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Classifying Coloring Graphs

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Abstract. Given a graph G, its k-coloring graph is the graph whose vertex set is the proper k-colorings of the vertices of G with two $k-$colorings adjacent if they differ at exactly one vertex. In this paper, we consider the question: Which graphs can be coloring graphs? In other words, given a graph H, do there exist G and k such that H is the k-coloring graph of G? We will answer this question for several classes of graphs and discuss important obstructions to being a coloring graph involving order, girth, and induced subgraphs.

1. Introduction

Let $G = (V, E)$ be a simple graph with finite vertex set. For $k \in \mathbb{N}$, a map $\alpha : V \to \{1, \ldots, k\}$ is called a *k-coloring* of G, and $\alpha(v)$ is called the *color* of v. If $\alpha(u) \neq \alpha(v)$ for all $uv \in E$, α is a *proper* k-coloring of G. For the purposes of this paper, all colorings are proper k-colorings. The *k-coloring graph* of G, denoted $\mathcal{C}_k(G)$, is the graph with the set of k-colorings of G as its vertex set and edges between colorings if and only if they differ at exactly one vertex.

The coloring graph naturally arises in theoretical physics when studying the Glauber dynamics of an anti-ferromagnetic Potts model at zero temperature [9, 15, 17, 20]. In this situation, the set of all proper k-colorings of a graph forms the state space for a Markov chain with iterations given by randomly recoloring a randomly selected vertex of G. When this process on $\mathcal{C}_k(G)$ exhibits rapid mixing, good estimates of the total number of proper k-colorings of G are obtained [15, 17, 20].

Motivated by the Markov chain connection, a graph G is said to be *k-mixing* if $\mathcal{C}_k(G)$ is connected. The question of when G is k-mixing as well as the computational complexity of answering that question have been extensively studied [1, 3, 4, 5, 10]. For instance, it has been shown that if the chromatic number $\chi(G) = k \in \{2, 3\}$, then G is not k-mixing. There are also examples of $k_1 < k_2$ such that G is k_1-mixing but not k_2-mixing. Mixing properties related to modifications of the coloring graph have also been studied (cf. [2, 13, 16]). Recent work considers when the coloring graph and its modifications contain a Hamiltonian cycle [6, 13].

The k-coloring graph is an example of a reconfiguration graph. A reconfiguration graph has as its vertices all feasible solutions to a given problem, and two solutions are adjacent if and only if one can be obtained from the other by one application of a specific reconfiguration rule (cf. [19]). For example, several different reconfiguration graphs have been proposed for studying domination in graphs (see

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[11, 14, 18]). In both knot theory and chemistry, the collection of perfect matchings of a graph is of interest where one transitions between perfect matchings by choosing the complementary set of edges around a face [7, 12]. The connectedness of this reconfiguration system known as the Z-transformation graph in chemistry or clock lattice in knot theory has been of particular interest.

In this article, we consider an inverse problem for coloring graphs: Which graphs are coloring graphs? In other words, given a graph H, does a pair k and G exist such that $\mathcal{C}_k(G) = H$? Following this introduction, there are four sections that each focus on tackling this question in a different way.

Section 2 focuses on preliminary results relied upon in later proofs as well as some of their immediate implications. We see in this section that all complete graphs are coloring graphs while almost no trees are coloring graphs. In Section 3 we address the question of forbidden subgraphs for coloring graphs. We show the class of coloring graphs does not have a forbidden subgraph characterization, and there are infinite families of minimal forbidden subgraphs. Section 4 explores girth of coloring graphs. We give a finite list of possible girths for $\mathcal{C}_k(G)$ when $k > \chi(G)$ but show that coloring graphs can have arbitrarily large girth when $k = \chi(G)$ even when restricting to graphs of chromatic number three. Section 5 shows how one can use the formula for order of coloring graphs together with our other results to completely determine which graphs of a particular order are coloring graphs.

We conclude this section with some basic notation to be used in the remainder of the paper. Notation is chosen to be consistent with Diestel's graph theory text and the work of Cereceda et al. on k-mixing [3, 8]. Let $V[G]$ denote the set of vertices in G, $E[G]$ denote the set of edges in G, and $|G|$ denote the *order of* G, which is the size of $V[G]$. The chromatic number, $\chi(G)$, is the minimum number of colors needed for a coloring of G; we may simply use χ when G is clear from context. We use the following notation for standard families of graphs with n vertices: paths P_n, cycles C_n, complete graphs K_n, and empty graphs I_n. Note that $P_1 = K_1 = I_1$ and $P_2 = K_2$. Additionally, we let $\sqcup$ denote the disjoint union of graphs and $\square$ be the Cartesian product of graphs, which we will define in the next section. Also note that we conflate the notion of equal and isomorphic graphs in our discussion and refer to vertices of $\mathcal{C}_k(G)$ and k-colorings of G interchangeably. The letters α, β, γ will typically denote colorings of G and hence vertices in $\mathcal{C}_k(G)$; we denote vertices in the base graph G by x, u, v, w.

2. Preliminaries

It is often sufficient to work with coloring graphs $\mathcal{C}_k(G)$ for which G is connected. Because coloring graphs for disconnected G have a product structure, they can be completely understood by looking at each of the connected components. This concept is implicitly used elsewhere in the literature [3, 4]. Since certain proofs in our work make extensive use of the Cartesian Product, we carefully define it here.

The *Cartesian product of graphs* G_1 *and* G_2, denoted by $G_1 \square G_2$, is defined as follows. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs with disjoint vertex and edge sets. Then $G_1 \square G_2$ is the graph with vertex set $\{(u, v) | u \in V_1 \text{ and } v \in V_2\}$ and edges between (u, v) and (u', v') whenever either (i) $u = u'$ and v is adjacent to v' in G_2 or (ii) $v = v'$ and u is adjacent to u' in G_1. See Figure 1 for an example. This definition extends naturally to products of more than two graphs. When we consider the n-fold product of G with itself, we denote this by $G^{\square n}$.

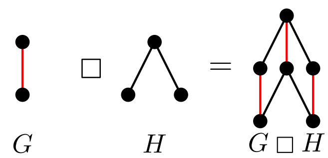

FIGURE 1. Cartesian product of graphs.

Colorings of $G = G_1 \sqcup G_2$ restrict to colorings of the components G_i for $i = 1, 2$. Moreover, a collection of colorings for the components of G naturally gives rise to a coloring of G. It follows that whenever G is disconnected, the Cartesian product of coloring graphs of the components precisely describes the structure of $\mathcal{C}_k(G)$, as described in Lemma 1. When $G = \sqcup G_i$, we use the product structure to denote vertices in $\mathcal{C}_k(G)$ by $(\alpha_1, \ldots, \alpha_n)$ where $\alpha_i \in V[\mathcal{C}_k(G_i)]$.

Lemma 1. *Let* $G = \sqcup_{i=1}^n G_i$. *Then* $\mathcal{C}_k(G) = \square_{i=1}^n \mathcal{C}_k(G_i)$.

Colorings with one or two colors are simple to understand. If G admits a 1-coloring then it must be edgeless, hence $G = I_{|G|}$. Additionally, since there is a unique way to 1-color a graph, it follows that $\mathcal{C}_1(G) = I_1$. Any graph G with at least one edge cannot be 1-colored so $\mathcal{C}_1(G)$ will not exist. Now, if a graph admits a 2-coloring then it is, by definition, bipartite. Any isolated vertex in this graph will contribute a K_2 in the 2-coloring graph, and any other connected component yields an I_2. Hence, as in [3], the 2-coloring graph of a bipartite graph G with m components and n isolated vertices is simply: $\mathcal{C}_2(G) = K_2^{\square n} \square I_2^{\square(m-n)}$. Moreover, all 2-coloring graphs have this structure.

Isomorphisms enhance our understanding of the degrees and orders of coloring graphs. Given a graph $G = (V, E)$, notice that a proper coloring partitions V into *independent sets* or sets that do not contain adjacent vertices. Two colorings are said to be *isomorphic* if they correspond to the same partition. For $k \geq i$, there are $\frac{k!}{(k-i)!} = k \cdot (k-1) \cdots (k-i+1)$ isomorphic k-colorings corresponding to a given partition into i non-empty, independent sets. If we then let m_i denote the number of ways to partition the vertex set of G into exactly i independent sets, the following formula expresses $|\mathcal{C}_k(G)|$ as a sum indexed by the isomorphism classes of colorings of G.

Lemma 2. $|\mathcal{C}_k(G)| = \sum_{i=1}^k m_i \left(\frac{k!}{(k-i)!}\right) = \sum_{i=1}^k m_i \cdot k(k-1)\cdots(k-i+1)$

There are many ramifications of this lemma; for instance, we see immediately that k divides the order of any k-coloring graph. Corollary 3 below gives two properties of order needed in later arguments. Section 5 uses this formula more extensively to determine which graphs of a particular order are coloring graphs.

