5-3-1996

Ideals of the Lipschitz class

Konstantin G. Kulev
University of Richmond

Ideals of the Lipschitz Class

Konstantin G. Kulev
Honors thesis[1]
Department of Mathematics and Computer Science
University of Richmond

May 3, 1996

[1]Under the direction of Prof. William T. Ross

Abstract

In this paper, a classification of the closed ideals of the Little Oh Lipschitz class of functions on the interval [0,1] is provided. The technique used to classify the ideals of the class of continuous functions is modified and applied to the Little Oh Lipschitz class. It is shown that every ideal of these two classes has the form $I = \{f : f|_E = 0\}$ for some closed set $E \subseteq [0,1]$. Furthermore, it is demonstrated that the same technique cannot be successfully applied to the classification of the closed ideals of the Big Oh Lipschitz class.

This paper is part of the requirements for honors in mathematics. The signatures below, by the advisor, a departmental reader, and a representative of the departmental honors committee, demonstrate that Konstantin Kulev has met all the requirements needed to receive honors in mathematics.

(advisor)

(reader)

Kathy Hoke

(honors committee representative)

IDEALS OF THE LIPSCHITZ CLASSES

KONSTANTIN KULEV

ABSTRACT. In this paper, a classification of the closed ideals of the Little Oh Lipschitz class of functions on the interval [0,1] is provided. The technique used to classify the ideals of the class of continuous functions is modified and applied to the Little Oh Lipschitz class. It is shown that every ideal of these two classes has the form $I = \{f : f|_E = 0\}$ for some closed set $E \subseteq [0,1]$. Furthermore, it is demonstrated that the same technique cannot be successfully applied to the classification of the closed ideals of the Big Oh Lipschitz class.

1. INTRODUCTION

In this paper we provide a complete classification of the *closed ideals* of the *Lipschitz classes* of functions. To classify the closed ideals of a certain set means to give a specific form which each closed ideal has; for example, as we will see, each closed ideal I of the set $C[0,1]$ of continuous functions is of the form $I = \{f \in C[0,1] : f|_E = 0\}$ for some closed set $E \subseteq [0,1]$. That is, given any ideal I of $C[0,1]$, there exists a closed set E such that $I = \{f \in C[0,1] : f|_E = 0\}$.

Section 2 introduces some definitions from the areas of real analysis and abstract algebra, while Section 3 provides the necessary definitions and ideas from topology. In Section 4 we consider the Whitney theorem, which provides a complete classification of the closed ideals of the m-times continuously differentiable functions. We prove an "easy" case of the theorem, thus formalizing the above classification of the closed ideals of the continuous functions on the interval $[0,1]$. In Section 5 we classify the ideals of the *Little Oh* Lipschitz class using the method applied in Section 4, with suitable modifications. Throughout the proof the reader should observe the sharp distinction between the *Big Oh* and *Little Oh* Lipschitz classes in order to understand the difficulties that arise when the same proof technique is applied in an attempt to classify the closed ideals of the *Big Oh* Lipschitz class.

2. PRELIMINARIES

First we shall define the classes of functions that we work with throughout this paper. We shall use the conventional notation $C^m[0,1]$ to denote the class of m-times continuously differentiable functions, i.e. the functions whose first m derivatives exist on the open interval $(0,1)$ and are continuous on the closed interval $[0,1]$. A class of slightly "smoother" than the continuous functions, yet not necessarily differentiable functions is the Lipschitz class, which is defined in the following manner:

Definition 2.1. *Given a real number* $0 < \alpha < 1$*, we say that a function* $f : [0,1] \to \mathbb{R}$ *belongs to the* Lipschitz class Λ_α *if there exists a constant* $C > 0$ *such that*

$$\sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha} \leq C. \tag{2.1}$$

The number α is called the *exponent.* We only consider values of α such that $0 < \alpha < 1$, since allowing $\alpha = 1$ creates certain difficulties when we introduce the Little Oh and Big Oh Lipschitz classes. For $\alpha > 1$, a simple application of the Mean Value Theorem shows that Λ_α consists only of the constant functions. For our investigations in this paper we need to introduce a special subset of the Lipschitz class, the Little Oh Lipschitz class. For convenience, we shall call the Lipschitz class Λ_α the *Big Oh Lipschitz class.*

Definition 2.2. *A function* f *belongs to the* Big Oh *Lipschitz class* Λ_α *if* $|f(x) - f(y)| = O(|x - y|^\alpha)$ *for all* $x, y \in [0,1]$*, which is equivalent to (*2.1*) for some constant* C*.*

A function f *belongs to the* Little Oh *Lipschitz class* λ_α *if* $|f(x) - f(y)| = o(|x - y|^\alpha)$ *for all* $x, y \in [0,1]$*, which is equivalent to*

$$\lim_{\delta \to 0} \sup_{0 < |x-y| < \delta} \frac{|f(x) - f(y)|}{|x - y|^\alpha} = 0. \tag{2.2}$$

It is clear that λ_α is contained in, but not equal to Λ_α, as the following example illustrates:

Example: Consider the function $f(x) = x^\alpha$. Clearly $f \in \Lambda_\alpha$, but we can show that $f \notin \lambda_\alpha$. Indeed, if we restrict $y = 0$, we make the supremum in equation (2.2) smaller than (or possibly equal to) the original one, so

$$\lim_{\delta \to 0} \sup_{0 < |x-y| < \delta} \frac{|f(x) - f(y)|}{|x - y|^\alpha} \geq \lim_{\delta \to 0} \sup_{0 < |x-0| < \delta} \frac{|f(x) - f(0)|}{|x - 0|^\alpha}$$

$$= \lim_{\delta \to 0} \sup_{0 < |x| < \delta} \frac{|x^\alpha|}{|x|^\alpha} = 1 \neq 0.$$

Therefore equation (2.2) does not hold and $f \notin \lambda_\alpha$.

Lemma 2.3. *If* $f \in C^1[0,1]$*, then* $f \in \lambda_\alpha$*.*

Proof. Let $f \in C^1[0,1]$, i.e. and let f be a function whose derivative exists and is continuous on $(0,1)$. Then, by the Mean Value Theorem, for each $x, y \in [0,1]$, $|f(x) - f(y)| = M|x - y|$, where $M = \sup_{c \in [0,1]} |f'(c)|$. Therefore

$$\lim_{\delta \to 0} \sup_{0 < |x-y| < \delta} \frac{|f(x) - f(y)|}{|x - y|^\alpha} = \lim_{\delta \to 0} \sup_{0 < |x-y| < \delta} \frac{M|x - y|}{|x - y|^\alpha}$$

$$= \lim_{\delta \to 0} \sup_{0<|x-y|<\delta} M|x-y|^{1-\alpha} = 0,$$

since $\alpha < 1$. Therefore $f \in \lambda_\alpha$ (and certainly $f \in \Lambda_\alpha$). $\square$

Next we provide the definitions of some basic abstract algebra notions which play a pivotal role in this paper. These can be found in any introductory text in abstract algebra (see, e.g., [2]). First we define a *ring*; however, since the full rigorous definition is rather long and will be omitted here, the next definition is more succinct and provides only an explanation of what a *commutative ring* is.

Definition 2.4. *If R is a set on which two binary operations are defined (usually called* addition *and* multiplication*), then R is called a* commutative ring *with respect to these operations if it has the closure property (i.e. $a, b \in R$ implies that $a + b, a \cdot b \in R$); if the associative, commutative and distributive laws hold; if there is an additive identity; and if every element in R has a unique additive inverse.*

Examples: The set of all integers $\mathbb{Z}$ under ordinary addition and multiplication, the set of m-times continuously differentiable functions $C^m[0,1]$, and the Big Oh and Little Oh Lipschitz classes Λ_α and λ_α under function addition and function multiplication are some examples of commutative rings. We shall prove that Λ_α and λ_α are indeed closed under addition and multiplication.

Theorem 2.5. *The Big Oh and Little Oh Lipschitz classes are closed under function addition and multiplication. That is,*

(1) *If $f, g \in \Lambda_\alpha$, then $f + g, f \cdot g \in \Lambda_\alpha$.*
(2) *If $f, g \in \lambda_\alpha$, then $f + g, f \cdot g \in \lambda_\alpha$.*

Proof. (1) We have that

$$\sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha} \leq C_f \quad \text{and} \quad \sup_{x \neq y} \frac{|g(x) - g(y)|}{|x-y|^\alpha} \leq C_g,$$

where C_f and C_g are constants. Then

$$\sup_{x \neq y} \frac{|(f+g)(x) - (f+g)(y)|}{|x-y|^\alpha} = \sup_{x \neq y} \frac{|f(x) - f(y) + g(x) - g(y)|}{|x-y|^\alpha}$$

$$\leq \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha} + \sup_{x \neq y} \frac{|g(x) - g(y)|}{|x-y|^\alpha} \leq C_f + C_g,$$

which is a constant. Therefore $f + g \in \Lambda_\alpha$.

