5-2006

Boundary behavior of Laplace transforms

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May 2006

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Abstract

In this thesis, we examine the boundary behavior of Laplace transforms (as analytic functions on the right and left half planes) of certain bounded functions. The types of bounded functions we consider are Fourier transforms of measures and almost periodic functions.

The signatures below, by the thesis advisor, a departmental reader, and the honors coordinator for mathematics, certify that this thesis, prepared by Timothy Ferguson, has been approved, as to style and content.

(thesis advisor)

(reader)

(honors coordinator)

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Contents

Overview

This thesis deals with boundary behavior of certain integral transforms. Specifically, we deal with the Laplace transform. Let

$$\mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$$

and

$$\mathbb{C}_- = \{z \in \mathbb{C} : \mathrm{Re}(z) < 0\}$$

denote the right-half complex plane and the left-half complex plane, respectively. If $\phi : \mathbb{R} \to \mathbb{C}$, then the Laplace transform of ϕ, denoted by $\mathfrak{L}[\phi]$ or f_ϕ, is a function from $\mathbb{C}_+$ to $\mathbb{C}$ and is defined by

$$f_\phi(z) = \int_0^\infty \phi(t)e^{-zt}dt.$$

A related integral transform of ϕ is the left Laplace transform, which is denoted by F_ϕ and is the function from $\mathbb{C}_-$ to $\mathbb{C}$ defined by

$$F_\phi(z) = -\int_{-\infty}^0 \phi(t)e^{-zt}dt.$$

When ϕ is bounded and continuous in $\mathbb{R}$, the functions f_ϕ and F_ϕ are analytic in $\mathbb{C}_+$ and $\mathbb{C}_-$ respectively.

However, just because f_ϕ is analytic in $\mathbb{C}_+$ and F_ϕ is analytic in $\mathbb{C}_-$, this does not mean that they are well behaved near the imaginary axis. An example relating to this is the following theorem, due to Poincaré. But to understand this theorem, we first need the following definition.

DEFINITION. We say that $f : \mathbb{C}_+ \to \mathbb{C}$ and $F : \mathbb{C}_- \to \mathbb{C}$ are *analytic continuations* of each other across some interval $\gamma \subset i\mathbb{R}$ if there is some analytic function g defined in a domain U which contains γ, and such that $g = f$ on $U \cup \mathbb{C}_+$ and $g = F$ on $U \cup \mathbb{C}_-$.

THEOREM (Poincaré, 1883, [14]). *Let $\{c_n\}_{n\geq 1}$ be a sequence of complex numbers such that*

$$\sum_{n=1}^\infty |c_n| < \infty$$

and let the sequence $\{x_n\}_{n\geq 1}$ be dense in $\mathbb{R}$. Then the function

$$f(z) = \sum_{n=1}^\infty \frac{c_n}{z - ix_n}$$

is analytic on $\mathbb{C}_+ \cup \mathbb{C}_-$ but does not have an analytic continuation across any arc of $i\mathbb{R}$.

It is not difficult to see that if

$$\phi(x) = \int_{\mathbb{R}} e^{ixt} d\sigma(t),$$

where

$$\sigma = \sum_{n=1}^{\infty} c_n \delta_{x_n}$$

is the discrete measure on $\mathbb{R}$ with a point mass at each x_n, then f_ϕ (F_ϕ) is the function in Poincaré's example, restricted to $\mathbb{C}_+$ (resp. $\mathbb{C}_-$). Thus, the transforms f_ϕ and F_ϕ are poorly behaved near $i\mathbb{R}$. However, one can show that in Poincaré's example, that the limits

$$\lim_{x \to 0^+} f_\phi(x+iy) \text{ and}$$
$$\lim_{x \to 0^-} F_\phi(x+iy)$$

exist and are equal for almost every $y \in \mathbb{R}$, in the sense of Lebesgue measure. A term coined by H.S. Shapiro (see [**17**]) says that f_ϕ and F_ϕ are *"pseudo-continuations"* of each other. The idea of pseudo-continuation has applications in several fields of mathematics and electrical engineering.

If σ is not a discrete measure but a general finite measure, and where as before we define

$$\phi(x) = \int_{\mathbb{R}} e^{ixt} d\sigma(t),$$

then the above limits for f_ϕ and F_ϕ also exist almost everywhere, but only in certain circumstances are those limits equal, i.e. f_ϕ and F_ϕ are pseudo-continuations of each other only in certain circumstances.

If σ is a discrete measure, then the associated function ϕ is a special type of function, known as an almost periodic function, although not all almost periodic functions are obtained in this way. If ϕ is an almost periodic function, we currently cannot say much about the existence or non-existence of the limits of f_ϕ and F_ϕ, as we did for when

$$\phi(x) = \int_{\mathbb{R}} e^{ixt} d\sigma(t).$$

We can also not say much about whether f_ϕ and F_ϕ are pseudo-continuations of each other. However, we can still say that f_ϕ and F_ϕ are "related" in a meaningful way, as did Bochner and Bohnenblust did in 1934 (see [4]). We denote this relation by saying that f_ϕ and F_ϕ are Bochner-Bohnenblust continuations of each other. One of the main results of this thesis is to extend Bochner-Bohnenblust continuation to a wider class of functions - the S^2-almost periodic functions of Besicovitch.

Although we do have some partial results which say that in certain cases f_ϕ and F_ϕ are pseudo-continuations of each other, we do not currently understand the precise relationship between Bochner-Bohnenblust continuation and pseudo-continuation. For example, we do not know whether every Bochner-Bohnenblust continuation is a pseudo-continuation.

Chapter 1 is a review of measure theory and some classical results from the early twentieth century about the boundary behavior of analytic functions. Chapter 1 also introduces almost periodic functions and S^2-almost periodic functions, and contains basic results about them. Chapter 2 is a slight detour into the investigation of boundary values of the Borel transform, which turns out to be closely related to the Laplace transform. Chapter 3 contains a discussion of the Laplace transform, as well as Bochner-Bohnenblust continuation. It also discusses the extension of Bochner-Bohnenblust continuation to S^2-almost periodic functions. Finally, Chapter 4 discusses some open questions and contains some partial results about the relationship between pseudo-continuation and Bochner-Bohnenblust continuation.

CHAPTER 1

Preliminaries

1.1. Basic Measure Theory

In this section, we follow [**16**]. A measure μ on the set $\mathbb{R}$ of real numbers is a function $\mu : Y \subset \mathcal{P}(\mathbb{R}) \to \mathbb{C} \cup \{\infty\}$ satisfying certain properties which we will specify later. Here $\mathcal{P}(\mathbb{R})$ denotes the power set of $\mathbb{R}$, and $\subset$ denotes being either a proper or improper subset. Measures are a way of assigning "size" to a set. In order to understand measures, we must first be clear on the type of domains they can be defined on. We need the following definition.

DEFINITION 1.1.1. A collection Σ of subsets of $\mathbb{R}$ is called a σ-algebra on $\mathbb{R}$ if the following conditions hold:

(1) $\mathbb{R} \in \Sigma$.
(2) $A \in \Sigma$ implies $A^c \in \Sigma$. (Here A^c is the complement of A in $\mathbb{R}$.)
(3) If $A_n \in \Sigma$ for all $n \in \mathbb{N}$, then $\bigcup_{n=1}^{\infty} A_n \in \Sigma$.

So a σ-algebra on $\mathbb{R}$ is just a collection of subsets of $\mathbb{R}$ which contains $\mathbb{R}$ and is closed under complements and countable unions. Note that, by De Morgan's laws, (2) and (3) imply that a σ-algebra is closed under countable intersection, and (1) and (2) imply that a σ-algebra contains the empty set $\emptyset$. An important fact is that for any collection of subsets of $\mathbb{R}$, there is a smallest σ-algebra containing every set in the collection.

DEFINITION 1.1.2. The *Borel sets* of $\mathbb{R}$ are be the elements of the smallest σ-algebra containing all the open subsets of $\mathbb{R}$.

We are now ready to give a precise definition of a measure.

DEFINITION 1.1.3. (1) Let Σ denote the σ-algebra of Borel sets of $\mathbb{R}$. A *positive Borel measure* is a function μ from Σ into $[0, \infty]$ which is countably additive. This last statement means that if $A_1, A_2, A_3, \ldots$ are pairwise disjoint Borel sets, then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

We also assume that $\mu(A) < \infty$ for some $A \in \Sigma$. (Otherwise, the measure is not interesting).

(2) A *complex measure* is the same as a positive measure, except that its range is contained in $\mathbb{C} \cup \{\infty\}$.
(3) A *finite complex measure* is a complex measure whose range *does not* contain ∞.

DEFINITION 1.1.4. $M(\mathbb{R})$ is the set of all finite complex measures on the Borel sets of $\mathbb{R}$. Let $M_+(\mathbb{R})$ denote the positive elements of $M(\mathbb{R})$.

From **[16]** we gather up some useful facts about measures.

PROPOSITION 1.1.5. *If $\mu \in M_+(\mathbb{R})$, we have the following:*

(1) $\mu(\emptyset) = 0$.
(2) $\mu(A_1 \cup A_2 \cup \cdots \cup A_n) = \mu(A_1) + \mu(A_2) + \cdots + \mu(A_n)$ *for* $A_1, A_2, \cdots, A_n$ *pairwise disjoint.*
(3) *If* $A_1 \subset A_2 \subset A_3 \subset \cdots$ *and* $A = \bigcup_{n=1}^{\infty} A_n$, *then we have that*

$$\lim_{n\to\infty} \mu(A_n) = \mu(A).$$

(4) *If* $A_1 \supset A_2 \supset A_3 \supset \cdots$, A_i *and* $A = \bigcap_{n=1}^{\infty} A_n$, *then*

$$\lim_{n\to\infty} \mu(A_n) = \mu(A).$$

(5)

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty} \mu(A_n),$$

whether the A_i *are disjoint or not.*
(6) $A \subset B$ *implies* $\mu(A) \le \mu(B)$.
(7) *If* $\mu \in M(\mathbb{R})$, *then (1)-(4) hold.*

THEOREM 1.1.6 (The Jordan Decomposition Theorem). *Let μ be a complex measure in $M(\mathbb{R})$. Then there are $\mu_1, \mu_2, \mu_3, \mu_4 \in M_+(\mathbb{R})$ so that*

$$\mu = (\mu_1 - \mu_2) + i(\mu_3 - \mu_4).$$

We now mention *Lebesgue measure*, which is undoubtedly the most important example of a Borel measure. It is a Borel measure on the real numbers, which means that it is defined on the Borel sets of the real numbers. Lebesgue measure is the unique measure which assigns to every open interval (a, b) and every closed interval $[a, b]$ the measure $b - a$. In fact, Lebesgue measure can be extended to be defined on more sets than just the Borel sets, but this is not important to us at the moment. We denote Lebesgue measure by m.

It will be useful to have a definition of the notion of the size of a measure. For positive measures, a natural value to take is just $\mu(\mathbb{R})$, but for complex measures this definition does not work, since we can have $\mu(\mathbb{R}) = 0$ but have $\mu(E) \neq 0$ for some set E, which cannot happen for positive measures. In order to try to define a type of norm for complex measures, we make the following definition.

DEFINITION 1.1.7. Let μ be a finite complex Borel measure on $\mathbb{R}$. The function $|\mu| : \Sigma \to [0, \infty]$ is defined by

$$|\mu|(E) = \sup \sum_{j=1}^{N} |\mu(E_i)|,$$

where the $\{E_i\}$ form a finite partition of E into Borel sets, and the supremum is taken over all possible finite partitions of E.

This function $|\mu|$ is called the *total variation measure* of μ. As one might guess from the name, it turns out that μ is actually a measure. We state this as a theorem.

THEOREM 1.1.8. *If $\mu \in M(\mathbb{R})$, then $|\mu| \in M_+(\mathbb{R})$.*

Now that we have defined $|\mu|$, we may define

$$\|\mu\| = |\mu|(\mathbb{R}).$$

This is called the *total variation* of μ. From the definition of $|\mu|$ we see that, for a given finite complex measure μ, $|\mu(E)| \leq \|\mu\|$ for all sets $E \in \Sigma$.

If μ and λ are two complex measures, and $c \in \mathbb{C}$, let us define $\mu + \lambda$ and $c\mu$ by

$$(\mu + \lambda)(E) = \mu(E) + \lambda(E)$$

and

$$(c\mu)(E) = c(\mu(E)).$$

If $\mu, \lambda \in M(\mathbb{R})$ then $c\mu$ and $\mu + \lambda$ are both in $M(\mathbb{R})$, so $M(\mathbb{R})$ is a complex vector space. We now have the following theorem.

