was proposed in the 1970's. The Victorian rule requires a positive vote from (i) both Ontario and Quebec, (ii) at least two of the four Atlantic provinces, and (iii) British Columbia and at least one Prairie province or all three Prairie Provinces. The current rule requires a positive vote from seven out of the ten provinces, including Ontario and/or Quebec. Game theory gives us a way to compare these proposals by assigning an index of power to each of the provinces under each of the different rules. Table 4.2 gives the Shapley value for each province under each of these rules.

Now suppose we want to consider the prospect that neighboring provinces are more likely to agree on a constitutional issue than two non-neighboring provinces. In other words, how would geographical location influence the power of the provinces? We can model this new situation as a graph, where the vertices are the provinces and two vertices are connected if the provinces share a geographical border. Since the graph shown below is the union of one cycle and one tree, we can apply Theorem 3.3 to describe the power of each province under the current rule with geographical location taken into account.

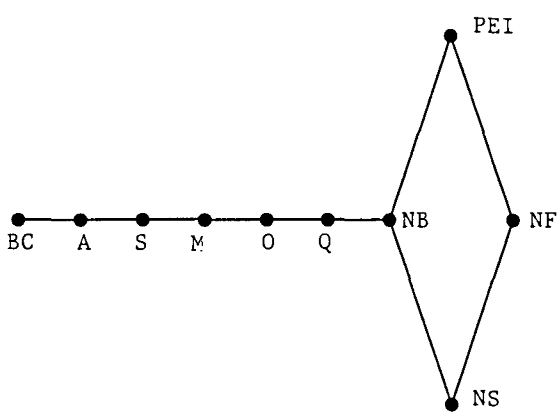


Table 4.1: Coefficients of Terms in the Arc MLE

	∅	f	ef	def	cdef	bcdef	abcdef
∅							1
g						1	-1
j						1	-1
gh					1	-1	0
ji					1	-1	0
ghi				1	-1	0	0
jih				1	-1	0	0
gj					1	-2	1
ghj				1	-2	1	0
gij				1	-2	1	0
ghij				-3	3	0	0

In order to solve the Canadian example, we need the original MLE for the Canadian game. Note that the smallest winning coaltition contains seven players. After determining the terms and coefficients in the original MLE, we grouped all of the terms who had equivalent minimal trees. Notice that if the terms or coalitions share the same minimal tree then they have the same arc MLE. Then we summed the original coefficients of the terms with the same arc MLE, which determined the coefficients of the terms in the arc MLE. Table 4.1 summarizes these coefficients.

To determine the power of each arc, we computed the Shapley values. Then we determined the position value of the players from the Shapley values. The power indices of the players under the rules mentioned above are displayed in Table 4.2.

Table 4.2: Population and Power Indices of Provinces Under Different Rules

		power indices of provinces under different rules			
Province	Population	Simple	Current	Victorian	Current with Graph
Brit. Columbia	0.120	0.10	0.092	0.1250	0.0029
Alberta	0.090	0.10	0.092	0.0417	0.0119
Saskatchewan	0.036	0.10	0.092	0.0417	0.0267
Manitoba	0.040	0.10	0.092	0.0417	0.1547
Ontario	0.369	0.10	0.130	0.3155	0.2738
Quebec	0.253	0.10	0.130	0.3155	0.2738
New Brunswick	0.027	0.10	0.092	0.0298	0.1726
Prince Ed. Isl.	0.025	0.10	0.092	0.0298	0.1904
New Foundland	0.021	0.10	0.092	0.0298	0.0238
Nova Scotia	0.033	0.10	0.092	0.0298	0.1904

Note that the last column indicates that the power indices determined by the position value under the current rule with the graph imposed appear most correlated to the population.

5. The Complete Graph

Why study the complete graph?

Consider the arc game on the complete graph, denoted by K_n. K_n has n players and all $\binom{n}{2}$ possible edges. Notice that this cooperative game does not restrict any communication between the players. Why would we want to study such a game? Recall that the Shapley Value of the arc game returns a power index for the edges in the graph. By computing the Shapley value of the arc game on a complete graph, we can determine the most and the least important lines of communication. Also, the position values for the original players computed from the Shapley values of the arc game on K_n are different from the Shapley value of the players in the original communication game. The result is a new index of power for all coalitional games [1].

Complexity of the Complete Graph

In order to determine the desired power indices we must compute the MLE for the unanimity games on the complete graph. Theorem 3.1 shows that we can write the MLE in terms of the unions of the minimal paths on the graph. When the graph contains several cycles, however, the determination of these minimal paths grows increasingly more difficult.