Also

$$\sup_{x \neq y} \frac{|(f \cdot g)(x) - (f \cdot g)(y)|}{|x-y|^\alpha} = \sup_{x \neq y} \frac{|f(x)g(x) - f(y)g(y)|}{|x-y|^\alpha}$$

$$= \sup_{x\neq y} \frac{|f(x)(g(x)-g(y)+g(y))-f(y)g(y)|}{|x-y|^\alpha} = \sup_{x\neq y} \frac{|f(x)(g(x)-g(y))+g(y)(f(x)-f(y))|}{|x-y|^\alpha}$$

$$\leq \sup_{x\neq y} \frac{|f(x)||g(x)-g(y)|}{|x-y|^\alpha} + \sup_{x\neq y} \frac{|g(y)||f(x)-f(y)|}{|x-y|^\alpha}$$

$$\leq \sup_{x\neq y} \frac{M_f|g(x)-g(y)|}{|x-y|^\alpha} + \sup_{x\neq y} \frac{M_g|f(x)-f(y)|}{|x-y|^\alpha} \leq M_f C_g + M_g C_f,$$

which is a constant. (Here we have used the fact that $f(x)$ and $g(y)$ are bounded by the constants M_f and M_g respectively on the interval $[0,1]$, since they both are continuous functions.) Therefore $f \cdot g \in \Lambda_\alpha$.

(2) Analogously, we have that

$$\lim_{\delta\to 0} \sup_{0<|x-y|<\delta} \frac{|f(x)-f(y)|}{|x-y|^\alpha} = 0 \text{ and } \lim_{\delta\to 0} \sup_{0<|x-y|<\delta} \frac{|g(x)-g(y)|}{|x-y|^\alpha} = 0.$$

Then

$$\lim_{\delta\to 0} \sup_{0<|x-y|<\delta} \frac{|(f+g)(x)-(f+g)(y)|}{|x-y|^\alpha} = \lim_{\delta\to 0} \sup_{0<|x-y|<\delta} \frac{|f(x)-f(y)+g(x)-g(y)|}{|x-y|^\alpha}$$

$$\leq \lim_{\delta\to 0} \sup_{0<|x-y|<\delta} \frac{|f(x)-f(y)|}{|x-y|^\alpha} + \lim_{\delta\to 0} \sup_{0<|x-y|<\delta} \frac{|g(x)-g(y)|}{|x-y|^\alpha} = 0+0 = 0.$$

Therefore $f+g \in \lambda_\alpha$.

For $f \cdot g$ we again use the same technique as in (1) to obtain

$$\lim_{\delta\to 0} \sup_{0<|x-y|<\delta} \frac{|(f\cdot g)(x)-(f\cdot g)(y)|}{|x-y|^\alpha} \leq$$

$$\leq \lim_{\delta\to 0} \sup_{0<|x-y|<\delta} \frac{M_f|g(x)-g(y)|}{|x-y|^\alpha} + \lim_{\delta\to 0} \sup_{0<|x-y|<\delta} \frac{M_g|f(x)-f(y)|}{|x-y|^\alpha}$$

$$= M_f \cdot 0 + M_g \cdot 0 = 0.$$

Therefore $f \cdot g \in \lambda_\alpha$. □

The definition of an *ideal* follows next.

Definition 2.6. *Let R be a commutative ring. A non-empty subset I of R is called an* ideal *of R if*

(1) *$a \pm b \in I$ for all $a, b \in I$ and*
(2) *$r \cdot a \in I$ for all $a \in I$ and $r \in R$.*

Examples: Since $\mathbb{Z}$ is a commutative ring, consider its subset $I = \{n \in \mathbb{Z} : n \text{ is even }\}$. Clearly I is an ideal, since it satisfies the above definition; specifically, I is closed under addition and subtraction and for any $r \in \mathbb{Z}$ and $a \in I$, $r \cdot a \in I$.

As another example consider the commutative ring $R = C[0,1]$ under function addition and multiplication and its subset $I = \{f \in C[0,1] : f(0) = 0\}$. Again I is an ideal, since it is closed under addition and subtraction and for any $g \in C[0,1]$ and $f \in I$, $g \cdot f \in I$.

Here we also define a *subring* and give the motivation for investigating only *ideals* in this paper as opposed to *subrings*, which may seem more natural to consider.

Definition 2.7. *Let R be a commutative ring. A non-empty subset S of R is called a* subring *of R if and only if*

(1) *$a + b \in S$ and $a \cdot b \in S$ for all $a, b \in S$ and*
(2) *if $a \in S$, then $-a \in S$ for all $a \in S$.*

Examples: Clearly every ideal is a subring, therefore all the examples above apply here as well. The converse, however, is not true and is easily illustrated by considering the constant functions as a subset of the commutative ring $R = C[0,1]$. That subset is a subring, but not an ideal of $C[0,1]$.

Easy examples can illustrate that the subrings cannot be subject to any classification similar to the one for the ideals, since they are a larger set that contains the ideals. That is why the restriction of considering only the ideals makes the classification possible. In the next section we will have the necessary definitions allowing us to explain why we focus only on *closed* ideals. Before we conclude this section and go on to the topological background for our results, we will introduce two theorems that will be used later. The first one is the well-known *Weierstrass M-test*, which guarantees the uniform convergence of a series of continuous functions that are bounded by constants, whose series converges.

Theorem 2.8. *(Weierstrass M-test): Let g_n be a continuous function on $[0,1]$ for $\forall n \in \mathbb{N}$. If $|g_n(x)| \leq M_n$ for $\forall x$ and $\sum_{n=1}^{\infty} M_n < \infty$, then $\sum_{n=1}^{\infty} g_n$ converges uniformly, and therefore is continuous.*

A proof of the Weierstrass M-test can be found on p. 339 of [1].

The second theorem allows the term by term differentiation of a uniformly convergent series of continuous functions, provided that all terms have continuous derivatives and that the series of derivatives converges uniformly.

Theorem 2.9. *Let $\{f_n\}$ be a sequence of continuously differentiable functions on the interval $[0,1]$ and let $h(x) = \sum_{n=1}^{\infty} f_n(x)$ and $g(x) = \sum_{n=1}^{\infty} f_n'(x)$. If $|f_n(x)| \leq M_n$ and $|f_n'(x)| \leq M_n'$ (M_n and M_n' are constants for all n), where both $\sum_{n=1}^{\infty} M_n$ and $\sum_{n=1}^{\infty} M_n'$ converge, then $h(x)$ is a continuous function on $[0,1]$, differentiable on $(0,1)$ and its derivative is $g(x)$.*

A proof of this theorem can also be found in [1].

3. Topology

In this section we place a topology on Λ_α and λ_α. We introduce the idea of a *metric space* and a *metric* associated with it, which allows us to define notions such as *open* and *closed sets*, *Cauchy sequences* and *completeness* of classes of functions. Thus we can use these concepts in a similar way as they are used with, say, the real numbers.

Definition 3.1. *Given a set S and a function $\rho : S \times S \to \mathbb{R}_+$, (S, ρ) is called a* metric space *with respect to the* metric ρ *if all of the following hold:*

(1) $\rho(x,y) \geq 0$
(2) $\rho(x,y) = 0$ *if and only if* $x = y$
(3) $\rho(x,y) = \rho(y,x)$
(4) $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$

for all $x, y, z \in S$.

Examples: A quite simple example is the set of complex numbers $\mathbb{C}$, which is a metric space with respect to the metric $\rho(x,y) = |x-y|$, since

(1) $|x-y| \geq 0$
(2) $|x-y| = 0$ if and only if $x = y$
(3) $|x-y| = |y-x|$
(4) $|x-z| \leq |x-y| + |y-z|$ by the triangle inequality

for all $x, y, z \in \mathbb{C}$.