THEOREM 1.1.9. *The set $M(\mathbb{R})$ with norm $\|\cdot\|$ forms a normed vector space. That is to say $M(\mathbb{R})$ forms a vector space and $\|\cdot\|$ satisfies the usual properties of a norm:*

(1) $\|\mu\| \geq 0$
(2) $\|\mu_1 + \mu_2\| \leq \|\mu_1\| + \|\mu_2\|$
(3) $\|c\mu\| = |c| \, \|\mu\|$.

Besides the Jordan composition, there is another common decomposition of measures. It is known as the *Lebesgue decomposition.* We first make the following preliminary definition.

DEFINITION 1.1.10. Let m be Lebesgue measure on $\mathbb{R}$ (although any positive measure would work) and let $\lambda \in M(\mathbb{R})$.

(1) If there is a Borel set E such that for any Borel set A we have that $\lambda(E \cap A) = \lambda(A)$, we say that λ is *concentrated* on E.
(2) If $m(E) = 0$ implies that $\lambda(E) = 0$ for every set $E \in \Sigma$, we say that λ is *absolutely continuous* with respect to m, and we write $\lambda \ll m$.
(3) If λ is concentrated on a set E with $m(E) = 0$ we say that λ is *singular with respect to m* or just *singular*, and we write $\lambda \perp m$.

An important observation is that if $\lambda \ll m$ and $\lambda \perp m$, then $\lambda = 0$.

We can now discuss two extremely important theorems in measure theory.

THEOREM 1.1.11 (The Lebesgue Decomposition Theorem). *Suppose $\lambda \in M(\mathbb{R})$. Then we may write*

$$\lambda = \lambda_a + \lambda_s,$$

where $\lambda_a, \lambda_s \in M(\mathbb{R})$, $\lambda_a \ll m$, and $\lambda_s \perp m$. Furthermore, this decomposition is unique.

THEOREM 1.1.12 (The Radon-Nikodym Theorem). *Suppose that $\lambda \in M(\mathbb{R})$ and that $\lambda \ll m$. Then there is a unique $h \in L^1(\mathbb{R})$ such that*

$$\lambda(E) = \int_E h \, dm$$

for any Borel set E.

In Theorem 1.1.11, we call λ_a the absolutely continuous part of λ and λ_s the singular part. We call the function h in Theorem 1.1.12 the Radon-Nikodym derivative of λ with respect to μ, and write

$$d\lambda = h\ dm$$

or

$$h = \frac{d\lambda}{dm}.$$

Note that the converse of the Radon-Nikodym theorem is also true. It states that if $h \in L^1(m)$ then the measure μ defined by

$$\mu(E) := \int_E h\ dm$$

is a finite measure in $\mathbb{R}$, and $\mu \ll m$.

We have just mentioned one notion of the derivative of a measure, the Radon-Nikodym derivative. There is another notion of the derivative of a measure which we will now discuss. It turns out that both of these notions are related.

DEFINITION 1.1.13. Let $\mu \in M(\mathbb{R})$ be a real measure. Then we define the *upper derivative of* μ *at* x by

$$(\overline{D}\mu)(x) = \limsup_{r \to 0^+} \frac{\mu((x-r, x+r))}{2r}.$$

If we replace sup by inf, we get the quantity $(\underline{D}\mu)(x)$, which is called the *lower derivative of* μ *at* x. Also, note that for all $x \in \mathbb{R}$,

$$-\infty \leq (\underline{D}\mu)(x) \leq (\overline{D}\mu)(x) \leq \infty.$$

DEFINITION 1.1.14. If $(\underline{D}\mu)(x)$ and $(\overline{D}\mu)(x)$ are finite and equal, we say that μ has a symmetric derivative at x and we set

$$(D\mu)(x) = (\underline{D}\mu)(x) = (\overline{D}\mu)(x).$$

If $\mu \in M(\mathbb{R})$ is a complex measure then $\mu = \mu_1 + i\mu_2$ for some $\mu_1, \mu_2 \in M(\mathbb{R})$, and we define $D\mu = D\mu_1 + iD\mu_2$ whenever the $D\mu_j$ exist.

We now have the following major theorem.

THEOREM 1.1.15. *Let* $\mu \in M(\mathbb{R})$.

(1) $D\mu$ *exists for* m*-a.e.* $x \in \mathbb{R}$.
(2) $D\mu \in L^1(\mathbb{R})$.
(3) *The Radon-Nikodym derivative of* μ *is equal to* $D\mu$ m*-a.e.*

Fact (3) is known as the Lebesgue differentiation theorem.

1.2. Integral Transforms

In this section, we discuss various integral transformations which will be useful later.

DEFINITION 1.2.1. Let $z = x + iy$ where $y > 0$.

(1) The *Poisson kernel* $P_z : \mathbb{R} \to \mathbb{R}$ is defined as

$$P_z(t) = \frac{1}{\pi} \operatorname{Im}\left(\frac{1}{t-z}\right) = \frac{1}{\pi} \frac{y}{(x-t)^2 + y^2}.$$

(2) The *conjugate Poisson kernel* $Q_z : \mathbb{R} \to \mathbb{R}$ is defined as

$$Q_z(t) = \frac{1}{\pi}\operatorname{Re}\left(\frac{1}{t-z}\right) = \frac{1}{\pi}\frac{x-t}{(x-t)^2+y^2}.$$

(3) The *Borel kernel* $B_z : \mathbb{R} \to \mathbb{C}$ is defined as

$$B_z(t) = \frac{1}{\pi}\frac{1}{t-z}.$$

Notice that $Q_z(t) + iP_z(t) = B_z(t)$. We now have:

DEFINITION 1.2.2. (1) The *Poisson transform* of a measure $\mu \in M(\mathbb{R})$ is the function $P\mu : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$ defined by

$$(P\mu)(z) = \int_{\mathbb{R}} P_z(t)d\mu(t).$$

(2) The *Conjugate Poisson transform* of a measure $\mu \in M(\mathbb{R})$ is the function $Q\mu : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$ defined by

$$(Q\mu)(z) = \int_{\mathbb{R}} Q_z(t)d\mu(t).$$

(3) The *Borel transform* of a measure $\mu \in M(\mathbb{R})$ is the function $B\mu : \mathbb{C}\setminus\mathbb{R} \to \mathbb{C}$ defined by

$$(B\mu)(z) = \int_{\mathbb{R}} B_z(t)d\mu(t).$$

The Poisson transform is also called the Poisson integral. The domain of all these transforms is $\mathbb{C} \setminus \mathbb{R}$. Clearly $B\mu$ is analytic on $\mathbb{C} \setminus \mathbb{R}$. We also have that $P\mu$ and $Q\mu$ are harmonic on $\mathbb{C} \setminus \mathbb{R}$ since they are the imaginary and real parts of $B\mu$, respectively.

One of the most important theorems about the Poisson kernel, called Fatou's Theorem, is as follows. It is found in [**16**] (Theorem 11.10) stated for the disc, but here we state it for the upper half plane.

THEOREM 1.2.3. *Suppose that $\mu \in M(\mathbb{R})$ and is real. Then for each $x \in \mathbb{R}$,*

$$(\underline{D}\mu)(x) \leq \liminf_{y\to 0^+}(P\mu)(x+iy) \leq \limsup_{y\to 0^+}(P\mu)(x+iy) \leq (\overline{D}\mu)(x).$$

Wherever $(D\mu)(x)$ exists and is finite, which occurs a.e., we have

$$\lim_{y\to 0^+}(P\mu)(x+iy) = (D\mu)(x).$$

We also define the Hilbert Transform, another extremely important integral transform. For $\epsilon > 0$, and for $\mu \in M(\mathbb{R})$, define

$$(H_\epsilon\mu)(x) = \int_{|x-t|\geq\epsilon} \frac{1}{x-t}d\mu(t) \tag{1.2.4}$$

Since $\mu \in M(\mathbb{R})$, $(H_\epsilon\mu)(x)$ is well defined for every x and for every $\epsilon > 0$. Set

$$(H\mu)(x) = \lim_{\epsilon\to 0}(H_\epsilon\mu)(x)$$

wherever it exists. In [**13**] we find that the $(H\mu)(x)$ exists a.e.[m]. The limit in the above equation is also called the principle value of the integral in equation (1.2.4) and is denoted by placing "*P.V.*" in front of the integral. We thus make the following definition.

DEFINITION 1.2.5. Let $\mu \in M(\mathbb{R})$. The function $(H\mu) : \mathbb{R} \to \mathbb{C}$ defined for m-a.e. $x \in \mathbb{R}$ by

$$(H\mu)(x) = P.V. \int \frac{1}{x-t} d\mu(t)$$

is called the *Hilbert Transform of* μ.

THEOREM 1.2.6 (Kolmogorov). *For* $\mu \in M(\mathbb{R})$,

$$m(\{x \in \mathbb{R} : |(H\mu)(x)| > \lambda\}) \leq \frac{C\|\mu\|}{\lambda}$$

for some constant C *independent of* μ *and* λ.

The two previous theorems may be found, stated slightly differently, in **[13]**.

THEOREM 1.2.7. *For* $\mu \in M(\mathbb{R})$,

$$\lim_{y \to 0^+} (Q\mu)(x+iy) = (H\mu)(x)$$

for m*-a.e.* x.

The main type of integral transformation we study is the *Laplace transform*. We will define it here, and will discuss it more later.

DEFINITION 1.2.8. Let $\phi : \mathbb{R} \to \mathbb{C}$ be a bounded function. Let $\mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$ denote the right half complex plane. Then the Laplace transform of ϕ, which we write as $\mathfrak{L}[\phi]$, is the function $\mathfrak{L}[\phi] : \mathbb{C}_+ \to \mathbb{C}$ defined by

$$(\mathfrak{L}[\phi])(z) = \int_0^\infty \phi(t) e^{-zt} dt.$$

Lastly, we define the Fourier transform of a measure.

DEFINITION 1.2.9. Let $\mu \in M(\mathbb{R})$. Then the Fourier transform of μ, written as $\widehat{\mu}$, is the function $\widehat{\mu} : \mathbb{R} \to \mathbb{C}$ defined by

$$\widehat{\mu}(x) = \int_{\mathbb{R}} e^{ixt} d\mu(t).$$

Notice that $|\widehat{\mu}(x)| \leq \|\mu\|$ for all $x \in \mathbb{R}$ and that a simple application of the Lebesgue dominated convergence theorem shows that $\widehat{\mu}$ is a continuous function on $\mathbb{R}$.

1.3. Fourier Series

In order to understand almost periodic functions, one must first understand periodic functions, especially since almost periodic functions have many properties which are analogous to properties of periodic functions. Also, knowledge of periodic functions is needed to prove some theorems about boundary values of analytic functions.

DEFINITION 1.3.1. A periodic function $f : \mathbb{R} \to \mathbb{C}$ of period k is a function such that $f(x+k) = f(x)$ for all $x \in \mathbb{R}$.

In particular, this definition implies that $f(x+nk) = f(x)$ for any integer n. From now on, all the periodic functions we will discuss will have period 2π, since all the results we prove easily generalize to arbitrary periods. Also, since they are periodic, it is only necessary to discuss their values on $[0, 2\pi)$. In addition, we assume all the functions discussed in this section are integrable with respect to m on $[0, 2\pi)$. The set of all functions on $[0, 2\pi)$ integrable with respect to m is denoted by $L^1(0, 2\pi)$.

DEFINITION 1.3.2. The *mean value* of $f \in L^1(0, 2\pi)$ is defined by

$$M(f) = \frac{1}{2\pi} \int_0^{2\pi} f(t)dt.$$

The most important of all concepts relating to periodic functions is the concept of a Fourier series. We first define a Fourier series as a formal series, without asserting anything about its convergence.

DEFINITION 1.3.3. Let $f \in L^1(0, 2\pi)$. The *Fourier series* which corresponds to f is defined by

$$\sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{inx},$$

where we write

$$f \sim \sum_{j=-\infty}^{\infty} \widehat{f}(n)e^{inx}$$

to indicate this correspondence. The *Fourier coefficient* $\widehat{f}(n)$ is defined by

$$\widehat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(x)e^{-inx}dx.$$

We use the notation

$$(f, g) = \frac{1}{2\pi} \int_0^{2\pi} f(x)\overline{g(x)}dx,$$

whenever this integral exists. Using this notation, we have that $\widehat{f}(n) = (f, e^{inx})$. Note that since we have assumed that f is integrable, the Fourier coefficients always exist.

Here are some standard facts about Fourier series.

PROPOSITION 1.3.4. *The set of functions* $\{e^{inx}\}_{n=-\infty}^{n=\infty}$ *is orthonormal, that is* $(e^{inx}, e^{imx}) = \delta_{nm}$.