Corollary 3.

(i) Let J be an isomorphism class of k-colorings. Then $k \big| |J|$.
(ii) The order of $\mathcal{C}_k(G)$ is either a power of k or divisible by $k(k-1)$.

We get other structural results by considering colorings that are isomorphic. Let α be a k-coloring of some graph G, and let σ be a permutation of the names of the k colors. Then $\sigma(\alpha)$ and α are isomorphic k-colorings of G. Hence, σ induces a graph automorphism on $\mathcal{C}_k(G)$. This automorphism imbues $\mathcal{C}_k(G)$ with some nice properties. In particular, two isomorphic colorings of G must have the same degree in $\mathcal{C}_k(G)$.

Let d_i denote the number of vertices of $\mathcal{C}_k(G)$ with degree i. Since all colorings in the same isomorphism class have the same degree, d_i counts a subset of isomorphism classes of k-colorings of G. It follows by Corollary 3 that k then divides d_i for all i. Hence $\sum_i id_i$, which is twice the number of edges in $\mathcal{C}_k(G)$, must also be divisible by k. This fact, summarized below, is useful later.

Corollary 4. *Given any graph G, the quantity $2|E\left[\mathcal{C}_k(G)\right]|$ is divisible by k.*

Notice that if $k > \chi(G)$, isomorphic colorings may be adjacent in the coloring graph. For instance, let α be a $\chi(G)$-coloring of a graph G. Then $(\alpha, \chi(G)+1)$ and $(\alpha, \chi(G)+2)$ are adjacent colorings in the $(\chi(G)+2)$-coloring graph of $G \sqcup I_1$. If $k = \chi(G)$, however, adjacent vertices in the coloring graph cannot be isomorphic. In fact, isomorphic colorings must be separated by a minimal distance of five as we see in the following lemma.

Lemma 5. *For $k = \chi(G) \geq 2$, the distance between any two distinct isomorphic colorings in $\mathcal{C}_k(G)$ is at least five.*

Proof. Let $k = \chi(G) = 2$. In this case G must have at least one edge uv and any isomorphism class of colorings has size 2. Consider some pair of isomorphic colorings $\{\alpha, \beta\}$ where $\alpha(u) = \beta(v) = 1$ and $\alpha(v) = \beta(u) = 2$. Since u and v are adjacent, they cannot be recolored one at a time. Hence no path exists between α and β.

Now let $k = \chi(G) \geq 3$, and consider a path $P = \alpha_1\alpha_2 \ldots \alpha_m$ between isomorphic colorings α_1 and α_m in $\mathcal{C}_k(G)$. Say that α_1 and α_m correspond to the partition of V into independent sets $V_1 \sqcup \cdots \sqcup V_k$. Since α_1 and α_m are isomorphic but distinct, all vertices in at least two of these independent sets, say V_1 and V_2, must be recolored as P is traversed. If $|V_1| + |V_2| > 4$ then P has length at least five, and we are done. Assume, then, that $|V_1| + |V_2| \leq 4$. In other words either V_1 or V_2 consists of a single vertex or both V_1 and V_2 contain two vertices. In each of these cases, we show P must still have length at least five.

For the first case, suppose, without loss of generality, $V_1 = \{u\}$ and $\alpha_1(u) = 1$. We claim two other vertices must be recolored before u is recolorable. Some other vertex w not adjacent to u must be recolored 1 or else there would be a proper coloring using only $k-1$ colors. At the same time, since $\chi(G) = k$, u is adjacent to at least one vertex in each of $V_2, \ldots, V_k$, and at least one of these neighbors of u must be recolored with a color other than 1 before u can be recolored. That is, if $\alpha_{i-1}\alpha_i$ is the first edge of P where vertex u is recolored, then $i \geq 4$. Since α_1 and α_m are isomorphic, the coloring of α_m corresponds to the same underlying partition of V. Hence, the argument above for α_1 can be applied to α_m to show $m - 3 \geq i - 1$. Combining these, $m \geq 6$.

In the latter case, suppose $V_1 = \{u, u'\}$ and $V_2 = \{v, v'\}$ with $\alpha_1(V_i) = i$, for $i = 1, 2$. If a vertex not in V_1 or V_2 is recolored along P or any vertex in V_1 or V_2 is recolored twice, then P will have length greater than 4. Hence, we only need to consider the case where P has length 4. In this case, P ends with α_5 where $\alpha_5(V_1) = 2$ and $\alpha_5(V_2) = 1$; i.e. the colors for V_i have been swapped. Such a path can only exist if there are no edges between vertices in V_1 and V_2. In that case $V_1 \cup V_2$ is an independent set, and G can be colored with $k-1$ colors contradicting the fact that $k = \chi(G)$. □

As seen in the previous proof, a path, or more generally a walk, in $\mathcal{C}_k(G)$ is a sequence of colorings of G where each subsequent coloring differs in the color of only one vertex of G. More formally, a walk $\alpha_1\alpha_2\ldots\alpha_m$ is such that $\alpha_i(v) = \alpha_{i+1}(v)$ for all but one $v \in V[G]$. It will be useful to label elements of $E[\mathcal{C}_k(G)]$ by the vertex of G that is recolored. Using this edge labeling, a *partition type* may be assigned to any walk of $\mathcal{C}_k(G)$ in the following way. Let n be the number of edges in the walk. Then $n \vdash n_1, n_2, \ldots, n_m$ where n_i is the number of times that vertex v_i is recolored. Since vertex labeling is arbitrary, we choose our labeling such that our partitions are descending.

For example, let C_n be a cycle in a coloring graph with partition type $n \vdash n_1, n_2, \ldots, n_m$. Then each n_i must be at least two since any vertex that changes color must eventually change back. Notice that this means the partition describing any three-cycle must have only one part, and so only one vertex changes color. Now, assume that the cycle C_n is induced in the coloring graph. If $n > 3$ is odd, then $m > 2$. To see this notice that if $m = 2$, the edge labeling on an induced cycle must alternate between v_1 and v_2 in order to avoid creating a chord. For n odd, this necessitates the change of at least one other vertex. Moreover, observe that this same alternating requirement gives that if n is even and $m = 2$, then $n \vdash \frac{n}{2}, \frac{n}{2}$.

It is natural to consider whether there are families of graphs that can or cannot be realized as coloring graphs. To begin, observe that $\mathcal{C}_k(I_1) = K_k$, so all complete graphs can be realized as coloring graphs. But for what other G is the coloring graph a complete graph? We know $\mathcal{C}_1(G) = I_1 = K_1$ for any empty graph $G = I_{|G|}$. The only complete 2-coloring graph is $\mathcal{C}_2(I_1) = P_2 = K_2$. Now consider $k > 2$. Then a complete coloring graph has the property that any three colorings span a copy of K_3. Using the partition argument from above, these colorings must differ at the same vertex. But we also know from isomorphisms that every vertex in G must be colored each of the k colors at some point. Combining these two facts gives us that G must simply contain one vertex and hence $G = I_1$. In turn, $\mathcal{C}_k(G) = K_k$ so $n = k$. This gives us the following theorem.

Theorem 6. *If $\mathcal{C}_k(G)$ is a complete graph, it must be K_k. Further, for $k > 1$, $\mathcal{C}_k(G) = K_k$ if and only if $G = I_1$.*

We just mentioned that $\mathcal{C}_1(I_1) = I_1 = P_1$ and $\mathcal{C}_2(I_1) = P_2$. It turns out that these are the only paths that can be realized as coloring graphs. In fact, these are the only trees that can be realized as coloring graphs.

Theorem 7. *The only trees that are coloring graphs are I_1 and P_2.*

Proof. The result is clear for 1- and 2-colorings since these are completely classified. Let $k \geq 3$, and assume that $\mathcal{C}_k(G)$ is a tree with n vertices. Corollary 3 shows that k must divide n. Additionally, Corollary 4 shows k also divides $2(n-1)$. Combining these we see that k must divide 2, which is a contradiction. □

We note that unlike trees, forests can be coloring graphs. For instance, let G be the 5-cycle with exactly one chord. Then $\mathcal{C}_3(G) = \sqcup_1^6 P_3$.

3. Forbidden and Permissible Graphs

In this section we further explore induced subgraphs of coloring graphs. A graph, H, will be called *permissible* if there exists a coloring graph that has H as an induced subgraph. If it is not possible for H to be realized as the induced subgraph of some coloring graph, it is called *forbidden*. A forbidden subgraph is called *minimal* if all of its proper induced subgraphs are permissible. Building on the discussion of complete graphs in the previous section, any collection of m colorings of G that differ at the same vertex create a clique of size m in $\mathcal{C}_k(G)$. Now instead start with a clique on m vertices in $\mathcal{C}_k(G)$. Then any triple of these vertices is isomorphic to K_3. By the partition type argument, the associated colorings must change at the same vertex of G. Hence, all of the colorings associated with the clique change the color of precisely the same vertex, giving the lemma below.

