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Problems in Harmonic Function Theory

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Abstract

Harmonic Function Theory is a field of differential mathematics that has both many theoretical constructs and physical connections, as well as its store of classical problems.

One such problem is the Dirichlet Problem. While the proof of the existence of a solution is well-founded on basic theory, and general methods for polynomial solutions have been well studied, much ground is still yet to be overturned. In this paper we focus on the examination, properties and computation methods and limitations, of solutions for rational boundary functions.

Anotner area that we shall study is the properties and generalizations of the zero sets of harmonic functions. Our study in this area has shown that these zero sets satisfy many strict criteria. Many familiar or simple curves do not satisfy such criteria themselves. In this paper we will present the criteria and how it is so restrictive.

This paper is part of the requirements for honors in mathematics. The signatures below, by the advisor, a departmental reader, and a representative of the departmental honors committee, demonstrate that Ronald A. Walker has met all the requirements needed to receive honors in mathematics.

(advisor)

(reader)

(honors committee representative)

Problems in Harmonic Function Theory

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Abstract

Harmonic Function Theory is a field of differential mathematics that has both many theoretical constructs and physical connections, as well as its store of classical problems.

One such problem is the Dirichlet Problem. While the proof of the existence of a solution is well-founded on basic theory, general methods for finding solutions are not extensively studied. Previously methods for solutions with polynomial boundary functions have been well studied, much ground is still yet to be overturned. In this paper we focus on the examination, properties and computation methods of solutions for rational boundary functions.

Another area that we shall study is the properties and generalizations of the zero sets of harmonic functions. Our study in this area has shown that these zero sets satisfy many strict criteria. Many familiar or simple curves do not satisfy such criteria themselves. In this paper we will present the criteria and how it is so restrictive.

1 Introduction

A real-valued function u is *harmonic* on the open set Ω in the complex plane C if $u \in C^2(\Omega)$ and

$$\Delta u = u_{xx} + u_{yy} = 0.$$

The differential operator Δ is called the *Laplacian*. Harmonic functions have physical significance in relation to several physical phenomena such as heat flow, electromagnetism, waves, elasticity, and fluid flow.

One classical problem in this area is the Dirichlet Problem: Given a real-valued continuous function f defined on the unit circle $S = \{(x,y) : x^2+y^2 = 1\}$ find a real-valued continuous function u defined on the closed unit disk, $\bar{B} = \{(x,y) : x^2 + y^2 \leq 1\}$, such that

1. $u(x,y) = f(x,y) \ \forall(x,y) \in S$

2. $u(x,y)$ is harmonic on $B = \{(x,y) : x^2 + y^2 < 1\}$.

There is always an unique solution (See Section 3), but finding the solution for a specific f is not always possible. Axler and Ramey have developed a method for finding the solution for polynomial boundary conditions in n variables. We have developed a method that finds the solution for rational functions in 2 variables.

Now we define the zero set of function to be the set of all points where the function is zero. A second problem in the study of harmonic functions is the following: Given a harmonic function on the plane, what are the characteristics of the function's zero set, or given an arbitrary set, can there be found an entire harmonic function that is zero on and only on the given set? By the characteristics of harmonic functions on the plane and their relation to entire functions on C, such sets must be a union of smooth, non-looping curves. This property plus a host of others makes the set of possible zero curves a distinct and very exclusive set.

2 Preliminaries

Harmonic functions have several properties that are of value in our investigation. The first of these is the Mean-Value Property.

Theorem 2.1 (The Mean-Value Property) *If u is harmonic on $\bar{B}(a,r)$ then*

$$u(a) = \oint_{|\zeta|=1} u(a + r\zeta) \frac{|d\zeta|}{2\pi}. \tag{1}$$

Essentially Theorem 2.1 states that $u(a)$ equals the average of the value of u over any sphere centered at a. Another property of Harmonic Functions is the Maximum Principle.

Theorem 2.2 (The Maximum Principle) *Given a real-valued harmonic function u on a bounded domain Ω, such that u is continuous on $\bar{\Omega}$, then u assumes its maximum and minimum values on $\partial\Omega$.*

A harmonic function can never have a local maximum or local minimum. Thereby any maxima or minima must occur on the boundary, as stated in the previous property. Now when the domain of a harmonic function is the entire plane, it becomes unbounded since it can have neither a maximum or minimum in the plane, unless the function is constant. This is expressed by the next property.

Theorem 2.3 (Liouville's Theorem) *A function that is positive and harmonic on the plane is constant.*

The above property is similar to that used for holomorphic functions. In fact there are many similarities between their properties. In the definition of a harmonic function we demanded that $u \in C^2(\Omega)$ and that $\Delta u = 0$. As it turns out $u \in C^\infty(\Omega)$.

Theorem 2.4 *If u is harmonic on Ω then u is infinitely differentiable on Ω.*

3 The Dirichlet Problem

3.1 The Poisson Integral

In theory the solution to the Dirichlet Problem with the boundary function f (which must be real-valued on S) involves the Poisson integral, which is given below.

$$P[f](z) = \oint_{|\zeta|=1} P(z,\zeta) f(\zeta) \frac{|d\zeta|}{2\pi}, \qquad z \in B. \tag{2}$$

$$P(z,\zeta) = \frac{1-|z|^2}{|z-\zeta|^2} \tag{3}$$

Then define the function u on $\bar{B}$ as follows

$$u(z) = \begin{cases} P[f](z) & \text{if } z \in B \\ f(z) & \text{if } z \in S \end{cases}$$

The function u meets all the criteria to solve the Dirichlet Problem. By certain classical estimates on $P[f]$ one can show that u is continuous on $\bar{B}$.

Clearly u equals f on S. Now a calculation shows that $\Delta P(z,\zeta) = 4\partial\bar{\partial}P(z,\zeta) = 0$ with respect to z. So by differentiation under the integral sign one can show that $\Delta P[f] = 0$ and thus u is harmonic on B. Now this solution is also unique, by the following reasoning. Given an v that is also a solution for the boundary condition f, then $v(z) - u(z)$ is harmonic and a solution for the boundary condition 0. Then by the Maximum Principle, $v(z) - u(z) = 0$. Therefore the solution to the Dirichlet Problem for f exists, and is unique, equaling $P[f]$.

3.2 The Schwarz Integral

By a result from complex analysis there exists a more general form of the Poisson Integral. This formula is known as the Schwarz Integral, and is given as follows; For a function g, which is real-valued and continuous on S.

$$S[g](z) = \frac{1}{2\pi i}\oint_{|\zeta|=1} \frac{\zeta+z}{\zeta-z}\frac{g(\zeta)}{\zeta}\, d\zeta, \qquad z \in B \tag{4}$$

The Poisson Integral is then just the real part of the Schwarz Integral, as stated in the following theorem.

Theorem 3.1 *If u is defined and continuous on S then*

$$P[g](z) = Re(S[g](z)). \tag{5}$$

Proof

By a slight alteration of (4),

$$S[g](z) = \oint_{|\zeta|=1} \frac{\zeta+z}{\zeta-z} g(\zeta)\, \frac{|d\zeta|}{2\pi}.$$

Now taking the real part of both sides (Note that g is real-valued.) one arrives at

$$\mathrm{Re}(S[g](z)) = \oint_{|\zeta|=1} \mathrm{Re}\left(\frac{\zeta+z}{\zeta-z}\right) g(\zeta)\, \frac{|d\zeta|}{2\pi}. \tag{6}$$

Now by a calculation

$$\frac{\zeta+z}{\zeta-z} = \frac{\zeta\bar{\zeta} - z\bar{z} + z\bar{\zeta} - \bar{z}\zeta}{(\zeta-z)\overline{(\zeta-z)}} = \frac{|\zeta|^2 - |z|^2 + 2i\,\mathrm{Im}(z\bar{\zeta})}{|z-\zeta|^2}.$$

In the integral, $|\zeta| = 1$, so

$$\mathrm{Re}\left(\frac{\zeta+z}{\zeta-z}\right) = \frac{1-|z|^2}{|z-\zeta|^2} = P(z,\zeta). \tag{7}$$

Thus substituting (7) into (6) yields (5) completing the proof.

$\square$

3.3 Rational Functions

With polynomial data, it was shown by Axler and Ramey that the Dirichlet solution is a polynomial as well. It would be nice if an analogous result carried over to rational functions. Now we can compute the Poisson Integral via the Schwarz Integral. Assume the function g is rational in terms of the complex variable z. Thus g is a meromorphic function (analytic with the exception of some poles). This means we can compute $S[g]$ using residue theory from complex analysis. By this pathway, we shall see that $S[g]$ will be rational. First we need the following lemma that show that the residues are rational.