THEOREM 1.3.5 (The Riemann-Lebesgue Lemma). *If* $f \in L^1(0, 2\pi)$, *then*

$$\lim_{|n| \to \infty} \widehat{f}(n) = 0.$$

A proof can be found in **[11]** (Theorem 30).

THEOREM 1.3.6 (Bessel's Inequality). *For* $f \in L^2(0, 2\pi)$, *we have*

$$\sum_{n=-N}^{N} |\widehat{f}(n)|^2 \leq \frac{1}{2\pi} \int_0^{2\pi} |f(t)|^2 dt.$$

We also have two other theorems, which can be found in **[16]** section 4.26.

THEOREM 1.3.7 (The Riesz-Fischer theorem). *If* $\{c_n\}_{n=-\infty}^{\infty}$ *is a sequence of complex numbers, and*

$$\sum_{j=-\infty}^{\infty} |c_n|^2 < \infty,$$

then there exists a periodic function $f \in L^2(0, 2\pi)$ *such that* $\widehat{f}(n) = c_n$ *for all* $n \in \mathbb{Z}$.

THEOREM 1.3.8 (Parseval's theorem). *If* $f, g \in L^2(0, 2\pi)$, *then*

$$\sum_{n=-\infty}^{\infty} \widehat{f}(n)\overline{\widehat{g}(n)} = (f, g).$$

An important question is that of the convergence of Fourier series, in various senses.

THEOREM 1.3.9. *Let* $f \in L^2(0, 2\pi)$. *Then the Fourier series of* f *converges to* f *in the "mean", or equivalently in the norm of* L^2. *That is,*

$$\lim_{N\to\infty} \int_0^{2\pi} \left| f(x) - \sum_{n=-N}^{N} \widehat{f}(n)e^{inx} \right|^2 dx = 0.$$

The above theorem tells us about convergence of Fourier series in the L^2 norm. However, it does not tell us about pointwise convergence, which is an interesting and very complicated question. We state, but do not prove, the following very deep theorems.

THEOREM 1.3.10 (Carleson). *The Fourier series of an* $L^2(0, 2\pi)$ *function converges to that function almost everywhere.*

THEOREM 1.3.11 (Hunt). *For* $p > 1$, *the Fourier series of an* $L^p(0, 2\pi)$ *function converges to that function almost everywhere.*

THEOREM 1.3.12 (Kolmogorov). *There exists a function in* $L^1(0, 2\pi)$ *whose Fourier series diverges everywhere.*

The first two are Theorem 12.8 in [1]. The result of Kolmogorov may be found in [12].

It is often more useful to look at other types of summability of Fourier series. Consider a series $\sum_{n=-\infty}^{\infty} a_n$ and define $S_N = \sum_{n=-N}^{N} a_n$. Then S_N is the N^{th} partial sum of the series. We now define

$$\sigma_N = \frac{1}{N+1} \sum_{n=0}^{N} S_n,$$

so that σ_N is the average of the first $N+1$ partial sums of the series.

DEFINITION 1.3.13. If for the series $\sum_{n=-\infty}^{\infty} a_n$ we have that

$$\lim_{N\to\infty} S_N = A,$$

then we say that the series converges to A *in the sense of Cauchy.*

DEFINITION 1.3.14. If for the series $\sum_{n=-\infty}^{\infty} a_n$ we have that

$$\lim_{N\to\infty} \sigma_N = A$$

then we say that the series is $(C, 1)$ *summable to* A or *Cesàro summable to* A.

If a series is $(C, 1)$ summable to some A, then we say it is $(C, 1)$ summable.

THEOREM 1.3.15. *If an infinite series converges to A (in the sense of Cauchy), then it is $(C,1)$ summable to A.*

This means that if a series converges and we compute its $(C,1)$ sum instead of its Cauchy sum, we will get the same result. Otherwise, it would not be sensible to speak of the $(C,1)$ sum as an actual sum of the series.

Another important method of summation is the method of Abel. We will need it later.

DEFINITION 1.3.16. Consider the series $\sum_{n=-\infty}^{\infty} a_n$ Define

$$\tau(r) = \sum_{n=-\infty}^{\infty} a_n r^{|n|} \tag{1.3.17}$$

for $0 < r < 1$, if the sum in (1.3.17) converges in the sense of Cauchy. Then we say that the series is *Abel summable to A* if

$$\lim_{r \to 1^-} \tau(r) = A.$$

EXAMPLE 1.3.18. Consider the series

$$\sum_{n=-\infty}^{\infty} a_n$$

where

$$a_n = \begin{cases} 0 & \text{if } n < 0 \\ (-1)^n & \text{if } n \geq 0 \end{cases}.$$

Its N^{th} partial sum S_N is 0 if N is odd and 1 if n is even. Thus,

$$\lim_{N \to \infty} S_N \text{ does not converge.}$$

Therefore, the series is not Cauchy summable. However, we have

$$\sigma_N = \frac{1}{N+1} \sum_{n=0}^{N} S_n = \frac{\lfloor N/2 \rfloor + 1}{N},$$

so

$$\lim_{N \to \infty} \sigma_N = \frac{1}{2}.$$

Thus

$$\sum_{n=-\infty}^{\infty} a_n = \frac{1}{2} \text{ in the sense of Cesàro.}$$

Lastly, we have

$$\tau(r) = \sum_{n=-\infty}^{\infty} a_n r^{|n|} = \sum_{n=0}^{\infty} (-1)^n r^n = \frac{1}{1+r}.$$

So

$$\lim_{r \to 1^-} \tau(r) = \frac{1}{2}$$

and thus

$$\sum_{n=-\infty}^{\infty} a_n = \frac{1}{2} \text{ in the sense of Abel.}$$

If a series is Abel summable to some A, then we say it is Abel summable. We then have the following theorem, which is Theorem 55 in [**10**].

THEOREM 1.3.19. *If an infinite series is* $(C,1)$ *summable to* A *then it is also Abel summable to* A.

COROLLARY 1.3.20. *If an infinite series is Cauchy summable to* A, *then it is Abel summable to* A.

This theorem states that Abel summation is at least as strong as Cesàro summation, which is at least as strong as Cauchy summation. In other words, the Abel sum of a series is defined and gives the same value as the Cesàro sum of the series, when the Cesàro sum of the series is defined. Also, the Cesàro sum of a series is defined and gives the same value as the Cauchy sum of the series, when the Cauchy sum is defined.

The (Cauchy) sum of a Fourier series at a point x is simply the Cauchy sum of the series

$$\sum_{n=-\infty}^{\infty} c_n e^{inx}. \tag{1.3.21}$$

The *Fejér sum* of a Fourier series at a point x is simply the $(C,1)$ sum of the series 1.3.21. The *Abel sum* of a Fourier series at a point x is simply the Abel sum of the series 1.3.21.

A Fourier series is said to be Fejér summable at a point if its Fejér sum exists at that point; the same is true for Cauchy summability. We now state three theorems about summability of Fourier series. The first two correspond to Theorem 73 in [**11**]. The third follows from the second theorem, and from Theorem 1.3.19.

THEOREM 1.3.22 (Fejér). *If* f *is continuous and periodic with period* 2π *then it is Fejér summable to itself for all* $x \in [0, 2\pi)$.

THEOREM 1.3.23 (Lebesgue). *If* $f \in L^1(0,2\pi)$ *then the Fourier series of* f *is Fejér summable to* f *almost everywhere.*

THEOREM 1.3.24 (Fatou). *If* $f \in L^1(0,2\pi)$, *then the Fourier series of* f *is Abel summable to* f *almost everywhere.*

Note that the above theorem of Lebesgue implies that if $\widehat{f}(n) = \widehat{g}(n)$ for all n and $f, g \in L^1(0,2\pi)$ then $f = g$ almost everywhere. This is true because the Fourier series for f will be the same as the Fourier series for g, and thus they will have the same Fejér sum almost everywhere. But this sum equals f almost everywhere and g almost everywhere, so f and g must be equal almost everywhere.

1.4. Almost Periodic Functions

Almost periodic functions, of the type we will be discussing, are functions from the real line to the complex plane. We will discuss some preliminary notions and then define them. In general, we follow [5] and [9].

DEFINITION 1.4.1. Let $f : \mathbb{R} \to \mathbb{C}$ be a function. An *ϵ-translation number* for f is a number τ such that for all $x \in \mathbb{R}$, we have

$$|f(x-\tau) - f(x)| < \epsilon. \tag{1.4.2}$$

We call the set of ϵ-translation numbers for f, $E_f(\epsilon)$.

DEFINITION 1.4.3. A set in $\mathbb{R}$ is called *relatively dense* if some number L exists such that every open interval of length L in $\mathbb{R}$ contains at least one element of the set.

EXAMPLE 1.4.4.

(1) The set of all multiples of 100 is relatively dense, since every open interval of length 101 contains an element of the set.
(2) The set of all squares of integers is not relatively dense, since the distance between n^2 and $(n+1)^2$ is $2n+1$, which approaches ∞ as $n \to \infty$.

DEFINITION 1.4.5. A *uniformly almost periodic function*, or *almost periodic function* for short, is a continuous function $f : \mathbb{R} \to \mathbb{C}$ such that for all $\epsilon > 0$, $E_f(\epsilon)$ is relatively dense.

Note that a continuous function f is periodic with period p if and only if the 0-translation numbers of f are all numbers of the form pn, where n is any integer. The almost periodic functions are "almost periodic" because they "almost" display periodicity, in the sense that for arbitrarily small ϵ they have in a sense regularly spaced ϵ-translation numbers.

PROPOSITION 1.4.6. *Any periodic function is almost periodic.*

Definition 1.4.1 characterizes almost periodic functions by their structural properties. It is desirable to find a more analytical characterization of them. Consider the class of all functions of the form

$$\sum_{n=0}^{N} a_n e^{i\lambda_n x}$$

where the λ_n are arbitrary real numbers. We call functions of this form *trigonometric polynomials.*

THEOREM 1.4.7. *Any trigonometric polynomial is almost periodic.*

PROOF. This theorem follows from Proposition 1.4.6 and part (3) of Theorem 1.4.9 (see below). □

Let the *uniform closure* of a class of functions be the set of all functions which can be uniformly approximated by functions in the class to an arbitrary degree of accuracy. Then we have the following:

THEOREM 1.4.8 (The Fundamental Theorem). *The set of uniform almost periodic functions is identical with the uniform closure of the trigonometric polynomials.*

Theorem 1.4.25 (see below) provides a proof of the Fundamental Theorem.

Given (uniform) almost periodic functions f and g, then using definition 1.4.5 one can show the following:

THEOREM 1.4.9. (1) *f is bounded.*
(2) *f^2 and $|f|$ are almost periodic.*
(3) *$f+g$ is almost periodic.*
(4) *fg is almost periodic.*

THEOREM 1.4.10. *Suppose $\{f_n\}$ is a sequence of almost periodic functions and $f_n \to f$ uniformly. Then f is almost periodic.*

COROLLARY 1.4.11. *The set of almost periodic functions with metric $d(f,g) = \sup\{|f(x) - g(x)| : x \in \mathbb{R}\}$ is a complete metric space.*

We also have the following very important theorem:

THEOREM 1.4.12. *Let f be an almost periodic function. Then the limit*

$$M(f) = \lim_{N\to\infty} \frac{1}{N} \int_0^N f(t)\, dt \tag{1.4.13}$$

exists and is finite.

We call the value $M(f)$ the *mean value* of f.

EXAMPLE 1.4.14. If f is a periodic function with period p, then

$$M(f) = \frac{1}{p} \int_0^p f(t) dt.$$

PROOF. Let $N > 0$ and define $K = K(N) = \lfloor N/p \rfloor$. Then we have

$$N = Kp + r$$

where r is a function of N and $0 \le r(N) < p$ for all N. Thus,

$$\begin{aligned}
\frac{1}{N}\int_0^N f(t)dt &= \frac{1}{N}\sum_{n=0}^{K-1}\int_{np}^{(n+1)p} f(t)dt + \frac{1}{N}\int_{Kp}^{Kp+r} f(t)dt \\
&= \frac{1}{N}\sum_{n=0}^{K-1}\int_0^p f(t)dt + \frac{1}{N}\int_{Kp}^{Kp+r} f(t)dt \\
&= \frac{K}{N}\int_0^p f(t)dt + \frac{1}{N}\int_0^r f(t)dt \\
&= \frac{N-r}{pN}\int_0^p f(t)dt + \frac{1}{N}\int_0^r f(t)dt
\end{aligned}$$

Now, since $0 \le r < p$, for all N, we have that as $N \to \infty$ the first term approaches

$$\frac{1}{p}\int_0^p f(t)dt,$$

and the second approaches 0. Thus,

$$M(f) = \lim_{N\to\infty} \frac{1}{N}\int_0^N f(t)dt = \frac{1}{p}\int_0^p f(t)dt.$$

□

The following is not difficult to see but will be important later.