Another example, which is not as trivial as the previous one, and is more relevant to the specifics of our paper, is $C^m[0,1]$. To define a suitable metric, we let

$$\|f\| = \sum_{k=0}^{m} \sup_{x\in[0,1]} \left| f^{(k)}(x) \right|$$

for any $f \in C^m[0,1]$. (We have used the symbol $f^{(i)}$ to denote the i^{th} derivative of the function f.) Since $f^{(k)}$ is continuous on $[0,1]$ for $\forall 0 \leq k \leq m$, then it is bounded, so all the suprema above exist. Now we define a metric on $C^m[0,1]$ by $d(f,g) = \|f-g\|$ and we state that explicitly:

Definition 3.2. *Given any two functions $f, g \in C^m[0,1]$, define the metric associated with them to be*

$$d(f,g) = \sum_{k=0}^{m} \sup_{x\in[0,1]} \left| f^{(k)}(x) - g^{(k)}(x) \right|.$$

It is an easy exercise to demonstrate that $d(f,g)$ satisfies all the conditions of Definition 3.1, so it is indeed a metric for $C^m[0,1]$.

We note that in the special case $m = 0$ the metric becomes simply $d(f,g) = \|f-g\| = \sup_{x\in[0,1]}|f(x)-g(x)|$.

It was crucial for $|f(x)-g(x)|$ to be bounded on $[0,1]$ for the supremum in the metric of $C[0,1]$ to exist. Since

$$\frac{|f(x)-f(y)|}{|x-y|^\alpha}$$

is always bounded for Λ_α and λ_α, it is natural to propose a metric in the following manner. First let

$$\|f\| = \sup_{x\neq y}\frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

Then define the metric by

$$\rho(f,g) = \|f-g\| = \sup_{x\neq y}\frac{|(f(x)-f(y))-(g(x)-g(y))|}{|x-y|^\alpha}.$$

However, it is easy to see that $\rho(f,g)$ cannot be a metric, since it does not satisfy condition (2) of Definition 3.1. Indeed, if we let $g\equiv 0$ and $f\equiv 1$, then $f\neq g$ and still $\rho(f,g)=0$, contradicting condition (2). In order to correct for this problem, we define the metric in the following way:

Definition 3.3. *Given any two functions $f,g\in\Lambda_\alpha$ (or λ_α respectively), the metric associated with them is given by*

$$d(f,g) = \sup_{x\in[0,1]}|f(x)-g(x)| + \sup_{x\neq y}\left(\frac{|(f(x)-f(y))-(g(x)-g(y))|}{|x-y|^\alpha}\right).$$

We will often denote $d(f,g)=\|f-g\|_\alpha$.

Using basic properties of suprema, it can be readily shown that $d(f,g)$, as defined above, satisfies the conditions of Definition 3.1 and therefore is a metric on Λ_α (respectively λ_α).

Next we define the notions of *open* and *closed sets.*

Definition 3.4. *Given a metric space (S,ρ) with $x\in S$ and $r\in\mathbb{R}$ fixed such that $r>0$, the set*

$$B(x;r) = \{y\in S : \rho(x,y) < r\}$$

is called the open ball *of radius r around x, while the set*

$$\bar{B}(x;r) = \{y\in S : \rho(x,y)\leq r\}$$

is called the closed ball *of radius r around x.*

Definition 3.5. *Given a metric space (S,ρ), the set $G\subset S$ is called an* open set *if, given any $x\in G$, there exists an $\epsilon_x>0$, such that $B(x;\epsilon_x)\subset G$.*

Definition 3.6. *Given a metric space (S, ρ), the set $F \subset S$ is called a* closed set *if $S \backslash F$ is open.*

Definition 3.7. *Let $A \subset S$, where (S, ρ) is a metric space. Then the* closure *of A, $cl(A)$, is defined by $cl(A) = \bigcap\{F \supset A : F \text{ is closed}\}$.*

The following theorem is well-known and we state it here without proof:

Theorem 3.8. *A set A is closed if and only if $A = cl(A)$.*

Next we consider sequences and completeness.

Definition 3.9. *If $\{x_n\}$ is a sequence in a given metric space (S, ρ), then $\{x_n\}$* converges *to $x \in S$, i.e. $\lim_{n\to\infty} x_n = x$ if for every $\epsilon > 0$, $\exists N \in \mathbb{N}$, such that $\rho(x, x_n) < \epsilon$ whenever $n \geq N$.*

Example: If $S = \mathbb{C}$, then $z = \lim_{n\to\infty} z_n$ means that for every $\epsilon > 0$, $\exists N \in \mathbb{N}$, such that $|z - z_n| < \epsilon$ whenever $n \geq N$.

The concept of a convergent sequence can be used to show that a set is closed in the following way:

Theorem 3.10. *A set $F \subset (S, \rho)$ is closed if and only if, for each sequence $\{x_n\}$ in F with $\lim_{n\to\infty} x_n = x$, we have $x \in F$.*

Since this is a well-known theorem from topology, we omit its proof here. (See [3] for a proof.)

Definition 3.11. *A sequence $\{x_n\}$ in (S, ρ) is called a* Cauchy sequence *if, $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$, such that $\rho(x_n, x_m) < \epsilon$, $\forall n, m \geq N$.*

There is a relation between convergent sequences and Cauchy sequences and it is as follows:

Theorem 3.12. *If a sequence $\{x_n\}$ in a metric space (S, ρ) converges, then it is a Cauchy sequence, i.e. $lim_{n\to\infty} x_n = x$ implies that $\{x_n\}$ is Cauchy.*

Proof. Let $\epsilon > 0$ be given and choose $N > 0 : \rho(x_n, x) < \frac{\epsilon}{2}$, $\forall n \geq N$. Then $\forall m, n > N$,

$$\rho(x_n, x_m) \leq \rho(x_n, x) + \rho(x_m, x) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

which shows the existence of the required $N \in \mathbb{N}$. $\square$

While a Cauchy sequence in a metric space is *not* necessarily convergent, there are some metric spaces in which *all* Cauchy sequences converge. This leads us to the following definition:

Definition 3.13. *A metric space (S, ρ) for which every Cauchy sequence converges is called* complete.

Examples: Simple examples of complete metric spaces include $\mathbb{R}$ and $\mathbb{R}^2$, if the metric is defined as the distance between two points. However, consider $(S,\rho) = (\mathbb{R}\backslash\{0\}, d(x,y) = |x-y|)$. Although it is a metric space, it does not have the completeness property, since the Cauchy sequence $x_n = \frac{1}{n}$ does not converge to a point in $\mathbb{R}\backslash\{0\}$.

The choice of metric is crucial for completeness, as illustrated by the following example:

Example: Let $S = C[0,1]$ and let

$$d_1(f,g) = \int_0^1 |f(x) - g(x)|dx \text{ and } d_2(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|,$$

both of which are metrics on S. It can be shown that (S, d_1) is *not* complete, while (S, d_2) *is* complete. In fact, for any m, $C^m[0,1]$ is well-known to be a complete metric space under the metric d_2.

It is also true that Λ_α and λ_α are complete metric spaces under the metric defined in Definition 3.3. We shall prove this fact, for which we need the following lemmas:

Lemma 3.14. *If $\{f_n\}$ is a Cauchy sequence in Λ_α, then* $\sup_n \|f_n\|_\alpha \le M < +\infty$.

Proof. Since f_n is Cauchy, if we choose $\epsilon = 1$, it follows that $\exists N$ such that for all $m,n \ge N$, $\|f_n - f_m\|_\alpha \le 1$. Then $\forall n \ge N$, $\|f_n\|_\alpha = \|f_n - f_N + f_N\|_\alpha \le \|f_n - f_N\|_\alpha + \|f_N\|_\alpha \le 1 + \|f_N\|_\alpha$.

Now let $C = \max_{1\le n\le N} \|f_n\|_\alpha$. Then let $M = max\{C, 1 + \|f_N\|_\alpha\}$. It follows that $\sup_n \|f_n\|_\alpha \le M$ and the proof is complete. □

Lemma 3.15. *If $\{f_n\}$ is a Cauchy sequence in Λ_α, then there exists a function $f \in \Lambda_\alpha$ such that $f_n(x)$ converges to $f(x)$ pointwise.*

Proof. First we show that such a function does exist and then we demonstrate that it is a Λ_α function. Fix $x_0 \in [0,1]$. Then

$$|f_n(x_0) - f_m(x_0)| \le \sup_{x\in[0,1]} |f_n(x) - f_m(x)| \le \|f_n - f_m\|_\alpha.$$

by Definition 3.3. Since f_n is Cauchy, it follows that $\|f_n - f_m\|_\alpha \to 0$ when $m,n \to +\infty$. Therefore $|f_n(x_0) - f_m(x_0)| \to 0$, and $\{f_n(x_0)\}$ is a Cauchy sequence of real numbers, so there exists a real number $f(x_0)$ to which that sequence converges. So $f_n(x_0) \to f(x_0)$ as $n \to \infty$.