Definitions: Let g be a rational function of z. Then

$$\mathcal{W}_g = \{r \mid |r| < 1 \text{ and } g(z) \text{ has a pole at } z = r\} \tag{8}$$

$$m_g(r) = \text{ the order of the pole of } g(z) \text{ at } z = r \tag{9}$$

$$c_{r,k} = \frac{1}{k!}\lim_{\zeta \to r}\left[\frac{d^k}{d\zeta^k}\left(g(\zeta)(\zeta - r)^{m_g(r)}\right)\right] \tag{10}$$

Lemma 3.2 *If g is a rational function of z, then for a fixed $r \in \mathcal{W}_g$ and $z \neq r$,*

$$Res(\frac{g(\zeta)}{\zeta - z}; r) = \sum_{k=0}^{m_g(r)-1} \frac{-c_{r,k}}{(z-r)^{m_g(r)-k}}. \tag{11}$$

Proof

If $r \in \mathcal{W}_g$ and $z \neq r$ then by a Laurent series argument,

$$\text{Res}(\frac{g(\zeta)}{\zeta - z}; r) = \lim_{\zeta \to r}\left[\frac{1}{(m_g(r)-1)!}\frac{d^{m_g(r)-1}}{d\zeta^{m_g(r)-1}}\left(\frac{g(\zeta)(\zeta - r)^{m_g(r)}}{\zeta - z}\right)\right]$$

$$= \sum_{k=0}^{m_g(r)-1}\left[\frac{\left(\begin{array}{c} m_g(r)-1 \\ k \end{array}\right)}{(m_g(r)-1)!}\lim_{\zeta \to r}\left[\frac{d^k}{d\zeta^k}\left(g(\zeta)(\zeta - r)^{m_g(r)}\right)\frac{d^{m_g(r)-1-k}}{d\zeta^{m_g(r)-1-k}}\left(\frac{-1}{z-\zeta}\right)\right]\right]$$

$$= \sum_{k=0}^{m_g(r)-1} \left[\frac{1}{k!(m_g(r)-1-k)!} \, k! \, c_{r,k} \, \lim_{\zeta \to r} \left(\frac{-(m_g(r)-1-k)!}{(z-\zeta)^{m_g(r)-k}} \right) \right]$$

$$= \sum_{k=0}^{m_g(r)-1} \frac{-c_{r,k}}{(z-r)^{m_g(r)-k}}.$$

□

Lemma 3.3 *Suppose g is a rational function of z, and $r \in \mathcal{W}_g$ then*

$$Res(\frac{g(\zeta)}{\zeta - r}; r) = \lim_{z \to r} \left(g(z) + \sum_{k=0}^{m_g(r)-1} \frac{-c_{r,k}}{(z-r)^{m_g(r)-k}} \right). \tag{12}$$

Proof

We shall show that the limit equals the residue. First take the Laurent series of $g(z)$ around r.

$$g(z) = \sum_{j=-m_g(r)}^{\infty} a_j (z-r)^j \tag{13}$$

Through the standard definitions of the coefficients in (13),

$$a_j = \frac{1}{2\pi i} \oint_C \frac{g(\zeta)}{(\zeta - r)^{j+1}} \, d\zeta,$$

where C is a positively oriented, closed contour, such that $\mathrm{int}(C) \cap \mathcal{W}_g = \{r\}$. Now by using residues,

$$a_j = \lim_{\zeta \to r} \left[\frac{1}{(m_g(r)+j)!} \frac{d^{(m_g(r)+j)}}{d\zeta^{(m_g(r)+j)}} \left(g(\zeta)(\zeta - r)^{m_g(r)} \right) \right] = c_{r,m_g(r)+j}.$$

Through substitution of coefficients into (13) and adjustment of indices we derive the following.

$$g(z) + \sum_{k=0}^{m_g(r)-1} \frac{-c_{r,k}}{(z-r)^{m_g(r)-k}} = \sum_{j=-m_g(r)}^{\infty} c_{r,m_g(r)+j}(z-r)^j - \sum_{j=-m_g(r)}^{-1} \frac{c_{r,m_g(r)+j}}{(z-r)^{-j}}$$

$$= \sum_{j=0}^{\infty} c_{r,m_g(r)+j}(z-r)^j.$$

Now by taking the limit of both sides as z approaches r,

$$\lim_{z\to r}\left(g(z)+\sum_{k=0}^{m_g(r)-1}\frac{-c_{r,k}}{(z-r)^{m_g(r)-k}}\right)=c_{r,m_g(r)}. \tag{14}$$

Computation of our original residue in (12) yields the following equality.

$$\mathrm{Res}(\frac{g(\zeta)}{\zeta-r};r)=c_{r,m_g(r)}. \tag{15}$$

Combining (15) with (14) completes the proof.

□

Theorem 3.4 *If $g(z)$ is a rational function of z with no poles on S, then $S[g](z)$ is a rational function of z with no poles on B.*

Proof

Using partial fractions, (4) can be restated as follows.

$$S[g](z)=\frac{1}{2\pi i}\oint_{|\zeta|=1}\frac{2g(\zeta)}{\zeta-z}\,d\zeta-\frac{1}{2\pi i}\oint_{|\zeta|=1}\frac{g(\zeta)}{\zeta}\,d\zeta.$$

Notice that the first term can be computed using residues and that the second term is constant with respect to z.

$$S[g](z)=2\sum_{r\in W_g\cup\{z\}}\mathrm{Res}(\frac{g(\zeta)}{\zeta-z};r)-a_g$$

$$a_g=\frac{1}{2\pi i}\oint_{|\zeta|=1}\frac{g(\zeta)}{\zeta}\,d\zeta. \tag{16}$$

Now when $z\notin W_g$

$$S[g](z)=2g(z)+2\sum_{r\in W_g}\mathrm{Res}(\frac{g(\zeta)}{\zeta-z};r)-a_g.$$

Now using Lemma 3.2

$$S[g](z)=2g(z)-2\sum_{r\in W_g}\left[\sum_{k=0}^{m_g(r)-1}\frac{c_{r,k}}{(z-r)^{m_g(r)-k}}\right]-a_g. \tag{17}$$

Clearly $S[g](z)$ is rational with respect to z, $\forall z \in B\backslash\mathcal{W}_g$. Now if $z_r \in \mathcal{W}_g$ then

$$S[g](z_r) = 2\mathrm{Res}(\frac{g(\zeta)}{\zeta - z_r}; z_r) + 2 \sum_{r\in\mathcal{W}_g\backslash\{z_r\}} \left(\mathrm{Res}(\frac{g(\zeta)}{\zeta - z_r}; r)\right) - a_g$$

$$= \lim_{z\to z_r} \left[2g(z) - 2\sum_{r\in\mathcal{W}_g}\left(\sum_{k=0}^{m_g(r)-1}\frac{c_{r,k}}{(z-r)^{m_g(r)-k}}\right)\right] - a_g.$$

Using the above equation and (17) the following is true.

$$S[g](z_r) = \lim_{z\to z_r} S[g](z)$$

Thus S[g](z) is a rational function with no poles on B.

□

Now we desire to carry this result over to functions of x and y. So now we express this in a closely related corollary.

Corollary 3.5 *Given a real-valued rational function, f, in terms of x and y, that is defined on S. Then $P[f](x,y)$ is rational in terms of x and y.*

Proof

Let the function h be defined as

$$h(z) = (\frac{z+\frac{1}{z}}{2}, \frac{z-\frac{1}{z}}{2i}). \tag{18}$$

Notice that on S, $h(x+iy) = (x,y)$. Thus on S, $(f\circ h)(x+iy) = f(x,y)$. Therefore $(f\circ h)$ is real-valued and the following equality holds.

$$P[f](x,y) = P[f\circ h](x+iy) = Re(S[f\circ h](x+iy)) \tag{19}$$

Since f is rational in x and y and h is rational in z, it follows that $(f\circ h)(z)$ is rational in z. By Theorem 3.4, $S[f\circ h](z)$ is rational in terms of z. Clearly it follows that $\mathrm{Re}(S[g](x+iy))$ is rational in terms of x and y. Thus it follows by (19) that $P[f]$ is a rational function of x and y.

□

3.4 Computational Implementation

Now given a function f in terms of x and y, then what is an efficient manner to compute $P[f]$? We implemented a computational method in Mathematica.