Lemma 8. *A set of m vertices in the coloring graph $\mathcal{C}_k(G)$ forms a clique if and only if the associated colorings of G differ at the same vertex of G.*

Notice that this same argument illustrates that m colorings form an induced star in $\mathcal{C}_k(G)$ if and only if each pendant coloring differs from the center coloring by a different vertex of G. Before further exploring families of permissible graphs, it is useful to see how to generate permissible graphs from other permissible graphs. An obvious beginning point is the disjoint union of two such graphs.

Theorem 9. *If H_1 and H_2 are permissible, then $H_1 \sqcup H_2$ is permissible. Alternately, if $H_1 \sqcup H_2$ is forbidden then either H_1 or H_2 is forbidden.*

Proof. Assume H_i is an induced subgraph of $\mathcal{C}_{k_i}(G_i)$ for $i = 1, 2$, and let $k = max\{k_i\}$. Recall that a coloring in $\mathcal{C}_{k+1}(G_1 \sqcup G_2) = \mathcal{C}_{k+1}(G_1) \square \mathcal{C}_{k+1}(G_2)$ is denoted by an ordered pair (α, β) where α is a coloring of G_1 and β a coloring of G_2. Let α_0 be a $(k+1)$-coloring of G_1 that uses color $k+1$. Similarly, select a $(k+1)$-coloring β_0 of G_2 that uses color $k+1$. Then $\{(\alpha, \beta_0) | \alpha \in V[H_1]\}$ and $\{(\alpha_0, \beta) | \beta \in V[H_2]\}$ respectively span induced copies of H_1 and H_2 in $\mathcal{C}_{k+1}(G_1 \sqcup G_2)$. By construction, the two copies are disjoint, so $\mathcal{C}_{k+1}(G_1 \sqcup G_2)$ contains an induced copy of $H_1 \sqcup H_2$. □

The color shifting technique of this proof can be applied in a somewhat different direction as well. If G_1 is an induced subgraph of G then any coloring of G_1 can be extended to a coloring of G using a sufficient number of extra colors. Hence, the coloring graph of any induced subgraph is itself induced inside of a coloring graph of the base graph with more colors.

We can also create new permissible graphs by taking the Cartesian product of permissible graphs as proven below. A nice consequence of this result is that if a forbidden graph is a product then it is not minimal.

Theorem 10. *If H_1 and H_2 are permissible, then $H_1 \square H_2$ is permissible. Alternately, if $H_1 \square H_2$ is forbidden then either H_1 or H_2 is forbidden.*

Proof. Assume H_i is an induced subgraph of $\mathcal{C}_{k_i}(G_i)$ for $i = 1, 2$, and let $k = k_1 + k_2$. Given any k_2-coloring β of G_2, define the shifted coloring β' by $\beta'(v) = \beta(v) + k_1$. Thus β' is a k-coloring that only uses colors in the set $\{k_1 + 1, \ldots, k_1 + k_2\}$. Now each pair (α, β') is a proper k-coloring of the graph $G_1 \sqcup G_2$. Therefore $V[H_1 \square H_2] = \{(\alpha, \beta') : \alpha \in V[H_1], \beta \in V[H_2]\} \subset V[\mathcal{C}_k(G_1 \sqcup G_2)]$. Moreover, since adjacent colorings differ at exactly one vertex, it follows that $(\alpha_1, \beta_1')(\alpha_2, \beta_2') \in E[\mathcal{C}_k(G_1 \sqcup G_2)]$ if and only if either $\alpha_1 = \alpha_2$ and $\beta_1 \beta_2 \in E[\mathcal{C}_{k_2}(G_2)]$ or $\beta_1 = \beta_2$ and $\alpha_1 \alpha_2 \in E[\mathcal{C}_{k_1}(G_1)]$. Hence, the product $H_1 \square H_2$ is induced in $\mathcal{C}_k(G_1) \square \mathcal{C}_k(G_2) = \mathcal{C}_k(G_1 \sqcup G_2)$. □

These two results alone yield a number of useful observations. Since we have shown P_2 is permissible, the product of any other permissible graph with P_2 is permissible. In particular, the rectangular $n \times 2$ grid $P_n \square P_2$ is permissible. Moreover, $P_n \square P_2$ contains an induced P_{n+1}. This inductive argument shows that all paths are permissible. Similarly, you can take any permissible graph and use this construction to add branches to a chosen vertex. Hence, all trees are permissible, and any tree added to any vertex in a permissible graph yields another permissible graph. This is of particular interest because, while no tree contains a forbidden subgraph, most trees are not coloring graphs.

Corollary 11. *All trees are permissible. Moreover, this implies that the class of coloring graphs does not have a forbidden subgraph characterization.*

Now we ask, are cycles permissible? We have seen that $K_3 = C_3$ is permissible. Notice that $P_2 \square P_2$ is C_4, and thus C_4 is permissible. Now, $C_4 \square P_2$ is permissible and this contains an induced C_6. In fact, $C_n \square P_2$ is permissible when C_n is permissible and, for $n > 3$, $C_n \square P_2$ contains an induced C_{n+2}. Examination shows that C_7 is induced in $\mathcal{C}_3(I_3)$. Putting this all together, we see that C_n is permissible for all $n \neq 5$.

The cycle C_5 is not generated by the described method, so it is reasonable to ask if C_5 is also permissible. Consider possible edge labeling partition types for an induced C_5. We know that each partition part must be at least 2, but we already argued that induced odd n-cycles for $n > 3$ must have at least three elements in their edge labeling partition. These two requirements cannot be simultaneously met, so we conclude C_5 is forbidden.

Corollary 12. *The graph C_n for $n \neq 5$ is permissible. The graph C_5 is forbidden.*

Whether a particular cycle can be a coloring graph is a more complex question we address in the last section.

Another interesting implication of Theorem 10 is that families of permissible graphs may also be created by subdividing edges. If $uv \in E[G]$ then the graph G' that arises from subdividing the edge uv consists of vertex set $V[G'] = V[G] \cup \{x\}$ and edge set $E[G'] = E[G] \backslash \{uv\} \cup \{ux, xv\}$.

Corollary 13. *Let H be a permissible graph containing a vertex of degree 2, whose neighbors are not adjacent. The graph H' obtained by subdividing both edges incident to the vertex of degree two is also permissible.*

Proof. Suppose the vertex of degree 2 is γ_2 with neighbors γ_1 and γ_3. Then H' is induced in $H \square P_2$ where P_2 consists of colorings α and β. A spanning set for H' is $S = \{(\sigma, \alpha) : \sigma \in V[H]\} \cup \{(\gamma_1, \beta), (\gamma_2, \beta), (\gamma_3, \beta)\} \backslash \{(\gamma_2, \alpha)\}$. □

Note that if γ_1 and γ_3 were adjacent, the construction would yield an extra undesired edge, thus the non-adjacency condition is necessary. By repeating this subdivision, a vertex of degree 2 may be replaced with a path of length $2m + 1$ each of whose interior vertices are of degree 2. This gives another method for obtaining new permissible graphs.

Theorem 14. *Suppose that H is induced in $\mathcal{C}_k(G)$. Let $\alpha_1, \ldots, \alpha_m$ be an induced P_m in H. Let $\{p_1, \ldots, p_{m+4}\}$ be the vertices of P_{m+4}. Define H' to be the graph with vertex set $V[H'] = V[H] \sqcup V[P_{m+4}]$ and edge set $E[H'] = E[H] \sqcup E[P_{m+4}] \sqcup \{p_1\alpha_1, p_{m+4}\alpha_m\}$. Then H' is an induced subgraph of $\mathcal{C}_k(G \sqcup I_2)$.*

Proof. A coloring of G extends to a coloring of $G \sqcup I_2$ by assigning a color to each of the two vertices of I_2. Now, H' is the induced graph spanned by $\{(\gamma, 1, 1) : \gamma \in V[H]\} \cup \{(\alpha_i, 2, 2) : 1 \leq i \leq m\} \cup \{(\alpha_1, 2, 1), (\alpha_m, 2, 1)\}$. □

Notice that this construction generalizes, as does that of Corollary 13, by adding new disjoint vertices to G and coloring them appropriately. In this way a path of length $m + 2n$ can always be added.