Now we have to show that $f \in \Lambda_\alpha$. Let $x_0, y_0 \in [0,1]$ and $x_0 \neq y_0$. Because of the pointwise convergence that we just demonstrated,

$$\frac{|f(x_0) - f(y_0)|}{|x_0 - y_0|^\alpha} = \lim_{n\to\infty} \left(\frac{|f_n(x_0) - f_n(y_0)|}{|x_0 - y_0|^\alpha} \right) \le \sup_n \left(\frac{|f_n(x_0) - f_n(y_0)|}{|x_0 - y_0|^\alpha} \right)$$

$$\le \sup_n \sup_{x_0 \neq y_0} \left(\frac{|f_n(x_0) - f_n(y_0)|}{|x_0 - y_0|^\alpha} \right) \le \sup_n \|f_n\|_\alpha \le M < +\infty$$

by Definition 3.3 and Lemma 3.14, where M is a constant, independent of x_0 and y_0. Therefore

$$\sup_{x_0 \neq y_0} \Big(\frac{|f(x_0) - f(y_0)|}{|x_0 - y_0|^\alpha} \Big) \leq M < +\infty.$$

Therefore, by Definition 2.2 $f \in \Lambda_\alpha$. □

Theorem 3.16. *The Big Oh Lipschitz class Λ_α is a complete metric space with respect to the metric in Definition* 3.3.

Proof. According to Definition 3.13 we have to show that every Cauchy sequence $\{f_n\}$ in Λ_α converges to a function $f \in \Lambda_\alpha$ under the Lipschitz metric. So let $\{f_n\} \subseteq \Lambda_\alpha$ be a Cauchy sequence, i.e. $\|f_n - f_m\|_\alpha \to 0$. We need to show that $\exists f \in \Lambda_\alpha$ such that $f_n \to f$ in the Λ_α metric, i.e. $\|f_n - f\|_\alpha \to 0$.

Let f be the function from Lemma 3.15 and let $x', x, y \in [0,1]$ be such that $x \neq y$. Then by Lemma 3.15

$$|f_n(x') - f(x')| + \frac{|(f_n(x) - f(x)) - (f_n(y) - f(y))|}{|x-y|^\alpha}$$

$$= \lim_{m\to\infty} \Big(|f_n(x') - f_m(x')| + \frac{|(f_n(x) - f_m(x)) - (f_n(y) - f_m(y))|}{|x-y|^\alpha} \Big) \leq \lim_{m\to\infty} \|f_n - f_m\|,$$

$$\text{so } \|f_n - f\| \leq \lim_{m\to\infty} \|f_n - f_m\|.$$

Since f_n is Cauchy in the Big Oh Lipschitz class Λ_α, then $\|f_n - f_m\|_\alpha \to 0$. Therefore Λ_α is a complete metric space. □

Now it is not difficult to show that λ_α is also a complete metric space.

Theorem 3.17. *The Little Oh Lipschitz class λ_α is a complete metric space with respect to the metric in Definition* 3.3.

Proof. Again we have to show that every Cauchy sequence $\{f_n\}$ in λ_α converges to a function $f \in \lambda_\alpha$ under the Lipschitz metric. So let $\{f_n\} \subseteq \lambda_\alpha$ be a Cauchy sequence. We need to show that $\exists f \in \lambda_\alpha$ such that $f_n \to f$ in the λ_α metric, i.e. $\|f_n - f\|_\alpha \to 0$.

Since $f_n \in \lambda_\alpha$ for $\forall n$, it is certainly true that $f_n \in \Lambda_\alpha$ for $\forall n$. By Theorem 3.16 the given sequence converges to a function $f \in \Lambda_\alpha$. We will prove that $f \in \lambda_\alpha$. Indeed, we have

$$\frac{|f(x) - f(y)|}{|x-y|^\alpha} = \frac{|f(x) - f_n(x) + f_n(x) - f(y) + f_n(y) - f_n(y)|}{|x-y|^\alpha}$$

$$\leq \frac{|(f(x) - f_n(x)) - (f(y) - f_n(y))|}{|x-y|^\alpha} + \frac{|f_n(x) - f_n(y)|}{|x-y|^\alpha}$$

$$\leq \sup_{x\neq y} \frac{|(f(x) - f_n(x)) - (f(y) - f_n(y))|}{|x-y|^\alpha} + \frac{|f_n(x) - f_n(y)|}{|x-y|^\alpha}$$

$$\leq \|f - f_n\|_\alpha + \frac{|f_n(x) - f_n(y)|}{|x-y|^\alpha},$$

$$\text{since } \sup_{x\neq y} \frac{|(f(x)-f_n(x))-(f(y)-f_n(y))|}{|x-y|^\alpha} = (*)$$

is only one part of the metric $\|f-f_n\|_\alpha$. Since $\|f_n - f\|_\alpha \to 0$, given any $\epsilon > 0$, we choose an n large enough, so that

$$\|f_n - f\|_\alpha \leq \epsilon$$

and then

$$(*) \leq \epsilon + \frac{|f_n(x)-f_n(y)|}{|x-y|^\alpha}.$$

Now take the $\lim_{\delta\to 0}\sup_{0<|x-y|<\delta}$ of the last expression. Since $f_n \in \lambda_\alpha$ for $\forall n$, the second part equals zero, so the limit equals ϵ. But ϵ was chosen arbitrarily, therefore that limit equals zero.

Therefore we have

$$\lim_{\delta\to 0}\sup_{0<|x-y|<\delta} \frac{|f(x)-f(y)|}{|x-y|^\alpha} = 0,$$

which means that indeed $f \in \lambda_\alpha$. □

We should emphasize here that the completeness properties of $C[0,1]$, Λ_α, and λ_α are of crucial importance for our investigations in this paper, since they guarantee that the closure of any subset of these three classes of functions belongs to the same class.

4. Ideals of Continuous Functions

In this section we consider the Whitney Theorem (see [4]) for the ideals of the class of m-times continuously differentiable functions $C^m[0,1]$. Since in this paper we are only interested in classifying the ideals of the class of continuous functions, we prove only a specific case of that theorem, namely when $m = 0$.

Before we state the Whitney theorem, we define a certain set $J \in C^m[0,1]$. Again, we use the symbol $f^{(i)}$ to denote the i^{th} derivative of the function f.

Definition 4.1. *Let $E_0 \subseteq E_1 \subseteq \cdots \subseteq E_m$ be closed sets in the interval [0,1]. Define $J(E_0, E_1, \ldots, E_m) \subseteq C^m[0,1]$ to be the following set:*

$$J(E_0, E_1, \ldots, E_m) = \{f \in C^m[0,1] : f^{(i)}|_{E_k} = 0, k = 0,1,2,\ldots,m, \forall i, 0 \leq i \leq k\}.$$

In particular, $J(E) = \{f \in C[0,1] : f|_E = 0\}$.

Remark: We note here that J is a closed set in $C^m[0,1]$. The proof requires the use of the Leibnitz formula

$$(f\cdot g)^n = \sum_{i=0}^{n} f^{(j)} g^{(n-j)} \binom{n}{j},$$

and we will not present it here, since it is rather tedious and since we refer to the Whitney theorem here only to provide the general background for our results.

Theorem 4.2. *(Whitney) Given any closed ideal $I \subseteq C^m[0,1]$, there exist closed sets $E_0 \subseteq E_1 \subseteq \cdots \subseteq E_m$, such that $I = J(E_0, E_1, \ldots, E_m)$, where J is the set in Definition* 4.1.

As we explained above, it is beyond the scope of this paper to present a proof of Whitney's theorem. The only reason we mention it here is its significance as a more general result of the theorem we are going to prove in this section. Before we proceed, however, we explain exactly where the complexity of its proof lies.

Let I be any closed ideal in $C^m[0,1]$. It is sufficient to show that there exist closed sets $E_0 \subseteq E_1 \subseteq \cdots \subseteq E_m$, such that

(1) $I \subseteq J(E_0, E_1, \ldots, E_m)$
(2) $J(E_0, E_1, \ldots, E_m) \subseteq I$

Let

$$E_0 = \{x : f(x) = 0, \forall f \in I\},$$

$$E_1 = \{x : f(x) = 0, f'(x) = 0, \forall f \in I\},$$

and in general

$$E_k = \{x : f^{(i)}(x) = 0, 0 \leq i \leq k, \forall f \in I\},$$

for all k such that $0 \leq k \leq m$.

