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ON WELL-POSEDNESS, STABILITY, AND BIFURCATION FOR THE AXISYMMETRIC SURFACE DIFFUSION FLOW[*]

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Abstract. We study the axisymmetric surface diffusion (ASD) flow, a fourth-order geometric evolution law. In particular, we prove that ASD generates a real analytic semiflow in the space of $(2 + \alpha)$-little-Hölder regular surfaces of revolution embedded in $\mathbb{R}^3$ and satisfying periodic boundary conditions. Further, we investigate the geometric properties of solutions to ASD. Utilizing a connection to axisymmetric surfaces with constant mean curvature, we characterize the equilibria of ASD. Then, focusing on the family of cylinders, we establish results regarding stability, instability, and bifurcation behavior, with the radius acting as a bifurcation parameter.

Key words. surface diffusion, well-posedness, periodic boundary conditions, maximal regularity, nonlinear stability, bifurcation, implicit function theorem

AMS subject classifications. Primary, 35K93, 53C44; Secondary, 35B35, 35B32, 46T20

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1. Introduction. The central focus of this article is the development of an analytic setting for the axisymmetric surface diffusion (ASD) flow with periodic boundary conditions. We establish well-posedness of ASD and investigate geometric properties of solutions, including characterizing equilibria and studying their stability, instability, and bifurcation behavior. We establish and take full advantage of *maximal regularity* for ASD. Most notably, with maximal regularity we gain access to the implicit function theorem, a very powerful tool in nonlinear analysis and dynamical systems theory. We begin with a motivation and derivation of the general surface diffusion flow, of which ASD is a special case, and we introduce the general outline of the paper.

The mathematical equations modeling surface diffusion go back to a paper by Mullins [55] from the 1950s, who was in turn motivated by earlier work of Herring [34]. Both of these authors investigate phenomena witnessed in sintering processes, a method by which objects are created by heating powdered material to a high temperature, while remaining below the melting point of the particular substance. When the applied temperature reaches a critical value, the atoms on the surfaces of individual particles will diffuse across to other particles, fusing the powder together into one solid object. In response to gradients of the chemical potential along the surface of this newly formed object, the surface atoms may undergo diffusive mass transport *on* the surface of the object, attempting to reduce the surface free energy. Given the right conditions—temperature, pressure, grain size, sample size, etc.—the mass flux due to this chemical potential will dominate the dynamics, and it is the resulting morphological evolution of the surface which the surface diffusion flow aims to model. We also note that the surface diffusion flow has been used to model the motion of surfaces in other physical processes (e.g., growth of crystals and nanostructures). The article [11] contains the formulation of the model which we present below, which is set in a more general framework than the original model developed by Mullins.

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1.1. The surface diffusion flow. From a mathematical perspective, the governing equation for motion via surface diffusion can be expressed for hypersurfaces in arbitrary space dimensions. In particular, let $\Gamma \subset \mathbb{R}^n$ be a closed, compact, immersed, oriented Riemannian manifold with codimension 1. Then we denote by $\mathcal{H} = \mathcal{H}(\Gamma)$ the (normalized) mean curvature on Γ, which is simply the sum of the principle curvatures on the hypersurface, and Δ_Γ denotes the Laplace–Beltrami operator, or *surface Laplacian*, on Γ. The motion of the surface Γ by surface diffusion is then governed by the equation

$$V = \Delta_\Gamma \mathcal{H},$$

where V denotes the normal velocity of the surface Γ. If Γ encloses a region Ω, we assume the unit normal field to be pointing outward. A solution to the surface diffusion problem on the interval $J \subset \mathbb{R}_+$, with $0 \in J$, is a family $\{\Gamma(t) : t \in J\}$ of closed, compact, immersed hypersurfaces in $\mathbb{R}^n$ which satisfy the equation

(1.1)
$$\begin{cases} V(\Gamma(t)) = \Delta_{\Gamma(t)} \mathcal{H}(\Gamma(t)), & t \in \dot{J} := J \setminus \{0\}, \\ \Gamma(0) = \Gamma_0, \end{cases}$$

for a given initial hypersurface Γ_0. It can be shown that solutions to (1.1) are volume-preserving, in the sense that the signed volume of the region Ω is preserved along solutions. Additionally, (1.1) is surface-area-reducing. It is also interesting to note that the surface diffusion flow can be viewed as the H^{-1}-gradient flow of the area functional, a fact that was first observed in [31]. This particular structure has been exploited in [52, 53] for devising numerical simulations.

For well-posedness of (1.1) we mention [26], where it is shown that (1.1) admits a unique local solution for any initial surface $\Gamma_0 \in C^{2+\alpha}$. Additionally, the authors of [26] show that any initial surface that is a small $C^{2+\alpha}$-perturbation of a sphere admits a global solution which converges to a sphere at an exponential rate. This result was improved in [27] to admit initial surfaces in the Besov space $B^s_{p,2}$ with $s = 5/2 - 4/p$ and $p > (2n + 10)/3$. For dimensions $n < 7$ note that this allows for initial surfaces which are less regular than C^2. An independent theory for existence of solutions to higher-order geometric evolution equations, which also applies to the surface diffusion flow, was developed in [37, 58]; see [54, 63] for a discussion of some limitations of these results.

More recent results for initial surfaces with low regularity are contained in [8, 42]. The author of [8] obtains existence and uniqueness of local solutions for various geometric evolution laws (including the surface diffusion flow), in the setting of entire graphs with initial regularity $C^{1+\alpha}$. Surface diffusion is also one of several evolution laws for which the authors of [42] establish solutions under very weak, and possibly optimal, regularity assumptions on initial data. In particular, their results guarantee existence and uniqueness of global analytic solutions, in the setting of entire graphs over $\mathbb{R}^n$, for Lipschitz initial data u_0 with small Lipschitz constant $\|\nabla u_0\|_{L_\infty(\mathbb{R}^n)}$.

For interesting new developments regarding lower bounds on the existence time of solutions to the surface diffusion flow in $\mathbb{R}^3$ and $\mathbb{R}^4$, we refer the reader to [54, 69, 70]. In particular, it is shown in [70] that the flow of a surface in $\mathbb{R}^3$ which is initially close to a sphere in L_2 (that is, the L_2-norm of the trace-free part of the second fundamental form is sufficiently small) is a family of embeddings that exists globally and converges at an exponential rate to a sphere. The results of [54, 69] regarding concentration of curvature along solutions may prove important in the analytic investigation of solutions approaching finite-time pinch-off.

In the context of geometric evolution equations, such as the mean curvature flow [35, 36], the surface diffusion flow, or the Willmore flow [43, 44], the underlying governing equations are often expressed by evolving a smooth family of immersions $X : M \times (0, T) \to \mathbb{R}^n$, where M is a fixed smooth oriented manifold and $\Gamma(t)$ is the image of M under $X(\cdot, t)$. In this formulation, the surface diffusion flow is given by

$$(1.2) \qquad\qquad \partial_t X = (\Delta H)\nu, \qquad X(\cdot, 0)M = \Gamma_0,$$

where ν is the normal to the surface $\Gamma(t)$. This formulation is invariant under the group of sufficiently smooth diffeomorphisms of M, and this implies that (1.2) is only weakly parabolic. A way to infer that (1.2) is not parabolic is to observe that if $X(\cdot, t)$ is a solution, then so is $X(\phi(\cdot), t)$, for any diffeomorphism ϕ. Given a smooth solution X one can therefore construct nonsmooth (i.e., non-C^∞) solutions by choosing ϕ to be nonsmooth. If (1.2) were parabolic, all solutions would have to be smooth, as was pointed out in [7] for the mean curvature flow.

Nevertheless, existence of unique smooth solutions for the mean curvature flow, for compact C^∞-initial surfaces $X(\cdot, 0)M$, can be derived by making use of the Nash–Moser implicit function theorem; see, for instance, [32, 33].

The Nash–Moser implicit function theorem may also lead to a successful treatment of the surface diffusion flow (1.2). However, there is an alternative approach to dealing with the motion of surfaces by curvature (for example, the mean and volume-preserving mean curvature flows, the surface diffusion flow, the Willmore flow) which removes the issue of randomness of a parameterization: if one fixes the parameterization as a graph in normal direction with respect to a reference manifold and then expresses the governing equations in terms of the graph function, the resulting equations are *quasilinear* and *strictly parabolic*. This approach has been employed in [26, 29, 30, 65], and also in [37]. In the particular case of the surface diffusion flow, one obtains a fourth-order quasilinear parabolic evolution equation. One can then apply well-established results for quasilinear parabolic equations. The theory in [2, 12], for instance, works for any quasilinear parabolic evolution equation, no matter whether it is cast as a more traditional PDE in Euclidean space or an evolution equation living on a manifold. This theory also renders access to well-known principles from dynamical systems.

The approach of parameterizing the unknown surface as a graph in normal direction has also been applied to a wide array of free boundary problems, including problems in phase transitions (where the graph parameterization and its extension into the bulk phases is often referred to as the Hanzawa transformation); see, for example, [61] and the references therein.

The literature on geometric evolution laws often considers the question of short-time existence standard and refers to the classical monographs [23, 24, 45]. However, when the setting is a manifold rather than Euclidean space, existence theory for parabolic (higher-order) equations does not belong to the standard theory and requires a proof, a point that is also acknowledged in [37, p. 61].

1.2. Axisymmetric surface diffusion (ASD). For the remainder of the paper, we focus our attention on the case of $\Gamma \subset \mathbb{R}^3$ an embedded surface which is symmetric about an axis of rotation (which we take to be the x-axis, without loss of generality) and satisfies prescribed periodic boundary conditions on some fixed interval L of periodicity (we take $L = [-\pi, \pi]$ and enforce 2π periodicity, without loss of generality). In particular, the axisymmetric surface Γ is characterized by the

parameterization

$$\Gamma = \Big\{ (x, r(x) \cos(\theta), r(x) \sin(\theta)) : \ x \in \mathbb{R}, \ \theta \in [-\pi, \pi] \Big\},$$

where the function $r : \mathbb{R} \to (0, \infty)$ is the *profile function* for the surface Γ. Conversely, a profile function $r : \mathbb{R} \to (0, \infty)$ generates an axisymmetric surface $\Gamma = \Gamma(r)$ via the parameterization given above.