Because the above results show that there are many ways to construct new graphs that will be permissible, one might speculate that there are only a limited number of forbidden subgraphs. We consider two families of graphs that supply us with infinitely many forbidden subgraphs. The first are (generalized) theta graphs, which are graphs consisting of paths whose only intersection are a common initial and terminal point. More precisely, let $x = (x_1, x_2, \ldots, x_n)$ be an n-tuple of nondecreasing positive integers. Define $T(x) = \cup_{i=1}^{n} P_{x_i+1}$ where the P_{x_i+1} are internally disjoint paths of length x_i starting and ending at fixed vertices u and v respectively. We call $T(x)$ the theta graph of type x. Theta graphs generalize cycles since $C_n = (m, n-m)$ for $0 < m < n$. Hence, we know that there is at least one forbidden generalized theta graph: C_5. Moreover, any theta graph containing an induced C_5 will be forbidden. It turns out that there are infinitely many other forbidden theta graphs, but each must contain one of five minimal forbidden theta graphs.

Theorem 15. *The only minimal forbidden theta graphs are* C_5, $T(1,2,2)$, $T(2,2,2)$, $T(3,3,3)$, $T(2,2,4)$.

Proof. First we argue that these five theta graphs are forbidden. We previously showed that C_5 is forbidden. In $T(1,2,2)$, the path of length 1 combines with each path of length 2 to create a K_3, and by Lemma 8, this means only one vertex can change color in each K_3. Since the path of length 1 is shared in both copies of K_3 all 4 colorings must in fact differ at the same vertex and this would produce an edge between the center vertex of each P_3. This forces $T(1,2,2)$ to be forbidden.

Notice that $T(2,2,2)$ has three induced copies of C_4, and each of these must have edge label partition type $4 \vdash 2,2$. We have shown the edge labels must alternate, say u, v, u, v. This is impossible to accomplish on all three copies of C_4 simultaneously. Thus $T(2,2,2)$ is also forbidden. Next consider $T(3,3,3)$, which has three induced copies of C_6. The only edge label partition types for an induced C_6 are $6 \vdash 3,3$ and $6 \vdash 2,2,2$, and again the labels must alternate. Any edge labeling that works for two of the three copies of C_6 will force a chord in the third. We conclude $T(3,3,3)$ is also forbidden. Finally, notice that $T(2,2,4)$ has an induced C_4, which must have partition type $4 \vdash 2,2$, and two induced copies of C_6, which can have partition types $6 \vdash 3,3$ or $6 \vdash 2,2,2$. Labels that achieve either of these on one C_6 will force two adjacent edges to get the same label on the other C_6, which implies a chord. Thus $T(2,2,4)$ is a forbidden induced subgraph as well.

Next we argue by construction that all theta graphs not containing induced copies of these five are permissible. Begin by examining $T(x)$ with $x = (x_1, x_2, \ldots, x_n)$ where $x_i \geq 4$ for all i. In the coloring graph $\mathcal{C}_2(I_3)$, a path of length 4 is induced by the colorings: $(1,1,1), (1,1,2), (2,1,2), (2,2,2), (2,2,1)$. So $T(4)$ is permissible. The graph $T(4,4)$ is induced in $\mathcal{C}_2(I_4)$ as follows. Extend each previous coloring γ of I_3 to a coloring on I_4 by $(\gamma, 1)$. Now a second length 4 path is induced by making the same color changes to the 4th vertex that were previously made on the 3rd. In other words, the new path is spanned by $(1,1,1,1)$, $(1,1,1,2)$, $(2,1,1,2)$, $(2,2,1,2)$, $(2,2,1,1)$. Notice that these paths share the first vertex $\alpha = (1,1,1,1)$ and the terminal vertex $\beta = (2,2,1,1)$. This construction can be extended naturally to add any desired number of paths of length 4. A path of length 5 can be induced in $\mathcal{C}_3(I_3)$ with the colorings: $(1,1,1),(1,1,2),(2,1,2),(2,1,3),(2,2,3),(2,2,1)$. The colorings can be similarly extended to $\mathcal{C}_3(I_r)$ to get more paths of length 5. Since the constructed paths of lengths 4 and 5 share an initial coloring $(1,1,\dots,1)$ and terminal coloring $(2,2,1,1,\dots,1)$ they can be combined into a permissible graph. Specifically, $\mathcal{C}_3(I_{n+2})$ will contain an induced $T(x_1,x_2,\dots,x_n)$, where each $x_i \in \{4,5\}$. Using Corollary 13 we see that any of these path lengths can be increased by 2. Hence all $T(x)$ are permissible when $x_i \geq 4$ for all i.

The remaining cases are when $x_1 = 1, 2$, or 3. For each of these values of x_1, and various values of $x_2, x_3 \leq 5$, $T(x_1,x_2,x_3)$ is induced in an appropriate $\mathcal{C}_k(I_r)$ by a construction similar to the case above and are left to the reader to verify. Theorem 14 and Corollary 13 are used to complete the proof. □

While there are only a few theta graphs that are minimal forbidden subgraphs, another family of graphs provides us with an infinite class of minimal forbidden subgraphs. To construct these, begin with the graph $P_n \square P_2$. Denote by α_i and β_i for $1 \leq i \leq n$ the top and bottom copies of P_n in $P_n \square P_2$ where i increases from left to right. We define $M_{n,p}$ to be $P_n \square P_2$ union the path $\alpha_1\beta_{n+p}\beta_{n+p-1}\cdots\beta_{n+1}\beta_n$. See Figure 2 for an illustration of $M_{n,p}$. Note that $M_{n,p}$ can be obtained as an induced subgraph of a Möbius ladder on $n+p$ rungs by removing the top rightmost p vertices from the ladder. We will show that there are infinitely many minimal forbidden $M_{n,p}$. Removing the last r vertices from the set $\{\alpha_i\}$ in $M_{n,p}$ leaves a copy of $M_{n-r,p+r}$. Thus if $M_{n,p}$ is forbidden so is $M_{n+1,p-1}$ (because $M_{n,p}$ is induced in $M_{n+1,p-1}$). Hence the set of minimal forbidden $M_{n,p}$ graphs comes from the cases where p is small. To begin, we classify exactly which $M_{n,p}$ are forbidden and permissible.

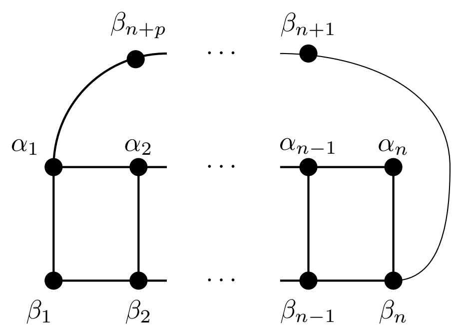

FIGURE 2. Vertex labels for $M_{n,p}$.

Lemma 16. *$M_{n,p}$ is forbidden if and only if $n \geq 1$ and $p \leq 3$.*

Proof. The graph $M_{n,p}$ contains induced 4-cycles $\alpha_i\alpha_{i+1}\beta_{i+1}\beta_i\alpha_i$ where $1 \leq i \leq n-1$. The only edge label partition type for such a cycle is $4 \vdash 2,2$ meaning two vertices change colors in an alternating fashion as the cycle is traversed. The

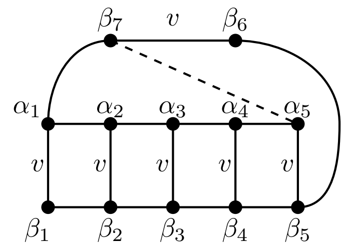

FIGURE 3. Forced faulty edge labelings for $M_{5,2}$

adjacency of these induced 4-cycles means that every vertical edge $\alpha_i\beta_i$ corresponds to a recoloring of the same vertex of G, say v. Because the 4-cycles are induced, no horizontal edge $\alpha_i\alpha_{i+1}$ or $\beta_i\beta_{i+1}$ for $1 \leq i \leq n-1$ corresponds to a recoloring of v since that would force a K_3 in $M_{n,p}$.

The graph $M_{n,p}$ also contains the induced cycle $\alpha_1\alpha_2\ldots\alpha_n\beta_n\beta_{n+1}\ldots\beta_{n+p}\alpha_1$. Since the edge $\alpha_n\beta_n$ represents a recoloring of v, some other edge in this cycle must also correspond to a recoloring of v. We cannot recolor v along two consecutive edges without forcing a chord, so neither $\beta_n\beta_{n+1}$ nor $\beta_{n+p}\alpha_1$ corresponds to recoloring v. Thus $M_{n,0}$ and $M_{n,1}$ are forbidden since they do not possess valid edge labelings.

In $M_{n,2}$ the only edge that could represent a recoloring of v is $\beta_{n+1}\beta_{n+2}$. This means v is recolored only twice along the induced cycle. Hence $\alpha_n(v) = \beta_{n+2}(v)$, and it follows that α_n and β_{n+2} differ at only one vertex - whichever one was recolored along edge $\beta_n\beta_{n+1}$. This would force the chord $\beta_{n+2}\alpha_n$ which is not in $M_{n,2}$, so we conclude that $M_{n,2}$ is forbidden. Figure 3 illustrates the faulty edge labelings.

