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Introduction to Model Spaces and their Operators

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Preliminaries

1.1 Measure and integral

1.1.1 Borel sets and measures

Most of the "measuring" in this book will take place on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Since we assume that the reader has a background in graduate analysis, we quickly review the standard definitions without much fanfare.

We let $m := d\theta/2\pi$ denote *Lebesgue measure* on $\mathbb{T}$, normalized so that $m(\mathbb{T}) = 1$. A subset of $\mathbb{T}$ is called a *Borel set* if it is contained in the *Borel σ-algebra*, the smallest σ-algebra of subsets of $\mathbb{T}$ that contains all of the open arcs of $\mathbb{T}$. A *Borel measure* on $\mathbb{T}$ is a countably additive function that assigns a complex number to each Borel subset of $\mathbb{T}$. Unless otherwise stated, our measures will always be finite. A Borel measure is *positive* if it assigns a non-negative number to each Borel set. We let $M(\mathbb{T})$ denote the set of all complex Borel measures on $\mathbb{T}$ and we let $M_+(\mathbb{T})$ denote the set of all positive Borel measures on $\mathbb{T}$. A function $f : \mathbb{T} \to \widehat{\mathbb{C}}$ (where $\widehat{\mathbb{C}}$ denotes the Riemann sphere $\mathbb{C} \cup \{\infty\}$) satisfying the condition that $f^{-1}(U)$ is a Borel set for any open set $U \subset \widehat{\mathbb{C}}$ is called a *Borel function*.

We often need to distinguish between the "support" and a "carrier" of a measure. For $\mu \in M_+(\mathbb{T})$, consider the union $\mathcal{U}$ of all the open subsets $U \subset \mathbb{T}$ for which $\mu(U) = 0$. The complement $\mathbb{T} \setminus \mathcal{U}$ is called the *support* of μ. On the other hand, a Borel set $E \subset \mathbb{T}$ for which

$$\mu(E \cap A) = \mu(A) \tag{1.1}$$

for all Borel subsets $A \subset \mathbb{T}$ is called a *carrier of μ*. The support of μ is certainly a carrier, but a carrier need not be the support. Indeed, a carrier of a measure might not even be closed. For example, if $f \geqslant 0$ is continuous and $d\mu = f dm$, then a carrier of μ is $\mathbb{T} \setminus f^{-1}(\{0\})$ (which is open) while the support of μ is the closure of this set. The support of a measure is unique while a carrier is not.