Part (1) is not difficult to show. First we note that, since E_k is closed for all k such that $0 \leq k \leq m$. Indeed, for any such k, E_k is the intersection of the following sets $A_0, A_1, \ldots, A_k$: $A_0 = \bigcap_{f \in I} f^{-1}(\{0\})$, $A_1 = \bigcap_{f \in I}(f')^{-1}(\{0\})$, $A_2 = \bigcap_{f \in I}(f'')^{-1}(\{0\})$, $\ldots$, $A_k = \bigcap_{f \in I}(f^{(k)})^{-1}(\{0\})$, which, in turn, are intersections of closed sets, since $f \in C^m$, so $(f^{(k)})^{-1}(\{0\})$ is closed for all f.

Now if $f \in I$, it follows that $f|_{E_0} = 0$, $f|_{E_1} = 0$ and $f'|_{E_1} = 0$, etc., so by Definition 4.1, $f \in J(E_0, E_1, \ldots, E_m)$. Therefore $I \subseteq J(E_0, E_1, \ldots, E_m)$. Finally, it is obvious that $E_0 \subseteq E_1 \subseteq \cdots \subseteq E_m$, so part (1) follows.

It is the proof of part (2) that is much more involved in this general case and is out of the scope of our investigations. That is why we only consider the special case $m = 0$ of the Whitney theorem, for which the proof of part (2) is not as complicated.

We state the special case $m = 0$ separately to emphasize that it is the main theorem of this section.

Theorem 4.3. *Given any closed ideal $I \subseteq C[0,1]$, there exists a closed set E such that $I = J(E)$, where $J(E)$ is the set in Definition* 4.1.

We will show that given any closed ideal $I \subseteq C[0,1]$, the set

$$E = \{x : f(x) = 0, \forall f \in I\}$$

is such that $I = J(E)$.

Before we present the proof of Theorem 4.3, we shall prove several lemmas, for which we need the following important definitions:

Definition 4.4. *The* support *of a function* $f : \mathbb{R} \to \mathbb{R}$ *is defined to be the closure of the set, on which* f *is non-zero, i.e.* $supp(f) = cl(\{x : f(x) \neq 0\})$.

Example: For instance, if $f(x) = x^n$ for $n = 1, 2, ...$ on $[0,1]$, then $supp(f) = cl((0;1]) = [0;1]$.

Definition 4.5. *Given a set* $A \subseteq [0,1]$, *we define the* distance function $v : [0,1] \to \mathbb{R}$ *to be* $v(x) = dist(x, A) = \inf_a\{|x-a| : a \in A\}$ *for every* $x \in [0,1]$.

Lemma 4.6. *Let* A *be a subset of* $[0,1]$, *and let* $v(x) = dist(x, A)$. *Then* $v(x)$ *is nonnegative and continuous on* $[0,1]$.

Proof. It is clear that $v(x) \geq 0$, since for any $x \in [0,1]$, $v(x)$ is the greatest lower bound of a set of nonnegative numbers $|x-a|$.

Since A is closed, $[0,1]\backslash A$ must be open, so it can be written as countable disjoint union of open intervals (see Theorem 10.1.9 on p.350 of [1]), i.e. $[0,1]\backslash A = \cup_{n=1}^{\infty}(a_n, b_n)$. To establish the continuity of $v(x)$, for every open interval (a_n, b_n) in $[0,1]\backslash A$, define the function $v_n(x) = dist(x, [0,1]\backslash(a_n, b_n))$. Clearly

$$v_n(x) = \begin{cases} 0 & \text{if } x \in [0, a_n] \cup [b_n, 1], \\ x - a_n & \text{if } x \in (a_n, (a_n + b_n)/2), \\ b_n - x & \text{if } x \in ((a_n + b_n)/2, b_n). \end{cases} \tag{4.1}$$

Therefore $v_n(x)$ is continuous, since it is continuous at the three joining points and its four segments are straight lines.

Now $v(x) = \sum_{n=1}^{\infty} v_n(x)$. We have that $v_n(x) \in C[0,1]$ and $|v_n(x)| \leq (b_n - a_n)/2 \; \forall x$ (since $v_n(x)$ has its maximum at the midpoint of $[a_n, b_n]$ and $v_n((a_n + b_n)/2) = (b_n - a_n)/2$). Since

$$\sum_{n=1}^{\infty} \frac{b_n - a_n}{2} \leq \frac{1}{2} L([0,1]) = \frac{1}{2},$$

where $L([0,1]) = 1$ denotes the length of the interval $[0,1]$, all the conditions of the Weierstrass M-Test (Theorem 2.8) are satisfied, and therefore $v(x)$ is continuous. □

Now we define a subset $J_0(E)$ of the set $J(E)$ by

$$J_0(E) = \{g \in C[0,1] : g = 0 \textit{ on an open neighborhood of } E\}. \tag{4.2}$$

Indeed, it is clear that $J_0(E) \subseteq J(E)$.

Lemma 4.7. *Let $J_0(E)$ be the set defined in equation (4.2) and let I be the closed ideal given in Theorem 4.3. Then $cl(J_0(E)) \subseteq I$, where the closure is taken with respect to the metric on the class of continuous functions from Definition 3.2.*

Proof. Let $g \in J_0(E)$, i.e. $g \in C[0,1]$ and $g \equiv 0$ on an open neighborhood of E. If we show that $g \in I$, we will have $J_0(E) \subseteq I$, but since I is closed, by Theorem 3.8, we must have $cl(J_0(E)) \subseteq I$.

We now show that $g \in I$. Notice that $supp(g) \cap E = \emptyset$. To convince ourselves of the validity of the last statement, assume that U is the open neighborhood of E, on which $g \equiv 0$ and that there is a point $x_0 \in supp(g) \cap E$. Certainly $x_0 \in U$ (since $x_0 \in E$), so by Definition 3.5 there exists an open ball $B(x_0, r) \subset U$. But then $g \equiv 0$ on $B(x_0, r)$ and therefore $x_0 \notin supp(g)$. So indeed $supp(g) \cap E = \emptyset$.

Next we will construct a function $w \in I$ such that $w \equiv 1$ on $supp(g)$. Then we will have $g = gw \in I$, since I is an ideal of $C[0,1]$.

Since $supp(g) \cap E = \emptyset$, given any $x \in supp(g)$, $x \notin E$, so there exists a function $h \in I$ such that $h(x) \neq 0$. Since h is continuous, $h(x) \neq 0$ on an open neighborhood $G_{x,h}$ of x. Then the set $\{G_{x,h} : x \in supp(g)\}$ is an open cover for $supp(g)$. But $supp(g)$ is compact (since it is closed and bounded), therefore there exists a finite subcover

$$\mathcal{G} = G_{x_1,h_1} \cup G_{x_2,h_2} \cup \cdots \cup G_{x_n,h_n}$$

containing $supp(g)$. The functions $h_1, h_2, \ldots, h_n \in I$ and they do not vanish simultaneously at any point of $supp(g)$. (If they did vanish at some $x_0 \in supp(g)$, then $x_0 \notin \mathcal{G}$, so $\mathcal{G}$ would not be a subcover for $supp(g)$.) Notice that $h_i \in I$ and, since I is an ideal, $h_i^2 \in I$ for all i such that $0 \leq i \leq n$ and therefore, the continuous function $u = h_1^2 + h_2^2 + \cdots + h_n^2 \in I$. Therefore $u \in I$ and $u \neq 0$ for all $x \in supp(g)$, so note that if $u(x) = 0$ then $x \notin supp(g)$, since then the functions $h_1, h_2, \ldots, h_n$ vanish simultaneously at x.

Now we will define a non-negative function $v \in C[0,1]$ such that

(1) $v \equiv 0$ on $supp(g)$ and
(2) $v(x) \neq 0$ whenever $u(x) = 0$.

Let $v(x) = dist(x, supp(g))$. By Lemma 4.6, $v(x) \geq 0, \forall x$ and is continuous. In addition, for any $x \in supp(g)$, we have that $dist(x, supp(g)) = 0$, so $v(x) = 0$ on $supp(g)$. Finally, as noted above, $u(x) = 0$ implies $x \notin supp(g)$, so $v(x) \neq 0$. Thus the function $v(x)$ satisfies all conditions that we wanted imposed on it.

Now it is easy to see that the function $1/(u+v) \in C[0,1]$, since $u + v \neq 0$ (u and v are never equal to zero simultaneously). Therefore, since $u \in I$, we have $u/(u+v) \in I$. Now let $w = u/(u+v)$. If $x \in supp(g)$, then $v(x) = 0$ and $u(x) \neq 0$, so $w(x) = 1$. Therefore, $g = wg \in I$ since I is an ideal, and this completes the proof of the lemma. □

Lemma 4.8. *Let $f \in C[0,1]$. For all $x \in \mathbb{R}$, define $f_+(x) = max[f(x),0]$ and $f_-(x) = min[f(x),0]$. Then $f = f_+ + f_-$ and both f_+ and f_- are continuous.*

Proof. First we notice that for any given x:
If $f(x) < 0$, then $f_+(x) = max[f(x),0] = 0$ and $f_-(x) = min[f(x),0] = f(x)$.
If $f(x) > 0$, then $f_+(x) = max[f(x),0] = f(x)$ and $f_-(x) = min[f(x),0] = 0$.
If $f(x) = 0$, then $f_+(x) = max[f(x),0] = 0$ and $f_-(x) = min[f(x),0] = 0$.
In any case, $f = f_+ + f_-$.