First we need to define a function that will compute S[g](z), given g. We factor g into rational form, $\frac{p}{q}$, where $p(z)$ and $q(z)$ are polynomials in z. To use residues we must compute the poles of g, which are the zeros of q. Note below that we are given our function g as an expression exp in terms of a variable var.

```
g=Together[exp];
qz=Solve[Denominator[g]==0,var];
```

If the zeros of q are unsolvable in Mathematica, then the poles of g cannot be found and therefore the Schwarz Integral cannot be computed through residues. This is a limitation found in Mathematica 2.0 and earlier versions. Mathematica 3.0 uses symbolic algebraic numbers to represent the roots of q, allowing us to do complete symbolic computation. Thus here we do a check to see if Mathematica has successfully found the roots..

```
If[HasRoots[qz], Module[ {ag,s,w={},comfact},
   segment of code discussed next
],
 Block[{},
    Message[Dirichlet::denominator];
    {}]]
```

When the zeros of q are found, then we compute $\mathcal{W}_g$, the set of z such that $q(z) = 0$ and $|z| < 1$.

```
qz=Rts[qz,var];
Do[w=Join[w,If[N[Abs[(qz[[i]])]]<1,{qz[[i]]},{}]],
 {i,1,Length[qz]}];
```

Note w represents the set $\mathcal{W}_g$ with multiplicity of zeros being counted. Now for the sake of later simplification we generate the polynomial comfact which has all the zeros, with multiplicity, given by $\mathcal{W}_g$. Note that comfact divides q. Furthermore we now convert w into the true set representation of $\mathcal{W}_g$.

```
comfact=Product[(var-w[[i]]),{i,1,Length[w]}];
w=Union[w];
```

Thus $S[g](z)$ is now readily computable. Calculation can be simplified by using a variant of the Schwarz integral $S^*[g]$.

$$S^*[g](z) = \lim_{\zeta \to z} \left[g(\zeta) - \sum_{r \in \mathcal{W}_g} \left(\sum_{k=0}^{m_g(r)-1} \frac{c_{r,m_g(r)}}{(\zeta - r)^{m_g(r)}} \right) \right] \tag{20}$$

```
Tail[exp_,var_,r_]:=Normal[Series[exp,{var,r,-1}]]
Tails[exp_,var_,rs_]:=Sum[Tail[exp,var,rs[[i]]],
   {i,1,Length[rs]}]
sstar=g-Tails[g,z,w];
```

However the above expression isn't quite the appropriate form. For notice in the definition of S^* that the limit is taken. However notice that the only trouble spots are at the poles of g. But S^* doesn't have any poles within B. Therefore the factors in the denominator that cause the poles in g, must cancel out with factors in the numerator. We force this computational simplification by dividing the numerator and denominator by `comfact`.

```
DoToBoth[f_,rat_]:=f[Numerator[rat]]/f[Denominator[rat]]
sstar=Together[sstar]
sstar=Together[Simplify[
   DoToBoth[(Apart[#/comfact])&,sstar]]];
```

Computation of $S^*[g](z)$ appears to be just as strenuous as for $S[g]$, with the exception of the a_g term. Now applying Theorem 2.1 to the computation of a_g in (16) one arrives with the following.

$$a_g = S^*[g](0) \tag{21}$$

Thus the final computation to yield the Schwarz Integral is

$$S[g](z) = 2S^*[g](z) - S^*[g](0)$$

```
ag=sstar/.{z->0};
sch=Simplify[2*sstar-ag];
```

Now below is a condensation of the previously discussed code into a Mathematica function defining SchwarzIntegral.

```
Dirichlet::denominator = "Denominator of generated
 fraction is unfactorable"

SchwarzIntegral[exp_,var_]:=Module[{g,qz},
 g=Together[Apart[exp]];
 qz=Solve[Denominator[g]==0,var];
 If[HasRoots[qz], Module[ {ag,s,w={},comfact},
   qz=Rts[qz,var];
   Do[w=Join[w,If[N[Abs[(qz[[i]])]]<1,{qz[[i]]},{}]],
     {i,1,Length[qz]}];
   comfact=Product[(var-w[[i]]),{i,1,Length[w]}];
   w=Union[w];
   s=Simplify[g-Tails[g,var,w]];
   s=Together[Apart[Together[s]]];
   s=Together[Simplify[DoToBoth[(Apart[#/comfact^2])&,s]]];
   ag=s/.{var->0};
   Simplify[2*s-ag]
 ],
   Block[{},
     Message[Dirichlet::denominator];
     {}]]
]
```

Note that intermediate products are simplified using the Simplfy, Together, and Apart commands in order to bring about a simpler final answer. Also notice DoToBoth[(Apart[#/comfact^2])&,s]. Dividing numerator and denominator by comfact^2, is done to help cancel all the factors of comfact that are present in the numerator and denominator. Note mathematical correctness is not affected.

With a function now defined to compute the Schwarz Integral, the computation of Dirichlet solution with boundary function f is rather straightforward. Noticing (19), the first thing needed is a computation of $g = f \circ h$.

```
R2toC[f_,z_]:=f[(z^2+1)/(2z),(z^2-1)/(2I*z)]
```

```
g=R2toC[f,z];
```

Now we must compute $S[g](z)$.

```
sg=SchwarzIntegral[g,z];
```

We now compute $Re(S[g](x+iy))$ to complete the calculation of the Dirichlet solution.

```
ans=(Block[{z=(Global‘x+I Global‘y)},sg+Conjugate[sg]])/2;
```

We then simplify the answer and return it to the user. In all the Mathematica code can be condensed into the following lines.

```
Dirichlet[f_]:=Module[{sg},
 sg=SchwarzIntegral[R2toC[f,z],z];
 ExpandDenominator[Together[(Block[
    {z=(Global‘x+I Global‘y)}, sg+Conjugate[sg]])/2]]
]
```

3.5 Examples of Solutions for Rational Boundary Functions

Now we present some examples of computation of Dirichlet solutions. The simplest rational function that one readily compute the Dirichlet solution for is $f(x,y) = \frac{1}{x^2+y^2}$. Clearly we can justify that $u = 1$ is the Dirichlet solution. Our Mathematica program quickly computes this solution, as all groups of $x^2 + y^2$ that appear, end up being simplied in `R2toC[f_,z_]` as if they were replaced by 1.

In[3]:= `f[x_,y_]:=1/(x^2+y^2)^10`

In[4]:= `Dirichlet[f]`

Out[4]= `1`

The next function we would consider would be the reciprocal of a monomial, as it is the simplest non-trivial example. So we consider $f(x,y) = \frac{1}{x+2}$.

In[5]:= `f[x_,y_]:=1/(x+2)`

In[6]:= `Dirichlet[f]`

Out[6]= `(-(3*x^2) - 2*Sqrt[3]*x^2 - 3*y^2 - 2*Sqrt[3]*y^2 +`
`26*Sqrt[3] + 45)/`

```
(3*Sqrt[3]*x^2 + 6*x^2 + 24*Sqrt[3]*x + 42*x +
  3*Sqrt[3]*y^2 + 6*y^2 + 45*Sqrt[3] + 78)
```

And verifying the answer

In[7]:= `Simplify[Laplacian[%]]`

Out[7]= `0`

In[8]:= `Simplify[%%/.{y^2->1-x^2}]`

Out[8]= `1/(2+x)`

Expressed in a more elegant way the above computation says that

$$P[f](x,y) = \frac{-3x^2 - 2\sqrt{3}x^2 - 3y^2 - 2\sqrt{3}y^2 + 26\sqrt{3} + 45}{3\sqrt{3}x^2 + 6x^2 + 24\sqrt{3}x + 42x + 3\sqrt{3}y^2 + 6y^2 + 45\sqrt{3} + 78}$$

And in a rearranged form we get

$$P[f](x,y) = \frac{-(3+2\sqrt{3})(x^2+y^2) + 45 + 26\sqrt{3}}{(6+3\sqrt{3})(x^2+y^2) + 78 + 45\sqrt{3} + (42+24\sqrt{3})x}$$

And if one now substitutes in 1 for $x^2 + y^2$, one can verify through a little bit of work that the above function equals $\frac{1}{2+x}$.

4 Zero Sets of Harmonic Functions

Definitions :

We shall always assume that u is a nonconstant, real-valued harmonic function on C, and that f is a corresponding holomorphic function such that $\mathrm{Re}(f) = u$.