This type of argument also holds for $M_{n,3}$. Either $\beta_{n+1}\beta_{n+2}$ or $\beta_{n+2}\beta_{n+3}$ must recolor v. The former case again forces $\beta_{n+2}\alpha_n$ to be a chord. By a symmetric argument, if $\beta_{n+2}\beta_{n+3}$ represented a recoloring of v, a chord $\beta_{n+2}\beta_1$ would be forced. Thus $M_{n,3}$ is also forbidden. Next we show that $M_{n,p}$ is permissible in all other cases, that is, whenever $p \geq 4$. We proceed by identifying colorings in $\mathcal{C}_k(I_3)$ with the vertices of $M_{n,4}$ and $M_{n,5}$ leaving it to the reader to verify that these span induced copies in $\mathcal{C}_k(I_3)$. This is sufficient since it then follows by Lemma 13, which allows the addition of 2 vertices to induced paths in permissible graphs, that $M_{n,4+2r}$ and $M_{n,5+2r}$ are also permissible for all $r \in \mathbb{N}$. In both $M_{n,4}$ and $M_{n,5}$, let α_i be the coloring $\left(1, \lceil\frac{i+1}{2}\rceil, \lfloor\frac{i+1}{2}\rfloor\right)$ for $1 \leq i \leq n$ and β_i the coloring $\left(2, \lceil\frac{i+1}{2}\rceil, \lfloor\frac{i+1}{2}\rfloor\right)$ for $1 \leq i \leq n+2$.

For $M_{n,5}$ vertex β_{n+3} is the coloring $\left(2, \lceil\frac{n+4}{2}\rceil, \lfloor\frac{n+4}{2}\rfloor\right)$, vertex β_{n+4} is the coloring $\left(1, \lceil\frac{n+4}{2}\rceil, \lfloor\frac{n+4}{2}\rfloor\right)$, and vertex β_{n+5} is the coloring $\left(1, 1, \lfloor\frac{n+4}{2}\rfloor\right)$. For $M_{n,4}$ vertex β_{n+3} is the coloring $\left(1, \lceil\frac{n+3}{2}\rceil, \lfloor\frac{n+3}{2}\rfloor\right)$, and vertex β_{n+4} is the coloring $\left(1, 1, \lfloor\frac{n+3}{2}\rfloor\right)$. One can check that these colorings span induced copies of $M_{n,4}$ and $M_{n,5}$ in $\mathcal{C}_k(I_3)$ for sufficiently large k. The examples of $M_{5,4}$ and $M_{5,5}$ are shown in Figure 4. $\square$

Unlike in the theta graph case, there are an infinite number of minimal forbidden graphs in this family. Hence, an infinite number of minimal forbidden subgraphs exist.

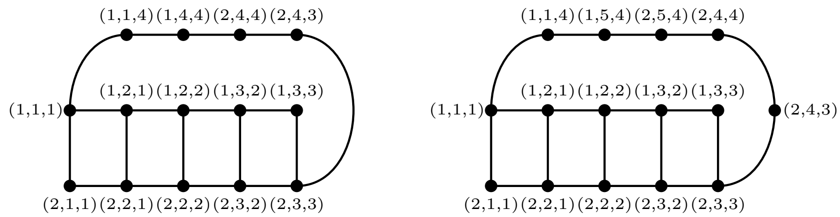

FIGURE 4. Colorings that give induced copies of $M_{5,4}$ and $M_{5,5}$.

Theorem 17. *For all $n \geq 2$, the graph $M_{n,3}$ is a minimal forbidden induced subgraph.*

Proof. We have shown that $M_{n,3}$ is forbidden, so it remains to show that the induced subgraphs of $M_{n,3}$ spanned by all but one vertex are permissible. This falls into five cases. Case 1: Removing $\alpha_1, \beta_n, \beta_{n+1}, \beta_{n+2}$, or β_{n+3} from $V[M_{n,3}]$ yields an induced grid graph $P_r \square P_2$ with one or two paths attached. By Theorem 10 and the discussion after it, this is permissible.

Case 2: Removing β_1 or α_n from $V[M_{n,3}]$ yields an induced copy of $M_{n-1,4}$ which is permissible.

Case 3: Removing β_2 or α_{n-1} from $V[M_{n,3}]$ yields an induced copy of $M_{n-2,5}$ with an attached leaf which is permissible.

Case 4: Removing α_2 or β_{n-1} from $V[M_{n,3}]$ yields an induced subgraph of $C_{n+4} \square P_2$. Since C_{n+4} is permissible when $n \geq 2$, this is permissible.

In the final case, consider removal of one of the vertices $\beta_3, \ldots, \beta_{n-2}$. The argument for removal of one of $\alpha_3, \ldots, \alpha_{n-2}$ is analogous. Let $3 \leq i \leq n-2$. Then $M_{n-i,3+i}$ is permissible. Take a copy of P_2 with vertices γ and σ. Then $M_{n-i,3+i} \square P_2$ is permissible, and every induced subgraph of it is also permissible. One can check that the induced subgraph of $M_{n-i,3+i}$ spanned by $V[M_{n-i,3+i}] \times \{\gamma\} \cup \{(\beta_{n-i+4}, \sigma), \ldots, (\beta_{n+3}, \sigma)\}$ is the same as the induced subgraph of $M_{n,3}$ obtained by removing β_i from $V[M_{n,3}]$. This is illustrated in Figure 5. $\square$

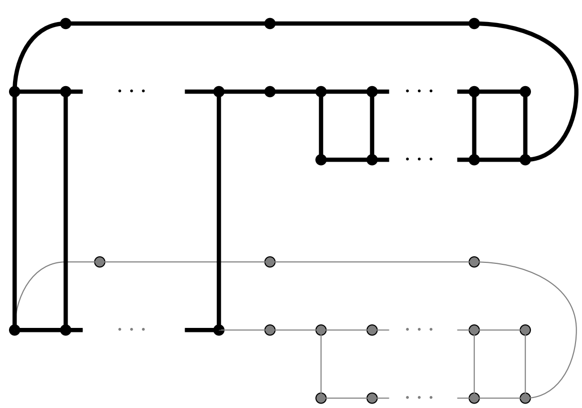

FIGURE 5. Case 5 of the proof of Theorem 17. Only the copies of P_2 lying in the pertinent induced subgraph are shown.

4. Girth of coloring graphs

Recall, the *girth* of a graph G, denoted $g(G)$, is the smallest number n such that C_n is an induced subgraph of G. If G is acyclic, then we say the girth of G is infinite. When $k > \chi(G)$, we can put tight bounds on the girth of $\mathcal{C}_k(G)$ as the following sequence of lemmas shows.

Lemma 18. *If $k > 1 + \chi(G)$ then $g\left(\mathcal{C}_k(G)\right) = 3$.*

Proof. Select a vertex $u \in G$ and any $\chi(G)$-coloring of G, say α. Since $k > 1+\chi(G)$, there are at least two colors, say 1 and 2, not assigned to neighbors of u. Define β and γ such that $\beta(v) = \gamma(v) = \alpha(v)$ for $v \neq u$, $\beta(u) = 1$ and $\gamma(u) = 2$. Then α, β, and γ differ in color only at u and are hence all adjacent in $\mathcal{C}_k(G)$, spanning a C_3 and illustrating that the girth of $\mathcal{C}_k(G)$ is 3. $\square$

This lemma completely classifies the possible coloring graph girths when more than $\chi(G) + 1$ colors are used. It also gives that any graph with chromatic number larger than 3 that has a vertex of degree 2 has a coloring graph with girth 3.

Lemma 19. *If $k = \chi(G) + 1$ and $\mathcal{C}_{k-1}(G)$ has an edge, then $g(\mathcal{C}_k(G)) = 3$.*

Proof. Let $k = \chi(G) + 1$ and suppose $\mathcal{C}_{k-1}(G)$ has an edge between two colorings, say α and β. Then α and β are also colorings in $\mathcal{C}_k(G)$ and they differ at only one vertex of G, say u. Since these are $k-1$ colorings, u is not colored k. Construct a new coloring γ of G such that $\gamma(v) := \alpha(v)$ for all $v \neq u$ and $\gamma(u) := k$. Then γ is adjacent to both α and γ, which are also adjacent. So $\{\alpha, \beta, \gamma\}$ spans C_3 and the girth of $\mathcal{C}_k(G)$ is 3. $\square$

To complete the analysis of coloring graphs that use more than $\chi(G)$ colors, we only need to address the case of $k = \chi(G) + 1$ and where $\mathcal{C}_\chi(G)$ is edgeless. It can be shown that $\mathcal{C}_{\chi(G)+1}(G)$ has girth at most 4 unless $G = K_{k-1}$. We address the complete graph first.