To show that the two functions are continuous, we notice that

$$f_+(x) = \frac{f(x) + |f(x)|}{2}, \text{ while } f_-(x) = \frac{f(x) - |f(x)|}{2}. \tag{4.3}$$

Since $|f|$ is continuous whenever f is continuous (this follows from a simple application of the backward triangle inequality), we have that both f_+ and f_- are continuous, since $f \in C[0,1]$. □

Given a function $f \in C[0,1]$, we define the following sequence of continuous functions $\{F_n\}$, for $n \in \mathbb{N}$:

$$F_n = (f_+ - \frac{1}{n})_+ + (f_- + \frac{1}{n})_-.$$

Lemma 4.9. *If $f \in J(E)$, then $F_n \in J_0(E), \forall n \in \mathbb{N}$.*

Proof. By several applications of Lemma 4.8, it follows that F_n is continuous. Now for every $n \in \mathbb{N}$ define the open set $E_n = \{x : -\frac{1}{2n} < f(x) < \frac{1}{2n}\}$. Clearly $E \subseteq E_n$, since $f \in J(E)$. We claim that $F_n = 0$ on E_n. Indeed, for any $x \in E_n$ we have
$f_+(x) < \frac{1}{2n} \implies f_+(x) - \frac{1}{n} < 0$, and
$f_-(x) > -\frac{1}{2n} \implies f_-(x) + \frac{1}{n} > 0$, so
$F_n(x) = (f_+(x) - \frac{1}{n})_+ + (f_-(x) + \frac{1}{n})_- = 0 + 0 = 0$.

Therefore $F_n = 0$ on E_n, which means that for every n, $F_n = 0$ on a neighborhood E_n of E, so $F_n \in J_0(E)$. □

Lemma 4.10. *If $\{F_n\}$ is the sequence defined above, then $\lim_{n\to\infty} F_n = f$ in $C[0,1]$.*

Proof. We will show that $(f_+ - \frac{1}{n})_+ \to f_+$ and $(f_- + \frac{1}{n})_- \to f_-$ in the $C[0,1]$ metric as $n \to \infty$. Then, by the triangle inequality it will follow that $F_n \to f_+ + f_- = f$ as $n \to \infty$. We will only prove the first part, i.e. that $(f_+ - \frac{1}{n})_+ \to f_+$, since the second part can be shown in an analogous way. Since $f_+ \geq 0$, assume that $f \geq 0$ to show $(f - \frac{1}{n})_+ \to f$ in $C[0,1]$. We have

$$(f - \frac{1}{n})_+(x) = \begin{cases} f(x) - \frac{1}{n} & \text{if } f(x) - \frac{1}{n} > 0, \\ 0 & \text{if } f(x) - \frac{1}{n} \leq 0. \end{cases}$$

Case 1: If x is such that $f(x) - \frac{1}{n} > 0$, then $|(f - \frac{1}{n})_+(x) - f(x)| = |f(x) - \frac{1}{n} - f(x)| = \frac{1}{n}$.

Case 2: If x is such that $f(x) - \frac{1}{n} \leq 0$, then $|(f - \frac{1}{n})_+(x) - f(x)| = |0 - f(x)| = |f(x)| = f(x) \leq \frac{1}{n}$.

In either case $|(f - \frac{1}{n})_+(x) - f(x)| \leq \frac{1}{n}$, therefore as $n \to \infty$, $(f - \frac{1}{n})_+ \to f$ in the metric of $C[0,1]$. $\square$

Now we are able to present the proof of our main theorem in this section:

Proof of Theorem 4.3. Analogously to Theorem 4.2, let I be any closed ideal in $C[0,1]$. It is sufficient to show that there exists a closed set E, such that

(1) $I \subseteq J(E)$, and
(2) $J(E) \subseteq I$

Let $E = \{x : f(x) = 0, \forall f \in I\}$.

Part (1) is again quite obvious. $E = \{x : f(x) = 0, \forall f \in I\} = \bigcap_{f \in I} f^{-1}(\{0\})$, so E is the intersection of closed sets, therefore it must be a closed set itself. Furthermore, by Definition 4.1, $J(E) = \{f \in C[0,1] : f|_E = 0\}$, so for any $f \in I$, $f|_E = 0$ and therefore $f \in J(E)$. Thus $f \in I$ implies $f \in J(E)$, or $I \subseteq J(E)$.

Now it remains to show part (2), namely that $J(E) \subseteq I$. We will proceed as follows. Consider the set $J_0(E) = \{g \in C[0,1] : g = 0$ on an open neighborhood of $E\}$. By Lemma 4.7 it follows that

$$cl(J_0(E)) \subseteq I, \tag{4.4}$$

where the closure is taken with respect to the norm on the class of continuous functions. Now we will show that

$$J(E) \subseteq cl(J_0(E)), \tag{4.5}$$

and then from (4.4) and (4.5) above it will follow that $J(E) \subseteq I$, which is exactly what we need to prove. So we proceed to show that indeed $J(E) \subseteq cl(J_0(E))$.

Let $f \in J(E)$ and consider the sequence of continuous functions $\{F_n\}$, defined by

$$F_n = (f_+ - \frac{1}{n})_+ + (f_- + \frac{1}{n})_- \text{ for } n \in \mathbb{N}.$$

By Lemma 4.9 $F_n \in J_0(E), \forall n \in \mathbb{N}$ and by Lemma 4.10 $F_n \to f$ in $C[0,1]$. Therefore $f \in cl(J_0(E))$, so $J(E) \subseteq cl(J_0(E))$. Therefore $J(E) \subseteq I$, and the proof is complete. $\square$

5. Ideals of the Lipschitz Classes

In this section we show that the classification of Section 4 applies to the functions of the Little Oh Lipschitz class λ_α as well. We state the analog of Theorem 4.3 for λ_α:

Theorem 5.1. *Given any closed ideal I in the* Little Oh *Lipschitz class λ_α, there exists a closed set E, such that $I = J(E)$, where $J(E) = \{f \in \lambda_\alpha : f|_E = 0\}$.*

Rather than writing out the entire proof of Theorem 5.1, we shall focus our attention on those specific parts of the argument of Section 4 that do not follow directly if $C[0,1]$ is substituted by λ_α, while only referring to Section 4 whenever the argument there applies to the Little Oh Lipschitz class in a straightforward manner.

Proof of Theorem 5.1. The first difficulty with the method applied in the previous section is that the *distance function* $v(x)$, in Definition 4.5, does not seem to belong to the Little Oh Lipschitz class, so it cannot be used in the same way as before. Therefore, in the proof of the following lemma we define a new function v and show that $v \in \lambda_\alpha$.

Lemma 5.2. *Given a closed set $A \subseteq [0,1]$, there exists a function $v \in \lambda_\alpha$, such that $v^{-1}(\{0\}) = A$, i.e $v = 0$ only on A and nowhere else.*

Proof. First consider the function $f : \mathbb{R} \to \mathbb{R}$, defined by

$$f(x) = \begin{cases} e^{-\frac{1}{(x-a)^2} - \frac{1}{(x-b)^2}} & \text{for } x \in [0,1] \backslash A, \\ 0 & \text{otherwise}, \end{cases} \tag{5.1}$$

where $a < x < b$. The maximum of f is achieved at the midpoint of the interval $x = (a+b)/2$ and

$$f\left(\frac{b+a}{2}\right) = e^{-\frac{8}{(b-a)^2}} \implies f(x) \le e^{-\frac{8}{(b-a)^2}}, \forall x.$$

We now define a sequence of functions $\{f_n(x)\}$ by

$$f_n(x) = e^{-\frac{1}{(x-a_n)^2} - \frac{1}{(x-b_n)^2}},$$

where, as before, $\cup_{n=1}^{\infty}(a_n, b_n) = [0,1]\backslash A$, so $0 \le a_n, b_n \le 1$ and $a_n < b_n$ for all n. An easy computation using L'Hopital's Rule shows that $f_n \in C^\infty$.