PROPOSITION 1.4.15. *If*

$$f(x) = a_0 + \sum_{n=1}^{N} e^{i\lambda_n x}$$

where $\lambda_n \neq 0$ for all n, then

$$M(f) = a_0.$$

In other words, the mean value of a trigonometric polynomial is equal to its constant term.

Any function of the form $e^{i\lambda x}$ is periodic and thus almost periodic. Let $e_\lambda(x) = e^{i\lambda x}$. Then, for any almost periodic function f, the function $e_\lambda f$ is almost periodic and we may define

$$a_f(\lambda) = M(fe_{-\lambda}), \tag{1.4.16}$$

When it is clear what function we are talking about, we will sometimes simply write $a(\lambda)$. We call this number the *Fourier coefficient corresponding to* λ for f.

THEOREM 1.4.17. *Let* $\{\lambda_n\}_{n=1}^N$ *be a finite set of real numbers. Then for an almost periodic function* f,

$$\sum_{n=1}^N |a_f(\lambda_n)|^2 \leq M(|f|^2).$$

Notice how this theorem resembles Bessel's inequality for periodic functions.

COROLLARY 1.4.18. *For any almost periodic function* f, $a_f(\lambda) = 0$ *for all* $\lambda \in \mathbb{R}$ *except possibly for a countable number of values of* λ.

The corollary must be true since otherwise we could obtain arbitrarily large values for $\sum_{n=1}^N |a(\lambda_n)|^2$ by picking appropriate sets of λ_n's.

DEFINITION 1.4.19. The *spectrum* of an almost periodic function is the set containing all λ for which $a(\lambda) \neq 0$. Let $\sigma(f)$ denote the spectrum.

Suppose we have ordered the spectrum of an almost periodic function as $\{\lambda_1, \lambda_2, \lambda_3, \ldots\}$. Then we may make the following definition.

DEFINITION 1.4.20. The *Fourier series* of an almost periodic function is the formal series

$$\sum_{n=1}^\infty a(\lambda_n)e^{i\lambda_n x}, \tag{1.4.21}$$

If f is an almost periodic function and $\sum_{n=1}^\infty a_n e^{i\lambda_n x}$ is its Fourier series then we write, as with the Fourier series of a periodic function,

$$f \sim \sum_{n=1}^\infty a_n e^{i\lambda_n x}.$$

We now have the following important theorems.

THEOREM 1.4.22 (Parseval's Theorem).

$$\sum_{n=1}^\infty |a(\lambda_n)|^2 = M(|f|^2), \tag{1.4.23}$$

where the sum is taken over $\sigma(f) = \{\lambda_n\}_{n=1}^\infty$.

THEOREM 1.4.24 (The Uniqueness Theorem). *If two almost periodic functions have the same Fourier series, they are equal to each other.*

We can now see many analogies between continuous periodic functions and almost periodic functions. For example, both types of functions have a mean value. However, the most important analogy is that both have a type of Fourier series. In addition, both types of Fourier series satisfy a Parseval's theorem and a uniqueness theorem. It is natural to ask whether there is some analog of Fejér summation for almost periodic functions. It turns out that there is, although it is more complicated than in the case of periodic functions. The following theorem establishes this "Fejér summation."

THEOREM 1.4.25. *Let f be an almost periodic function with Fourier series*

$$\sum_{n=1}^{\infty} a(\lambda_n)e^{i\lambda_n x}.$$

Then there are rational numbers $r_k^{(m)}$ such that

(1) *for all k and m, $0 \leq r_k^{(m)} \leq 1$,*
(2) *$r_k^{(m)} \to 1$, as $m \to \infty$, with k fixed*
(3) *the sequence of functions $s_m \to f$ uniformly as $m \to \infty$, where*

$$s_m(x) = \sum_{k=1}^{n} a(\lambda_k) r_k^{(m)} e^{i\lambda_k x}.$$

Here n is a function of m.

Notice how the Fundamental Theorem follows from this Fejér summation.

1.5. A Generalization of Almost Periodic Functions

The definitions and results of this section can be found in [**3**].

DEFINITION 1.5.1. Let f and g be two Lebesgue measurable functions. Let $l > 0$ and $p \geq 1$. Define the S_l^p distance between f and g as

$$D_{S_l^p} = \|f - g\|_{S_l^p} = \sup_{x\in\mathbb{R}} \left\{ \frac{1}{l} \int_x^{x+l} |f(t) - g(t)|^p dt \right\}^{1/p}.$$

If either l or p are equal to one we omit writing them when convenient.

PROPOSITION 1.5.2. *The S_l^p distance defines a metric.*

DEFINITION 1.5.3. The set of all S_l^p-almost periodic functions is the closure of the trigonometric polynomials under the S_l^p metric.

PROPOSITION 1.5.4. *For a measurable function f, the following are equivalent:*

(1) *The function f is in closure of the trigonometric polynomials under the S_l^p metric, i.e. it is S_l^p-almost periodic.*
(2) *The function f is in closure of the uniformly almost periodic functions under the S_l^p metric.*
(3) *For any $\epsilon > 0$ the set of all S_l^p ϵ-translation numbers of f is relatively dense, where me make the definition that if for some $\epsilon > 0$ and $\tau \in \mathbb{R}$ we have $D_{S_l^p}(f(x+\tau), f(x)) < \epsilon$, then τ is an S_l^p ϵ-translation number of f.*

In fact if f is an S_l^p almost periodic function then for any other $l' > 0$ it is an $S_{l'}^p$ almost periodic function, so from now on we just speak of S^p-almost periodic functions.

For S^p-almost periodic functions, the mean value always exists, as in Theorem 1.4.12. These functions have Fourier series, which contain at most a countable number of non-zero terms, as in Definition 1.4.20 and Corollary 1.4.18. If f is a S^p-almost periodic function, it has Fejér sums which converge to it in the S_l^p norm for any l, analogous to Theorem 1.4.25. There is also a uniqueness theorem for the S_l^p Fourier series, like Theorem 1.4.24. Note that to S_l^p almost periodic functions are considered identical if they are equal in value a.e.

1.6. Some Classical Boundary Value Theorems

We have stated that we will be investigating the boundary values of Laplace transforms, but we have not discussed precisely what these boundary values are and how they are defined. In this section, we define two types of boundary values, the *radial limit* and the *non-tangential limit*. Since we are talking about Laplace transforms, we are strictly speaking only worried about the boundary values for functions in the right and left half plane. However, we will for now usually discuss these limits for the unit disk, since proving things about them is simpler in this case. However, all the results, with appropriate modification of definitions, will hold for the right and left half planes, which can be seen by conformal mapping of the disk onto the right half plane.

DEFINITION 1.6.1. If $f : \mathbb{D} \to \mathbb{C}$, we define the radial limit of f at the point $e^{i\theta}$, where $\theta \in \mathbb{R}$, as $\lim_{r\to 1^-} f(re^{i\theta})$, whenever the limit exists.

Since we are studying these limits, it is important to know conditions for their existence. The following theorem is helpful:

THEOREM 1.6.2 (Fatou). *Let $f : \mathbb{D} \to \mathbb{C}$ be bounded and analytic. Then for almost all $\theta \in \mathbb{R}$, the radial limit of f at $e^{i\theta}$ exists and is finite.*

PROOF. The following proof is from [8]. First, write $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Now, writing $z = re^{i\theta}$ we have

$$|f(z)|^2 = \sum_{n=0}^{\infty} a_n r^n e^{in\theta} \sum_{n=0}^{\infty} \overline{a_n} r^n e^{-in\theta} = \sum_{n=0}^{\infty} c_n(r,\theta),$$

where we have set

$$c_n(r,\theta) = \left(\sum_{j=0}^{n} a_j \overline{a_{n-j}} e^{-in\theta} \right) r^n.$$

Now, for $0 \le r < 1$, the power series for f is absolutely convergent no matter what θ is, so that for a given r we have

$$\sum_{n=0}^{\infty} |a_n| r^n \sum_{n=0}^{\infty} |a_n| r^n = \sum_{n=0}^{\infty} c_n'(r) = M,$$

where we have set

$$c_n'(r) = \left(\sum_{j=0}^{n} |a_j||a_{n-j}| \right) r^n.$$

But we have $c_n'(r) \ge |c_n(r,\theta)|$ for all n, so that each term of the sum of the $c_n(r,\theta)$ is less than a corresponding term of an absolutely convergent series, and this absolutely convergent series is independent of θ. Thus $\sum_{n=0}^{\infty} c_n(r,\theta)$ is uniformly convergent in θ. Now upon integrating term by term in θ, only terms of the form $\int_0^{2\pi} |a_n|^2 r^{2n} = 2\pi |a_n|^2 r^{2n}$ do not vanish, so we have

$$\sum_{n=0}^{\infty} |a_n|^2 r^{2n} = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \leq \sup\{|f(z)|^2 : z \in \mathbb{D}\}.$$

This holds for all $r < 1$. Now, by the Lebesgue monotone convergence theorem,

$$\lim_{r \to 1^-} \sum_{n=0}^{\infty} |a_n|^2 r^{2n} = \sum_{n=0}^{\infty} \lim_{r \to 1^-} |a_n|^2 r^{2n} = \sum_{n=0}^{\infty} |a_n|^2,$$

so

$$\sum_{n=0}^{\infty} |a_n|^2 \leq \sup\{|f(z)|^2 : z \in \mathbb{D}\}.$$

But now by the Riesz-Fischer theorem, we have that the series

$$\sum_{n=0}^{\infty} a_n e^{in\theta}$$

is the Fourier series for some function $f \in L^2(0, 2\pi)$. Now, this series is $(C, 1)$ summable to f almost everywhere, so it must be Abel summable to f almost everywhere by Theorem 1.3.19, so we have that

$$\lim_{r \to \infty} \sum_{n=0}^{\infty} a_n r^n e^{in\theta} = \lim_{r \to \infty} f(re^{i\theta})$$

exists for almost all θ. □

It is useful to know whether or not radial limits are unique. That is, if two bounded functions have the same radial limits, we wish to know whether they are equal.

THEOREM 1.6.3 (F. and M. Riesz). *Let f be analytic and bounded in $\mathbb{D}$, and suppose f has its radial limits equal to zero on some set of positive measure. Then $f = 0$ identically on $\mathbb{D}$.*

Thus, radial limits of bounded analytic functions are unique. However, we have from [2]:

THEOREM 1.6.4 (Bagemihl and Seidel). (1) *There exists a non-zero analytic function in $\mathbb{D}$ with radial limits equal to 0 almost everywhere.*
(2) *There exists a non-zero analytic function in $\mathbb{D}$ with radial limits equal to ∞ almost everywhere.*

The proof is by construction. Note that by the Riesz theorem the function in part (1) must be unbounded. The first part of the theorem says two different unbounded functions can have the same radial limits almost everywhere. The second part illustrates that the hypothesis of boundedness is required in the theorem of Fatou.

Even though if f is unbounded the radial limits of f equaling zero almost everywhere does not imply that $f = 0$, we do not know about the case where f equals zero everywhere. The next theorem deals with this case.

THEOREM 1.6.5. *If a function f analytic in $\mathbb{D}$ has radial limits equal to* 0 everywhere *then it is equal to* 0.

PROOF. We shall prove the theorem for the right half plane, which implies that it holds for $\mathbb{D}$ by conformal mapping. Let S be the finite square $0 < x < 1, 0 < y < 1$. Now, look at the family of functions $\{f_y : 0 < y < 1\}$, where $f_y(x) = f(x + iy)$, where $(x, y) \in S$. Also, define $f_y(0)$ to be the radial limit of f at y. Then each f_y is continuous on $[0, 1]$ and equals 0 at $x = 0$, since the radial limit is 0 everywhere. Now let

$$E_k = \{y : \max_{0 \leq x \leq 1} |f_y(x)| \leq k\},$$

where k is a positive integer. Then

$$\bigcup_{k=1}^{\infty} E_k = [0, 1]$$

(this is false if the limit only exists a.e.) Now if $y_0 \in {E_k}^c$, where $k \in \mathbb{N}$, then $|f(x_0 + iy_0)| > k$ for some $x_0 > 0$. But letting $z_0 = x_0 + iy_0$, we see that for all z in some ball about z_0, $|f(z)| > k$. Then if we take all the imaginary coordinates of points in the ball, we get an open interval about y_0 such that for all y in the interval, $|f(x + iy)| > k$ for some x. So all the y in this interval will belong to ${E_k}^c$. This shows that ${E_k}^c$ is open, and thus that E_k is closed.