Lemma 20. *For $k > 2$, $g(\mathcal{C}_k(K_{k-1})) = 6$.*

Proof. Suppose $V[K_{k-1}] = \{v_1, \ldots v_{k-1}\}$. Define the following proper colorings γ_i, $i = 1, \ldots, 6$.

Coloring	$\gamma_i(v_1)$	$\gamma_i(v_2)$	$\gamma_i(v_j), j > 2$
γ_1	1	2	j
γ_2	1	k	j
γ_3	2	k	j
γ_4	2	1	j
γ_5	k	1	j
γ_6	k	2	j

Then $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6\}$ span an induced C_6 in $\mathcal{C}_k(G)$, hence $g(\mathcal{C}_k(G)) \leq 6$.

It remains to show that $\mathcal{C}_k(K_{k-1})$ does not contain any smaller cycles. A C_3 can only occur if one vertex can change to two other colors but the base graph K_{k-1} is complete so there is only one possible color change for any vertex in K_{k-1}. By Corollary 12 there is no induced C_5. Hence the only possible cycle is C_4 and it must have edge label partition type 2,2. Call the two vertices that change color u and v. Then both u and v must each take a new color in this C_4. Hence two new colors are required but again, there is only one spare color available. Thus the girth must be six. $\square$

Lemma 21. *Assume $k = \chi(G) + 1$ and $G \neq K_{k-1}$. Then $g(\mathcal{C}_k(G)) \leq 4$.*

Proof. Assume G is not K_{k-1}. Then there exist vertices u and v of $V[G]$ which are not adjacent. Let α be a $k-1$ coloring of G. Define three new colorings γ_i by $\gamma_i(w) = \alpha(w)$ for all $w \neq u, v$ and:

Coloring	$\gamma_i(u)$	$\gamma_i(v)$
γ_1	$\alpha(u)$	k
γ_2	k	k
γ_3	k	$\alpha(v)$

Then the set $\{\alpha, \gamma_i : 1 \leq i \leq 3\}$ spans an induced C_4 so the $g(\mathcal{C}_k(G)) \leq 4$. □

Combining these lemmas gives the girth classification for $k > \chi(G)$, summarized in the theorem below. Note that the lemmas actually give more detailed information about each of the girth cases.

Theorem 22. *If $k > \chi(G)$ then $g(\mathcal{C}_k(G)) \in \{3, 4, 6\}$.*

We can finally answer the question: Are cycles coloring graphs?

Theorem 23. *C_3, C_4 and C_6 are the only cycles that occur as coloring graphs.*

Proof. We have shown $\mathcal{C}_3(I_1) = C_3$ and $\mathcal{C}_2(I_2) = C_4$. We leave it to the reader to verify $\mathcal{C}_3(P_2) = C_6$. The only 1-coloring graph is I_1, and the bipartite classification shows the only 2-coloring graph that is a cycle is C_4. When $k \geq 3$ and $k \geq \chi(G)+1$, we have shown $g(\mathcal{C}_k(G))$ is 3, 4, or 6; so in this case, if $\mathcal{C}_k(G) = C_n$, it follows that $n = 3, 4$, or 6. For the remainder of the argument, assume $k = \chi(G) \geq 3$ and $\mathcal{C}_k(G) = C_n$.

Let $P = \alpha_1 \dots \alpha_q$ be a longest induced path in $\mathcal{C}_k(G)$ such that no two vertices are isomorphic colorings. Because $\mathcal{C}_k(G)$ is a cycle of length at least $k!q$, the coloring α_q must be adjacent to another coloring β where $\beta \neq \alpha_i$ for all $1 \leq i \leq q$. Since P is maximal, β must be isomorphic to some α_i. However, isomorphic colorings are distance at least 5, so $1 \leq i \leq q-4$.

Let σ be the permutation for which $\sigma(\alpha_i) = \beta$. If $i > 1$, the coloring $\beta = \sigma(\alpha_i)$ must be adjacent to $\sigma(\alpha_{i-1})$ and $\sigma(\alpha_{i+1})$. Because P consists of non isomorphic colorings and $q \neq i \pm 1$, it follows that $\alpha_q \neq \sigma(\alpha_{i\pm1})$. Since β is adjacent to α_q and has degree 2 in the coloring graph, this cannot happen. We conclude $\beta = \sigma(\alpha_1)$. Let the order of permutation σ be m. Then the following is a closed walk in $\mathcal{C}_k(G)$.

$$\alpha_1 \dots \alpha_q \sigma(\alpha_1) \dots \sigma(\alpha_q) \sigma^2(\alpha_1) \dots \dots \sigma^{m-2}(\alpha_q) \sigma^{m-1}(\alpha_1) \dots \sigma^{m-1}(\alpha_q) \alpha_1$$

Since $k = \chi(G)$, no two permutations of α_i yield the same coloring. The symmetric group on n elements is not cyclic for $n \geq 3$, and so the walk above does not contain all colorings of G. Thus, we have shown that when $k = \chi(G) \geq 3$, the coloring graph cannot be a cycle. □

The question of girth when $k = \chi(G)$ is more complex. We show that in this case, there are infinitely many possible girths. The following definition will be useful. If α is a k-coloring define V_α to be the set of vertices that can be recolored in α, that is, $x \in V_\alpha$ if and only if there exists another χ-coloring α_x such that $\alpha(v) = \alpha_x(v)$ for all vertices of $V[G]$ except $\alpha(x) \neq \alpha_x(x)$. We show that each vertex that can be recolored can only change to one other color, that every vertex that can be recolored can only change to the same other color, and other properties about elements of this set.

Lemma 24. *Let G be a graph with coloring graph $\mathscr{C}_\chi(G)$ of girth greater than 4. Let α be any χ-coloring of G.*

1. *For every $v \in V[G]$, there is at most one other coloring of G adjacent to α in $\mathcal{C}_\chi(G)$ which recolors v.*
2. *The induced subgraph of G spanned by V_α is a clique. Additionally, $|V_\alpha| \leq \chi(G) - 1$.*
3. *Let $\alpha\beta$, $\alpha\gamma \in E[\mathcal{C}_\chi(G)]$ correspond to recoloring vertices u and v respectively. Then $\beta(u) = \gamma(v)$.*
4. *Suppose $\alpha_1, \ldots, \alpha_t$ are colorings of G that form a cycle in $\mathscr{C}_\chi(G)$, where $\alpha_i\alpha_{i+1}$ recolors v_i. Then $v_1, v_2, \ldots, v_t$ is a closed walk in G.*

Proof. 1. The first observation is trivial since if there were two such colorings they would form an induced C_3 but the girth is greater than 4.

2. Let $u, v \in V_\alpha$ such that $uv \notin E[G]$. Since $u, v \in V_\alpha$ there exist colorings β and γ which recolor u and v respectively. Because $uv \notin E[G]$, $u \in V_\gamma$ and $v \in V_\beta$. Moreover, there exists a proper coloring μ such that $\mu(w) = \alpha(w) = \beta(w)$ for all $w \neq u, v$, $\mu(u) = \beta(u)$ and $\mu(v) = \gamma(v)$. Then $\{\alpha, \beta, \gamma, \mu\}$ span an induced C_4 in $\mathcal{C}_k(G)$, which contradicts the girth requirement. Hence $uv \in E[G]$ and V_α spans a clique in G. Because V_α is a χ-colorable clique, each of whose vertices is recolorable, $|V_\alpha| \leq \chi(G) - 1$.

3. Let $\alpha\beta$, $\alpha\gamma \in E[\mathcal{C}_k(G)]$ correspond to recolorings of u and v respectively. If $\beta(u) \neq \gamma(v)$, then again we can define the coloring μ as above so that $\{\alpha, \beta, \gamma, \mu\}$ span an induced C_4 in $\mathcal{C}_k(G)$, again contradicting the girth requirement.

4. It follows from part 2 of this lemma that v_i, v_{i+1} are adjacent for all i. Thus the v_i form a walk. In fact $v_t, v_1 \in V_{\alpha_1}$, so these vertices are adjacent as well and $v_1, \ldots, v_t$ forms a closed walk in G. $\square$

The following theorem gives all the possibilities for $g(\mathcal{C}_\chi(G))$ in the case where $\chi(G) = 3$ and G is connected.

Theorem 25. *Let G be a connected graph with $\chi(G) = 3$. Then $g(\mathcal{C}_3(G)) \in \{\infty, 4\} \cup \{15, 24, \ldots, 9m+6, \ldots\}$ and each of these girths can be achieved. Moreover, if $4 < g(\mathcal{C}_3(G)) < \infty$, then $|G| = 3m + 2$ for some $m > 0$ and $\mathcal{C}_3(G) = C_{9m+6} \sqcup C_{9m+6}$.*

Proof. First notice that $g(\mathcal{C}_3(K_3)) = \infty$, and if G has two vertices of degree 1 then $g(\mathcal{C}_3(G)) = 4$ since those two vertices can be independently recolored to create a C_4. Hence we know these girths can be obtained. Since $\chi(G) = 3$, and is connected it is clear $\mathcal{C}_3(G)$ is triangle free.