In the complete graph with n vertices, there are $\binom{n}{2}$ edges; hence there are $2^{\binom{n}{2}}$ coalitions of edges and terms in the MLE whose coefficients must be computed. The number of possible coalitions, or terms in the MLE, grows exponentially, thus for $n \geq 5$ the computations are unreasonable by hand. Dr. Kathy Hoke has designed a *Mathematica* program which carries out these computations (see Appendix A). The output of the program includes the terms in the MLE with non-zero coefficients. We used the program to compute the MLE for the unanimity games, μ_s^A, on the complete graph with $n = 3, 4$, and 5, however the speed of the program prohibits any further computations of the entire MLE for $n \geq 6$.

Symmetric Advantages

When studying the output of the program for $n = 4$ and 5, the symmetry found within the edges on the complete graph provided a natural way to group the non-zero terms in the MLE. We attempted to classify the terms in the MLE using the following definitions.

Definition: Consider the unanimity game μ_s^A. An *in arc* is an arc whose adjacent vertices are both in the special set s. A *between arc* is defined as one which touches only one vertex in s with a vertex not in s. An *out arc* is an arc whose adjacent vertices are not in s.

Each term in the MLE is a combination of *in arcs*, *out arcs*, and/or *between arcs*. The terms which contain only *in edges* clearly had different coefficients than other terms. We detected that the coefficient of the term in the MLE depended on exactly how many *in, between*, and *out edges* were in the set, which is difficult to compute for a reasonable n.

Coefficient of Last Term

The coefficient of the last term of the MLE contains the most interesting pattern. The last term takes complete advantage of the symmetry of the complete graph. Manipulation of the *Mathematica* program led to the successful computation of the last term in the MLE for the unanimity games on the complete graph when $n = 6$. An interesting pattern emerged from these coefficients. This pattern is shown and generalized in the Table 5.1. Note the last column of the Table 5.1 includes our conjectures for the graph with n vertices.

Conjecture 5.1: Given the unanimity game μ_s^A on n players, the coefficient of the last term in the MLE can be computed from the following equation:

$$\Gamma_s(\text{a}ll) = \pm(|s| - 1) * (n - 2)!$$

where s is any special set of edges defined by μ_s^A.

Table 5.1: Coefficients of Last Term in the Arc MLE

	n = 3	n = 4	n = 5	n = 6	n = n
μ^A_{12}	-1	-2	6	24	± (n-2)!
μ^A_{123}	-2	-4	12	48	± 2(n-2)!
μ^A_{1234}	–	–	18	72	± 3(n-2)!
μ^A_{12345}	–	–	–	96	± 4(n-2)!
μ^A_s	–	–	–	–	± (|s|-1)(n-2)!

Another interesting approach to the study of the coefficient of the last term in the arc MLE on the complete graph with n players is to consider the set of winning coalitions that were winning on the graph with $n-1$ players, separately from the set of winning coalitions on the graph that were not winning on the $n-1$ player graph. In other words, let v_n be a vertex not in s and divide the set of winning coalitions on the n player graph into the following two groups:

• (i) all the sets of edges that would still be winning if we removed v_n and all the edges connecting it and

• (ii) all the sets of edges that would not be winning when we remove v_n and all of its adjacent edges.

The following theorem represents an attempt to simplify the computation of the last coefficients in the MLE for the arc unanimity games.

Theorem 5.1: Given the unanimity game μ^A_s on n players, the coefficient of the last term in the arc MLE can be computed by considering only the sets W of edges such that $v_n \in vert(W)$ and if v_n and its edges are removed from W, then W loses.

Proof 5.1: Let W^n_k be the number of winning arc sets of size k on the graph with n vertices. Let $\Gamma_S(A)$ be the coefficient of the last term in the MLE(μ^A_S), where A is the set of all arcs in the graph G. Then,

$$\Gamma_S(A) = \sum_{k=1}^{\binom{n}{2}} (-1)^{\binom{n}{2}-k} W^n_k$$

The set of edges that are winning if v_n and its edges are removed has the form:

(winning set of edges fromK_{n-1})$\cup$(J edges adjacent to v_n) $J = 0, 1, ..., n-1$

The number of these sets can be written as:

$$W_k^{n-1}\binom{n-1}{J}$$

In order to determine the value that these sets contribute to the coefficient of the last term in the MLE we must alternatively sum the number of winning sets of each size of this type. We simplify this computation in the following way:

$$\sum_{k=1}^{\binom{n}{2}}\sum_{j=0}^{n-1}(-1)^{\binom{n}{2}-(k+J)}W_k^{n-1}\binom{n-1}{J}$$

$$= \sum_{k=1}^{\binom{n}{2}}(-1)^{\binom{n}{2}-(k)}W_k^{n-1}\sum_{j=0}^{n-1}(-1)^{-J}\binom{n-1}{J}$$