But then the Baire category theorem says that some E_k must contain an interval, say $[y_1, y_2]$, where $y_1 < y_2$ (see [**16**], section 5.7). And therefore in the box $S' = [0, 1) \times [y_1, y_2]$, we will have that f is bounded and has radial limit equal to 0 on a set of positive measure. Thus we may apply the Riesz Uniqueness theorem, Theorem 1.6.3, and conclude that $f = 0$ identically in S'. But f is analytic so it must be equal to zero everywhere. □

We now discuss angular sectors in $\mathbb{D}$. Let ξ be a point on the boundary of $\mathbb{D}$ and let β be the argument of the ray from ξ to the origin. Then the angular sector of $\mathbb{D}$ at ξ with angle α is the set $\{z \in \mathbb{D} : |\arg(z - \xi) - \beta| < \alpha/2\}$. We denote this set by $\Gamma_\alpha(\xi)$. It is a triangular shaped region with vertex at ξ.

DEFINITION 1.6.6. $f : \mathbb{D} \to \mathbb{C}$ has a non-tangential limit at ξ if for each α with $0 < \alpha < \pi$ we have

$$\lim_{\substack{z \to \xi \\ z \in \Gamma_\alpha(\xi)}} f(z) = L.$$

THEOREM 1.6.7 (Lindelöf). *Let f be an analytic function bounded in some disk. Let a be a point on the boundary of the disk. If $f(z)$ converges to some number b as z approaches the point a along some Jordan arc, then it has a non-tangential limit at a, and the value of this limit is b.*

COROLLARY 1.6.8 (Fatou). *Let $f : \mathbb{D} \to \mathbb{C}$ be bounded and analytic. Then for almost all $\theta \in \mathbb{R}$, the non-tangential limit of f at $e^{i\theta}$ exists and is finite.*

PROOF. Apply the above theorem of Lindelöf and the Theorem of Fatou on the existence of radial limits almost everywhere (Theorem 1.6.2). □

Note that unlike Theorem 1.6.3, the following theorem applies to unbounded analytic functions as well as bounded ones.

THEOREM 1.6.9 (Privalov). *If a function f analytic in $\mathbb{D}$ has non-tangential limits equal to 0 on some set of positive measure, then it is equal to 0 identically in $\mathbb{D}$.*

1.7. Continuation of Analytic Functions

Sometimes we have two analytic functions, defined on different domains, and we wish to determine whether they are in some sense the same function. The simplest way to do this is through the notion of analytic continuation.

DEFINITION 1.7.1. We say that $f : \mathbb{C}_+ \to \mathbb{C}$ and $F : \mathbb{C}_- \to \mathbb{C}$ are *analytic continuations* of each other across some arc $\gamma \subset i\mathbb{R}$ if there is some analytic function g defined in a domain U which contains γ, and such that $g = f$ on $U \cup \mathbb{C}_+$ and $g = F$ on $U \cup \mathbb{C}_-$.

Here, $\mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0$ is the right half complex plane and $\mathbb{C}_- = \{z \in \mathbb{C} : \mathrm{Re}(z) < 0$ is the left half complex plane. If f and F are analytic continuations of each other, then they are in some sense the "same" function, just defined on different domains.

There is still another type of continuation, pseudo-continuation.

DEFINITION 1.7.2. Let $f : \mathbb{C}_+ \to \mathbb{C}$ and $F : \mathbb{C}_- \to \mathbb{C}$ be analytic. If the non-tangential limits of f and of F exist and are equal a.e., we say that f and F are *pseudo-continuations* of each other.

PROPOSITION 1.7.3. *Pseudo-continuation is compatible with analytic continuation, in the sense that if $f : \mathbb{C}_+ \to \mathbb{C}$ has a pseudo-continuation $F : \mathbb{C}_- \to \mathbb{C}$ and f has an analytic continuation G across an arc of $i\mathbb{R}$ in a neighborhood Ω of the arc, then $G = F$ on Ω.*

PROOF. If there is some function G which is an analytic continuation of f, then G must have the same non-tangential limits as f in $(i\mathbb{R}) \cap \Gamma$, and thus G must have the same non-tangential limit as F almost everywhere, so they are equal by Theorem 1.6.9 (Privalov's uniqueness theorem). Here, U is as in definition 1.7.1. □

This theorem provides the reason whereby we may call pseudo-continuation a "continuation," which we would not want to do if it were not compatible with analytic continuation. There are other types of continuation, for example Bochner-Bohnenblust continuation, which we will define later. Since it is a continuation, it is also compatible with analytic continuation. We will prove in the next chapter that the example of Poincaré mentioned in the introduction, namely

$$f(z) = \sum_{j=1}^{\infty} \frac{c_n}{z - ix_n},$$

has the properties that $f|_{\mathbb{C}_+}$ and $f|_{\mathbb{H}_-}$ are pseudo-continuations of each other.

CHAPTER 2

The Borel Transform

2.1. The Borel Transform

As a preliminary to other results, we prove some theorems about the Borel transforms of measures. Borel transformations are useful since, as we will see later, they are related to Laplace transforms. Let

$$\mathbb{H}_+ = \{z : \operatorname{Im} z > 0\}$$

be the upper half plane. Recall that for $\sigma \in M(\mathbb{R})$, the Borel transform of σ is defined as

$$(B\sigma)(z) = \int \frac{d\sigma(t)}{z-t}.$$

THEOREM 2.1.1. *Let $\sigma \in M_+(\mathbb{R})$. Then*

$$B\sigma(\mathbb{H}_+) \subset \mathbb{H}_+,$$

PROOF. Letting $z = x + iy$, we have that

$$\operatorname{Im}\left(\frac{1}{z-t}\right) = \frac{y}{(x-t)^2+y^2}$$

which is greater than zero if $\operatorname{Im} z = y > 0$. So when $\operatorname{Im} z > 0$,

$$\operatorname{Im}(B\sigma)(z) = \int \operatorname{Im}\left(\frac{d\sigma(t)}{z-t}\right) = \int \operatorname{Im}\left(\frac{1}{z-t}\right) d\sigma(t) > 0.$$

□

THEOREM 2.1.2. *Let $\sigma \in M(\mathbb{R})$. Then the non-tangential limits of $(B\sigma)$ exist m-a.e.*

PROOF. We follow **[18]**. If $\sigma \in M_+(\mathbb{R})$, Theorem 2.1.1 says that $B\sigma$ is an analytic map of $\mathbb{H}_+$ into $\mathbb{H}_+$. Now, let

$$g = \frac{-i(z+1)}{z-1}.$$

Then g is a conformal map from $\mathbb{D}$ to $\mathbb{H}_+$. Set

$$f = g^{-1} \circ (B\sigma) \circ g,$$

Then f maps $\mathbb{D}$ to $\mathbb{D}$ and so has finite non-tangential limits almost everywhere by Fatou's theorem, Theorem 1.6.2. Therefore, $B\sigma$ must have non-tangential (but possibly infinite) limits a.e. So we must show that $B\sigma$ does not have infinite non-tangential limits on a set of positive measure. But $B\sigma$ can only have an infinite non-tangential limit at a point if f has non-tangential limit of 1 there, since the only pole of g is at 1. And f cannot have non-tangential limit of 1 on a set of positive measure unless f is identically 1, by the Riesz theorem (Theorem 1.6.3).

But this is impossible impossible since 1 is not in the range of g. So $B\sigma$ has finite non-tangential limits almost everywhere.

Now, if $\sigma \in M(\mathbb{R})$, write

$$\sigma = (\sigma_1 - \sigma_2) + i(\sigma_3 - \sigma_4),$$

using the Jordan decomposition, where each $\sigma_i \in M_+(\mathbb{R})$. Now, since each $B\sigma_i$ has non-tangential limits a.e., so does $B\sigma$. □

The previous result is, in a sense, not particular to Borel transforms. In fact, more is true.

THEOREM 2.1.3. *Suppose $f : \mathbb{C}_+ \to D$ is analytic, where D is the complex plane with some ray omitted. Then f has non-tangential limits almost everywhere.*

PROOF. By a translation and a rotation, we may assume that $D = \mathbb{C}\setminus(-\infty, 0]$. Define

$$g(z) = [f(z)]^2.$$

Then $g : \mathbb{C}_+ \to \mathbb{C}_+$, and by following a similar procedure to that in Theorem 2.1.2, we see that the non-tangential limits of g exist almost everywhere. Now, if g has non-tangential limits equal to zero on a set of positive measure, then g is equal to zero identically by Theorem 1.6.3, so f is also zero identically, and thus f clearly has non-tangential limits almost everywhere. But if g is not the zero function, then g has non-zero non-tangential limits almost everywhere, so $\log|g(z)|$ must have finite non-tangential limits almost everywhere. But then

$$f(z) = e^{\frac{1}{2}\log|g(z)| + \frac{i}{2}\operatorname{Arg} g(z)}$$

must have non-tangential limits almost everywhere. □

Notice how the Borel Transform $(B\sigma)|_{|\mathbb{H}_+}$ has an analytic continuation to $(B\sigma)_{|\mathbb{H}_-}$ across any arc that avoids the support of the measure σ. We also have the following result.

THEOREM 2.1.4. *Let $\sigma \in M(\mathbb{R})$. Then for all $x \in \mathbb{R}$ we have that*

$$\lim_{y\to 0} y(B\sigma)(x+iy) = -i\sigma(\{x\}).$$

PROOF. For a fixed x,

$$y(B\sigma)(x+iy) = y\int_{\mathbb{R}} \frac{d\sigma(t)}{x+iy-t} = \int_{\mathbb{R}} \frac{y}{(x-t)+iy} d\sigma(t).$$

Now, we have that

$$\left|\frac{y}{(x-t)+iy}\right| = \frac{|y|}{\sqrt{(x-t)^2+y^2}} \leq 1$$

and that

$$\lim_{y\to 0} \frac{y}{(x-t)+iy} = \begin{cases} 0 & \text{if } t \neq x \\ -i & \text{if } t = x \end{cases}.$$

Thus, by the Lebesgue dominated convergence theorem,

$$\lim_{y\to 0} \int_{\mathbb{R}} \frac{y}{(x-t)+iy} d\sigma(t) = \int_{\mathbb{R}} -i\chi_{\{x\}}(t) d\sigma(t) = -i\sigma(\{x\}).$$

□

This theorem implies that the non-tangential limit of a Borel transform can be infinite on a dense set, e.g. for the measure

$$\sigma = \sum_{n=1}^{\infty} 2^{-n}\delta_{\alpha_n},$$

where $\alpha_1, \alpha_2, \ldots$ is an enumeration of the rationals. However, amazingly, the non-tangential limit must still exist almost everywhere for any Borel transform. Also, the Borel transform of a measure σ cannot have an analytic continuation across any arc which contains a point mass of σ.

Now that we have investigated analytic continuation of the Borel transform, we wish to investigate pseudo-continuations. We begin with the following theorem. Recall that

$$(D\sigma)(x) = \lim_{r\to\infty} \frac{\sigma(x-r, x+r)}{2r}$$

wherever this limit exists (which is almost everywhere).

THEOREM 2.1.5. *If $\sigma \in M(\mathbb{R})$ then*

$$\lim_{y\to 0+} ((B\sigma)(x+iy) - (B\sigma)(x-iy)) = 2i(D\sigma)(x) \text{ a.e.}$$

PROOF.

$$\begin{aligned}
(B\sigma)(x+iy) - (B\sigma)(x-iy) &= \frac{1}{\pi}\int \left(\frac{d\sigma(t)}{t-(x+iy)} - \frac{d\sigma(t)}{t-(x-iy)}\right) \\
&= \frac{1}{\pi}\int \frac{[(t-x)+iy] - [(t-x)-iy]d\sigma(t)}{(t-x)^2+y^2} \\
&= \frac{1}{\pi}\int \frac{2iyd\sigma(t)}{(t-x)^2+y^2} \\
&= 2i\int P_{x+iy}(t)d\sigma(t)
\end{aligned}$$

and

$$\lim_{x\to 0+} \int P_{y+ix}d\sigma(t) = (D\sigma)(y),$$

by Theorem 1.2.3. □

COROLLARY 2.1.6. *If $\sigma \in M(\mathbb{R})$ then $(B\sigma)_{|\mathbb{H}_-}$ is a pseudo-continuation of $(B\sigma)_{|\mathbb{H}_+}$ if and only if $\sigma \perp m$.*

PROOF. Recall that

$$D\sigma = \frac{d\sigma}{dx} \text{ almost everywhere,}$$

where $d\sigma/dx$ is the Radon-Nikodym derivative of σ (see Theorem 1.1.12). Thus, by Theorem 2.1.5,

$$(B\sigma)_{|\mathbb{H}_+} = (B\sigma)_{|\mathbb{H}_-}$$

precisely when $d\sigma/dx = 0$ almost everywhere, i.e. when $\sigma \perp m$. □

We also wish to see whether, when $(B\sigma)_{|\mathbb{H}_+}$ has an analytic continuation to $\mathbb{H}_-$, we must have that this analytic continuation is $(B\sigma)_{|\mathbb{H}_-}$. To investigate this question, we use the Borel transform to give an extension of the Cauchy integral formula to the upper half plane, which is interesting in its own right.