For the rest of the proof assume the girth of the 3-coloring graph is at least 5. We will show that the girth must be of the form $9m + 6$. Let $\alpha_1, \alpha_2 \ldots \alpha_n, \alpha_{n+1} = \alpha_1$ be a shortest cycle in $\mathcal{C}_3(G)$ and $\alpha_i\alpha_{i+1}$ recolor $v_i \in V[G]$. By Lemma 24, part 4, $v_1, v_2, \ldots, v_{n+1}$ forms a closed walk in G.

We now argue what the coloring α_1 is on all vertices of the closed walk. Without loss of generality, $\alpha_1(v_1) = 1, \alpha_1(v_2) = 2$, then all neighbors of v_1 must also be color 2 in α_1 so that v_1 can be recolored, so in particular $\alpha_1(v_n) = 2$. Now α_2 differs from α_1 only at the vertex v_1, and in α_2 all neighbors of v_2 must be the same color so that v_2 can be recolored, so in particular $\alpha_2(v_3) = \alpha_1(v_3) = 3$. Continuing this line of reasoning, we see that α_1 must color the vertices $v_1, \ldots, v_n$ as $123123\ldots12$. A priori, the walk $v_1, v_2, \ldots, v_n$ may not be a simple cycle. We next establish that it is a walk that goes 3 times around a cycle of length $3m+2$. Observe that the first few colorings along the cycle in $\mathcal{C}_3(G)$ must be as follows on the vertices $v_1, \ldots, v_{3m+2}$.

	v_1	v_2	v_3	v_4	v_5	v_6	$\ldots$	v_{3m}	v_{3m+1}	v_{3m+2}	$\ldots$
α_1	1	2	3	1	2	3	$\ldots$	3	1	2	$\ldots$
α_2	3	2	3	1	2	3	$\ldots$	3	1	2	$\ldots$
α_3	3	1	3	1	2	3	$\ldots$	3	1	2	$\ldots$
α_4	3	1	2	1	2	3	$\ldots$	3	1	2	$\ldots$
α_5	3	1	2	3	2	3	$\ldots$	3	1	2	$\ldots$
$\ldots$							$\ldots$				
α_{3m+1}	3	1	2	3	1	2	$\ldots$	2	1	2	$\ldots$
α_{3m+2}	3	1	2	3	1	2	$\ldots$	2	3	2	$\ldots$
α_{3m+3}	3	1	2	3	1	2	$\ldots$	2	3	1	$\ldots$

If two v_i represented the same vertex they must get the same color under each α_j. Examining these colorings we can observe that each vertex in the set $\{v_1, \ldots, v_{3m+2}\}$ must be distinct. For example, α_1 shows that $v_1 \neq v_{3r+1}$ and $v_1 \neq v_{3r}$ for $r = 0, \ldots, m$, while α_2 shows that $v_1 \neq v_{3r+2}$ for $r = 1, \ldots, m$.

Note that α_1 and α_{3m+3} are isomorphic colorings on these $3m+2$ vertices (the color classes are the same independent sets). Thus if v_{3m+2} and v_n are the same vertex then continuing this pattern of colorings forces $3(3m+2)$ different colorings before returning to α_1. Thus the α_i form a cycle of length $9m+6$. By Lemma 24 part 2, for each α_i, $|V_{\alpha_i}| \leq 2$ and each vertex $v \in V_{\alpha_i}$ can only be recolored with one other color (part 1 of lemma). Thus in fact each α_i has degree exactly 2 in $\mathcal{C}_3(G)$ and the α_i form a connected component in $\mathcal{C}_3(G)$.

Next we show there are no other vertices in the base graph. Suppose $V[G] \neq \{v_1, \ldots, v_{3m+2}\}$. Because G is connected there must be an $x \in V[G] \backslash \{v_1, \ldots, v_{3m+2}\}$ such that xv_r is an edge for some $r = \{1, \ldots, 3m+2\}$. Each α_i is a proper coloring of $V[G]$. Since $x \in V[G] \backslash \{v_1, \ldots, v_{3m+2}\}$, $\alpha_i(x) = \alpha_j(x)$ for all α_i, α_j. But now, there exist i, j, k such that $\alpha_i(v_r) = 1, \alpha_j(v_r) = 2, \alpha_k(v_r) = 3$ so that there is no color for x that can be used in all colorings. Thus G has no other vertices.

Thus far we have proven that if $4 < g\left(\mathcal{C}_3(G)\right) < \infty$ then the base graph must have $|V[G]| = 3m+2$ and $g\left(\mathcal{C}_3(G)\right) = 9m+6$. It remains to show that such a graph actually exists. This is accomplished with the family of circulant graphs G_m, with $V[G_m] = \{v_1, \ldots, v_{3m+2}\}$, and $E[G_m] = \{v_iv_j : |i-j| = 3k+1, 1 \leq k \leq m\}$. It is clear that $|V[G_m]| = 3m+2$ and $\chi(G_m) = 3$. An inductive argument can be used to show that the only proper 3-colorings of G_m up to isomorphism are those given by the α_i. Thus $\mathcal{C}_3(G_m) = C_{9m+6} \sqcup C_{9m+6}$. $\square$

Note that in the above proof we have given only one possible graph that achieves each girth $g = 9m+6$ for $m \geq 1$. It is possible that a different set of chords on a $3m+2$ cycle would give a base graph with the same coloring graph. We observe that things get considerably more complicated when $\chi(G) > 3$. We show only some possible girths using circulants similar to those in the proof of Theorem 25.

Theorem 26. *Given $k \geq 3$, there exists a graph S_k with $\chi(S_k) = k$ and $g(\mathcal{C}_k(S_k)) = k(2k-1)$.*

Proof. Given $k \geq 3$, denote by S_k the graph on cyclically ordered vertex set $V = \{v_1, \ldots, v_{2k-1}\}$ with edges connecting each vertex to the next $k-2$ (modulo $2k-1$)

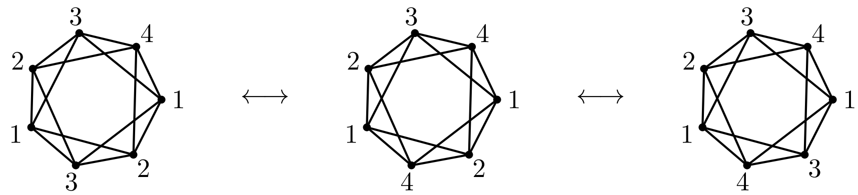

FIGURE 6. Part of a cycle in $\mathcal{C}_4(S_4)$.

vertices. The size of a maximum independent set in this graph is two. It follows that $\chi(S_k) \geq k$. Define $\alpha : V[S_k] \to \{1, \ldots, k\}$ as follows.

$$\alpha(v_i) = \begin{cases} i & \text{if } 1 \leq i \leq k-1 \\ i-k+1 & \text{if } k \leq i \leq 2k-2 \\ k & \text{if } i = 2k-1. \end{cases}$$

This is a proper k-coloring of S_k, so $\chi(S_k) = k$. The leftmost coloring in Figure 6 shows S_4 colored according to α.

There are $2k-1$ non-isomorphic k-colorings of S_k – one for each choice of vertex to be uniquely colored. We call this uniquely colored vertex the exceptional vertex and its color the exceptional color. Note that there are $(2k-1)k!$ vertices in $\mathcal{C}_k(S_k)$. Given any k-coloring of S_k, the only two vertices that can be recolored are those that are distance two from the exceptional vertex. They can only be recolored by the exceptional color. Hence every vertex in $\mathcal{C}_k(S_k)$ has degree 2. One can check that, after $k(2k-1)$ recolorings, one arrives back at the original coloring. Hence $\mathcal{C}_k(S_k)$ is the disjoint union of $(k-1)!$ cycles of size $k(2k-1)$. □

The existence of a coloring graph with a given girth for some χ ensures the existence of other coloring graphs of the same girth for any $\chi' > \chi$, as described below.