Notice that the second summation in the expression above is simply the alternating sum of binomial coefficients, which is identically equal to zero. So,

$$= \sum_{k=1}^{\binom{n}{2}}(-1)^{\binom{n}{2}-(k)}W_k^{n-1} * (0)$$

$$= 0$$

Therefore, to compute the coefficient of the last term in the arc MLE, we need only consider the set of arcs who 'need' player n to be winning. □

The following theorem, known as the *minimal spanning tree theorem*, will help to determine which coalititions are winning on n but not winning on $n-1$.

Theorem 5.2: (Minimal Spanning Tree) A graph with n vertices and at least $\binom{n-1}{2+1}$ edges, must be connected.

Hence, we need only compute the number of winning coalititions of size k, such that $1 \leq k \leq \binom{n-1}{2} + 2$, that were not winning on the graph with $n-1$ vertices.

Symmetric Shapley Values

Ultimately, we are interested in the Shapley value of the arcs in the complete graph. Again, the symmetry in the graph led to interesting patterns in the Shapley value. We found that there exist at most three different Shapley values among the players in any μ_s^A on the complete graph. These three power indices for the edges are directly determined by the position of the arc as either *in, out,* or *between.*

Note that on the complete graph for the game, μ_s^A, where $|s| = 2$, there is only one *in arc.* Computing the Shapley value for the *in arc* for small values of n led us to the following conjecture:

Conjecture 5.2: Given μ_s^A, where $|s| = 2$ and i is the *in arc*,

$$\phi_i(\mu_s^A) = \frac{2}{n}$$

6. Further Study

The MLE(μ_S^A) has been simplified in the cases when the graph is a tree, simple cycle, and series of simple cycles. The communication game applies to an infinite number of graphs. One interesting approach, however, would be to determine a practical application, form the corresponding graph, and apply the appropriate communication game. This may lead to interesting generalizations of the communication game on various graphs, as well as more practical applications.

The study of the simplification of the MLE on the complete graph provides another interesting direction for further research. In Section 5, we reduced the number of terms we needed to consider; however, the remaining terms in the MLE proved difficult to determine. Although the complete graph requires exponential computations, its symmetry offers many advantages and may hold the key to future work.

Appendix A

Arc MLE Algorithm:

```
<<DiscreteMath`Combinatorica`
n = 5 (*number of vertices in graph*)
l = 10 (*number of arcs in graph, n choose 2*)
m=ToUnorderedPairs[K[n]]
Do[kk=Binomial[l,k];s=RandomKSubset[m,k];
    Do[coeff=0;
    ss=NextKSubset[m,s];s=ss;
        Do[ii=Binomial[k,i];t=RandomKSubset[ss,i];
            Do[tt=NextKSubset[ss,t];t=tt;
                tt=Append[tt,{4,4}];tt=Append[tt,{5,5}];
                g=FromUnorderedPairs[tt];
                g1=g[[1]];
                g2=g1.g1;g3=g2.g1;g4=g3.g1;
                gg=g1+g2+g3+g4;
                If[gg[[1,2]]gg[[2,3]]gg[[3,4]]!=0,
                   coeff=coeff+(-1)^(k-i)],
                    (*for mu_s, s={1,2,3,4}*)
            {n,ii}],
        {i,k}];
        If[coeff!=0,ss>>>games;coeff>>>games],
    {j,kk}],
{k,l}]
```

■ Coefficient of Last Term Algorithm:

```
<<DiscreteMath`Combinatorica`
n = 4 (*number of vertices in graph*)
l = 6 (*number of arcs in graph, n choose 2*)
ss=ToUnorderedPairs[K[n]];
k = 1;
coeff2=0;
coeff1=0;
s=ss;
Do[ii=Binomial[k,i];t=RandomKSubset[ss,i];
    Do[tt=NextKSubset[ss,t];t=tt;
        tt=Append[tt,{4,4}];
        g=FromUnorderedPairs[tt];
        g1=g[[1]];
        g2=g1.g1;g3=g2.g1;
        gg=g1+g2+g3;
        If[gg[[1,2]]!=0,
            coeff2=coeff2+(-1)^(k-i)],
            (*for mu_s, s={1,2}*)
        If[gg[[1,2]]gg[[2,3]]!=0,
            coeff1=coeff1+(-1)^(k-i)];
            (*for mu_s, s={1,2,3}*)
    {n,ii}],
{i,k}];
Print[ss];Print[coeff1];Print[coeff2];
```

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