THEOREM 2.1.7. *Suppose f is an analytic function in $\mathbb{H}_+$ and continuous in $\mathbb{H}_+ \cup \mathbb{R}$. Further, suppose that f satisfies*

$$|f(z)| < \frac{C}{|z|^\alpha + 1} \text{ for all } z \in \mathbb{H}_+ \cup \mathbb{R},$$

where C is a constant and $\alpha > 1$. Then for any $z \in \mathbb{H}_+$ we have

$$f(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(x)}{x - z} dx.$$

PROOF. Let $z \in \mathbb{H}_+$ and suppose that f is an analytic function in $\mathbb{H}_+$ which is continuous on $\mathbb{H}_+ \cup \mathbb{R}$, and satisfies

$$|f(z)| < \frac{C}{|z|^\alpha + 1} \text{ for all } z \in \mathbb{H}_+ \cup \mathbb{R},$$

where C is a constant and $\alpha > 1$. Then, by the Cauchy integral formula,

$$f(z) = \frac{1}{2\pi i} \int_{-N}^{N} \frac{f(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi i} \int_{\gamma_N} \frac{f(\zeta)}{\zeta - z} d\zeta, \tag{2.1.8}$$

where $N > |z|$ and γ_N is the upper half of the circle with radius N and center 0. (So γ_N is a semi-circular path connecting $-N$ and N.) Now, on γ_N the value of $|f(\zeta)|$ is bounded above by $C'/(N^\alpha + 1)$, which is bounded above by $C'/(N^\alpha)$, where C' is a constant. Also, $|1/(\zeta - z)| < 2/N$ for $N > |2z|$. Thus, on γ_N

$$\left|\frac{f(\zeta)}{\zeta - z}\right| < \frac{C''}{N^{\alpha+1}},$$

where C'' is a constant. The length of γ_N is $2\pi N$, so

$$\begin{aligned}
\left|\frac{1}{2\pi i} \int_{\gamma_N} \frac{f(\zeta)}{\zeta - z} d\zeta\right| &\le \frac{1}{2\pi} \int_{\gamma_N} \left|\frac{f(\zeta)}{\zeta - z}\right| |d\zeta| \\
&< \frac{1}{2\pi} \int_{\gamma_N} \frac{C''}{N^{\alpha+1}} |d\zeta| \\
&= 2\pi N \frac{1}{2\pi} \frac{C''}{N^{\alpha+1}} \\
&= \frac{C''}{N^\alpha}.
\end{aligned}$$

Letting $N \to \infty$, we see that

$$\lim_{N \to \infty} \frac{1}{2\pi i} \int_{\gamma_N} \frac{f(\zeta)}{\zeta - z} d\zeta = 0.$$

Note that f is integrable on the real line, since $\alpha > 1$. Now, let $N \to \infty$ in (2.1.8) to see that the theorem is true. □

We now can prove the following theorem about analytic continuation of Borel transforms.

THEOREM 2.1.9. *There exists a $\sigma \in M(\mathbb{R})$ such*

$$f : \mathbb{H}_+ \to \mathbb{C} \text{ given by } f(z) = (B\sigma)(z)$$

does not analytically continue to

$$g : \mathbb{H}_- \to \mathbb{C} \text{ given by } g(z) = (B\sigma)(z),$$

even though f is continuous on $\mathbb{H}_+ \cup \mathbb{R}$ and g is continuous on $\mathbb{H}_- \cup \mathbb{R}$.

PROOF. Let

$$d\sigma = \phi\, dm$$

where

$$\phi(x) = (x+i)^{-3/2}.$$

In general, for complex z, define

$$\phi(z) = (z+i)^{-3/2},$$

where we use a branch of the square root function so that ϕ is analytic in $\mathbb{H}_+$. We know that $\sigma \in M(\mathbb{R})$ because $\phi|_{\mathbb{R}} \in L^1(\mathbb{R})$. Then using (2.1.7), we can see that, for $z \in \mathbb{H}_+$,

$$(B\sigma)(z) = \frac{1}{\pi i}\int_{\mathbb{R}} \frac{\phi(x)}{x-z}dx = 2\phi(z).$$

Let $f = (B\sigma)_{|\mathbb{H}_+}$ and $g = (B\sigma)_{|\mathbb{H}_-}$. It is clear that f does not analytically continue from the upper half plane to a function analytic in the lower half plane, since $\phi(z)$ must have a branch cut somewhere in the lower half plane. But g is analytic in $\mathbb{H}_-$, since it is a Borel transform. Thus, f cannot be analytically continued to g. □

We know that $\lim_{y\to 0^+}(B\sigma)(x+iy)$ exists for almost every $x \in \mathbb{R}$, and even that if $\sigma \perp m$, that it equals $\lim_{y\to 0^-}(B\sigma)(x+iy)$ almost everywhere, but we have not yet discussed the acutal value of the limit. We can prove the following (which we will do later in a slightly different setting, see Theorem 3.4.3).

THEOREM 2.1.10. *Suppose that*

$$\sigma = \sum_{j=1}^{\infty} a_j \delta_{\lambda_j}$$

where

$$\sum_{j=1}^{\infty} |a_j| < \infty$$

and $\lambda_j \in \mathbb{R}$ *for all* $j \in \mathbb{N}$. *If* $x_0 \in \mathbb{R}$ *is such that*

$$\sum_{j=1}^{\infty} \frac{|a_j|}{|x_0 - \lambda_j|} < \infty$$

then

$$\lim_{y\to 0^+}(B\sigma)(x_0+iy) = \sum_{j=1}^{\infty} \frac{a_j}{x_0-\lambda_j}.$$

The previous theorem applies only at a specific point. We now give a global theorem.

THEOREM 2.1.11. *Suppose that*

$$\sum_{j=1}^{\infty} |a_j| < \infty$$

and that there exists a sequence $\{b_n\}_{n=1}^{\infty} \subset (0,1)$ *such that*

(1)

$$\sum_{j=1}^{\infty} b_j = 1$$

(2)
$$\sum_{j=1}^{\infty} \frac{|a_j|}{b_j} < \infty.$$

If $\lambda_j \in \mathbb{R}$ *for every* $j \in \mathbb{N}$ *and*
$$\sigma = \sum_{j=1}^{\infty} a_j \delta_{\lambda_j}$$
then
$$\lim_{y \to 0+} (B\sigma)(x+iy) = \sum_{j=1}^{\infty} \frac{a_j}{x - \lambda_j}$$
for almost every $x \in \mathbb{R}$.

PROOF. By Theorem 2.1.10, it suffices to show that
$$g(x) = \sum_{j=0}^{\infty} \frac{|a_n|}{|x - \lambda_n|} < \infty \text{ a.e.}$$
Note that
$$m\left(\{x : \frac{1}{|x-\lambda|} > c\}\right) = \frac{2}{\lambda}.$$
Now,
$$g(x) = \sum_{j=1}^{\infty} \frac{|a_j|}{|x - a_j|} > n$$
only if for some j,
$$\frac{|a_j|}{|x - a_j|} > nb_j.$$
But this implies that
$$\{x : g(x) > n\} \subset \bigcup_{j=1}^{\infty} \left\{x : \frac{|c_j|}{|x - \lambda_j|} > nb_j\right\},$$
so
$$m(\{x : g(x) > n\}) \leq \sum_{j=1}^{\infty} m\left(\left\{x : \frac{|c_j|}{|x - \lambda_j|} > nb_j\right\}\right) = \sum_{j=1}^{\infty} \frac{2}{n} \frac{|a_j|}{b_j} = \frac{2}{n} \sum_{j=1}^{\infty} \frac{|a_j|}{b_j}.$$
Now, as $n \to \infty$ we see that
$$m(\{x : g(x) = \infty\}) \leq 0.$$
□

CHAPTER 3

The Laplace Transform

3.1. The Laplace Transform

Let $\phi : \mathbb{R} \to \mathbb{C}$ be a bounded measurable function. Recall that we define the Laplace transform of ϕ, which we write as $\mathfrak{L}[\phi]$, as the function $\mathfrak{L}[\phi] : \mathbb{C}_+ \to \mathbb{C}$, where

$$\mathfrak{L}[\phi](z) = \int_0^\infty \phi(t)e^{-zt}dt.$$

Recall that $\mathbb{C}_+ = \{z \in \mathbb{C} : \text{Re}(z) > 0\}$ is the right half plane. We also write f_ϕ for $\mathfrak{L}[\phi]$.

The Laplace transform has the following basic properties.

THEOREM 3.1.1. (1) *The Laplace transform is linear, i.e.*

$$\mathfrak{L}[c_1\phi + c_2\theta] = c_1\mathfrak{L}[\phi] + c_2\mathfrak{L}[\theta]$$

where ϕ and θ are functions and c_1 and c_2 are scalars.

(2) *If ϕ is bounded, then $\mathfrak{L}[\phi](z)$ exists for all z such that* $\text{Re}(z) > 0$.

(3) *If ϕ is bounded, $\mathfrak{L}[\phi]$ is analytic for* $\text{Re}\, z > 0$.

PROOF. The first assertion is obvious. For the second, notice that

$$\left|\int_0^\infty \phi(z)e^{-zt}dt\right| \leq \|\phi\|_\infty \int_0^\infty e^{-\text{Re}(tz)}dt = \|\phi\|_\infty \int_0^\infty e^{-t\,\text{Re}(z)}dt < \infty$$

as long as $\text{Re}\, z > 0$. The third can be shown to be true by differentiating under the integral by Leibniz's rule. □

Since almost periodic functions are continuous and bounded on $[0, \infty)$ the previous theorem implies that the Laplace transform of any almost periodic function exists for $\text{Re}(z) > 0$.

We have the following result which relates Borel transforms to Laplace transforms. Recall that for $\sigma \in M(\mathbb{R})$ we define the Fourier transform of σ as the function $\widehat{\sigma} : \mathbb{R} \to \mathbb{C}$ defined by

$$\widehat{\sigma}(x) = \int_{\mathbb{R}} e^{ixt}d\sigma(t).$$

THEOREM 3.1.2. *Let $\sigma \in M(\mathbb{R})$. Then*

$$(\mathfrak{L}\hat{\sigma})(z) = \int_{\mathbb{R}} \frac{d\sigma(x)}{z - ix} = i\pi(B\sigma)(-iz).$$

PROOF. We have

$$(\mathfrak{L}\hat{\sigma})(z) = \int_0^\infty e^{-zt}\int_{\mathbb{R}} e^{ixt}d\sigma(x)dt = \int_0^\infty \int_{\mathbb{R}} e^{-zt}e^{ixt}d\sigma(x)dt$$
$$= \int_{\mathbb{R}}\int_0^\infty e^{-zt}e^{ixt}dtd\sigma(x) = \int_{\mathbb{R}} \frac{1}{z - ix}d\sigma(x),$$

where we applied Fubini's theorem, which is permissible since σ is a finite measure. □

THEOREM 3.1.3 (Uniqueness Theorem). *Suppose that ϕ is a bounded continuous function on $[0,\infty)$ and that $\mathfrak{L}[\phi] = 0$. Then $\phi = 0$.*

PROOF. Using the substitution $w = e^{-t}$, we have

$$\mathfrak{L}[\phi](z) = \int_0^\infty \phi(t)e^{-zt}dt = \int_0^1 w^z\phi(-\log(w))\frac{dw}{w} = \int_0^1 w^{z-1}\phi(-\log(w))dw.$$

The previous identity along with the linearity of the integral, says that

$$\int_0^1 P(w)\phi(-\log w)dw = 0$$

for any polynomial P. Since the polynomials are uniformly dense in the set of continuous functions on $[0,1]$, we see that

$$\int_0^1 g(w)\phi(-\log w)dw = 0$$

for all g continuous on $[0,1]$. It now follows that

$$\int_0^1 [(\phi(-\log w)]^2 dw = 0$$

and so

$$\phi(-\log w) = 0 \text{ for all } w > 0.$$

□

Thus, the Laplace transform is unique on the space of continuous bounded functions.

EXAMPLE 3.1.4. If $\phi = e^{i\lambda x}$ then

$$f_\phi(z) = \frac{1}{z-\lambda}.$$

DEFINITION 3.1.5. Analogous to f_ϕ, we define $F_\phi : \mathbb{C}_- \to \mathbb{C}$ by

$$F_\phi(z) = -\int_{-\infty}^0 \phi(t)e^{-zt}dt,$$

where $\text{Re}(z) < 0$.