Theorem 27. *Let G be a graph with $\chi(G) = k$ and $g(\mathcal{C}_k(G)) = r$. Then for any $k' > k$ there exists a connected graph G' with $\chi(G') = k'$ such that r is the girth of $\mathcal{C}_{k'}(G')$.*

Proof. Suppose $V[G] = \{v_1, \ldots, v_n\}$. Let G' be the union of G and a complete graph on k' vertices with additional edges defined as follows. Let the vertex set of G' be given by $V[G'] = V[G] \cup \{u_1, \ldots, u_{k'}\}$ and $E[G'] = E[G] \cup \{u_iu_j : i \neq j\} \cup \{v_iu_j : 1 \leq i \leq n, k+1 \leq j \leq k'\}$. Let α be a k'-coloring of G'. Then $\alpha|_G$ is a k-coloring of G using exactly the k colors used on $\{u_1, \ldots, u_k\}$ and $V_\alpha \subset V[G]$. Thus $\mathcal{C}_{k'}(G')$ contains a copy of $\mathcal{C}_k(G)$. Hence $\mathcal{C}_{k'}(G')$ consists of $k'!$ disjoint copies of $\mathcal{C}_k(G)$, and the girths of $\mathcal{C}_{k'}(G')$ and $\mathcal{C}_k(G)$ are equal. Notice the equality follows since the addition of the complete graph on k' vertices to the base graph G can not reduce the girth when k' colors are used. □

5. Implications of order

In Section 2, we presented a formula for the order of $\mathcal{C}_k(G)$ in terms of isomorphism classes of k-colorings of G. In this section, we give examples of how this formula can be utilized to analyze which graphs of a particular order, say n, are coloring graphs. We include general results about order, example computations characterizing all coloring graphs of particular orders, and a census of all coloring graphs of order up to 12.

Let m_i be the number of isomorphism classes of k-colorings that partition $V[G]$ into exactly i independent sets where $1 \leq i \leq k$. Then Lemma 2 gives the following formula for the order of a coloring graph.

$$|\mathcal{C}_k(G)| = \sum_{i=1}^{k} m_i \cdot k(k-1)\dots(k-i+1) \tag{1}$$

For a given n, there is at least one coloring graph of that order since $\mathcal{C}_n(I_1) = K_n$. More generally, for each pair $r, s \in \mathbb{N}$ such that $n = r^s$ we get a coloring graph of order n via the construction $\mathcal{C}_r(I_s) = \square_{i=1}^{s} K_r$. This characterizes all coloring graphs that come from empty graphs.

We have previously seen one divisibility consequence of Equation 1: k must divide $|\mathcal{C}_k(G)|$. Recall that we also proved $\mathcal{C}_k(G_1 \sqcup G_2) = \mathcal{C}_k(G_1) \square \mathcal{C}_k(G_2)$. Putting these two facts together, yields the following lemma.

Lemma 28. *If G has m components, then the order of $\mathcal{C}_k(G)$ is divisible by k^m. Said another way, if n is squarefree and $|\mathcal{C}_k(G)| = n$, then G is connected.*

If a graph G has at least one edge, we can say more. It follows that $\chi(G) > 1$, $m_1 = 0$, and hence $|\mathcal{C}_k(G)|$ is divisible by $k(k-1)$. This means that, unless $k = 2$ or G is edgeless, the order of a coloring graph must have two consecutive nonunital factors. These observations lead to the following two lemmas.

Lemma 29. *If n is a power of 2, then a coloring graph of order n has one of the following two structures.*

- *$\mathcal{C}_r(I_s) = \square_{i=1}^{s} K_r$ where $n = r^s$ (Here r is a power of 2.)*
- *$\mathcal{C}_2(G) = K_2^{\square a} \square I_2^{\square s=b}$ where $n = 2^a 2^b$ and G is bipartite with $a + b$ components a of which are isolated vertices*

Lemma 30. *If n is not a power of 2 and there does not exist $k > 2$ with $k(k-1)$ divides n, then the only coloring graphs of order n are $\mathcal{C}_r(I_s) = \square_{i=1}^{s} K_r$ where $n = r^s$.*

If a given n cannot be placed into one of the above categories, it is still possible to use Equation 1 together with other results from this article to help determine which graphs of order n are coloring graphs. In particular, we can use the factorization of n to rule out k values for which $|\mathcal{C}_k(G)| \neq n$ regardless of the structure of G. Of course, as n becomes larger, this becomes more difficult. As an example of such an argument, we classify all coloring graphs of order $n = 6$.

Lemma 31. *The coloring graphs of order 6 are I_6, K_6, and C_6.*

Proof. Assume that $|\mathcal{C}_k(G)| = 6$. Since k divides 6, we see that $k = 2, 3$, or 6. (Note that $k \neq 1$ since $\mathcal{C}_1(G) = I_1$ or $\emptyset$.) If $k = 6$, then it follows from the order formula that $m_1 = 1$ and $m_i = 0$ for all $i > 1$. This means that G is a single vertex, and $\mathcal{C}_6(G) = K_6$. Since 6 is not a power of 2 it follows from the characterization of 2-coloring graphs that $k \neq 2$.

Hence any other coloring graphs of order 6 must occur when $k = 3$. From Equation 1, if $k = 3$ we have $6 = |\mathcal{C}_3(G)| = 3m_1 + 6m_2 + 6m_3$. If $m_1 \neq 0$, that would imply G is edgeless and the order of $\mathcal{C}_3(G)$ is a power of 3. Since 6 is not a power of 3, it follows that $m_1 = 0$.

If $m_2 = 1$ and $m_3 = 0$, then G must have only two vertices joined by an edge. We can check that, indeed, $\mathcal{C}_3(P_2) = C_6$. If $m_2 = 0$ and $m_3 = 1$, this means that $\chi(G) = 3$. Since isomorphic colorings are not adjacent in the coloring graph and $m_3 = 1$, it follows that $\mathcal{C}_3(G)$ must be edgeless. Furthermore, $\mathcal{C}_3(K_3) = I_6$, so this is a coloring graph. □

We provide another more involved example classifying coloring graphs of order 12.

Lemma 32. *The coloring graphs of order 12 are* $I_{12}, K_{12}, \sqcup_{i=1}^{6} P_2, \mathcal{C}_3(P_3)$, *and* $\mathcal{C}_4(P_2)$.

Proof. Because 12 is not a power of 2, we know $k \neq 2$, so we may assume $k \geq 3$. Since k^2 does not divide 12 for any $k \geq 3$, we conclude that G is connected. If $m_1 \neq 0$, then G is edgeless. Since G is connected, in this case we conclude that $\mathcal{C}_k(G) = \mathcal{C}_{12}(I_1) = K_{12}$.

Now assume $m_1 = 0$, and G therefore has at least one edge. Substitution into Equation 1 yields

$$12 = m_2 \cdot k(k-1) + m_3 \cdot k(k-1)(k-2) + \cdots + m_k \cdot k!.$$

If $k \geq 5$, then the right-hand side of the above equation is too large. Thus $k = 3$ or $k = 4$.

Suppose $k = 4$. Then we have $12 = 12m_2 + 24m_3 + 24m_4$, and the only possibility is $m_2 = 1$ and $m_3 = m_4 = 0$. Therefore G must be connected and bipartite, and cannot have more than 2 vertices because otherwise we would not have $m_3 \neq 0$. The only possibility is $G = P_2$. In this case, the coloring graph is $\mathcal{C}_4(P_2)$.

Suppose $k = 3$. Then we see that $12 = 6(m_2 + m_3)$. If $m_2 = 2$ and $m_3 = 0$, it follows that G can only have two vertices. The only possible G would therefore be P_2, but then $m_2 \neq 2$. Hence we cannot have a 3-coloring graph with $m_2 = 2$ and $m_3 = 0$.

Suppose $m_2 = m_3 = 1$. Since $m_3 = 1$, we know G contains at least three vertices. Also, $m_2 \neq 0$ implies G is 2-colorable. If G, which must be connected, contains four or more vertices, one can show that $m_3 > 1$. Therefore, we know $G = P_3$, and the coloring graph in this case is $\mathcal{C}_3(P_3)$.

Suppose $m_2 = 0$ and $m_3 = 2$. This means $\chi(G) = 3$. In this case, by Lemma 5, there are not enough isomorphism classes for isomorphic colorings to be path connected. Hence $\mathcal{C}_k(G)$ is either edgeless or consists of six copies of P_2. We can realize both of these. Indeed, the 3-coloring graph of two copies of K_3 glued at one vertex is I_{12}, and the 3-coloring graph of K_3 with one additional edge is $\sqcup_{i=1}^{6} P_2$. □

We conclude with a chart applying our results to classify all coloring graphs up to order 12.

Vertices	**Possible values of k**	**Number of graphs**	**Members**
1	1	1	K_1
2	2	2	I_2, K_2
3	3	1	K_3
4	2,4	4	I_4, M_4, C_4, K_4
5	5	1	K_5
6	3,6	3	I_6, C_6, K_6
7	7	1	K_7
8	2, 8	5	I_8, M_8, $(K_2)^{\square 3}$, $(K_2)^{\square 2} \square I_2$, K_8
9	3, 9	2	$K_3 \square K_3$, K_9
10	10	1	K_{10}
11	11	1	K_{11}
12	3,4	5	$I_{12}, K_{12}, \sqcup_{i=1}^{6} P_2, \mathcal{C}_3(P_3), \mathcal{C}_4(P_2)$

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