Given u, we define its zero set to be

$$\mathcal{Z}(u) = \{z : u(z) = 0\} = u^{-1}(\{0\}) \tag{22}$$

4.1 General Characteristics

The zero sets of harmonic functions compose an interesting set. With exception to the constant functions, the zero set of a harmonic function is a set of curves. These zero curves of harmonic functions have several characteristics that set them apart from curves in general. First we shall show that the zero set of a non-constant harmonic function can be aptly named the zero curves.

Theorem 4.1 *$\mathcal{Z}(u)$ is locally expressible by an infinitely differentiable curve with the only exceptions are a set of isolated points where $u_x = u_y = 0$.*

Proof

Since u is harmonic on the plane and non-constant, $\mathcal{Z}(u)$ is non-empty, by Theorem 2.3. All points in $\mathcal{Z}(u)$ must fall into one of the following cases.

Case 1: $u_y(p) \neq 0$.

Since u_y is continuous and $u_y(p) \neq 0$,

$$\exists \varepsilon > 0, \text{ s.t. } \forall z \in B(p, \varepsilon), u_y(z) \neq 0.$$

By the Implicit Function Theorem $\mathcal{Z}(u)$ is defined in the neighborhood $B(p, \varepsilon)$ by a function, $y = g(x)$, with the derivative

$$\frac{dy}{dx} = \frac{-u_x}{u_y}$$

Notice by Theorem 2.4 that the above equation is infinitely differentiable in the neighborhood $B(p, \varepsilon)$. Thus $\forall p \in \mathcal{Z}(u)$, s.t. $u_y(p) \neq 0, \mathcal{Z}(u)$ is expressible locally by an infinitely differentiable curve.

Case 2: $u_x(p) \neq 0$.

By a similar argument in Case 1,

$$\exists \varepsilon > 0, \text{ s.t. } \forall z \in B(p, \varepsilon), u_x(z) \neq 0.$$

Again by the Implicit Function Theorem $\mathcal{Z}(u)$ is defined in the neighborhood $B(p, \varepsilon)$ by a function, $y = g(x)$, with the derivative

$$\frac{dx}{dy} = \frac{-u_y}{u_x}.$$

Notice by a similar argument presented in Case 1, $\forall p \in \mathcal{Z}(u)$, s.t. $u_x(p) \neq 0, \mathcal{Z}(u)$ is expressible locally by an infinitely differentiable curve.

Case 3: $u_x(p) = u_y(p) = 0$.

There exists an entire function, f, such that $u(x, y) = \mathrm{Re}(f(x + iy))$. By the Cauchy Riemann equations, $u_x(p) = u_y(p) = 0$, if and only if $f'(p) = 0$. Since f' is entire, its zeros are isolated. Thus

$$\forall p \in \mathcal{Z}(u), \text{ s.t. } u_x(p) = u_y(p) = 0, \exists \varepsilon, \nexists z \in B(p, \varepsilon) \backslash p, u_x(z) = u_y(z) = 0.$$

Thus points which do not have a unique locally defined (infinitely differentiable) curve for $\mathcal{Z}(u)$ are isolated.

□

The points in the zero set where $f' = 0$ are currently left with no further description. As we shall see, they are cross-points of infinite differentiable curves and have other special properties. But first we'll explore several other properties of the zero curves of a harmonic function.

Theorem 4.2 *$\mathcal{Z}(u)$ has no endpoints.*

Proof

Assume for the purpose of contradiction that $\mathcal{Z}(u)$ had an endpoint, p. Then on any sufficiently small circle, ∂B_ε, around p with radius ε, there would exist only one point, p_ε, in the set $\mathcal{Z}(u) \cap \partial B_\varepsilon$ and thus $u(p_\varepsilon) = 0$. By continuity of u, the rest of the points in ∂B_ε must be postive (or negative). But by Theorem 2.1, this means $u(p) > 0$ (or < 0), contradicting that $u(p) = 0$. Thus $\mathcal{Z}(u)$ has no endpoints.

□

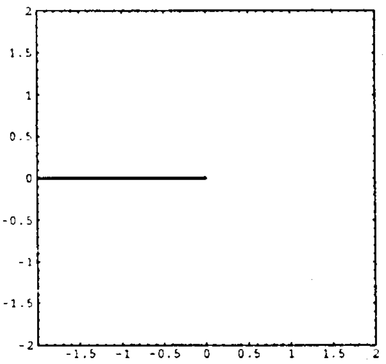
Theorem 4.3 *$\mathcal{Z}(u)$ forms no closed loops.*

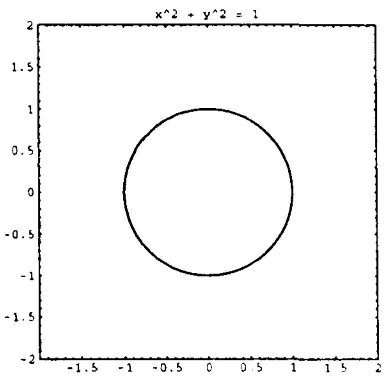
Proof

Assume for the purpose of contradiction that $\mathcal{Z}(u)$ forms a closed loop. Therefore the interior of the loop is a bounded, open set and thus by Theorem 2.2, the the function $u = 0$ for all points in the interior of the loop. Since the interior of the closed loop forms an open set on which $u = 0$, then $u = 0$ on the entire plane, contradicting that u is non-constant. Thus $\mathcal{Z}(u)$ form no closed loops.

□

These properties further restricts which curves may be zero curves of a harmonic function.





(1) is unallowable for it has an endpoint (Theorem 4.2). (2) is unallowable for it forms a closed loop (Theorem 4.3).

4.2 Cross Points

The cross points of the zero curves of harmonic functions have various properties. We can see these properties in the simple cases, which will later aid us in production of a proof.

Theorem 4.4 *$\mathcal{Z}(Re(a(z-z_0)^m))$, $m \geq 1, a \neq 0$, is composed of m lines ($2m$ rays) that "intersect" at equal angles of $\frac{\pi}{m}$.*

Proof

Let $u = \mathrm{Re}(a(z-z_0)^m)$, $m \neq 0, a \neq 0$. Define z in terms of polar coordinates around z_0.

$$z = z_0 + re^{i\theta}, \quad ,r > 0, 0 \leq \theta < 2\pi.$$

Now substitute in the definition of z into $u = 0$ to achieve the solution for all $z \in \mathcal{Z}(u)$ in terms of r and θ.

$$\mathrm{Re}(|a|r^m e^{i\arg(a)+mi\theta}) = 0.$$

After a calculation the previous equation implies

$$\cos(m\theta + \arg(a)) = 0.$$

Solution of the above equation yields

$$\theta = \frac{\pi - 2\arg(a)}{2m} + \frac{k\pi}{m}, \quad \left\lceil \frac{\arg(a)}{\pi} - \frac{1}{2} \right\rceil \leq k < \left\lceil \frac{\arg(a)}{\pi} - \frac{1}{2} \right\rceil + 2m.$$

There are $2m$ solutions for θ, independent of r. Notice there are no restrictions on r. Thus $\mathcal{Z}(u)$ is a set of $2m$ rays originating at z_0. By the polar coordinate solution one can also tell that they are at equal angles of $\frac{\pi}{m}$, thus each ray has an corresponding ray at an angle of π. Thus the $2m$ rays are truly m lines that intersect at z_0 with equal angles.

□

We see cross points are related to the order of the zero of the holomorphic function at that point. We generalize this for all cross points.

Theorem 4.5 *Given that $p \in \mathcal{Z}(Re(f))$, the following are equivalent:*

1. *There are m incident curves at p.*
2. *$\forall k \leq m-1, f^{(k)}(p) = 0$ and $f^{(m)}(p) \neq 0$.*
3. *There are exactly $2m$ "rays" in $\mathcal{Z}(Re(f))$ with origin p which intersect with equal angles of $\frac{\pi}{m}$.*

Proof

2. $\Rightarrow$ 1. and 3.
Let $p \in \mathcal{Z}(\mathrm{Re}(f))$. Then

$$\exists!\, m, m \geq 1, \text{ s.t. } \forall k \leq m-1, f^{(k)}(p) = 0 \text{ and } f^{(m)}(p) \neq 0.$$

If $m = 1$ then $f'(p) \neq 0$, and thus by the Implicit Function Theorem, there exists one and only one curve in $\mathcal{Z}(\mathrm{Re}(f))$ passing through p. And since it is locally differentiable the measure of the angles between the two "rays" is π.