Note that if $\phi : \mathbb{R} \to \mathbb{C}$, then, letting $x = -t$ we have

$$F_\phi(z) = \int_{-\infty}^0 \phi(t)e^{-zt}dt = -\int_0^\infty \phi(-x)e^{-(-z)x}dx = -f_\psi(-z)$$

where

$$\psi(t) = \phi(-t) \text{ for all } t \in \mathbb{R}.$$

The left Laplace transform has the same basic properties of the Laplace transform, as found in Theorems 3.1.1 and 3.1.3, with appropriate modification of domain. We also have the following:

THEOREM 3.1.6. *Let $\sigma \in M(\mathbb{R})$. Then for* $\text{Re}\, z < 0$

$$F_{\hat\sigma}(z) = \int_{\mathbb{R}} \frac{d\sigma(x)}{z - ix} = i\pi(B\sigma)(-iz).$$

PROOF. We have

$$F_{\widehat{\sigma}}(z) = -\int_{-\infty}^{0} e^{-zt} \int_{\mathbb{R}} e^{ixt} d\sigma(x) dt = -\int_{-\infty}^{0} \int_{\mathbb{R}} e^{-zt} e^{ixt} d\sigma(x) dt$$
$$= -\int_{\mathbb{R}} \int_{-\infty}^{0} e^{-zt} e^{ixt} dt \; d\sigma(x) = \int_{\mathbb{R}} \frac{1}{z - ix} d\sigma(x),$$

where we applied Fubini's theorem, which is permissible since σ is a finite measure. □

EXAMPLE 3.1.7. If $\phi = e^{i\lambda x}$ then

$$F_{\phi}(z) = \frac{1}{z - \lambda}.$$

3.2. Bochner-Bohnenblust Continuation

We now define Bochner-Bohnenblust continuation. In this section, we shall reprove a 1934 theorem of Bochner and Bohnenblust, thereby showing that Bochner-Bohnenblust continuation is compatible with analytic continuation. In the next section, we shall define an extension of Bochner-Bohnenblust continuation and prove that it also is compatible with analytic continuation.

DEFINITION 3.2.1. Let $f : \mathbb{C}_+ \to \mathbb{C}$ and $F : \mathbb{C}_- \to \mathbb{C}$ be analytic. If there is an almost periodic function ϕ such that

$$f(z) = f_{\phi}(z) \text{ for } \operatorname{Re}(z) > 0 \text{ and}$$
$$F(z) = F_{\phi}(z) \text{ for } \operatorname{Re}(z) < 0$$

then we say that f and F are *Bochner-Bohnenblust* continuations of each other.

Note that f and F can be Bochner-Bohnenblust continuations of each other even if they are not analytic continuations of each other. For example, let $\sigma \in M(\mathbb{R})$ be defined by

$$\sigma = \sum_{n=1}^{\infty} 2^{-n} \delta_{\lambda_n}$$

where $\lambda_1, \lambda_2, \ldots$ is an enumeration of the rationals. Let

$$f = f_{\widehat{\sigma}} \text{ and } F = F_{\widehat{\sigma}}.$$

Since

$$\widehat{\sigma}(x) = \sum_{n=1}^{\infty} 2^{-n} e^{i\lambda_n x},$$

$\widehat{\sigma}$ is an almost periodic function and so f and F are Bochner-Bohnenblust continuations of each other. But by theorems 2.1.4 and 3.1.2, both f and F have infinite non-tangential limits on a dense set of $i\mathbb{R}$, so neither can be analytically continued across $i\mathbb{R}$. But, by theorems 2.1.5 and 3.1.2, f and F are pseudo-continuations of each other.

We need the following theorem.

THEOREM 3.2.2. *Suppose that $\phi : \mathbb{R}_+ \to \mathbb{R}$ is a measurable function, and that*

$$|\phi(x)| < \epsilon \text{ for all } x > 0.$$

Then

$$|f_{\phi}(z)| < \frac{\epsilon}{\operatorname{Re} z} \text{ for all } z \in \mathbb{C}_+.$$

PROOF. We have that

$$\left|\int_0^\infty \phi(t)e^{-zt}dt\right| \leq \int_0^\infty |\phi(t)e^{-zt}|dt \leq \int_0^\infty \epsilon e^{-\operatorname{Re}(z)t}dt = \frac{\epsilon}{\operatorname{Re}(z)}.$$

□

Note that a similar result holds for the left Laplace transform.

We are now almost ready to prove that Bochner-Bohnenblust continuation is compatible with analytic continuation. However, we first prove the following lemma. Recall that

$$a_\phi(\lambda) = M(\phi e_{-\lambda}) = \lim_{N\to\infty} \frac{1}{N}\int_0^N \phi(t)e^{-i\lambda t}dt.$$

Also recall that

$$\sigma(\phi) = \{y : a_\phi(y) \neq 0\}$$

is called the spectrum of ϕ and is at most a countable set.

LEMMA 3.2.3. *For all* $y \in \mathbb{R}$,

$$\lim_{x\to 0+} xf_\phi(x+iy) = \lim_{x\to 0-} xF_\phi(x+iy) = a_\phi(y).$$

PROOF. We restrict ourselves to the proof of the lemma for f_ϕ; the proof for F_ϕ is similar. Let $\epsilon > 0$ be given. Let

$$\phi_n(t) = \sum_{k=1}^K b_k^{(n)} e^{it\lambda_k}$$

be one of the Fejér polynomials for ϕ such that

$$|\phi(x) - \phi_n(x)| < \epsilon \text{ for all } x \in \mathbb{R}.$$

(See Theorem 1.4.25, and take $b_k^{(n)} = a(\lambda_n)r_k^{(n)}$.) Here, K is the integer required by Theorem 1.4.25. Note that, every λ_k is in the spectrum of ϕ. Now by Example 3.1.4,

$$f_\phi(z) = \sum_{k=1}^K \frac{b_k^{(n)}}{z - i\lambda_k} + \int_0^\infty (f(t) - f_n(t))e^{-(x+iy)t}dt$$

and thus

$$\begin{aligned} |xf_\phi(x+iy) - a(y)| &\leq \left|\sum_{k=1}^K \frac{xb_k^{(n)}}{x + i(y-\lambda_k)} - a(y)\right| + x\left(\frac{\epsilon}{x}\right) \\ &= \left|\sum_{k=1}^K \frac{xb_k^{(n)}}{x + i(y-\lambda_k)} - a(y)\right| + \epsilon, \end{aligned}$$

where we have used the triangle inequality and Theorem 3.2.2.

Now, if y is not in the spectrum of ϕ, then $a(y) = 0$, and also as $x \to 0+$, we will have

$$\sum_{k=1}^K \frac{x}{x + i(y-\lambda_k)} \to 0.$$

So in this case

$$\limsup_{x\to 0+} |xf_\phi(x+iy) - a(y)| \leq \epsilon.$$

Since ϵ was arbitrary the limit must equal 0.

If y is in the spectrum of ϕ, then all terms of the sum approach 0 as $x \to 0+$, except the term for which $\lambda_k = y$, this term will be equal to $b_k^{(n)}$ for all x. Thus we have

$$\limsup_{x \to 0+} |x f_\phi(x+iy) - a(y)| \le |b_k^{(n)} - a(y)| + \epsilon \le 2\epsilon,$$

for large enough choices of n, since $r_k^{(n)} \to 1$ as $n \to \infty$, which means that $b_k^{(n)} \to a(y)$ as $n \to \infty$. So once again the limit must equal zero, since ϵ was arbitrary. □

We now show that Bochner-Bohnenblust continuation is compatible with analytic continuation. This is a reformulation of the 1934 theorem by Bochner and Bohnenblust[4].

THEOREM 3.2.4. *Bochner-Bohnenblust continuation is compatible with analytic continuation. That is, if f_ϕ has an analytic continuation across some sub-arc of $i\mathbb{R}$, this analytic continuation must equal F_ϕ.*

PROOF. Suppose that f_ϕ has an analytic continuation across (ia, ib). Then we must have that $a(y) = 0$ for all $y \in (a, b)$, by the previous lemma. Now, we can approximate ϕ by

$$\phi_n(t) = \sum_{k=1}^{K} b_k^{(n)} e^{it\lambda_k},$$

where for each n we have $|\phi(x) - \phi_n(x)| < 1/n$ for all $x \in \mathbb{R}$. Now, let

$$R_n(z) := \sum_{k=1}^{K} \frac{b_k^{(n)}}{z - i\lambda_k} = \mathfrak{L}[\phi_n](z).$$

Then we have, for all $z \in (-1, 1) \times (a, b)$,

$$|f_\phi(z) - R_n(z)| \le \frac{C}{n} \frac{1}{|\mathrm{Re}(z)|} \text{ if } \mathrm{Re}\, z > 0, \text{ and}$$
$$|F_\phi(z) - R_n(z)| \le \frac{C}{n} \frac{1}{|\mathrm{Re}(z)|} \text{ if } \mathrm{Re}\, z < 0.$$

by Theorem 3.2.2. The previous two inequalities gives us the fact that

$$|R_n(z)| \le \frac{C'}{|z|}$$

in some box S whose intersection with the imaginary axis is a subset of (a, b), since both f_ϕ and F_ϕ are bounded in some such box. Furthermore, each $R_n(z)$ must be analytic in S because its set of singularities is a subset of $i\sigma(\phi)$ (where $\sigma(\phi)$ is the spectrum of ϕ and we assume that $\sigma(\phi) \cap (a, b) = \emptyset$). Thus, by a classical and technical theorem of Beurling (see [**15**, p. 95]), the R_n form a normal family on S and thus by Montel's theorem ([**7**, p. 201]) there is a subsequence of these R_n converging uniformly on compact subsets of S to some analytic function. But by (3.3) this subsequence, if it converges uniformly at z must converge to f_ϕ if $\mathrm{Re}(z) > 0$ and F_ϕ if $\mathrm{Re}(z) < 0$. Thus the analytic continuation of f_ϕ must be F_ϕ. □

3.3. An Extension of Bochner-Bohnenblust Continuation

Recall the definition of the S^2 (i.e. S_1^2) norm given in Definition 1.5.1. We wish to extend the definition of Bochner-Bohnenblust continuation using this norm.

DEFINITION 3.3.1. Let $f : \mathbb{C}_+ \to \mathbb{C}$ and $F : \mathbb{C}_- \to \mathbb{C}$. If there is an S^2-almost periodic function ϕ such that

$$f(z) = f_\phi(z) \text{ for } \mathrm{Re}(z) > 0 \text{ and}$$
$$F(z) = F_\phi(z) \text{ for } \mathrm{Re}(z) < 0$$

then we say that f and F are *extended-Bochner-Bohnenblust* continuations of each other.

We have the following theorem, analogous to Theorem 3.2.2.

THEOREM 3.3.2. *Suppose that ϕ is a function with S^2 norm less than ∞. Then there is a constant C independent of ϕ such that, for* $\mathrm{Re}(z) > 0$,

$$|\mathfrak{L}[\phi](z)| < \begin{cases} \frac{2\|\phi\|_{S^2}}{\mathrm{Re}(z)} & \text{if } 0 \le \mathrm{Re}(z) \le 1 \\ \|\phi\|_{S^2} & \text{if } \mathrm{Re}(z) > 1 \end{cases}. \tag{3.3.3}$$

PROOF. We have, letting $x = \mathrm{Re}(z)$,

$$\begin{aligned}|\mathfrak{L}[\phi](z)| &\le \int_0^\infty |\phi(t)e^{-zt}|\,dt = \sum_{N=0}^\infty \int_N^{N+1} |\phi(t)e^{-zt}|\,dt \\ &\le \sum_{N=0}^\infty \left[\int_N^{N+1} |\phi(t)|^2\,dt\right]^{1/2} \left[\int_N^{N+1} |e^{-zt}|^2\,dt\right]^{1/2}\end{aligned}$$

by the Cauchy-Schwarz inequality. Now, using the fact that e^{-t} is a decreasing function we see that

$$\left[\int_N^{N+1} |e^{-zt}|^2\right]^{1/2} dt \le ((e^{-N})^2)^{1/2} = e^{-N}.$$

Also, from the definition of the S^2 norm we have that

$$\left[\int_N^{N+1} |\phi(t)|^2 dt\right]^{1/2} \le \|\phi\|_{S^2}.$$

Thus, we find

$$\begin{aligned}|\mathfrak{L}[\phi](z)| &\le \sum_{N=0}^\infty \left[\int_N^{N+1} |\phi(t)|^2\,dt\right]^{1/2} \left[\int_N^{N+1} |e^{-zt}|^2\,dt\right]^{1/2} \\ &\le \|\phi\|_{S^2} \sum_{N=0}^\infty e^{-Nx} = \frac{\|\phi\|_{S^2}}{1-e^{-x}}\end{aligned} \tag{3.3.4}$$

Now, for $0 \le x \le 1$, we have that

$$1 - e^{-x} \ge \frac{x}{2},$$

so

$$\frac{\|\phi\|_{S^2}}{1-e^{-x}} \le \frac{2\|\phi\|_{S^2}}{x}$$

and thus Inequality (3.3.4) implies that

$$|\mathfrak{L}[\phi](z)| \leq \frac{2\|\phi\|_{S^2}}{x} \text{ for } 0 \leq x \leq 1.$$

Also,

$$1 - e^{-x} < 1 \text{ for all } x,$$

so Inequality (3.3.4) implies that

$$|\mathfrak{L}[\phi](z)| \leq \|\phi\|_{S^2} \text{ for all } x \geq 0.$$

This proves the result. □

Note that, once again, a similar result holds for F_ϕ.