We thus recast the surface diffusion problem as an evolution equation for the profile functions $r = r(t)$. In particular, one can see that the surface $\Gamma(r)$ inherits the Riemannian metric

$$g = (1 + r_x^2) \, dx \wedge dx + r^2 \, d\theta \wedge d\theta$$

from the embedding $\Gamma \hookrightarrow \mathbb{R}^3$, with respect to the surface coordinates (x, θ), where the subscript $f_{x_i} := \partial_{x_i} f$ denotes the derivative of f with respect to the indicated variable x_i. It follows that the (normalized) mean curvature $\mathcal{H}(r)$ of the surface is given by $\mathcal{H}(r) = \kappa_1 + \kappa_2$, where

$$\kappa_1 = \frac{1}{r \sqrt{1 + r_x^2}} \quad \text{and} \quad \kappa_2 = \frac{-r_{xx}}{(1 + r_x^2)^{3/2}}$$

are the *azimuthal* and *axial* principle curvatures, respectively, on $\Gamma(r)$. Meanwhile, the Laplace–Beltrami operator on Γ and the normal velocity of $\Gamma = \Gamma(t)$ are

$$\Delta_{\Gamma(r)} = \frac{1}{r \sqrt{1 + r_x^2}} \left(\partial_x \left[\frac{r}{\sqrt{1 + r_x^2}} \partial_x \right] + \partial_\theta \left[\frac{\sqrt{1 + r_x^2}}{r} \partial_\theta \right] \right),$$

$$V(t) = \frac{r_t}{\sqrt{1 + r_x^2}} .$$

Finally, substituting these terms into (1.1) and simplifying, we arrive at the expression

$$\text{(1.3)} \quad \begin{cases} r_t = \dfrac{1}{r} \partial_x \left[\dfrac{r}{\sqrt{1 + r_x^2}} \, \partial_x \left(\dfrac{1}{r \sqrt{1 + r_x^2}} - \dfrac{r_{xx}}{(1 + r_x^2)^{\frac{3}{2}}} \right) \right], & t > 0, \ x \in \mathbb{R}, \\[4mm] r(t, x + 2\pi) = r(t, x), & t \geq 0, \ x \in \mathbb{R}, \\[2mm] r(0, x) = r_0(x), & x \in \mathbb{R}, \end{cases}$$

for the periodic axisymmetric surface diffusion problem. To simplify notation in what follows, we define the operator

$$\text{(1.4)} \qquad\qquad G(r) := \frac{1}{r} \, \partial_x \left[\frac{r}{\sqrt{1 + r_x^2}} \, \partial_x \mathcal{H}(r) \right],$$

which is formally equivalent to the right-hand side of the first equation in (1.3). The main results of this paper address
    (a) existence, uniqueness, and regularity of solutions for (1.3);
    (b) nonlinear stability and instability of equilibria for (1.3);
    (c) bifurcation of equilibria from the family of cylinders, with the radius serving as bifurcation parameter.

As mentioned in the introduction, we develop and take full advantage of maximal regularity for ASD. In this setting, the results in (a) follow in a straightforward way from [12]. Existence results could also be based on the approach developed in [8, 42], but we prefer to work within the well-established framework of continuous maximal regularity. It provides a general and flexible setting for investigating further qualitative properties of solutions. The novelty of the results in (b) is analysis of the *nonlinear* structure of solutions. Corresponding results for *linear* stability and instability of equilibria are contained in [10], where a precise characterization of the eigenvalues of the linearized problem is given. Based on a formal center manifold analysis, the authors in [10] predict subcritical bifurcation of equilibria at the critical value of radius $r_\star = 1$, but no analytical proof is provided. Thus, our result in (c) appears to be the first rigorous proof of bifurcation. In addition, we show that the bifurcating equilibria (which are shown to coincide with the Delaunay unduloids) are *nonlinearly* unstable. We note that previous results show *linear* instability of unduloids, and we refer the reader to Remark 10 for a more detailed discussion.

The publication [10] has served as a source of inspiration for our investigations. It provides an excellent overview of the complex qualitative behavior of ASD, with results supported by analytic arguments and numerical computations.

The first investigations of evolution of an axisymmetric surface via surface diffusion can be traced back to the work of Mullins and Nichols [56, 57] in 1965, where one can already see some of the benefits of this special setting. Taking advantage of the symmetry of the problem, they developed an adequate scheme for numerical methods, and they already predicted the finite-time pinch-off of tube-like surfaces via surface diffusion, a feature similar to the mean curvature flow and a natural phenomenon to study in exactly this axisymmetric setting. Research continued to focus on pinch-off behavior using numerical methods; cf. [10, 13, 14, 15, 19, 49, 50], wherein many schemes were developed to handle the continuation of solutions after the change of topology at the moment of pinch-off. Unlike the related behavior for the mean curvature flow, pinch-off for the surface diffusion flow remains a numerical observation that has yet to be verified analytically.

Much research has also focused on the numerical investigation of stability and instability of cylinders under perturbations of various types; see [10, 13, 15], for instance. In an important construction from [15], the authors observe destabilization of a particular perturbation of a cylinder (i.e., divergence from the cylinder) due to second-order effects of the flow, whereas the first-order (linear) theory predicts asymptotic stability. In fact, their formulation produces conditions under which a perturbation will destabilize due to nth-order effects, where $(n-1)$st-order analysis predicts stability. This result highlights the importance of studying the full nonlinear behavior of solutions to ASD.

We proceed with an outline of the article and description of our main results. In section 2, we establish existence of solutions to (1.3) in the framework of continuous maximal regularity. In particular, we have existence and uniqueness of maximal solutions for initial surfaces which are $(2 + \alpha)$-little-Hölder continuous. Solutions are also analytic in time and space, for positive time, with a prescribed singularity at time $t = 0$. Additionally, we state conditions for global existence of the semiflow induced by (1.3). We rely on the theory developed in [46] and the well-posedness results for quasilinear equations with maximal regularity provided in [12]. We include comments on how we prove these well-posedness results in an appendix.

In section 3, we characterize the equilibria of ASD using results of Delaunay [21] and Kenmotsu [39] regarding constant mean curvature surfaces in the axisymmetric

setting. We conclude that all equilibria of (1.3) must fall into the family of undulary curves, which includes all constant functions $r(x) \equiv r_\star > 0$ (corresponding to the cylinder of radius $r_\star$) and the two-parameter family of nontrivial undulary curves $R(B, k)$.

In section 4, we prove that the family of cylinders with radius $r_\star > 1$ are asymptotically, exponentially stable under a large class of nonlinear perturbations, which maintain the same axis of symmetry and satisfy the prescribed periodic boundary conditions. In particular, given $r_\star > 1$, we prove that any sufficiently small $(2 + \alpha)$-little-Hölder regular perturbation produces a global solution which converges exponentially fast to the cylinder of radius $r_\star + \eta > 1$. The value η is determined by the volume enclosed by the perturbation, which may differ from the volume of the original cylinder. In proving this result, we note that the spectrum of the linearized equation at $r_\star$ is contained in the *left half* of the complex plane, though it will always contain 0 as an eigenvalue. By reducing the equation, essentially to the setting of volume-preserving perturbations of a cylinder, we are able to eliminate the zero eigenvalue. We then prove nonlinear stability in the reduced setting, utilizing maximal regularity methods on exponentially weighted function spaces, and we transfer the result back to the (full) problem via a *lifting* operator.

In section 5, we prove nonlinear instability of cylinders with radius $0 < r_\star < 1$. We take nonlinear instability to be the logical negation of stability, which one may interpret as the existence of at least one *unstable perturbation*; see Theorem 5.1 for a precise statement. This result makes use of a contradiction technique reminiscent of results from the theory of ordinary differential equations; cf. [62]. By isolating the linearization of the governing equation, one takes advantage of a *spectral gap* and associated spectral projections in order to derive necessary conditions for *stable perturbations*, which in turn lead to a contradiction.

We note that previous instability results for ASD have focused primarily on classifying stable and unstable *eigenmodes* of equilibria, which gives precise results on the behavior of solutions associated with unstable perturbations. However, these methods are limited to the behavior of solutions under the linearized flow.

Finally, in section 6 we apply classic methods of Crandall and Rabinowitz [16] to verify the subcritical bifurcation structure of all points of intersection between the family of cylinders and the disjoint branches of unduloids. In particular, taking the inverse of the radius $\lambda = 1/r_\star$ as a bifurcation parameter, we verify the existence of continuous families of nontrivial equilibria which branch off of the family of cylinders at radii $r_\star = 1/\ell$ for all $\ell \in \mathbb{N}$. We conclude that each of these branches corresponds to the branch $R(B, \ell)$ of $2\pi/\ell$-periodic undulary curves. Working in the reduced setting established in section 4, it turns out that eigenvalues associated with the linearized problem are not simple, and hence we cannot directly apply the results of [16]. However, restricting attention to surfaces which are even (symmetric about the surface $[x = 0]$), we eliminate redundant eigenvalues, similar to a method used by Escher and Matioc [25]. In this even function setting, we have simple eigenvalues and derive bifurcation results, which we apply back to the full problem via a posteriori symmetries of equilibria.

Using eigenvalue perturbation methods, we are also able to conclude nonlinear instability of nontrivial unduloids, using the same techniques as in section 5. We once more refer the reader to Remark 10 for more information.

In future work we plan to investigate well-posedness of ASD under weaker regularity assumptions on the initial data. This will allow for a better understanding of

global existence, and obstructions thereof. In particular, we conjecture that solutions developing singularities will have to go through a pinch-off.

We also plan to consider nonaxisymmetric surfaces. In particular, we plan to investigate the stability of cylinders under nonaxisymmetric perturbations.

Other interesting questions involve the existence and nature of unstable families of perturbations and reformulations of the problem to allow for different boundary conditions and immersed surfaces. In particular, reformulating ASD in terms of parametrically defined curves in $\mathbb{R}^2$ would allow for consideration of immersed surfaces of revolution, a setting within which the branches of $2\pi/\ell$-periodic nodary curves would be added to the collection of equilibria. See section 3 for a definition and graphs of nodary curves.

Throughout the paper we will use the following notation: If E and F are arbitrary Banach spaces, $\mathbb{B}_E(a, r)$ denotes the open ball in E with center a and radius $r > 0$ and $\mathcal{L}(E, F)$ consists of all bounded linear operators from E into F. For $U \subset E$ an open set, we denote by $C^\omega(U, F)$ the space of all real analytic mappings from U into F.

1.3. Maximal regularity. We briefly introduce (continuous) maximal regularity, also called *optimal regularity* in the literature. Maximal regularity has received a lot of attention in connection with parabolic partial differential equations and evolution laws; cf. [3, 4, 5, 12, 41, 48, 60, 64]. Although maximal regularity can be developed in a more general setting, we will focus on the setting of *continuous* maximal regularity and direct the interested reader to the references [3, 48] for a general development of the theory.

Let $\mu \in (0, 1]$, $J := [0, T]$ for some $T > 0$, and let E be a (real or complex) Banach space. Following the notation of [12], we define spaces of continuous functions on $\dot{J} := J \setminus \{0\}$ with prescribed singularity at 0. Namely, define

$$BUC_{1-\mu}(J, E) := \left\{ u \in C(\dot{J}, E) : [t \mapsto t^{1-\mu}u(t)] \in BUC(\dot{J}, E) \text{ and} \right.$$

(1.5)
$$\left. \lim_{t \to 0^+} t^{1-\mu}\|u(t)\|_E = 0 \right\}, \quad \mu \in (0, 1),$$

$$\|u\|_{B_{1-\mu}} := \sup_{t \in J} t^{1-\mu}\|u(t)\|_E,$$

where BUC denotes the space consisting of bounded, uniformly continuous functions. It is easy to verify that $BUC_{1-\mu}(J, E)$ is a Banach space when equipped with the norm $\|\cdot\|_{B_{1-\mu}}$. Moreover, we define the subspace

$$BUC^1_{1-\mu}(J, E) := \left\{ u \in C^1(\dot{J}, E) : u, \dot{u} \in BUC_{1-\mu}(J, E) \right\}, \quad \mu \in (0, 1),$$

and we set

$$BUC_0(J, E) := BUC(J, E), \qquad BUC^1_0(J, E) := BUC^1(J, E).$$

Now, if E_1 and E_0 are a pair of Banach spaces such that E_1 is continuously embedded in E_0, denoted $E_1 \hookrightarrow E_0$, we set

$$\mathbb{E}_0(J) := BUC_{1-\mu}(J, E_0), \qquad \mu \in (0, 1],$$
$$\mathbb{E}_1(J) := BUC^1_{1-\mu}(J, E_0) \cap BUC_{1-\mu}(J, E_1),$$

where $\mathbb{E}_1(J)$ is a Banach space with the norm

$$\|u\|_{\mathbb{E}_1(J)} := \sup_{t \in \dot{J}} t^{1-\mu} \Big(\|\dot{u}(t)\|_{E_0} + \|u(t)\|_{E_1} \Big).$$