If $m > 1$ then we use the following reasoning. The Taylor series of f around p gives

$$f(z) = (z-p)^m \sum_{j=0}^{\infty} a_{m+j}(z-p)^j, \quad a_0 \neq 0.$$

Notice that in a sufficiently small neighborhood of p that

$$f(z) \approx a_0(z-p)^m.$$

It follows that

$$\mathcal{Z}(\mathrm{Re}(f)) \approx \mathcal{Z}(\mathrm{Re}(a_0(z-p)^m)).$$

Thus for all $m > 1$, since $a_0 \neq 0$, we know the nature of $\mathcal{Z}(\text{Re}(a_0(z-p)^m))$ by Theorem 4.4. Thus we can conclude that the nature of $\mathcal{Z}(\text{Re}(f))$ around p is the same, in that it has m incident curves or $2m$ "rays" with equal intersection angles of $\frac{\pi}{m}$.

1. $\Rightarrow$ 2.

Assume for the sake of contradiction there is a counterexample. Thus there is a f and a $p \in \mathcal{Z}(\text{Re}(f))$ with m curves incident upon p and that m does not satisfy 2. However since f is entire and nonconstant,

$$\exists n, \text{ s.t. } \forall k \leq n-1, f^{(k)}(p) = 0 \text{ and } f^{(n)}(p) \neq 0.$$

Since 2. implies 1., this means that $\mathcal{Z}(\text{Re}(f))$ has n incident curves upon p. Therefore $n = m$, but this means m satisfies 2., thus contradiction and thus there is no counterexample.

3. $\Rightarrow$ 1.

Each "ray" has a corresponding "ray" that is m angles away. Thus the angle measure between these two "rays" is $m\frac{\pi}{m} = \pi$, which is a straight angle. Thus each pair of opposite "rays" forms a "line" or curve, thus there are m curves incident upon p.

□

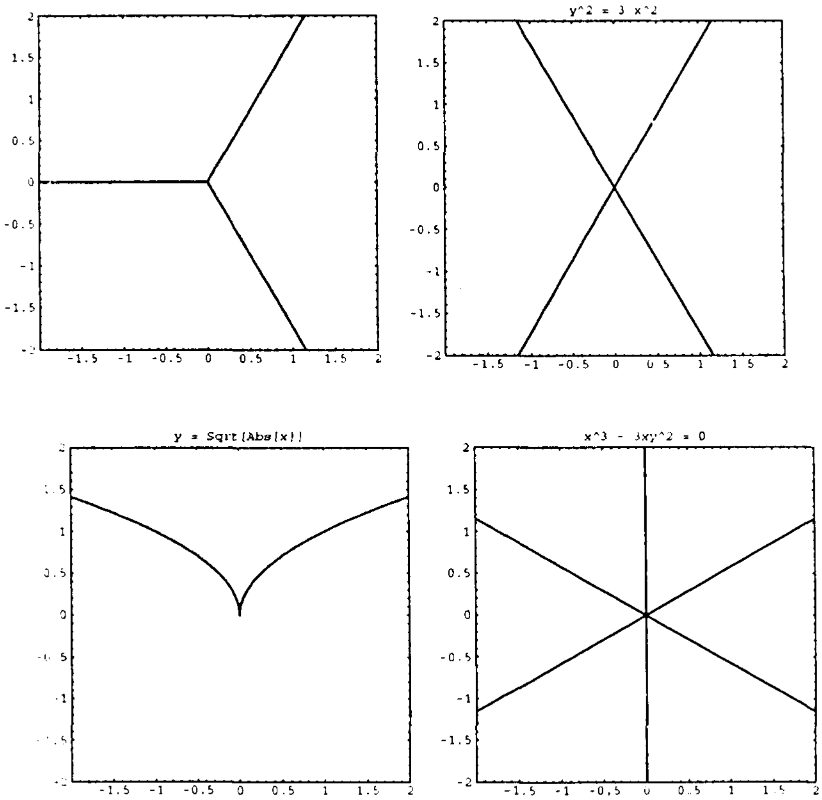
Corollary 4.6 *A cross point occurs at $p \in \mathcal{Z}(Re(f))$ if and only if $f'(p) = 0$.*

Proof

A cross point occurs at p if and only if there are $m, m \geq 2$ curves incident upon p. Then by Theorem 4.5, since $1 \leq m-1$, $f'(p) = 0$.

□

The above theorem provides a strict guideline for all cross points. A cross point must always be composed of curves intersecting at equal angles. This also states that there are no crosspoints in $\mathcal{Z}(\text{Re}(f))$ which have an odd number of "rays".



(3) is unallowable as it has only an odd number of rays. (4) and (5) are unallowable as both do not have equal angles between curves. (6) is $\mathcal{Z}(\mathrm{Re}(z^3))$ and satisfies Theorem 4.5.

In fact there is one more corollary of interest which we now demonstrate.

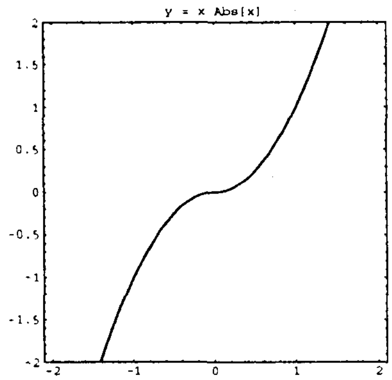
Corollary 4.7 *$\mathcal{Z}(u)$ is the union of one or more C^∞ curves.*

Proof

Using Theorem 4.1, we see that in any neighborhood of any $p \in \mathcal{Z}(u)$ not containing one of the cross points (where $u_x = u_y = 0$), that $\mathcal{Z}(u)$ is expressible as the union of one (or more non-intersecting) C^∞ curves. Now we need to show that this is true for any cross point within some neighborhood. Now there is a function f such that $u = \mathrm{Re}(f)$. We know that near p, $f \approx a_m(z-p)^m$, and thus their zero sets have similar natures. By Theorem 4.4 we see that $\mathcal{Z}(\mathrm{Re}(f))$ is similarly a union of m curves, each which are composed of 2 infinitely differentiable "rays", which adjoin with equal derivatives at p. This shows that $\mathcal{Z}(u)$ is the union of m C^1 curves in the neighborhood of p. It can be further worked out that $\mathcal{Z}(u)$ is the union of m C^∞ curves in the neighborhood of p, and thus $\mathcal{Z}(u)$ is globally a union of C^∞ curves.

□

Another argument can be developed to show that $\mathcal{Z}(u)$ must be a union of analytic curves. The eliminates many contours and curves that by our previous criteria would not have excluded.



(7) is unallowable since the second and greater derivatives are not defined and continuous at the origin (Corollary 4.7).

Now $\mathcal{Z}(u)$ is expressible as the union of one or more C^∞ curves, we provide some definitions here for later use.

Definitions:

$$\mathcal{Z}(u) = \bigcup_{j=1}^{n(\mathcal{Z}(u))} \Gamma_j(u). \tag{23}$$

$\Gamma_j(u) \subseteq \mathcal{Z}(u)$ is a distinct C^∞ curve in the plane with no endpoints.

$n(\mathcal{Z}(u))$ is the number of distinct C^∞ curves in $\mathcal{Z}(u)$.

4.3 Zero Curves at ∞

Another distinguishing characteristic of harmonic zero sets, is their behavior in the neighborhood of the point at ∞. Previous theorems have shown that zeros curves cannot end or form loops in the plane. Thus it makes sense that they must be incident upon or "end" at ∞.

Theorem 4.8 *$\forall j \leq n(\mathcal{Z}(u)), \Gamma_j(u)$ is incident upon ∞ from both ends.*

Proof

We will show that given an $\Gamma_j(u)$ and a direction along that curve that we will encounter the point at ∞ in following the curve in that direction. Given an arbitrary $\Gamma_j(u)$ and a direction to follow along it then choose a point p that is on the curve and is not a cross point. Note that at p, $\Gamma_j(u)$ forms part (or all) of the boundary for a connected region R where $u > 0$ (and also for one where $u < 0$). Note this region's boundary is in $\mathcal{Z}(u)$. Note that by Theorem 4.3, that R cannot be bounded. Thus the boundary of R is incident upon the point at ∞. Now following $\Gamma_j(u)$ from p we must follow along the boundary of R until it hits a cross point or until it reaches the point at ∞. If the latter is true then we are done for this particular $\Gamma_j(u)$ and this particular direction. Thus we need to consider what happens when $\Gamma_j(u)$ encounters a cross point.