We showed above that $f_n(x)$ is bounded by $e^{-\frac{8}{(b_n-a_n)^2}}$, which is a constant for all n, so we have

$$|f_n(x)| = \left| e^{-\frac{1}{(x-a_n)^2} - \frac{1}{(x-b_n)^2}} \right| \le \left| e^{-\frac{8}{(b_n-a_n)^2}} \right| = M_n, \forall x.$$

First, we will use the Weierstrass M-test to show that the series $\sum_{n=1}^{\infty} f_n(x)$ is a continuous function on [0,1], so we need to show that $\sum_{n=1}^{\infty} M_n$ is finite. Indeed, since $e^x > x$ for all $x > 1$ it follows that $e^{-x} < 1/x$ and therefore

$$e^{-\frac{8}{(b_n-a_n)^2}} < \frac{(b_n - a_n)^2}{8} \le \frac{b_n - a_n}{8},$$

since $b_n - a_n \leq 1$. Therefore for all n

$$M_n = e^{-\frac{8}{(b_n-a_n)^2}} < \frac{b_n - a_n}{8} \implies \sum_{n=1}^{\infty} M_n < \frac{1}{8}\sum_{n=1}^{\infty}(b_n - a_n) \leq \frac{1}{8},$$

since $\sum_n^{\infty}(b_n - a_n) \leq 1$. So the sum $\sum_{n=1} M_n$ is bounded and therefore by the Weierstrass M-test (Theorem 2.8) the series $\sum_{n=1}^{\infty} f_n(x)$ is a continuous function.

Now we compute the derivative of $f_n(x)$ to get

$$|f_n'(x)| = \left(\frac{2}{(x-a_n)^3} + \frac{2}{(x-b_n)^3}\right) e^{-\frac{1}{(x-a)^2} - \frac{1}{(x-b)^2}}.$$

Let $B_n = max\{n^2, \sup_{x\in(a_n,b_n)} |f_n'(x)|\}$ and let $M_n' = 1/B_n^2$. Then $\forall x \in \mathbb{R}$,

$$|M_n' f_n'(x)| \leq M_n' \| f_n'(x)\| \leq M_n' B_n = \frac{1}{B_n^2} B_n = \frac{1}{B_n} \leq \frac{1}{n^2}.$$

Therefore $\sum_{n=1}^{\infty} M_n' f_n'(x)$ converges uniformly.

Define the function $v(x) = \sum_{n=1}^{\infty} M_n' f_n(x)$. We have that

$$|M_n' f_n(x)| \leq M_n' e^{-\frac{8}{(b_n-a_n)^2}} \leq \frac{1}{n^4} e^{-\frac{8}{(b_n-a_n)^2}} \leq \frac{1}{n^4},$$

and $\sum_{n=1}^{\infty} \frac{1}{n^4}$ converges. Therefore, by Theorem 2.9, v is differentiable on the interval (0,1). But by Lemma 2.3 it follows that $v \in \lambda_\alpha$.

It is also clear that v is identically equal to zero only on A and nowhere else, therefore it satisfies all the conditions imposed on it. □

Now, using the function $v(x)$ from Lemma 5.2, we can prove the analog of Lemma 4.7. We present the proof of the lemma in its entirety, in spite of the fact that it is completely analogous to the one in the previous section.

Lemma 5.3. *Let $J_0(E) = \{g \in \lambda_\alpha : g = 0$ on an open neighborhood of $E\}$. Then $cl(J_0(E)) \subseteq I$, where the closure is taken with respect to the norm on the Little Oh Lipschitz class.*

Proof. Again, it is clear that $J_0(E) \subseteq J(E)$. Let $g \in J_0(E)$, i.e. $g \in \lambda_\alpha$ and $g \equiv 0$ on an open neighborhood of E. If we show that $g \in I$, we will have $J_0(E) \subseteq I$, but since I is closed, we must have $cl(J_0(E)) \subseteq I$.

We now show that $g \in I$. Again, notice that $supp(g) \cap E = \emptyset$. We will construct a function $w \in I$ such that $w \equiv 1$ on $supp(g)$. Then we will have $g = gw \in I$, since I is an ideal of λ_α.

Since $supp(g) \cap E = \emptyset$, given any $x \in supp(g)$, $x \notin E$, so there exists a function $h \in I$ such that $h(x) \neq 0$. Since $h \in \lambda_\alpha$ and therefore is continuous, $h(x) \neq 0$ on a neighborhood $G_{x,h}$ of x. Then the set $\{G_{x,h} : x \in supp(g)\}$ is an open cover for $supp(g)$. But $supp(g)$ is compact (since it is closed and bounded), therefore there exists a finite subcover

$$\mathcal{G} = G_{x_1,h_1} \cup G_{x_2,h_2} \cup \cdots \cup G_{x_n,h_n}$$

containing $supp(g)$. The functions $h_1, h_2, \ldots, h_n \in I$ and they do not vanish simultaneously at any point of $supp(g)$. (If they did vanish at some $x_0 \in supp(g)$, then $x_0 \notin \mathcal{G}$, so $\mathcal{G}$ would not be a subcover for $supp(g)$.) Notice that $h_i \in I$ and, since I is an ideal, $h_i^2 \in I$ for all i such that $1 \leq i \leq n$ and therefore, $u = h_1^2 + h_2^2 + \cdots + h_n^2$ is a Little Oh Lipschitz function that belongs to I. Therefore $u \in I$ and $u \neq 0$ for all $x \in supp(g)$, so note that if $u(x) = 0$ then $x \notin supp(g)$, since then the functions $h_1, h_2, \ldots, h_n$ vanish simultaneously at x.

Now we will define a non-negative function $v \in \lambda_\alpha$ such that

(1) $v \equiv 0$ on $supp(g)$ and
(2) $v(x) \neq 0$ whenever $u(x) = 0$.

Let v be the function in Lemma 5.2, and let $A = supp(g)$. By Lemma 5.2, $v(x) \geq 0, \forall x$ and $v(x) \in \lambda_\alpha$. In addition, for any $x \in supp(g)$, $v(x) = 0$. Finally, as noted above, $u(x) = 0$ implies $x \notin supp(g)$, so $v(x) \neq 0$. Thus the function $v(x)$ satisfies all conditions that we wanted imposed on it.

Now it is easy to see that the function $1/(u+v) \in \lambda_\alpha$, since $u + v \neq 0$ (u and v are never equal to zero simultaneously). Therefore, since $u \in I$, we have $u/(u+v) \in I$. Now let $w = u/(u+v)$. If $x \in supp(g)$, then $v(x) = 0$ and $u(x) \neq 0$, so $w(x) = 1$. Therefore $g = wg \in I$ since I is an ideal, and this completes the proof of the lemma. □

The next step is the analog of Lemma 4.8, the proof of which is the same as in the previous section. We only need to use the following fact:

Lemma 5.4. *If $f \in \lambda_\alpha$, then $|f| \in \lambda_\alpha$.*

Proof. From the backward triangle inequality $\big|\, |f| - |g| \,\big| \leq |f - g|$ and from the fact that $f \in \lambda_\alpha$, we have

$$\frac{\big|\, |f|(x) - |f|(y) \,\big|}{|x-y|^\alpha} \leq \frac{|f(x) - f(y)|}{|x-y|^\alpha},$$

therefore

$$\lim_{\delta \to 0} \sup_{0<|x-y|<\delta} \frac{\big|\, |f|(x) - |f|(y) \,\big|}{|x-y|^\alpha} \leq \lim_{\delta \to 0} \sup_{0<|x-y|<\delta} \frac{|f(x) - f(y)|}{|x-y|^\alpha} = 0.$$

Since the first limit above cannot be negative, it follows that it equals zero. Therefore $|f| \in \lambda_\alpha$. □

Now in a similar fashion as before, given a function $f \in \lambda_\alpha$, we define the following sequence of Little Oh Lipschitz functions $\{F_n\}$, for $n \in \mathbb{N}$:

$$F_n = (f_+ - \frac{1}{n})_+ + (f_- + \frac{1}{n})_-.$$

Note that by equation (4.3) and Lemma 5.4, $F_n \in \lambda_\alpha, \forall n$.

Lemma 5.5. *If $f \in J(E)$, then $F_n \in J_0(E), \forall n \in \mathbb{N}$.*

The proof is strictly analogous to the proof of Lemma 4.9, since f is still a continuous function in this case.