It follows that the trace operator $\gamma : \mathbb{E}_1(J) \to E_0$, defined by $\gamma v := v(0)$, is well defined, and we denote by $\gamma \mathbb{E}_1$ the image of γ in E_0, which is itself a Banach space when equipped with the norm

$$\|x\|_{\gamma \mathbb{E}_1} := \inf \Big\{ \|v\|_{\mathbb{E}_1(J)} : v \in \mathbb{E}_1(J) \, \text{and} \, \gamma v = x \Big\}.$$

For a bounded linear operator $B \in \mathcal{L}(E_1, E_0)$ which is closed as an operator on E_0, we say $\big(\mathbb{E}_0(J), \mathbb{E}_1(J)\big)$ is a *pair of maximal regularity* for B and write $B \in \mathcal{MR}_\mu(E_1, E_0)$ if

$$\left(\frac{d}{dt} + B, \, \gamma \right) \in \mathcal{L}_{isom}(\mathbb{E}_1(J), \mathbb{E}_0(J) \times \gamma \mathbb{E}_1),$$

where $\mathcal{L}_{isom}$ denotes the space of bounded linear isomorphisms. In particular, $\big(\mathbb{E}_0(J), \mathbb{E}_1(J)\big)$ is a pair of maximal regularity for B if and only if for every $(f, u_0) \in \mathbb{E}_0(J) \times \gamma \mathbb{E}_1$, there exists a unique solution $u \in \mathbb{E}_1(J)$ to the inhomogeneous Cauchy problem

$$\begin{cases} \dot{u}(t) + Bu(t) = f(t), & t \in \dot{J}, \\ u(0) = u_0. \end{cases}$$

Moreover, in the current setting, it follows that $\gamma \mathbb{E}_1 \doteq (E_0, E_1)^0_{\mu,\infty}$, i.e., the trace space $\gamma \mathbb{E}_1$ is topologically equivalent to the noted continuous interpolation spaces of Da Prato and Grisvard; cf. [3, 12, 17, 48].

2. Well-posedness of (1.3). When considering the surface diffusion problem, the underlying Banach spaces E_0 and E_1 in the formulation of maximal regularity will be spatial regularity classes which describe the properties of the profile functions $r(t)$. We proceed by defining these regularity classes. We define the one-dimensional torus $\mathbb{T} := [-\pi, \pi]$, where the points $-\pi$ and π are identified, which is equipped with the topology generated by the metric

$$d_{\mathbb{T}}(x, y) := \min\{|x - y|, 2\pi - |x - y|\}, \qquad x, y \in \mathbb{T}.$$

There is a natural equivalence between functions defined on $\mathbb{T}$ and 2π-periodic functions on $\mathbb{R}$ which preserves properties of (Hölder) continuity and differentiability. In particular, we will be working with the so-called periodic little-Hölder spaces $h^\sigma(\mathbb{T})$ for $\sigma \in \mathbb{R}_+ \setminus \mathbb{Z}$. Definitions and basic properties of periodic little-Hölder spaces, as well as details on the connection between spaces of functions on $\mathbb{T}$ and 2π-periodic functions on $\mathbb{R}$, can be found in [46] and the references therein. For the reader's convenience, we provide a brief definition of $h^\sigma(\mathbb{T})$ below.

For $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, denote by $C^k(\mathbb{T})$ the Banach space of k-times continuously differentiable functions $f : \mathbb{T} \to \mathbb{R}$, equipped with the norm

$$\|f\|_{C^k(\mathbb{T})} := \sum_{j=0}^{k} \|f^{(j)}\|_{C(\mathbb{T})} := \sum_{j=0}^{k} \left(\sup_{x \in \mathbb{T}} |f^{(j)}(x)| \right).$$

Moreover, for $\alpha \in (0,1)$ and $k \in \mathbb{N}_0$, we define the space $C^{k+\alpha}(\mathbb{T})$ to be those functions $f \in C^k(\mathbb{T})$ such that the α-Hölder seminorm

$$\left[f^{(k)} \right]_{\alpha,\mathbb{T}} := \sup_{\substack{x,y \in \mathbb{T} \\ x \neq y}} \frac{|f^{(k)}(x) - f^{(k)}(y)|}{d_{\mathbb{T}}^\alpha(x,y)}$$

is finite. It follows that $C^{k+\alpha}(\mathbb{T})$ is a Banach space when equipped with the norm

$$\|f\|_{C^{k+\alpha}(\mathbb{T})} := \|f\|_{C^k(\mathbb{T})} + [f^{(k)}]_{\alpha,\mathbb{T}}.$$

Finally, we define the periodic little-Hölder space

$$h^{k+\alpha}(\mathbb{T}) := \left\{ f \in C^{k+\alpha}(\mathbb{T}) : \lim_{\delta \to 0} \sup_{\substack{x,y \in \mathbb{T} \\ 0 < d_{\mathbb{T}}(x,y) < \delta}} \frac{|f^{(k)}(x) - f^{(k)}(y)|}{d_{\mathbb{T}}^\alpha(x,y)} = 0 \right\}$$

for $k \in \mathbb{N}_0$ and $\alpha \in (0,1)$, which is a Banach algebra with pointwise multiplication of functions and equipped with the norm $\|\cdot\|_{h^{k+\alpha}} := \|\cdot\|_{C^{k+\alpha}(\mathbb{T})}$ inherited from $C^{k+\alpha}(\mathbb{T})$. For equivalent definitions and more properties of the periodic little-Hölder spaces, see [46, section 1].

In order to make explicit the quasilinear structure of (1.3), we reformulate the problem. By expanding the governing equation we arrive at the formally equivalent problem

$$(2.1) \qquad \begin{cases} \partial_t r(t,x) + \left[\mathcal{A}(r(t)) r(t) \right](x) = f(r(t,x)), & t > 0, \ x \in \mathbb{T}, \\ r(0,x) = r_0(x), & x \in \mathbb{T}, \end{cases}$$

where, for appropriately chosen functions ρ,

$$(2.2) \qquad \mathcal{A}(\rho) := \frac{1}{(1+\rho_x^2)^2} \, \partial_x^4 + \frac{2\rho_x \left(1 + \rho_x^2 - 5\rho\rho_{xx} \right)}{\rho \left(1 + \rho_x^2 \right)^3} \, \partial_x^3$$

is a fourth-order differential operator with variable coefficients over $\mathbb{T}$ and

$$(2.3) \qquad f(\rho) := \frac{\rho_x^2 - 1}{\rho^2 (1+\rho_x^2)^2} \, \rho_{xx} + \frac{6\rho_x^2 - 1}{\rho(1+\rho_x^2)^3} \, \rho_{xx}^2 + \frac{3 - 15\rho_x^2}{(1+\rho_x^2)^4} \, \rho_{xx}^3 + \frac{\rho_x^2}{\rho^3(1+\rho_x^2)}$$

is an $\mathbb{R}$-valued function over $\mathbb{T}$. Looking at these formal expressions, one can deduce several properties that the functions ρ must satisfy in order to get good mapping properties for f and $\mathcal{A}$. In particular, we want to choose ρ such that $\rho(x) \neq 0$ for all $x \in \mathbb{T}$; also we want that the spatial derivatives ρ_x and ρ_{xx} make sense and the products ρ^2, ρ^3, $\rho\rho_x^2$, etc., have desired regularity properties. With these conditions in mind, we proceed with our well-posedness result.

2.1. Existence and uniqueness of solutions. We collect statements of well-posedness results and refer the reader to the appendix for comments on their proof. Fix $\alpha \in (0,1)$ and define the spaces of $\mathbb{R}$-valued little-Hölder continuous functions

$$(2.4) \qquad E_0 := h^\alpha(\mathbb{T}), \quad E_1 := h^{4+\alpha}(\mathbb{T}), \quad \text{and} \quad E_\mu := (E_0, E_1)_{\mu,\infty}^0,$$

where $(\cdot, \cdot)_{\mu,\infty}^0$, for $\mu \in (0,1)$, denotes the continuous interpolation functor of Da Prato and Grisvard; cf. [17] or [3]. It is well known that the little-Hölder spaces are stable under this interpolation method; in particular we know that

$$E_\mu = h^{4\mu+\alpha}(\mathbb{T}) \quad \text{(up to equivalent norms)} \qquad \text{for} \quad 4\mu + \alpha \notin \mathbb{Z};$$

cf. [46, 48]. Further, let V be the set of functions $r : \mathbb{T} \to \mathbb{R}$ such that $r(x) > 0$ for all $x \in \mathbb{T}$ and define $V_\mu := V \cap E_\mu$ for $\mu \in [0, 1]$. We note that V_μ is an open subset of E_μ for all $\mu \in [0, 1]$.

Before we can properly state a result on maximal solutions, we need to introduce one more space of functions from an interval $J \subset \mathbb{R}_+$ to a Banach space E, with prescribed singularity at zero. Namely, if $J = [0, a)$ for $a > 0$, i.e., J is a right-open interval containing 0, then we set

$$C_{1-\mu}(J, E) := \{u \in C(\dot{J}, E) : u \in BUC_{1-\mu}([0, T], E), \quad T < \sup J\},$$
$$C^1_{1-\mu}(J, E) := \{u \in C^1(\dot{J}, E) : u, \dot{u} \in C_{1-\mu}(J, E)\}, \qquad \mu \in (0, 1],$$

which we equip with the natural Fréchet topologies induced by $BUC_{1-\mu}([0, T], E)$ and $BUC^1_{1-\mu}([0, T], E)$, respectively.

We list some important properties of the mappings $\mathcal{A}$ and f, introduced in (2.2) and (2.3).

LEMMA 2.1. *Let $\mu \in [1/2, 1]$. Then*

$$(\mathcal{A}, f) \in C^\omega \left(V_\mu, \ \mathcal{MR}_\nu(E_1, E_0) \times E_0 \right) \qquad for \quad \nu \in (0, 1],$$

where C^ω denotes the space of real analytic mappings between Banach spaces.

PROPOSITION 2.2 (existence and uniqueness). *Fix $\alpha \in (0, 1)$ and take $\mu \in [1/2, 1]$ so that $4\mu + \alpha \notin \mathbb{Z}$. For each initial value $r_0 \in V_\mu := h^{4\mu+\alpha}(\mathbb{T}) \cap [r > 0]$, there exists a unique maximal solution*

$$r(\cdot, r_0) \in C^1_{1-\mu}(J(r_0), h^\alpha(\mathbb{T})) \cap C_{1-\mu}(J(r_0), h^{4+\alpha}(\mathbb{T})),$$

where $J(r_0) = [0, t^+(r_0)) \subseteq \mathbb{R}_+$ denotes the maximal interval of existence for initial data r_0. Further, it follows that

$$\mathcal{D} := \bigcup_{r_0 \in V_\mu} J(r_0) \times \{r_0\}$$

is open in $\mathbb{R}_+ \times V_\mu$ and $\varphi : [(t, r_0) \mapsto r(t, r_0)]$ is an analytic semiflow on V_μ; i.e., using the notation $\varphi^t(r_0) := \varphi(t, r_0)$, the mapping φ satisfies the conditions

(2.5)
- $\varphi \in C(\mathcal{D}, V_\mu)$;
- $\varphi^0 = id_{V_\mu}$;
- $\varphi^{s+t}(r_0) = \varphi^t \circ \varphi^s(r_0)$ *for $0 \le s < t^+(r_0)$ and $0 \le t < t^+(\varphi^s(r_0))$;*
- $\varphi(t, \cdot) \in C^\omega(\mathcal{D}_t, V_\mu)$ *for $t \in \mathbb{R}_+$ with $\mathcal{D}_t := \{r \in V_\mu : (t, r) \in \mathcal{D}\} \ne \emptyset$.*

The results in [12] also give the following conditions for global solutions.