By Theorem 4.5 we see that $\Gamma_j(u)$ has a corresponding curve to follow through the cross point. Note that $\Gamma_j(u)$ is no longer following along the boundary for region R, but is now following the boundary for R_1, which is another region where u is positive. Again the previous argument holds that $\Gamma_j(u)$ must encounter a cross point or the point at ∞. Again we only need to continue if a cross point was encountered.

By repeating this argument inductively we see that if $\Gamma_j(u)$ meets only a finite number of cross points, then after meeting those points it will have to follow the boundary of some region where u is positive to the point at ∞. Note that by Theorem 4.2 that the curve must continue and not have an endpoint. Thus the only remaining case is if $\Gamma_j(u)$ encounters an infinite number of cross points. But cross points must accumulate at ∞ (since $u = \text{Re}(f)$, and $f' = 0$ at the cross points). Thus the curve must be incident on the point at ∞.

Hence for all j, $\Gamma_j(u)$ in incident upon ∞ from both directions of the curve.

□

This is important because the point at ∞ provides us a way with "counting" curves.

Corollary 4.9 *There are $n(\mathcal{Z}(u))$ curves, $2n(\mathcal{Z}(u))$ "rays", incident upon the point at ∞.*

Proof

There are, by definition, exactly $n(\mathcal{Z}(u))$ curves in $\mathcal{Z}(u)$. By Theorem 4.8 each one is incident upon ∞ once and only once from both ends, thus the corollary holds.

$\square$

The point at ∞ has characteristics similar to those of the cross points. Comparison to corresponding holomorphic functions is key in the study of the curves' behavior around ∞. When there is a pole at ∞ (when f is a polynomial), then the behavior near ∞ is exactly the same as near a cross point.

Lemma 4.10 *$\mathcal{Z}(Re(az^{-m}))$, where $m \geq 1$, is composed of $2m$ rays incident upon $z = 0$ with angles $\frac{\pi}{m}$.*

Proof

Define z in polar coordinates.

$$z = re^{i\theta}, \quad r > 0, \ 0 \leq \theta < 2\pi.$$

Then the solution for $\mathcal{Z}(\mathrm{Re}(az^{-m}))$ is

$$\cos(-m\theta + arg(a)) = 0$$

This the following solution in polar coordinates.

$$\theta = \frac{2\arg(a) - \pi}{2m} + \frac{k\pi}{m}, \quad \left\lceil \frac{1}{2} - \frac{\arg(a)}{\pi} \right\rceil \leq k < \left\lceil \frac{1}{2} - \frac{\arg(a)}{\pi} \right\rceil + 2m.$$

Just as in Theorem 4.4, we see that this means that the zero set is composed of m lines (minus the point at the origin) all incident upon the origin and having equal angles of $\frac{\pi}{m}$.

$\square$

Theorem 4.11 *If f has a pole of order m at the point at ∞, then there are m curves, $2m$ "rays", incident upon the point at ∞, or more simply $n(\mathcal{Z}(Re(f))) = m$.*

Proof

If f has a pole of order m at the point at ∞, then f has the following form.

$$f(z) = \sum_{j=0}^{m} a_j z^j.$$

Note that in a close enough neighborhood of the point at ∞,

$$f(z) \approx a_m z^m.$$

The behavior of $f(z)$ around the point at ∞ then correlates to the behavior of $f(\frac{1}{z})$ around 0. Thus

$$f(\frac{1}{z}) \approx a_m z^{-m}.$$

Now by Lemma 4.10, we now know that the behavior corresponds to m incident curves. Now by Corollary 4.9 this proves that $n(\mathcal{Z}(\mathrm{Re}(f))) = m$.

□

However if f has an essential singularity at ∞, then such a nice property doesn't exist However zero curves for such a f can be distinguished from the zero curves for functions with a pole at infinity. We will now develop a description of this characteristic, but we will need to use the following lemma.

Lemma 4.12 *Given a differentiable curve $\gamma(t) = (\gamma_1(t), \gamma_2(t))$, with the properties that $Re(f(\gamma(t))) = 0$, for all $t \in [a, b]$, and that $f(\gamma(a)) = f(\gamma(b))$, then there exists a c such that there is a cross point in $\mathcal{Z}(Re(f))$ at $\gamma(c)$*

Proof

Define $f := u + iv$. Notice that since u is constantly zero along the curve that

$$\frac{d}{dt}(u(\gamma(t))) = 0, \quad t \in (a, b). \tag{24}$$

Now define $g(t) = v(\gamma(t))$. Then $g(a) = g(b)$. Now by the Mean-Value Theorem,

$$\exists c \in (a, b), \text{ s.t. } g'(c) = 0.$$

Therefore by the above equation and (24), at $t = c$,

$$v_x(\gamma)\gamma_1' + v_y(\gamma)\gamma_2' = 0.$$

$$u_x(\gamma)\gamma_1' + u_y(\gamma)\gamma_2' = 0.$$

Since γ is a differentiable curve, $(\gamma_1', \gamma_2') \neq (0,0)$. Therefore by linear algebra, at $\gamma(c)$,

$$u_x v_y = u_y v_x.$$

Then by the Cauchy-Riemann equations

$$(u_x)^2 + (u_y)^2 = 0.$$

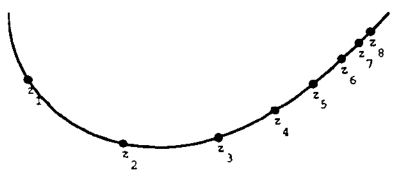
So at $\gamma(c)$, $u_x = u_y = 0$ and thus $f'(\gamma(c)) = 0$. Therefore by Corollary 4.6 there must be a cross point at $\gamma(c)$.

□

Theorem 4.13 *If f has an essential singularity at the point at ∞, then $n(\mathcal{Z}(Re(f))) = \infty$.*

Proof

For the purpose of contradiction, assume that there is a f that has an essential singularity at the point at ∞ so that $n(\mathcal{Z}(\mathrm{Re}(f)) = M$, where M is a finite number. Then there are M zero curves incident upon the point at ∞. Now by the Great Picard Theorem, there is an infinite number of points in any neighborhood of ∞ where the value of f is 0 (or i). Now by the Pigeonhole Principle there must be a Γ_k such that contains an infinite number of the points where f is 0 (or i). Enumerate these points as z_j, in order of occurance as one starts at some point in Γ_k and heads toward the point at ∞ along Γ_k.



Now by Lemma 4.12 between each pair of consecutive points, z_j and z_{j+1}, there is a cross point in $\mathcal{Z}(\text{Re}(f))$, involving Γ_k being intersected by at least the curve Γ_{k_j}. Note by Theorem 4.3 that the following statement must hold true.

$$\forall a, b, \text{ s.t. } a \neq b, \ k_a \neq k_b.$$

Now if we examine $n(\mathcal{Z}(\text{Re}(f)))$, we arrive with the following.

$$n(\mathcal{Z}(\text{Re}(f))) > |\{j \in \mathcal{N} : \Gamma_{k_j}\}| = |\{j \in \mathcal{N} : k_j\}| = |\mathcal{N}| = \infty.$$

Therefore contradiction, and thus the theorem is proven.

□

Now one should also note that the converses to the above theorems also hold true, and now we can prove both, with most the work already done in the previous theorems.

Theorem 4.14 *If $n(\mathcal{Z}(Re(f))) = m$, where m is finite, then f has a pole of order m at the point at ∞ (i.e. f is a polynomial of degree m.)*

Proof

Assume for the sake of contradiction that f did not have a pole of order m at the point at ∞. Since f is non-constant it cannot have a removable singularity there. Suppose that there was a pole of order n, $n \neq m$ at the point at ∞, then by Theorem 4.11, $n(\mathcal{Z}(\text{Re}(f))) = n$. But this implies that $n = m$. Thus the only remaining choice is that f has an essential singularity at the point at ∞. Thus by Theorem 4.13, $n(\mathcal{Z}(\text{Re}(f))) = \infty$. Thus contradiction