To prove extended-Bochner-Bohnenblust continuation is compatible with analytic continuation, we need the following lemma. It is analogous to Lemma 3.2.3.

LEMMA 3.3.5. *Let ϕ be an S^2-almost periodic function. Then*

$$\lim_{x \to 0+} x f_\phi(x+iy) = \lim_{x \to 0-} x F_\phi(x+iy) = a_\phi(y).$$

PROOF. We restrict ourselves to the proof of the lemma for f_ϕ; the proof for F_ϕ is similar. Let $\epsilon > 0$ be given. Let

$$\phi_n(t) = \sum_{k=1}^{K} b_k^{(n)} e^{it\lambda_k}$$

be one of the Fejér polynomials for ϕ that is within $\epsilon/2$ of it in the S^2 norm. Thus, every λ_k is in the spectrum of ϕ. Now we have

$$f_\phi(s) = \sum_{k=1}^{K} \frac{b_k^{(n)}}{s - i\lambda_k} + \int_0^\infty (f(t) - f_n(t))e^{-(x+iy)t}dt.$$

Note the S^2 norm of $f(t) - f_n(t)$ is less than $\epsilon/2$, so by Theorem 3.3.2 we have that

$$\int_0^\infty (f(t) - f_n(t))e^{-zt}dt < \frac{\epsilon}{x}.$$

for sufficiently small x. Thus, by applying the triangle inequality we see that, for sufficiently small x,

$$\begin{aligned}|xf_\phi(x+iy) - a(y)| &\leq \left|\sum_{k=1}^{K} \frac{xb_k^{(n)}}{x+i(y-\lambda_k)} - a(y)\right| + x\left(\frac{C\epsilon}{x}\right) \\ &= \left|\sum_{k=1}^{K} \frac{xb_k^{(n)}}{x+i(y-\lambda_k)} - a(y)\right| + \epsilon.\end{aligned}$$

The rest of the proof is as in Lemma 3.2.3.

□

Now that we have Lemma 3.3.5 we have the following important theorem. It is analogous to 3.2.4.

THEOREM 3.3.6. *Extended-Bochner-Bohnenblust continuation is compatible with analytic continuation. That is, if f_ϕ has an analytic continuation across some sub-arc of $i\mathbb{R}$, this analytic continuation must equal F_ϕ.*

PROOF. Suppose that f_ϕ has an analytic continuation across (ia, ib). Then we must have that $a(y) = 0$ for all $y \in (a, b)$, by the lemma. Now, we can approximate ϕ by

$$\phi_n(t) = \sum_{k=1}^{K} b_k^{(n)} e^{it\lambda_k},$$

where for each n we have $\|\phi - \phi_n\|_{S^2} < 1/n$. Now, let

$$R_n(z) := \sum_{k=1}^{K} \frac{b_k^{(n)}}{z - i\lambda_k} = \mathfrak{L}[\phi_n](z).$$

Then we have

$$|f_\phi(z) - R_n(z)| \leq \frac{C}{n} \frac{1}{|\operatorname{Re}(z)|} \text{ for } 0 < \operatorname{Re}(z) \leq 1$$

$$|F_\phi(z) - R_n(z)| \leq \frac{C}{n} \frac{1}{|\operatorname{Re}(z)|} \text{ for } -1 \leq \operatorname{Re}(z) < 0$$

by Theorem 3.3.2. The previous two inequalities gives us the fact that

$$|R_n(z)| \leq \frac{C'}{|z|} \text{ for all } z \in S$$

where S is some box whose intersection with the imaginary axis is a subset of (a, b), since both f_ϕ and F_ϕ are bounded in some such box. The rest of the proof goes like the proof of Theorem 3.2.4. □

3.4. Some Specific Results about Non-tangential Limits

We now state some results about specific values of non-tangential limits of Laplace transforms. The first shows that the Laplace transform of an almost periodic function ϕ may not have a non-tangential limit at a point iy even if $y \notin \sigma(\phi)$. Recall that $\sigma(\phi)$ denotes the spectrum of ϕ, (see Definition 1.4.19).

THEOREM 3.4.1. *Let a_n and λ_n be sequences of real numbers such that each $a_n > 0$,*

$$\sum_{n=1}^{\infty} |a_n| < \infty$$

and

$$\limsup_{n\to\infty} \frac{a_n}{n\lambda_n^2} = \infty.$$

Then if ϕ is the almost periodic function defined by

$$\phi(x) = \sum_{j=1}^{\infty} a_j e^{i\lambda_j x},$$

The Laplace transform f_ϕ does not have a non-tangential limit at 0.

PROOF. Define

$$\phi_n(x) = \sum_{j=1}^{n} a_j e^{i\lambda_j x}.$$

Now, note that if $\operatorname{Re}(z) > 0$,

$$\operatorname{Re}\left(\frac{1}{z - i\lambda}\right) = \frac{\operatorname{Re}(z)}{|z - i\lambda|^2} > 0.$$

Thus,

$$\text{Re}(f_{\phi_n}(z)) > \text{Re}(f_{\phi_m}(z))$$

for $n > m$ and for all $z \in \mathbb{C}_+$. Now let $z_n = 1/n$. Then

$$\text{Re}(f_\phi(z_n)) > (\text{Re}\, f_{\phi_n}(z_n)) \geq \frac{a_n/n}{\lambda_n^2 + 1/n^2} = \frac{a_n n}{\lambda_n^2 n^2 + 1}$$

so we have that

$$\limsup_{n\to\infty} \text{Re}(f_\phi(z_n)) \geq \limsup_{n\to\infty} \frac{a_n}{\lambda_n^2 n}$$

□

So, for example if we choose

$$a_n = \frac{1}{2^n}$$

and

$$\lambda_n = \frac{1}{2^n}$$

we will have

$$\limsup_{n\to\infty} \frac{a_n}{\lambda_n^2 n} = \limsup_{n\to\infty} \frac{2^n}{n} = \infty.$$

So then

$$\limsup_{n\to\infty} \text{Re}(f_\phi(z_n)) = \infty.$$

Thus, in this case f_ϕ will not have a non-tangential limit at zero, even though 0 is not in the spectrum of ϕ.

We will now discuss conditions under which we can actually compute certain boundary values of Laplace transforms.

LEMMA 3.4.2. *Let $\{\phi_n\}$ be a sequence of bounded continuous functions on $\mathbb{R}^+ = [0, \infty)$ which converge uniformly to a bounded function ϕ. Then*

$$f_{\phi_n} \to f_\phi$$

uniformly on compact subsets of $\mathbb{C}_+$.

Now, for any $z \in \mathbb{C}_+$, we can use the lemma to conclude that, for any almost periodic function ϕ,

$$\lim_{n\to\infty} f_{\phi_n}(z) = f_\phi(z),$$

where the ϕ_n are almost periodic and approach ϕ uniformly.

THEOREM 3.4.3. *Suppose that*

$$\phi(x) = \sum_{j=1}^{\infty} a_j e^{i\lambda_j x}$$

where

$$\sum_{j=1}^{\infty} |a_j| < \infty.$$

If $y_0 \in \mathbb{R}$ is such that

$$\sum_{j=1}^{\infty} \frac{|a_j|}{|iy_0 - i\lambda_j|} < \infty$$

then

$$\lim_{x\to 0^+}(f_\phi)(x+iy_0)=\sum_{j=1}^{\infty}\frac{a_j}{iy_0-i\lambda_j}.$$

PROOF. Define

$$\phi_n(x)=\sum_{j=1}^{n}a_je^{i\lambda_jx}.$$

Now, applying the Lemma, we have for all $z\in\mathbb{C}_+$ that

$$f_\phi(z)=\lim_{n\to\infty}f_{\phi_n}(z)=\sum_{j=1}^{\infty}\frac{a_j}{z-i\lambda_j}.$$

So, letting $z=x+iy$, we have

$$\left|\frac{a_j}{z-i\lambda_j}\right|=\frac{|a_j|}{|z-i\lambda_j|}=\frac{|a_j|}{|x+i(y-\lambda_j|)}\le\frac{|a_j|}{|y-\lambda_j|}.$$

So if

$$\sum_{n=1}^{\infty}\left|\frac{a_j}{y-\lambda_j}\right|<\infty$$

then

$$|f_\phi(z)|=\left|\sum_{j=1}^{\infty}\frac{a_j}{z-i\lambda_j}\right|<\sum_{n=1}^{\infty}\left|\frac{a_j}{y-\lambda_j}\right|<\infty.$$

for all $z\in\mathbb{C}_+$. Now we apply the dominated convergence theorem to see that

$$\lim_{x\to 0^+}f_\phi(x+iy)=\lim_{x\to 0^+}\sum_{j=1}^{\infty}\frac{a_j}{x+iy-i\lambda_j}=\sum_{j=1}^{\infty}\lim_{x\to 0^+}\frac{a_j}{x+iy-i\lambda_j}=\sum_{j=1}^{\infty}\frac{ia_j}{\lambda_j-y}.$$

□

We now give a global theorem, analogous to Theorem 2.1.11

THEOREM 3.4.4. *Suppose that*

$$\sum_{j=1}^{\infty}|a_j|<\infty$$

and that there exists a sequence $\{b_n\}_{n=1}^{\infty}\subset(0,1)$ *such that*

(1)

$$\sum_{j=1}^{\infty}b_j=1$$

(2)

$$\sum_{j=1}^{\infty}\frac{|a_j|}{b_j}<\infty.$$

Define ϕ *to be the almost periodic function*

$$\phi=\sum_{j=1}^{\infty}a_je^{i\lambda_jx}.$$

Then

$$\lim_{x \to 0^+} (f_\phi)(x+iy) = \sum_{j=1}^{\infty} \frac{a_j}{y - i\lambda_j}$$

for almost every $x \in \mathbb{R}$.

CHAPTER 4

The Big Question

4.1. The Big Question

For many almost periodic functions ϕ, f_ϕ and F_ϕ are pseudo-continuations of each other.

EXAMPLE 4.1.1. If

$$\phi \sim \sum_{n=1}^{\infty} a_n e^{i\lambda_n x}$$

where

$$\sum_{n=1}^{\infty} |a_n| < \infty,$$

then f_ϕ and F_ϕ are pseudo-continuations of each other.

This is true since the condition

$$\sum_{n=1}^{\infty} |a_n| < \infty,$$

says that the measure

$$\sigma = \sum_{n=1}^{\infty} a_n \delta_{\lambda_n}$$

is a finite measure which is singular with respect to m, and since $\phi = \widehat{\sigma}$. Thus, Theorem 3.1.2 and Corollary 2.1.6 imply that f_ϕ and F_ϕ are pseudo-continuations of each other.

The following question now arises.

QUESTION 4.1.2. Is every Bochner-Bohnenblust continuation also a pseudo-continuation? In other words, for every almost periodic function ϕ, is it the case that f_ϕ and F_ϕ are pseudo-continuations of each other.

This question is the main one to which all results of this thesis relate. If every Bochner-Bohnenblust continuation is a pseudo-continuation, then Bochner-Bohnenblust continuation is not really a new type of continuation at all. However, if some almost periodic function ϕ exists so that f_ϕ and F_ϕ are not pseudo-continuations of each other, then Bochner-Bohnenblust continuation is a distinct type of continuation from pseudo-continuation.

Intuitively, one would expect that there is some almost periodic function ϕ for which f_ϕ and F_ϕ are not pseudo-continuations of each other. There are two possible ways to show this.

(1) Show that for some ϕ, f_ϕ has non-existent non-tangential limits on a set of positive measure.

(2) Show that for some ϕ, even though both f_ϕ and F_ϕ may have non-tangential limits almost everywhere, their non-tangential limits will be unequal on a set of positive measure.

Unfortunately, both approaches are more difficult then they might seem, even for extended-Bochner-Bohnenblust continuation.

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