PROPOSITION 2.3 (global solutions). *Let $r_0 \in V_\mu := h^{4\mu+\alpha}(\mathbb{T}) \cap [r > 0]$ for $\mu \in (1/2, 1]$, such that $4\mu + \alpha \notin \mathbb{Z}$, and suppose there exists $0 < M < \infty$ so that, for all $t \in J(r_0)$,*

- $r(t, r_0)(x) \ge 1/M$ *for all $x \in \mathbb{T}$ and*
- $\|r(t, r_0)\|_{h^{4\mu+\alpha}(\mathbb{T})} \le M.$

Then it must hold that $t^+(r_0) = \infty$, so that $r(\cdot, r_0)$ is a global solution. Conversely, if $r_0 \in V_\mu$ and $t^+(r_0) < \infty$, i.e., the solution breaks down in finite time, then one, or both, of the conditions stated must fail to hold.

We can also state the following result regarding analyticity of the maximal solutions $r(\cdot, r_0)$ in both space and time.

PROPOSITION 2.4 (regularity of solutions). *Under the same assumptions as in Proposition* 2.2, *it follows that*

(2.6) $r(\cdot, r_0) \in C^\omega((0, t^+(r_0)) \times \mathbb{T})$ *for all* $r_0 \in V_\mu, \quad \mu \in [1/2, 1].$

Proof. Here we rely on an idea that goes back to Masuda [51] and Angenent [5, 6] of introducing parameters and using the implicit function theorem to obtain regularity results for solutions; see also [28]. The technical details are included in the appendix.   □

Remark 1. The preceding results can be slightly weakened to allow for arbitrary values of $\mu \in (1/2, 1]$, i.e., without eliminating the possibility that $4\mu + \alpha \in \mathbb{Z}$, by taking initial data from the continuous interpolation spaces $(E_0, E_1)^0_{\mu,\infty}$, which coincide with the Zygmund spaces over $\mathbb{T}$.

3. Characterizing the equilibria of ASD. We begin our analysis of the long-time behavior of solutions by characterizing and describing the equilibria of (1.3). For this characterization, we make use of a well-known, strict Lyapunov functional for the surface diffusion flow, namely, the surface area functional, and a characterization of surfaces of revolution with prescribed mean curvature, as presented by Kenmotsu [39].

Recalling the operator G, as expressed by (1.4), and taking it to be defined on $V_1 \subset h^{4+\alpha}(\mathbb{T})$, one will see that the set of equilibria of (1.3) coincides with the null set of G. Although, from the well-posedness results of the previous section, we know that we can consider (1.3) with initial conditions in $h^{2+\alpha}(\mathbb{T})$, upon which the operator G is not defined, one immediately sees that all equilibria must be in $h^{4+\alpha}(\mathbb{T})$ (in fact, by Proposition 2.4, we can conclude that equilibria are in $C^\omega(\mathbb{T})$). More specifically, if we define equilibria to be those elements $\bar{r} \in V_{1/2} = V \cap h^{2+\alpha}(\mathbb{T})$, such that the maximal solution $r(\cdot, \bar{r})$ satisfies

$$r(t, \bar{r}) = \bar{r}, \qquad t > 0,$$

then it follows immediately that $\bar{r} \in h^{4+\alpha}(\mathbb{T})$ and $G(\bar{r}) = 0$. Now, we proceed by characterizing the elements of the null set of G.

Consider the functional

$$S(r) := \int_{\mathbb{T}} r(x) \sqrt{1 + r_x^2(x)} dx,$$

which corresponds to the surface area of $\Gamma(r)$. If $r = r(\cdot, r_0)$ is a solution to (1.3) on the interval $J(r_0)$, then (suppressing the variable of integration)

$$\partial_t S(r(t)) = \int_{\mathbb{T}} \left[\sqrt{1 + r_x^2(t)} + \frac{r(t) r_x(t)}{\sqrt{1 + r_x^2(t)}} \, \partial_x \right] G(r(t)) \, dx$$

$$= \int_{\mathbb{T}} \partial_x \left(\frac{r(t)}{\sqrt{1 + r_x^2(t)}} \, \partial_x \mathcal{H}(r(t)) \right) \mathcal{H}(r(t)) \, dx$$

$$= -\int_{\mathbb{T}} \frac{r(t)}{\sqrt{1 + r_x^2(t)}} \left(\partial_x \mathcal{H}(r(t)) \right)^2 \, dx, \qquad t \in J(r_0) \setminus \{0\},$$

where we use integration by parts twice and eliminate boundary terms because of periodicity. Notice that the expression is nonpositive for all times $t \in J(r_0) \setminus \{0\}$.

If $\bar{r}$ is an equilibrium of (1.3), it follows that $\partial_x \mathcal{H}(\bar{r})$ is identically zero on $\mathbb{T}$. Meanwhile, by definition of the operator G, $G(\bar{r}) = 0$ whenever $\partial_x \mathcal{H}(\bar{r}) = 0$. Hence, we conclude that $S(r)$ is a strict Lyapunov functional for (1.3), as claimed, and we also see that the equilibria of (1.3) are exactly those functions $\bar{r} \in h^{4+\alpha}(\mathbb{T})$ for which the mean curvature function $\mathcal{H}(\bar{r})$ is constant on $\mathbb{T}$.

The axisymmetric surfaces with constant mean curvature have been characterized explicitly by Kenmotsu in [39]. All equilibria of (1.3) are so-called *undulary* curves, and the *unduloid* surfaces, which are generated by the undulary curves by revolution about the axis of symmetry, are stationary solutions of the original surface diffusion problem (1.1).

THEOREM 3.1 (see Delaunay [21] and Kenmotsu [39]). *Any complete surface of revolution with constant mean curvature $\mathcal{H}$ is either a sphere, a catenoid, or a surface whose profile curve is given (up to translation along the axis of symmetry) by the parametric expression, parameterized by the arclength parameter $s \in \mathbb{R}$,*

$$(3.1) \quad R(s; \mathcal{H}, B) := \left(\int_{\pi/2\mathcal{H}}^{s} \frac{1 + B\sin(\mathcal{H}t)}{\sqrt{1 + B^2 + 2B\sin(\mathcal{H}t)}} \, dt, \frac{\sqrt{1 + B^2 + 2B\sin(\mathcal{H}s)}}{|\mathcal{H}|} \right).$$

Remark 2. We can immediately draw several conclusions from Theorem 3.1 and characterize the equilibria of (1.3). We use the notation $R(\mathcal{H}, B)$ to denote the curve in $\mathbb{R}^2$ with parametric expression $R(\cdot\,; \mathcal{H}, B)$.

(a) Although the curves $R(\mathcal{H}, B)$ are well defined for arbitrary values $B \in \mathbb{R}$ and $\mathcal{H} \neq 0$, it is not difficult to see that, up to translations along the x-axis, we may restrict our attention to values $\mathcal{H} > 0$ and $B \geq 0$; cf. [39, section 2]. However, in what follows we will consider the unduloids in the setting of even functions on $\mathbb{T}$, for which we will benefit by allowing $B < 0$.

(b) When $|B| = 1$, $R(\mathcal{H}, B)$ corresponds to a family of spheres controlled by the parameter $\mathcal{H}$. The spheres are a well-known family of stable equilibria for the surface diffusion flow (cf. [26, 70]); however, their profile curves are outside of our current setting because they fail to be continuously differentiable functions on all of $\mathbb{T}$. Moreover, we note that the spheres represented by $R(\mathcal{H}, \pm 1)$ are in fact a *connected* family of spheres, or a *chain of pearls* (see Figure 1),[1] for which even general techniques
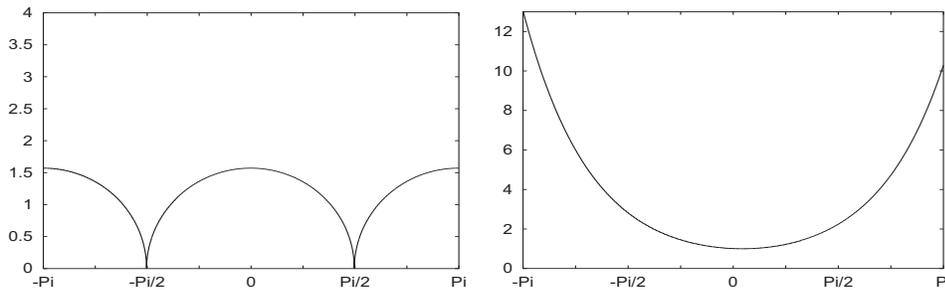

FIG. 1. *Profile curves for a family of spheres and a catenoid, respectively.*

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[1] All figures were generated with GNU Octave, version 3.4.3, © 2011 John W. Eaton, and GNU-PLOT, version 4.4 patchlevel 3, © 2010 Thomas Williams, Colin Kelley.
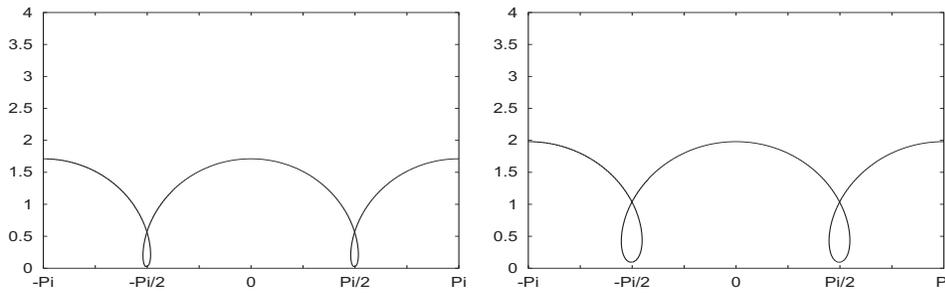
FIG. 2. *π-periodic nodary curves with $B = 1.03$ and $B = 1.1$, respectively.*

for (1.1) break down, as the manifold is singular at the points of intersection. These families of connected spheres may be interesting objects to investigate in a weaker formulation of ASD, but they fall outside of the current setting.

(c) Catenoids, or more precisely the generating catenary curves (which are essentially just the hyperbolic cosine, up to scaling), fail to satisfy periodic boundary conditions; cf. Figure 1.

(d) In case $|B| > 1$, the curve $R(\mathcal{H}, B)$ is called a *nodary* (see Figure 2), which cannot be realized as the graph of a function over the x-axis and hence falls outside the current setting. A reformulation of (1.3) to allow for immersed surfaces would permit nodary curves as equilibria. Such an extension of the current setting may prove beneficial to the investigation of pinch-off, as it may likely be easier to handle concerns regarding concentration of curvature for solutions near nodary curves, rather than embedded undularies.