□

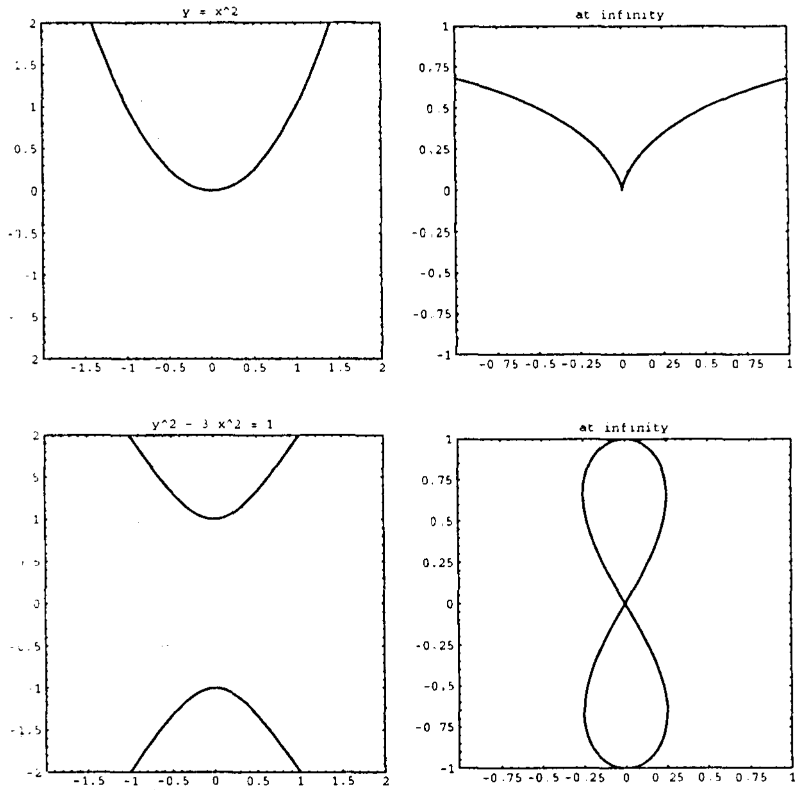
Theorem 4.15 *If $n(\mathcal{Z}(Re(f))) = \infty$, then f has an essential singularity at the point at ∞.*

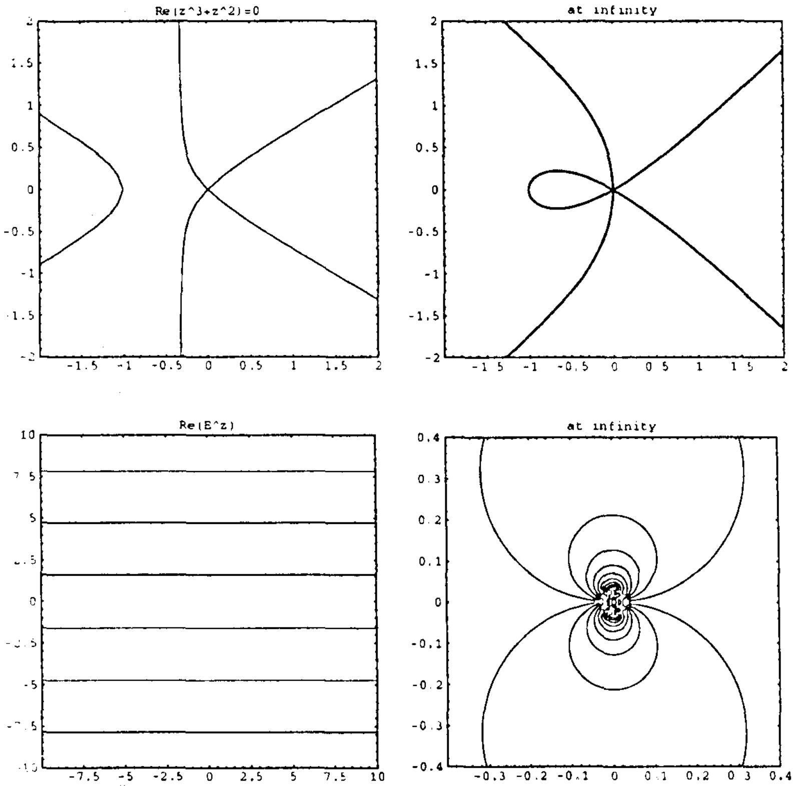
Proof

Assume for the sake of contradiction, that f didn't have a essential singularity at the point at ∞. Thus f must have a pole of some finite order m at the point at ∞, as f is non-constant. By Theorem 4.11 this means that $n(\mathcal{Z}(\text{Re}(f))) = m$. Thus contradiction, and this completes the proof.

□

Now with these theorems many classes of curves can be excluded from being considered the zero set of a harmonic function. Functions which have a finite number of lines approaching the point at ∞ must have equiangular asymptotes.





(8) and (9) are unallowable as the angles of their asymptotes (at ∞) are not equal. (10) is valid, and demonstrates the properties at ∞, (Theorem 4.11), and the properties of a cross point (Theorem 4.4). (11) is also valid and demonstrates Theorem 4.13.

4.4 Exclusiveness of Equations

The set of zero sets of harmonic functions is indeed a very restrictive one. As we have seen many simple candidates are quickly eliminated by the many properties that zero curves of harmonic functions satisfy. We now turn from a geometric examination to a more algebraic examination. If we give the zero set as the zero set of some function g, the question is, does there exist a harmonic function with the same zero set as g. If $\Delta g \equiv 0$, then the answer is clearly "yes". If $\Delta g \not\equiv 0$, then the answer is undetermined. For example, x^2y, xy^2, and xy all have the same zero set, yet the first two functions are not harmonic while the third is. Notice by Theorem 4.14, that if there is a harmonic polynomial u, such that $\mathcal{Z}(u) = \mathcal{Z}(g)$, then the degree of u must be $n(\mathcal{Z}(g))$. So if we put the restriction on g that $\deg(g) = n(\mathcal{Z}(g))$, and then $\Delta g \neq 0$ is a sufficient fact to eliminate $\mathcal{Z}(g)$ as a zero set of a harmonic function.

Before proving the above statement though, we will prove what yields to be a strong theorem for our purposes. First we put forth the following lemma.

Lemma 4.16 *Given a polynomial p in x and y of degree m.*

$$p(x,y) = \sum_{i=0}^{m} \sum_{j=0}^{m-i} a_{i,j}(p) x^i y^j. \tag{25}$$

There exists a function $h[p,\mu,b]$, which is a polynomial of degree m in x, such that p and $h[p,\mu,b]$ are equal on the line $y = \mu x + b$, that is $p(x,\mu x + b) = h[p,\mu,b](x)$.

Proof

$$\begin{aligned}
p(x,\mu x + b) &= \sum_{i=0}^{m} \sum_{j=0}^{m-i} \left(a_{i,j}(p) x^i (\mu x + b)^j\right) \\
&= \sum_{i=0}^{m} \sum_{j=0}^{m-i} \left(a_{i,j}(p) x^i \sum_{k=0}^{j} \binom{j}{k} (\mu x)^k b^{j-k}\right) \\
&= \sum_{i=0}^{m} \sum_{j=0}^{m-i} \sum_{k=0}^{j} \left(a_{i,j}(p) \binom{j}{k} \mu^k b^{j-k} x^{i+k}\right) \\
&= \sum_{i=0}^{m} \sum_{j=i}^{m} \sum_{k=i}^{j} \left(a_{i,j-i}(p) \binom{j-i}{k-i} \mu^{k-i} b^{j-k} x^k\right) \\
&= \sum_{k=0}^{m} \sum_{i=0}^{k} \sum_{j=k}^{m} \left(a_{i,j-i}(p) \binom{j-i}{k-i} \mu^{k-i} b^{j-k} x^k\right) \\
&= \sum_{k=0}^{m} d_k(p,\mu,b) x^k
\end{aligned}$$

where $d_k(p,\mu,b)$ is defined as follows

$$d_k(p,\mu,b) = \sum_{i=0}^{k} \sum_{j=k}^{m} a_{i,j-i}(p) \binom{j-i}{k-i} \mu^{k-i} b^{j-k}. \tag{26}$$

Let $h[p,\mu,b] = \sum_{k=0}^{m} d_k(p,\mu,b) x^k$ and the proof is finished.

$\square$

Now that we have worked out the above calculation we generate the following theorem.