The proof of the following lemma, however, is not analogous to the corresponding proof in Section 4. Before we present the new proof, we need to establish the following fact:

Lemma 5.6. *Let $\{a_n\}$ be a sequence of numbers with the following property: Given any subsequence $\{a_{n_k}\}$ of $\{a_n\}$, there exists a further subsequence $\{a_{n_{k_m}}\}$, which converges to zero. Then $a_n \to 0$.*

Proof. Suppose that a sequence $\{a_n\}$ has the given property, but does not converge to zero. Then there exists a subsequence $\{|a_{n_k}|\}$ bounded away from zero. But by hypothesis, there exists a further subsequence $\{a_{n_{k_m}}\}$ that converges to zero and this is a contradiction. □

Remark: Note that if a function $g \in \lambda_\alpha$, then $|g(x)-g(y)| = o(|x-y|^\alpha)$ and so the function

$$H(x,y) = \begin{cases} \frac{|g(x)-g(y)|}{|x-y|^\alpha} & \text{if } x \neq y, \\ 0 & \text{if } x = y \end{cases}$$

is continuous on $[0,1]$. Note that this is not true when $g \in \Lambda_\alpha$.

Now we are ready to prove the analogous lemma:

Lemma 5.7. *If $\{F_n\}$ is the sequence defined above, then $\lim_{n\to\infty} F_n = f$ in λ_α.*

Proof. Similarly to the parallel proof of the previous section, we will show that $(f_+ - \frac{1}{n})_+ \to f_+$ and $(f_- + \frac{1}{n})_- \to f_-$ as $n \to \infty$. Then it will follow that $F_n \to f_+ + f_- = f$ as $n \to \infty$. Again, we will only prove the first part, i.e. that $(f_+ - \frac{1}{n})_+ \to f_+$, since the second part can be shown in an analogous way. Since $f_+ \geq 0$, assume that $f \geq 0$ to show $f_n = (f - \frac{1}{n})_+ \to f$ in λ_α.

First we note that $f_n = (f - \frac{1}{n})_+ = max[0, f - \frac{1}{n}] = \frac{1}{2}(f(x) - \frac{1}{n} + |f(x) - \frac{1}{n}|)$, so $(f - \frac{1}{n})_+ \in \lambda_\alpha$.

To show that $f_n \to f$ in the λ_α metric, we recall that

$$\|f_n - f\|_{\lambda_\alpha} = \sup_{x\in[0,1]} |f_n(x) - f(x)| + \sup_{x\neq y} \frac{|(f_n(x) - f(x)) - (f_n(y) - f(y))|}{|x-y|^\alpha}.$$

We have already demonstrated that the first part of the above metric $\sup_{x\in[0,1]} |f_n(x) - f(x)|$ converges to zero as $n \to \infty$ (see Lemma 4.10). It remains to show that the second part, namely

$$\sup_{x\neq y} \frac{|(f_n(x) - f(x)) - (f_n(y) - f(y))|}{|x-y|^\alpha} = \sup_{x\neq y}(*)$$

also converges to zero as $n \to \infty$.

This time we consider four different cases for (x, y). So let

$$A(n) = \{(x,y) : f(x) - \frac{1}{n} \geq 0, f(y) - \frac{1}{n} \geq 0\},$$

$$B(n) = \{(x,y) : f(x) - \frac{1}{n} \leq 0, f(y) - \frac{1}{n} \leq 0\},$$

$$C(n) = \{(x,y) : f(x) - \frac{1}{n} \geq 0, f(y) - \frac{1}{n} \leq 0\},$$

$$D(n) = \{(x,y) : f(x) - \frac{1}{n} \leq 0, f(y) - \frac{1}{n} \geq 0\}.$$

Since

$$\sup_{x \neq y}(*) = \sup_{\substack{x \neq y \\ A \cup B \cup C \cup D}} (*),$$

it suffices to show that each of the four suprema converges to zero as $n \to \infty$. Since there is a symmetry between C and D, we only need to consider three cases, for A, B and C only:

Case 1: On $A(n)$, x and y are such that $f(x) - \frac{1}{n} \geq 0$ and $f(y) - \frac{1}{n} \geq 0$, so $f_n(x) = f(x) - \frac{1}{n}$ and $f_n(y) = f(y) - \frac{1}{n}$. Then it follows that

$$\sup_{A(n)} \frac{|(f_n(x) - f(x)) - (f_n(y) - f(y))|}{|x-y|^\alpha} = \frac{|(f(x) - \frac{1}{n} - f(x)) - (f(y) - \frac{1}{n} - f(y))|}{|x-y|^\alpha} = 0.$$

Case 2: On $B(n)$, x and y are such that $f(x) - \frac{1}{n} \leq 0$ and $f(y) - \frac{1}{n} \leq 0$, so $f_n(x) = f_n(y) = 0$. Then by the remark preceding this lemma it follows that

$$\sup_{B(n)} \frac{|(f_n(x) - f(x)) - (f_n(y) - f(y))|}{|x-y|^\alpha} = \sup_{B(n)} \frac{|f(x) - f(y)|}{|x-y|^\alpha} = \frac{|f(x_n) - f(y_n)|}{|x_n - y_n|^\alpha} = (**),$$

for some $(x_n, y_n) \in B(n)$, since $B(n)$ is compact and $f \in \lambda_\alpha$. Given any subsequence $\{x_{n_k} - y_{n_k}\}$, either $|x_{n_k} - y_{n_k}| \to 0$, in which case $(**) \to 0$, or there exists a further subsequence $|x_{n_{k_m}} - y_{n_{k_m}}| \geq \delta > 0$, i.e. bounded away from zero. But then $(**)$ still goes to zero, since

$$|f(x_{n_{k_m}})| \leq \frac{1}{n_{k_m}} \text{ and } |f(y_{n_{k_m}})| \leq \frac{1}{n_{k_m}}.$$

So by Lemma 5.7 $\sup_{B(n)}(**) \to 0$.

Case 3: On $C(n)$, x and y are such that $f(x) - \frac{1}{n} \geq 0$ and $f(y) - \frac{1}{n} \leq 0$, so $f_n(x) = f(x) - \frac{1}{n}$ and $f_n(y) = 0$. Then again by the remark above it follows that

$$\sup_{C(n)} \frac{|(f_n(x) - f(x)) - (f_n(y) - f(y))|}{|x-y|^\alpha} = \sup_{C(n)} \frac{|-f(x) - (f(y) - \frac{1}{n} - f(n)|}{|x-y|^\alpha} =$$

$$= \sup_{C(n)} \frac{|f(x) - \frac{1}{n}|}{|x-y|^\alpha} = \frac{|f(x_n) - \frac{1}{n}|}{|x_n - y_n|^\alpha} = (***),$$

for some $(x_n, y_n) \in C(n)$. Note that

$$\frac{|f(x) - \frac{1}{n}|}{|x-y|^\alpha} \leq \frac{|f(x) - f(y)|}{|x-y|^\alpha}.$$

Now let $\{x_{n_k} - y_{n_k}\}$ be any subsequence. Then either $|x_{n_k} - y_{n_k}| \to 0$, in which case we have $\sup_{C(n)}(***) \to 0$, or there exists a further subsequence $|x_{n_{k_m}} - x_{n_{k_m}}| \geq \delta > 0$, i.e. bounded away from zero. But then

$$\sup_{C(n)}(***) \leq \frac{\frac{1}{n_{k_m}} + \frac{1}{n_{k_m}}}{\delta} \to 0.$$

Therefore in any case the supremum approaches zero, so it follows that $(f - \frac{1}{n})_+ \to f$ as $n \to \infty$, and this completes the proof. □

Now that we have shown that all the analogous lemmas (with the adjustments we have made), still hold for the Little Oh Lipschitz class λ_α, the central theorem of this section follows in precisely the same way as before, so we can consider its proof complete. □

As the reader might have observed, at several critical points in this paper we have used the fact that λ_α (as opposed to Λ_α) has been the class of functions under consideration. We do not see how to overcome these serious difficulties in the attempt to classify the closed ideals of the Big Oh Lipschitz class. In fact, this leads to the conjecture that our main theorem (Theorem 5.1) is false for Λ_α.

References

1. R.G. Bartle, D.R. Sherbert, *Introduction to Real Analysis*, John Wiley & Sons, Inc., 1992.
2. J.A. Beachy, W.D. Blair, *Abstract Algebra with a Concrete Introduction*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1990.
3. J. B. Conway, *Functions of One Complex Variable*, 2nd ed., Springer-Verlag, New York, 1978.
4. B. Malgrange, *Ideals of Differentiable Functions*, Tata Inst. Fund. Res., Bombay, and Oxford Univ. Press, London, 1967.
5. H. Whitney, 'On ideals of differentiable functions', Amer. Journ. Math. **70** (1948), 635 - 658.

Department of Mathematics, University of Richmond, Richmond, Virginia, 23173
E-mail address: kgk0o@aurora.urich.edu