(e) For values $0 \leq |B| < 1$, $R(\mathcal{H}, B)$ is the family of *undulary* curves, which generate the *unduloid* surfaces. The undulary curves are representable as graphs of functions over the x-axis, which are strictly positive for B in the given range (see Figure 3). In fact, the case $B = 0$ corresponds to the cylinder of radius $1/\mathcal{H}$. Hence, by Theorem 3.1 above, we conclude that *all equilibria of* (1.3) *fall into the family of undulary curves.*

(f) Notice that the curve $R(\mathcal{H}, B)$ is always periodic in both the parameter s and the spacial variable x. In order to ensure that the curve satisfies the 2π-periodic boundary conditions enforced in (1.3) (which we emphasize is a condition regarding
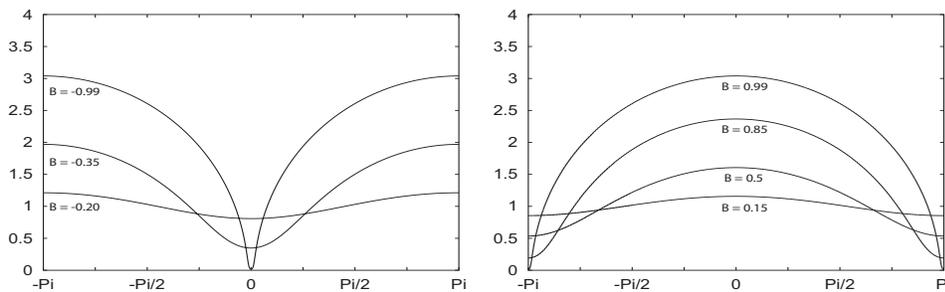

FIG. 3. *Families of 2π-periodic undulary curves with selected parameter values from $B = -.99$ to $B = 0.99$, as indicated.*

periodicity over the variable x and not the arclength parameter s), we must impose further conditions on the parameters $\mathcal{H}$ and B; here we avoid $B = 0$ because the curve $R(\mathcal{H}, 0)$ trivially satisfies periodic boundary conditions. In particular, for $B \neq 0$, if $\mathcal{H}$ and B satisfy the relationship

$$(3.2) \qquad \frac{\pi \mathcal{H}}{k} = \int_{\pi/2}^{3\pi/2} \frac{1 + B \sin t}{\sqrt{1 + B^2 + 2B \sin t}} \, dt \,,$$

then the curve $R(\mathcal{H}, B)$ is $2\pi/k$-periodic in the x variable for $k \in \mathbb{N}$. In what follows, we will use the notation $R(B, k)$ to denote the $2\pi/k$-periodic undulary curve with free parameter $-1 < B < 1$ and parameter $\mathcal{H} = \mathcal{H}(B)$ fixed according to (3.2).

(g) The role of the parameters B and k is clearly seen in the context of Delaunay's construction. By rolling an ellipse with eccentricity B along the x-axis, the path traced out by one focus is an undulary curve. Here $B < 0$ corresponds to a reassignment of major and minor axes in the associated ellipse. Further, it is clear that the ellipses are restricted to those with circumference $2\pi/k$, to match periodic boundary conditions.

4. Stability of cylinders with large radius. As seen above, the constant function $r(x) \equiv r_\star$, for $r_\star > 0$, is an equilibrium of (2.1). Moreover, the constant function $r(x) \equiv r_\star$ is associated to the cylinder $\Gamma(r_\star)$ with radius $r_\star$, which is a stationary solution of the original surface diffusion problem (1.1). In this section, we establish tools for and carry out the investigation of nonlinear stability for these equilibria.

4.1. Preliminary analysis and definitions. Throughout this analysis, we consider an arbitrary $r_\star > 0$ and $\sigma \in \mathbb{R}_+ \setminus \mathbb{Z}$, unless otherwise stated. Focusing on the properties of solutions near $r_\star$, we shift our equations, including the *shifted* operator

$$G_\star(\rho) := G(\rho + r_\star) = \frac{1}{\rho + r_\star} \, \partial_x \left[\frac{\rho + r_\star}{\sqrt{1 + \rho_x^2}} \, \partial_x \mathcal{H}(\rho + r_\star) \right],$$

which maps $\rho \in E_1 \cap U_\star$ to E_0, where we consider $\rho = r - r_\star$, and is in the regularity class C^ω by Lemma 2.1; here we take $U_\star := V - r_\star := \{\rho - r_\star : \rho \in V\}$. Now we consider the surface diffusion problem shifted by $r_\star$,

$$(4.1) \qquad \begin{cases} \rho_t(t, x) = G_\star(\rho(t, x)), & x \in \mathbb{T}, \ t > 0, \\ \rho(0, x) = \rho_0(x), & x \in \mathbb{T}, \end{cases}$$

where $\rho_0 := r_0 - r_\star$. We say that

$$\rho = \rho(\cdot, \rho_0) \in C^1(\dot{J}, E_0) \cap C(\dot{J}, E_1) \cap C(J, E_\mu \cap U_\star)$$

is a solution to (4.1), with initial data $\rho_0 \in E_\mu \cap U_\star$, on the interval $J \subset \mathbb{R}_+$ if ρ satisfies (4.1) pointwise, for $t > 0$, and $\rho(0) = \rho_0$. We investigate the properties of $G_\star$ around 0 in order to gain information about the stability of $r_\star$ in (1.3).

Define the functional

$$F_\star(\rho) = F_\star(\rho; r_\star) := \int_{\mathbb{T}} \left(\rho(x) + r_\star \right)^2 dx,$$

which corresponds to the volume enclosed by the surface $\Gamma(\rho + r_\star)$. It follows from the analyticity of multiplication and integration on little-Hölder spaces that $F_\star$ is of

class C^ω from $h^\sigma(\mathbb{T})$ to $\mathbb{R}$, $\sigma \in \mathbb{R}_+ \setminus \mathbb{Z}$. The Fréchet derivative of $F_\star$ is

$$(4.2) \quad DF_\star(\rho) : \left[h \longmapsto 2 \int_{\mathbb{T}} \big(\rho(x) + r_\star\big)h(x)dx \right] \in \mathcal{L}\big(h^\sigma(\mathbb{T}), \mathbb{R}\big), \qquad \rho \in h^\sigma(\mathbb{T}, \mathbb{R}).$$

Moreover, it holds that $F_\star(\rho)$ is conserved along solutions to (4.1). Indeed, if $\rho = \rho(\cdot, \rho_0)$ is a solution to (4.1), then

$$\frac{1}{2}\frac{d}{dt}F_\star(\rho(t)) = \int_{\mathbb{T}} \big(\rho(t) + r_\star\big)\rho_t(t)dx = \int_{\mathbb{T}} \partial_x \left[\frac{\big(\rho(t) + r_\star\big)}{\sqrt{1 + \rho_x^2(t)}} \, \partial_x \mathcal{H}(\rho(t) + r_\star) \right] dx = 0$$

for $t \in J(\rho_0) \setminus \{0\}$, where the last equality holds by periodicity. Thus, conservation of $F_\star$ along the solution ρ follows by continuity of $F_\star$ and convergence of ρ to the initial data ρ_0 in E_μ. From these properties, it follows that

$$(4.3) \qquad \mathcal{M}_\eta^\sigma := \left\{ \rho \in h^\sigma(\mathbb{T}) : F_\star(\rho) = F_\star(\eta) \right\}, \qquad \eta \in \mathbb{R}, \ \sigma \in \mathbb{R}_+ \setminus \mathbb{Z},$$

is a family of invariant level sets for (4.1). The following techniques are motivated by results of Prokert [59] and Vondenhoff [68], whereby one can take advantage of invariant manifolds in order to derive stability results.

First, we introduce the mapping

$$P_0\rho := \rho - \frac{1}{2\pi} \int_{\mathbb{T}} \rho(x)dx,$$

which defines a projection on $h^\sigma(\mathbb{T})$. We denote by $h_0^\sigma(\mathbb{T})$ the image $P_0\big(h^\sigma(\mathbb{T})\big)$, which exactly coincides with the zero-mean functions on $\mathbb{T}$ in the regularity class $h^\sigma(\mathbb{T})$, and we have the topological decomposition

$$h^\sigma(\mathbb{T}) = h_0^\sigma(\mathbb{T}) \oplus (1 - P_0)\big(h^\sigma(\mathbb{T})\big) \cong h_0^\sigma(\mathbb{T}) \oplus \mathbb{R}.$$

In what follows, we equate the constant function $[\eta(x) \equiv \eta] \in (1 - P_0)\big(h^\sigma(\mathbb{T})\big)$ with the value $\eta \in \mathbb{R}$, and we denote each simply as η.

Consider the operator

$$\Phi(\rho, \tilde{\rho}, \eta) := \Big(P_0\rho - \tilde{\rho}, \ F_\star(\rho) - F_\star(\eta) \Big),$$

which maps $h^\sigma(\mathbb{T}) \times h_0^\sigma(\mathbb{T}) \times \mathbb{R}$ to $h_0^\sigma(\mathbb{T}) \times \mathbb{R}$ and is of class C^ω, by regularity of the mappings $F_\star$ and P_0. Notice that $\Phi(0, 0, 0) = (0, 0)$ and, using (4.2),

$$(4.4) \qquad D_1\Phi(0, 0, 0) = \Big(P_0, \ 4\pi r_\star(1 - P_0) \Big) \in \mathcal{L}_{isom}(h^\sigma(\mathbb{T}), h_0^\sigma(\mathbb{T}) \times \mathbb{R}),$$

i.e., the Fréchet derivative of Φ with respect to the first variable, at the origin, is a linear isomorphism. Hence, it follows from the implicit function theorem that there exist neighborhoods $(0, 0) \in U = U_0 \times U_1 \subset h_0^\sigma(\mathbb{T}) \times \mathbb{R}$ and $0 \in U_2 \subset h^\sigma(\mathbb{T})$ and a C^ω function $\psi : U \to U_2$ such that, for all $(\rho, \tilde{\rho}, \eta) \in U_2 \times U$,

$$\Phi(\rho, \tilde{\rho}, \eta) = (0, 0) \qquad \text{if and only if} \qquad \rho = \psi(\tilde{\rho}, \eta).$$

Remark 3. We can immediately state the following properties of ψ, which follow directly from its definition and elucidate the relationship between P_0 and ψ.

(a) $P_0\psi(\tilde{\rho}, \eta) = \tilde{\rho}$ for all $(\tilde{\rho}, \eta) \in U$.

(b) Given $\rho \in \psi(U) \cap \mathcal{M}_\eta^\sigma$, it follows that $\psi(P_0\rho, \eta) = \rho$.

(c) $\psi(0, \eta) = \eta$, for $\eta \in U_1$. This and the preceding remark follow from the fact that $F_\star(\eta)$ is injective when restricted to $\eta \in (-r_\star, \infty) \subset \mathbb{R}$.

(d) It follows from the identity $\Phi(\psi(\tilde{\rho}, \eta), \tilde{\rho}, \eta) = (0, 0)$ and differentiating with respect to $\tilde{\rho}$ that $D_1\Phi(\psi(0, \eta), 0, \eta)D_1\psi(0, \eta)h - (h, 0) = (0, 0)$. From this observation, and the fact that $D_1\Phi(\eta, 0, \eta) = (P_0, 4\pi(r_\star + \eta)(1 - P_0))$, it follows that

$$D_1\psi(0, \eta)h = h, \qquad h \in h_0^\sigma(\mathbb{T}), \ \eta \in U_1.$$

(e) $\psi(U_0, \eta) \subset \mathcal{M}_\eta^\sigma$ for $\eta \in U_1$. Hence, $\psi(\cdot, \eta)$ can be taken as a (local) parameterization of $\mathcal{M}_\eta^\sigma$. Moreover, from the preceding remark and the bijectivity of $\psi(\cdot, \eta)$ from U_0 to $\mathcal{M}_\eta^\sigma \cap U_2$, we can see that $\mathcal{M}_\eta^\sigma \cap U_2$ is a Banach manifold over $h_0^\sigma(\mathbb{T})$ anchored at the point $\eta \in \mathbb{R}$.