Theorem 4.17 *Let f and g be functions of x and y with degree m with $\mathcal{Z}(f) = \mathcal{Z}(g)$. If there exists m parallel lines indexed as $\mathcal{L}_j$, so that $\forall j$, $1 \leq j \leq m$, that $|\mathcal{L}_j \cap \mathcal{Z}(f)| = m$, then $g = cf$ where c is a constant.*

Proof

First we consider the case that the lines are not vertical. Then all the lines have the same slope which we define to be μ. The lines can be indexed as $\mathcal{L}_j = \mathcal{L}(b_j) = \{(x,y) : y = \mu x + b_j\}$. Now for any b such that $|\mathcal{L}_j \cap \mathcal{Z}(f)| = m$, then define the point (x_i, y_i) as the ith zero point in $\mathcal{L}(b) \cap \mathcal{Z}(f)$. Choose (x', y') to be an element of $\mathcal{L}(b) \backslash \mathcal{Z}(f)$. Define

$$c_b = \frac{g(x',y')}{f(x',y')}.$$

By Lemma 4.16 $h[f,\mu,b](x,y) = f(x,y)$, and similarly for g. Putting this equation for the m points (x_i, y_i) and the point (x',y') gives

$$\begin{pmatrix} 1 & x' & \cdots & x'^m \\ 1 & x_1 & \cdots & x_1^m \\ 1 & x_2 & \cdots & x_2^m \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_m & \cdots & x_m^m \end{pmatrix} \begin{pmatrix} d_0(f,\mu,b) \\ d_1(f,\mu,b) \\ d_2(f,\mu,b) \\ \vdots \\ d_m(f,\mu,b) \end{pmatrix} = \begin{pmatrix} f(x',y') \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Note the above square matrix is a Vandermonde matrix. Since x', and x_i, for all i, are distinct, the matrix above is invertible. Hence

$$\begin{pmatrix} d_0(g,\mu,b) \\ d_1(g,\mu,b) \\ d_2(g,\mu,b) \\ \vdots \\ d_m(g,\mu,b) \end{pmatrix} = \begin{pmatrix} 1 & x' & \cdots & x'^m \\ 1 & x_1 & \cdots & x_1^m \\ 1 & x_2 & \cdots & x_2^m \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_m & \cdots & x_m^m \end{pmatrix}^{-1} \begin{pmatrix} g(x',y') \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & x' & \cdots & x'^m \\ 1 & x_1 & \cdots & x_1^m \\ 1 & x_2 & \cdots & x_2^m \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_m & \cdots & x_m^m \end{pmatrix}^{-1} \begin{pmatrix} f(x',y') \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} c_b$$

$$= c_b \begin{pmatrix} d_0(f,\mu,b) \\ d_1(f,\mu,b) \\ d_2(f,\mu,b) \\ \vdots \\ d_m(f,\mu,b) \end{pmatrix}$$

Hence for all j and k, $d_k(g,\mu,b_j) = c_{b_j} d_k(f,\mu,b_j)$. Now using (26), we have that for all j that

$$d_m(f,\mu,b_j) = \sum_{i=0}^{m} a_{i,m-i}(f)\mu^{m-i}.$$

So for any j and j', $d_m(f,\mu,b_j) = d_m(f,\mu,b_{j'})$ (and similiarly $d_m(g,\mu,b_j) = d_m(g,\mu,b_{j'})$). Now

$$c_{b_j} = \frac{d_m(g,\mu,b_j)}{d_m(f,\mu,b_j)} = \frac{d_m(g,\mu,b_1)}{d_m(f,\mu,b_1)} = c_{b_1}.$$

Let us define $c = c_{b_1}$. Then for all k,

$$d_k(g,\mu,b_j) = c d_k(f,\mu,b_j). \tag{27}$$

It remains to show that $a_{i,j}(g) = ca_{i,j}(f)$, for all i and j. We will prove this using induction. Our induction statement on n is that the statement $a_{i,j}(g) = ca_{i,j}(f)$ is true for all i and j such that $0 \le i < n$ and $0 \le j \le m - i$. The basis case for $n = 0$ holds clearly as there are no i such that $0 \le i < 0$. Now assume the statement is true for $n = k$. Notice by (27) and (26) that

$$0 = d_k(g,\mu,b) - cd_k(f,\mu,b)$$

$$= \sum_{i=0}^{k}\sum_{j=k}^{m}\left[(a_{i,j-i}(g) - ca_{i,j-i}(f)) \begin{pmatrix} j-i \\ k-i \end{pmatrix} \mu^{k-i}b^{j-k}\right]$$

$$= \sum_{j=k}^{m} \left[(a_{k,j-k}(g) - ca_{k,j-k}(f))b^{j-k}\right]$$

$$+\sum_{i=0}^{k-1}\sum_{j=k}^{m}\left[(a_{i,j-i}(g) - ca_{i,j-i}(f))\begin{pmatrix} j-i \\ k-i \end{pmatrix}\mu^{k-i}b^{j-k}\right].$$

By our induction hypothesis, $a_{i,j}(g) - ca_{i,j}(f) = 0$ for all i and j such that $0 \le i < k$ and $0 \le j \le m - i$. So the second summation equals zero, and so

$$\sum_{j=0}^{m-k}(a_{k,j}(g) - ca_{k,j}(f))b^j = 0.$$

Now the above is true for b_j, for all j such that $1 \le j \le m - k + 1$. Setting this up in matrix form we get

$$\begin{pmatrix} 1 & b_1 & \cdots & b_1^{m-k} \\ 1 & b_2 & \cdots & b_2^{m-k} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & b_{m-k+1} & \cdots & b_{m-k+1}^{m-k} \end{pmatrix}\begin{pmatrix} a_{k,0}(g) - ca_{k,0}(f) \\ a_{k,1}(g) - ca_{k,1}(f) \\ \vdots \\ a_{k,m-k}(g) - ca_{k,m-k}(f) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

The above matrix is a Vandermonde matrix and hence invertible since all b_j are distinct. Thus $a_{k,j}(g) - ca_{k,j}(f) = 0$ for all j such that $0 \le j \le m - k$. Therefore the induction statement holds for $n = k + 1$. By finite induction the statement, $a_{i,j}(g) = ca_{i,j}(f)$, is proven for all i and j such that $i, j \ge 0$ and $i + j \le m$. Therefore $g(x, y) = cf(x, y)$.

Now in the case that the parallel lines used are vertical, then apply the above proof to $f(y, x)$ and $g(y, x)$, using horizontal lines.

□

From this we develop two corollaries relating to the zero sets of harmonic functions.

Corollary 4.18 *Given an harmonic polynomial u of degree m. Then any polynomial p of degree m such that $\mathcal{Z}(p) = \mathcal{Z}(u)$, is equal to a non-zero multiple of u.*

Proof

Since $\mathcal{Z}(u)$ has m asymptotes, choose a line through the origin that is not parallel to any of the asymptotes. This line is of the form $\{(x, y) : Ax + By = 0\}$. Now there exists a D such that for all $d > D$, that the line $\mathcal{L}(d) = \{(x, y) :$

$Ax + By = d\}$ has m intersection points with $\mathcal{Z}(p)$. Hence we can choose m parallel lines that satisfy the condition for Theorem 4.17. Thus $p = cu$.

□

Corollary 4.19 *Given a polynomial p of degree m with m lines in $\mathcal{Z}(p)$ incident upon ∞ (or m asymptotes), then there exists a harmonic function u such that $\mathcal{Z}(u) = \mathcal{Z}(p)$ if and only if $\Delta p \equiv 0$.*

Proof

If $\Delta p \equiv 0$ then p is a harmonic function.

If there exists a harmonic function u, such that $\mathcal{Z}(p) = \mathcal{Z}(u)$ then by Corollary 4.18, $p = cu$. Since u is harmonic, $\Delta p \equiv c\Delta u \equiv 0$.

□

Now an alternative way to attempt to show the above corollaries is directly through the Hilbert-Nullstellensatz Theorem. Through this theorem we can say that there must exist numbers $m, n \geq 1$ and polynomials r and s such that $p^m = ru$ and $u^n = sp$. However it is not clear how having the degree of u and p being equal implies that m and n equal 1 (and therefore that r and s are constant). Plus the fact that u is harmonic is not used in any of this construction. Thus this method doesn't seem to yield the above corollaries.

Most of previous work has examined the properties of zero sets in two dimensions. One question is do these translate into higher dimensions. With the utility of complex numbers now unavailable. With the exception of cross points and cross lines, the Implicit Function Theorem will hold that the zero sets in any dimension are locally infinitely differentiable hypersurfaces. General properties of harmonic functions were used to prove Theorem 4.2 and Theorem 4.3, so these hold for any dimension. Examination of cross-points requires severe alteration. Most importantly we can no longer say that a zero set is union of one or more infinitely differentiable hypersurfaces. In fact there are definite counter-examples to this. However at cross points, cross lines, etc., where the zero set is locally a union of infinitely differentiable hypersurfaces, we can show that certain principles of symmetry must exist. By the Schwarz Reflection Principle, the zero set but be locally symmetric about each surface at the cross point. Many of the properties may exist at all cross points but either a different condition is needed other than "equal angles", or an appropriate definition of angle measure needs to be used.