(f) For $(\tilde{\rho}, \eta) \in U$, we have the representation

$$\psi(\tilde{\rho}, \eta) = \Big(P_0 + (1 - P_0)\Big)\psi(\tilde{\rho}, \eta) = \tilde{\rho} + \frac{1}{2\pi}\int_\mathbb{T} \psi(\tilde{\rho}, \eta)(x)dx,$$

and so we can see that $\mathcal{M}_\eta^\sigma \cap U_2$ can be realized (locally) as the graph of an $\mathbb{R}$-valued analytic function over the zero-mean functions $\tilde{\rho} \in h_0^\sigma(\mathbb{T})$.

(g) Although $\psi(\cdot, \eta)$ depends upon the parameter σ, a priori, it follows easily from the preceding representation that

$$\psi(\cdot, \eta) : U_0 \cap h_0^{\tilde{\sigma}}(\mathbb{T}) \to h^{\tilde{\sigma}}(\mathbb{T}), \quad \tilde{\sigma} \in \mathbb{R}_+ \setminus \mathbb{Z},$$

so that ψ preserves the spacial regularity of functions regardless of the regularity parameter σ with which ψ was constructed. However, notice that the neighborhood U_0 will remain intrinsically linked with the parameter which was used to construct ψ.

With the established invariance and local structure of the sets $\mathcal{M}_\eta^\sigma$, it follows that the dynamics governing solutions to (1.3) reside in the tangent space to the manifold $\mathcal{M}_\eta^\sigma \cap U_2$. Hence, if we reduce (1.3) to a local system *on* $\mathcal{M}_\eta^\sigma \cap U_2$, then we will have captured all of the dynamics of the problem. Remark 3(d) is the first observation toward this reduced formulation. In fact, one can make use of the properties established in Remark 3 to prove the following, even more general, result regarding the properties of the the tangent vectors to $\mathcal{M}_\eta^\sigma$. Although we use other tools to connect the reduced problem (4.5) below to the full problem (1.3), this remark provides good intuition into the nature of these manifolds.

Remark 4. Given $(\tilde{\rho}, \eta) \in U$ it follows that $D_1\psi(\tilde{\rho}, \eta) \circ P_0 = id_{T_{\psi(\tilde{\rho}, \eta)}\mathcal{M}_\eta^\sigma}$, where $T_\rho\mathcal{M}_\eta^\sigma$ denotes the tangent space to the manifold $\mathcal{M}_\eta^\sigma$ at the point ρ.

4.2. The reduced problem. Fix $\alpha \in (0, 1)$ and denote the spaces

$$F_0 := h_0^\alpha(\mathbb{T}), \quad F_1 := h_0^{4+\alpha}(\mathbb{T}), \quad \text{and} \quad F_\mu := (F_0, F_1)_{\mu,\infty}^0, \quad \mu \in (0, 1),$$

so that $F_\mu = P_0E_\mu$ for $\mu \in [0, 1]$. Define the operator

$$\mathcal{G}_\star(\tilde{\rho}, \eta) = \mathcal{G}_\star(\tilde{\rho}, \eta; r_\star) := P_0 G\big(\psi(\tilde{\rho}, \eta) + r_\star\big),$$

which is defined for all $(\tilde{\rho}, \eta) \in U \subset F_0 \times \mathbb{R}$ with $\tilde{\rho} \in U_0 \cap F_1$.

Now we consider the *reduced* problem for the zero-mean functions

(4.5)
$$\begin{cases} \tilde{\rho}_t(t, x) = \mathcal{G}_\star(\tilde{\rho}(t, x), \eta), & t > 0, \ x \in \mathbb{T}, \\ \tilde{\rho}(0, x) = \tilde{\rho}_0(x), & x \in \mathbb{T}, \end{cases}$$

where $\tilde{\rho}_0 := P_0 r_0 = P_0(r_0 - r_\star)$. One will note that we should insist on $\psi(\tilde{\rho}, \eta)(x) > -r_\star$ for all $x \in \mathbb{T}$ in order to guarantee that $G(\psi(\tilde{\rho}, \eta) + r_\star)$ is well defined. However, we can assume, without loss of generality, that the neighborhood U is chosen small enough to ensure this property holds for all $(\tilde{\rho}, \eta) \in U$.

Remark 5. Throughout most of the analysis that follows, we will treat the parameter η as a free parameter, although it has a very specific interpretation in relation to (2.1). If one is given initial data r_0 close to $r_\star$, then the parameter η is chosen so that

$$F_\star(\eta) = F_\star(r_0 - r_\star).$$

(a) Essentially, this parameter allows for the possibility that the volume enclosed by the surface $\Gamma(r_0)$ differs from that of the cylinder $\Gamma(r_\star)$, thereby allowing us to handle non-volume-preserving perturbations r_0 of the cylinder $r_\star$.

(b) From a more general viewpoint, one can see that the family $\{\mathcal{M}_\eta^\sigma \cap \psi(U) : \eta \in U_1\}$ forms a dimension 1 foliation of a neighborhood of the positive real axis $\mathbb{R}_+ \subset h^\sigma(\mathbb{T})$ and the parameter η separates the leaves of the foliation.

For $\mu \in (0, 1]$ and closed intervals $J \subseteq \mathbb{R}_+$ with $0 \in J$, define the spaces

$$\mathbb{E}_0(J) := BUC_{1-\mu}(J, E_0),$$
$$\mathbb{E}_1(J) := BUC_{1-\mu}^1(J, E_0) \cap BUC_{1-\mu}(J, E_1)$$

and

$$\mathbb{F}_0(J) := BUC_{1-\mu}(J, F_0),$$
$$\mathbb{F}_1(J) := BUC_{1-\mu}^1(J, F_0) \cap BUC_{1-\mu}(J, F_1),$$

within which we will discuss solutions to the shifted problem (4.1) and the reduced problem (4.5), respectively.

In order to connect these two problems, we will make use of the *lifting* map ψ, defined in the previous section. To ensure that ψ is well defined on $\mathbb{F}_1(J)$, we must restrict our attention to functions which map into an appropriate neighborhood $U_0 \subset F_0$ of 0. In particular, we assume that U_0 is given so that

$$\psi(\cdot, \eta) : U_0 \subset F_0 \to E_0, \quad \eta \in U_1,$$

is in the regularity class C^ω and, without loss of generality, we assume that U_0 is given sufficiently small so that ψ and the derivative $D_1\psi$ are bounded on $U = U_0 \times U_1$. More precisely, U_0 is chosen sufficiently small so that there exists a constant $N > 0$ for which the inequalities

$$(4.6) \qquad \|\psi(\tilde{\rho}, \eta)\|_{E_0} \leq N \quad \text{and} \quad \|D_1\psi(\tilde{\rho}, \eta)\|_{\mathcal{L}(F_0, E_0)} \leq N$$

hold for all $(\tilde{\rho}, \eta) \in U = U_0 \times U_1$.

LEMMA 4.1. *Fix $\eta \in U_1$ and $J := [0, T]$ for $T > 0$. Then*

$$\psi(\cdot, \eta) : \mathbb{F}_1(J) \cap C(J, U_0) \longrightarrow \mathbb{E}_1(J), \qquad with \quad \psi(\tilde{\rho}, \eta)(t) := \psi(\tilde{\rho}(t), \eta).$$

Moreover, if $\tilde{\rho}_0 \in F_\mu$ and $\tilde{\rho} = \tilde{\rho}(\cdot, \tilde{\rho}_0) \in \mathbb{F}_1(J) \cap C(J, U_0)$ is a solution to (4.5), for some $\mu \in [1/2, 1]$, then $\rho := \psi(\tilde{\rho}, \eta)$ is the unique solution on the interval J to (4.1), with initial data $\rho_0 := \psi(\tilde{\rho}_0, \eta) \in E_\mu$.

Proof. First notice that the embeddings

(4.7) $\mathbb{F}_1(J) \hookrightarrow BUC(J, F_\mu) \hookrightarrow BUC(J, F_0), \qquad \mu \in [1/2, 1],$

follow from [3, Theorem III.2.3.3] and the continuous embedding of little-Hölder spaces, respectively.

To see that the mapping property for $\psi(\cdot, \eta)$ holds, let $\tilde{\rho} \in \mathbb{F}_1(J) \cap C(J, U_0)$. Uniform continuity and differentiability of the function $\psi(\tilde{\rho}(\cdot), \eta)$ follow from the regularity of ψ and $\tilde{\rho}$ and compactness of the interval J. Hence we focus on demonstrating that $\psi(\tilde{\rho}(\cdot), \eta)$ satisfies the boundedness conditions for $\mathbb{E}_1(J)$. In the case $\mu \in [1/2, 1)$, it follows from Remark 3(f) and (4.6) that, for $t \in \dot{J}$,

(4.8)
$$
\begin{aligned}
t^{1-\mu} \|\psi(\tilde{\rho}(t), \eta)\|_{E_1} &\leq t^{1-\mu} \|\tilde{\rho}(t)\|_{F_1} + \frac{t^{1-\mu}}{2\pi} \int_{\mathbb{T}} |\psi(\tilde{\rho}(t), \eta)(x)| dx \\
&\leq \|\tilde{\rho}\|_{\mathbb{F}_1(J)} + t^{1-\mu} \|\psi(\tilde{\rho}(t), \eta)\|_{C(\mathbb{T})} \\
&\leq \|\tilde{\rho}\|_{\mathbb{F}_1(J)} + T^{1-\mu} N,
\end{aligned}
$$

$$
\lim_{t \to 0} t^{1-\mu} \|\psi(\tilde{\rho}(t), \eta)\|_{E_1} = 0.
$$

From (4.8) we conclude that $\psi(\tilde{\rho}, \eta) \in BUC_{1-\mu}(J, E_1)$. Meanwhile, looking at the time derivative of $\psi(\tilde{\rho}, \eta)$, we note that $\partial_t \psi(\tilde{\rho}(t), \eta) = D_1 \psi(\tilde{\rho}(t), \eta) \partial_t \tilde{\rho}(t)$, and so we again make use of (4.6) to see that

$$
\begin{aligned}
t^{1-\mu} \|\partial_t \psi(\tilde{\rho}(t), \eta)\|_{E_0} &\leq \|D_1 \psi(\tilde{\rho}(t), \eta)\|_{\mathcal{L}(F_0, E_0)} t^{1-\mu} \|\partial_t \tilde{\rho}(t)\|_{F_0} \\
&\leq N \|\tilde{\rho}\|_{\mathbb{F}_1(J)} < \infty,
\end{aligned}
$$

$$
\lim_{t \to 0} t^{1-\mu} \|\partial_t \psi(\tilde{\rho}(t), \eta)\|_{E_0} = 0.
$$

Hence, making use of the embedding $E_1 \hookrightarrow E_0$, we see that $\psi(\tilde{\rho}, \eta) \in \mathbb{E}_1(J)$, as desired. Meanwhile, when $\mu = 1$ we again get continuity and differentiability from the regularity of the mappings $\tilde{\rho}$ and ψ.

To see that the second part of the lemma holds, observe by (4.7) that $\rho_0 := \psi(\tilde{\rho}_0, \eta) \in E_\mu \cap U_\star$. Hence, by Proposition 2.2, there exists a unique maximal solution

$$
r(\cdot, \rho_0) \in C^1_{1-\mu}(J(\rho_0), E_0) \cap C_{1-\mu}(J(\rho_0), E_1)
$$

to (4.1) on some maximal interval of existence $J(\rho_0) = [0, t^+(\rho_0))$. Now, define $\rho(\cdot) := \psi(\tilde{\rho}(\cdot), \eta)$ as indicated, and it suffices to show that $\rho_t(t) = G_\star(\rho(t))$ for $t \in \dot{J} := (0, T]$, since this will imply that $\rho(t) = r(t, \rho_0)$ by uniqueness and maximality of the solution $r(\cdot, \rho_0)$. Proceeding, let $t \in \dot{J}$ and consider the auxiliary problem

$$
\begin{cases}
\dot{\gamma}(\tau) = G_\star(\gamma(\tau)) & \text{for } \tau \in [0, \varepsilon], \\
\gamma(0) = \rho(t),
\end{cases}
$$

which has a unique solution $\gamma \in C^1([0, \varepsilon], E_0) \cap C([0, \varepsilon], E_1)$ by Proposition 2.2, provided we choose $\varepsilon > 0$ sufficiently small for the particular value $\rho(t) \in E_1$. Notice, by the regularity of γ, we have

$$
\dot{\gamma}(0) = G_\star(\gamma(0)) = G_\star(\rho(t)).
$$

Further, note that $\rho(t) \in \mathcal{M}_\eta^{4+\alpha}$, from which we conclude that $\gamma(\tau) \in \mathcal{M}_\eta^{4+\alpha}$, and by Remark 3 we have the representation $\gamma(\tau) = \psi(P_0\gamma(\tau), \eta)$, $\tau \in [0, \varepsilon]$. Finally, we see that

$$G_\star(\rho(t)) = \dot{\gamma}(0) = \partial_\tau \left(\psi(P_0\gamma(\tau), \eta) \right)\Big|_{\tau=0} = D_1\psi(P_0\gamma(0), \eta)P_0\dot{\gamma}(0)$$

(4.9)
$$= D_1\psi(P_0\rho(t), \eta)P_0 G_\star(\rho(t)) = D_1\psi(\tilde{\rho}(t), \eta)\mathcal{G}_\star(\tilde{\rho}(t), \eta)$$

$$= \partial_t \left(\psi(\tilde{\rho}(t), \eta) \right) = \rho_t(t),$$

which concludes the proof. □

We also get the following results, which further illuminate the relationship between the mappings $G_\star$ and $\mathcal{G}_\star$ and explicitly connect the equilibria of the two problems (4.1) and (4.5).

LEMMA 4.2. *For any $\rho \in \mathcal{M}_\eta^{4+\alpha} \cap U_2$, it follows that*

(4.10)
$$G_\star(\rho) = D_1\psi(P_0\rho, \eta)P_0 G_\star(\rho)$$

and

(4.11)
$$DG_\star(\rho)h = D_1^2\psi(P_0\rho, \eta)[P_0h, P_0 G_\star(\rho)] + D_1\psi(P_0\rho, \eta)P_0 DG_\star(\rho)h$$

for $h \in E_1$.

Proof. The first claim was justified in the proof of Lemma 4.1 above and is expressed in (4.9). Meanwhile, the second claim follows immediately by differentiation. □

PROPOSITION 4.3. *If $(\tilde{\rho}, \eta) \in U$, then $(\tilde{\rho}, \eta)$ is an equilibrium of (4.5) if and only if $\psi(\tilde{\rho}, \eta)$ is an equilibrium of (4.1), i.e.,*

$$\mathcal{G}_\star(\tilde{\rho}, \eta) = 0 \quad \Longleftrightarrow \quad G_\star(\psi(\tilde{\rho}, \eta)) = 0.$$

Moreover, if $\mathcal{G}_\star(\tilde{\rho}, \eta) = 0$, then it follows that

(4.12)
$$DG_\star(\psi(\tilde{\rho}, \eta))h = D_1\psi(\tilde{\rho}, \eta)P_0 DG_\star(\psi(\tilde{\rho}, \eta))h, \qquad h \in E_1,$$

and

(4.13)
$$DG_\star(\psi(\tilde{\rho}, \eta))D_1\psi(\tilde{\rho}, \eta)\tilde{h} = D_1\psi(\tilde{\rho}, \eta)D_1\mathcal{G}_\star(\tilde{\rho}, \eta)\tilde{h}, \qquad \tilde{h} \in F_1.$$

Proof. The first claim follows from the definition of $\mathcal{G}_\star$ and (4.10), while (4.12) is a consequence of (4.11) and (4.13) follows from (4.11) and the chain rule:

$$DG_\star(\psi(\tilde{\rho}, \eta))D_1\psi(\tilde{\rho}, \eta)\tilde{h} = D_1\psi(\tilde{\rho}, \eta)P_0 DG_\star(\psi(\tilde{\rho}, \eta))D_1\psi(\tilde{\rho}, \eta)\tilde{h}$$

$$= D_1\psi(\tilde{\rho}, \eta)D_1\mathcal{G}_\star(\tilde{\rho}, \eta)\tilde{h}. □$$

4.3. Mapping properties of $D_1\mathcal{G}_\star(0, \eta)$. Notice that the points $(0, \eta) \in U$ are equilibria of (4.5), and they correspond to the cylinders $\Gamma(r_\star + \eta)$. We are interested in the spectral properties of the linearization of $\mathcal{G}_\star$ about these equilibria. In particular, we compute the Fréchet derivative

$$D_1\mathcal{G}_\star(0, \eta)h = P_0 DG_\star(\psi(0, \eta)) D_1\psi(0, \eta)h = P_0 DG_\star(\eta) D_1\psi(0, \eta)h$$

for $h \in F_1$. Hence, by Remark 3(d) we derive the formula

(4.14)
$$D_1\mathcal{G}_\star(0, \eta) = P_0 DG_\star(\eta)\big|_{F_1} = DG_\star(\eta)\big|_{F_1},$$

where the last equality is verified by application of the divergence theorem to the linearization

(4.15)
$$DG_\star(\eta) = -\partial_x^2\left(\frac{1}{(r_\star + \eta)^2} + \partial_x^2\right).$$

Utilizing the Fourier series representation of functions in $h^\sigma(\mathbb{T})$ (cf. [46, Propositions 1.2 and 1.3]), we find the eigenvalues of this linearized operator. In particular, for $h \in E_1$,

$$(\lambda - DG_\star(\eta))\,h = \left(\lambda + \partial_x^2\left(\frac{1}{(r_\star + \eta)^2} + \partial_x^2\right)\right)\sum_{k\in\mathbb{Z}}\hat{h}(k)e_k$$

$$= \sum_{k\in\mathbb{Z}}\left(\lambda - k^2\left(\frac{1}{(r_\star + \eta)^2} - k^2\right)\right)\hat{h}(k)e_k$$

(4.16) $\implies$ $\sigma_p(DG_\star(\eta)) = \left\{k^2\left(\frac{1}{(r_\star + \eta)^2} - k^2\right) : k \in \mathbb{Z}\right\}.$

Noting that the embedding $E_1 \hookrightarrow E_0$ is compact, it follows that the resolvent $R(\lambda) := (\lambda - DG_\star(\eta))^{-1}$ is a compact operator, λ in the resolvent set $\rho(DG_\star(\eta))$. It follows from classic theory of linear operators that the spectrum $\sigma(DG_\star(\eta))$ consists entirely of isolated eigenvalues of finite multiplicity; see Kato [38, Theorem III.6.29], for instance. Hence, $\sigma_p(DG_\star(\eta)) = \sigma(DG_\star(\eta))$.

Remark 6. If $r_\star + \eta > 1$, then $\sigma(DG_\star(\eta)) \subset (-\infty, 0]$; however, the spectrum will always contain 0. The presence of this 0 eigenvalue can be seen as a consequence of the fact that the equilibria $r_\star + \eta$ are not isolated in the space E_1. Hence, by passing to the operator $\mathcal{G}_\star$, which acts on an open subset of the zero-mean functions F_1, we eliminate the nontrivial equilibria (since the only constant function in F_1 is the zero function) and thereby eliminate the zero eigenvalue. In particular, one easily computes that

(4.17) $\sigma(D_1\mathcal{G}_\star(0, \eta)) = \left\{k^2\left(\frac{1}{(r_\star + \eta)^2} - k^2\right) : k \in \mathbb{Z}\setminus\{0\}\right\},$ $\eta \in U_1.$

Before we return to problem (1.3), we state the following maximal regularity result for the linearization $D_1\mathcal{G}_\star(0, \eta)$. For this result, we define the exponentially weighted maximal regularity spaces

$$\mathbb{F}_j(\mathbb{R}_+, \omega) := \left\{f : (0, \infty) \to F_0 \,\middle|\, [t \mapsto e^{\omega t}f(t)] \in \mathbb{F}_j(\mathbb{R}_+)\right\},\qquad \omega \in \mathbb{R},\, j = 0, 1,$$

which are Banach spaces when equipped with the norms $\|u\|_{\mathbb{F}_j(\mathbb{R}_+, \omega)} := \|e^{\omega t}u\|_{\mathbb{F}_j(\mathbb{R}_+)}$.

THEOREM 4.4. *Suppose $r_\star > 1$ and $\mu \in (0, 1]$. There exist nonzero positive constants $\delta = \delta(r_\star)$ and $\omega = \omega(r_\star, \delta)$ such that*

$$\left(\mathbb{F}_0(\mathbb{R}_+, \omega),\ \mathbb{F}_1(\mathbb{R}_+, \omega)\right)$$

is a pair of maximal regularity for $-D_1\mathcal{G}_\star(0, \eta)$ for any $\eta \in (-\delta, \delta)$. That is,

$$(\partial_t - D_1\mathcal{G}_\star(0, \eta), \gamma) \in \mathcal{L}_{isom}\left(\mathbb{F}_1(\mathbb{R}_+, \omega), \mathbb{F}_0(\mathbb{R}_+, \omega) \times h_0^{4\mu+\alpha}(\mathbb{T})\right)$$

holds uniformly for $\eta \in (-\delta, \delta)$.

Proof. Fix $\delta > 0$ so that $(-\delta, \delta) \subset U_1 \cap (1 - r_\star, \infty)$. Following the notation and definitions of [46], it is clear from the representation (4.15) that $-DG_\star(\eta)$ is a uniformly elliptic operator from which we see, by [46, Theorem 4.4], that $DG_\star(\eta)$ generates an analytic semigroup on $h^\alpha(\mathbb{T}, \mathbb{C})$ with domain $h^{4+\alpha}(\mathbb{T}, \mathbb{C})$. Since $h_0^\alpha(\mathbb{T}, \mathbb{C})$ inherits the topology of $h^\alpha(\mathbb{T}, \mathbb{C})$ and the projection P_0 commutes with $DG_\star(\eta)$, the analogous resolvent estimates hold for $D_1\mathcal{G}_\star(0, \eta)$, and so we see that $D_1\mathcal{G}_\star(0, \eta)$ generates an analytic semigroup on $h_0^\alpha(\mathbb{T}, \mathbb{C})$ with domain $h_0^{4+\alpha}(\mathbb{T}, \mathbb{C})$. Moreover, from (4.17) it holds that $type(D_1\mathcal{G}_\star(0, \eta)) < 0$ for all $\eta \in (-\delta, \delta)$, where $type(B)$ denotes the spectral type of the semigroup generator B. In particular, we have

$$type(D_1\mathcal{G}_\star(0, \eta)) < \frac{1 - (r_\star - \delta)^2}{(r_\star - \delta)^2} < 0, \qquad \eta \in (-\delta, \delta).$$

Now, choose $\omega \in \left(0, \frac{(r_\star - \delta)^2 - 1}{(r_\star - \delta)^2}\right)$, and the remainder of the result follows from [3, Theorem III.3.4.1 and Remarks 3.4.2(b)] and the restriction of maximal regularity from the complex-valued spaces $h_0^\sigma(\mathbb{T}, \mathbb{C})$ to the subspaces $h_0^\sigma(\mathbb{T})$. $\square$

4.4. Exponential stability of cylinders with radius $r_\star > 1$. Our main result of this section establishes exponential asymptotic stability of the family of cylinders, by which we mean that small perturbations of a cylinder $\Gamma(r_\star)$ will have global solutions which converge exponentially fast to a cylinder $\Gamma(r_\star + \eta)$, where $r_\star \neq r_\star + \eta$ in general. Before formulating our result, we recall that $\mathbb{B}_E(a, \varepsilon)$ denotes the open ball with center a and radius ε, in the normed vector space E. In particular, $\mathbb{B}_{h^{2+\alpha}}(r, \varepsilon)$ consists of all functions in $h^{2+\alpha}(\mathbb{T})$ which are close to r in the $C^{2+\alpha}$ topology.

THEOREM 4.5 (exponential stability). *Fix $\alpha \in (0, 1)$, $\mu \in [1/2, 1]$, so that $4\mu + \alpha \notin \mathbb{Z}$, and $r_\star > 1$. There exist nonzero positive constants $\varepsilon = \varepsilon(r_\star)$, $\delta = \delta(r_\star)$, and $\omega = \omega(r_\star, \delta)$, such that problem (2.1) with initial data $r_0 \in \mathbb{B}_{h^{4\mu+\alpha}}(r_\star, \varepsilon)$ has a unique global solution*

$$r(\cdot, r_0) \in C^1_{1-\mu}(\mathbb{R}_+, h^\alpha(\mathbb{T})) \cap C_{1-\mu}(\mathbb{R}_+, h^{4+\alpha}(\mathbb{T})),$$

and there exist $\eta = \eta(r_0) \in (-\delta, \delta)$ and $M = M(\alpha) > 0$ for which the bound

$$t^{1-\mu}\|r(t, r_0) - (r_\star + \eta)\|_{h^{4+\alpha}} + \|r(t, r_0) - (r_\star + \eta)\|_{h^{4\mu+\alpha}} \le e^{-\omega t} M \|r_0 - r_\star\|_{h^{4\mu+\alpha}}$$

holds uniformly for $t > 0$.

Proof. (i) Let $\delta, \omega > 0$ be the constants given by Theorem 4.4, and consider the operator

$$\mathcal{K}(\tilde{\rho}, \tilde{\rho}_0, \eta) := \left(\partial_t \tilde{\rho} - \mathcal{G}_\star(\tilde{\rho}, \eta), \ \gamma\tilde{\rho} - \tilde{\rho}_0\right),$$

acting on $\mathbb{U} := (\mathbb{F}_1(\mathbb{R}_+, \omega) \cap C(\mathbb{R}_+, U_0)) \times (U_0 \cap F_\mu) \times U_1$, which is open in the Banach space $\mathbb{F}_1(\mathbb{R}_+, \omega) \times F_\mu \times \mathbb{R}$.

First, we show that $\mathcal{K}$ maps $\mathbb{U}$ into $\mathbb{F}_0(\mathbb{R}_+, \omega) \times F_\mu$. Notice that

$$\gamma : \mathbb{F}_1(\mathbb{R}_+, \omega) \to (F_0, F_1)^0_{\mu, \infty}$$

follows from [12, Lemma 2.2(a)], so $\gamma\tilde{\rho} \in F_\mu$. Meanwhile, ∂_t maps $\mathbb{F}_1(\mathbb{R}_+, \omega)$ into $\mathbb{F}_0(\mathbb{R}_+, \omega)$ by definition of the spaces $BUC^1_{1-\mu}(J, E)$. Finally, to see that $\mathcal{G}_\star(\cdot, \eta)$ maps $\mathbb{U}$ into $\mathbb{F}_0(\mathbb{R}_+, \omega)$, choose $\tilde{\rho} \in \mathbb{U}$ and notice that $\tilde{\rho}(t) \in U_0 \cap h_0^{2+\alpha}(\mathbb{T})$, for $t > 0$,

from the embeddings (4.7). Utilizing the explicit quasilinear representation of the operator G, as given by (2.2)–(2.3), whereby

$$\mathcal{G}_\star(\tilde{\rho}(t), \eta) = P_0\Big(-\mathcal{A}\big(\psi(\tilde{\rho}(t), \eta) + r_\star\big)(\psi(\tilde{\rho}(t), \eta) + r_\star) + f\big(\psi(\tilde{\rho}(t), \eta) + r_\star\big)\Big),$$

one will easily conclude the desired mapping property for the operator $\mathcal{G}_\star$. For instance, we have seen that $\mathcal{A}(\rho)\rho = b_1(\rho)\partial_x^4\rho + b_2(\rho)\partial_x^3\rho$, where the functions b_i depend only on ρ, ρ_x, and ρ_{xx}, $i = 1, 2$. Hence, it follows that

$$e^{\omega t} t^{1-\mu} \left\| \mathcal{A}\big(\psi(\tilde{\rho}(t), \eta) + r_\star\big)(\psi(\tilde{\rho}(t), \eta) + r_\star) \right\|_{E_0}$$
$$\leq e^{\omega t} t^{1-\mu} \left\| \partial_x^4 \psi(\tilde{\rho}(t), \eta) \right\|_{E_0} \left\| b_1\big(\psi(\tilde{\rho}(t), \eta) + r_\star\big) \right\|_{E_0}$$
$$+ e^{\omega t} t^{1-\mu} \left\| \partial_x^3 \psi(\tilde{\rho}(t), \eta) \right\|_{E_0} \left\| b_2\big(\psi(\tilde{\rho}(t), \eta) + r_\star\big) \right\|_{E_0}$$

for $t > 0$. From here, we take advantage of the boundedness of $\psi(\tilde{\rho}(t), \eta)$ in the topology of $F_{1/2}$, in conjunction with the explicit formulas for b_i, in order to bound the terms $\|b_i(\psi(\tilde{\rho}(t), \eta) + r_\star)\|_{E_0}$, uniformly in t. Meanwhile, the representation given by Remark 3(d) and the fact that $\tilde{\rho} \in \mathbb{F}_1(\mathbb{R}_+, \omega)$ yield the bounds

$$e^{\omega t} t^{1-\mu} \|\partial_x^k \psi(\tilde{\rho}(t), \eta)\|_{E_0} = e^{\omega t} t^{1-\mu} \|\partial_x^k \tilde{\rho}(t)\|_{F_0} \leq \|e^{\omega t} \tilde{\rho}\|_{\mathbb{F}_1(\mathbb{R}_+)}, \qquad k = 1, \ldots, 4.$$

Analogous methods work for the remaining terms of the function $G_\star\big(\psi(\tilde{\rho}(t), \eta)\big)$, since we can always isolate an element of the form $\partial_x^k \psi(\tilde{\rho}(t), \eta)$, and bound the remaining elements using boundedness in $F_{1/2}$. We conclude the result by noting that the linear projection P_0 adds no complexity to acquiring the necessary bounds.

Regarding the regularity of $\mathcal{K}$, it can be shown that $\mathcal{G}_\star$ is C^ω via substitution operators, and the derivative ∂_t and the trace operator, γ are linear. Hence, it follows that

$$\mathcal{K} \in C^\omega\Big(\mathbb{U}, \mathbb{F}_0(\mathbb{R}_+, \omega) \times F_\mu\Big).$$

Meanwhile, notice that $\mathcal{K}(0, 0, 0) = (0, 0)$ and

$$D_1\mathcal{K}(0, 0, 0) = \Big(\partial_t - D_1\mathcal{G}_\star(0, 0), \gamma\Big) \in \mathcal{L}_{isom}\Big(\mathbb{F}_1(\mathbb{R}_+, \omega), \mathbb{F}_0(\mathbb{R}_+, \omega) \times F_\mu\Big)$$

by Theorem 4.4. Hence, we conclude from the implicit function theorem that there exist an open neighborhood $0 \in \tilde{U} \subset F_\mu \times \mathbb{R}$ and a C^ω mapping $\kappa : \tilde{U} \to \mathbb{F}_1(\mathbb{R}_+, \omega)$ such that

$$\mathcal{K}(\kappa(\tilde{\rho}_0, \eta), \tilde{\rho}_0, \eta) = (0, 0) \quad \text{for all} \quad (\tilde{\rho}_0, \eta) \in \tilde{U}.$$

In particular, $\kappa(\tilde{\rho}_0, \eta)$ is a global solution to (4.5) with parameter η and initial data $\tilde{\rho}_0 \in F_\mu$, where we assume, without loss of generality, that $\tilde{U} \subseteq U$.

(ii) Choose $\varepsilon > 0$ so that for every $r_0 \in \mathbb{B}_{E_\mu}(r_\star, \varepsilon)$, there exists $\eta \in (-r_\star, \infty)$ for which

$$(P_0 r_0, \eta) \in \tilde{U} \quad \text{and} \quad F_\star(r_0 - r_\star; r_\star) = F_\star(\eta; r_\star).$$

The existence of such a constant ε is guaranteed by the continuity of P_0 and $F_\star$, injectivity of $F_\star(\eta; r_\star)$ for $\eta \in (-r_\star, \infty)$, and the fact that $P_0 r_\star = 0$.

Let $r_0 \in \mathbb{B}_{E_\mu}(r_\star, \varepsilon)$ and fix $\eta = \eta(r_0)$ as mentioned so that $F_\star(r_0 - r_\star) = F_\star(\eta)$. Define the function

$$(4.18) \qquad\qquad r := \psi(\kappa(P_0 r_0, \eta), \eta) + r_\star,$$

where $\psi(\kappa(P_0 r_0, \eta), \eta)(t) := \psi(\kappa(P_0 r_0, \eta)(t), \eta)$, and we will demonstrate that r satisfies the desired properties claimed in the theorem.

To see that r is the unique global solution to (2.1) with initial data r_0, first fix $T > 0$ and consider the interval $J := [0, T]$. By the choice of $\varepsilon > 0$ we know that $(P_0 r_0, \eta) \in \tilde{U}$, and so it follows from part (i) above that $\kappa(P_0 r_0, \eta) \in \mathbb{F}_1(\mathbb{R}_+, \omega)$. From this we see that $\kappa(P_0 r_0, \eta) \in \mathbb{F}_1(J)$ is a solution to (4.5) with initial data $P_0 r_0 \in F_\mu$. Thus it follows, by Lemma 4.1, that $r \in \mathbb{E}_1(J)$ is the solution on J to the problem (2.1) with initial data

$$\psi(P_0 r_0, \eta) + r_\star = \psi(P_0(r_0 - r_\star), \eta) + r_\star = r_0,$$

where we use Remark 3(b) and the fact that $r_0 - r_\star \in \mathcal{M}_\eta^{4\mu + \alpha}$. The claim now follows by the fact that $T > 0$ was arbitrary and by definition of the Fréchet spaces $C_{1-\mu}(\mathbb{R}_+, E)$.

To see that r satisfies the exponential bounds in the second part of the claim, first notice that $\kappa(0, \eta) \equiv 0$ for $\eta \in U_1$. Then, by Remark 3, and application of the mean value theorem, the expression

$$r(t) - (r_\star + \eta) = \psi(\kappa(P_0 r_0, \eta)(t), \eta) - \eta = \psi(\kappa(P_0 r_0, \eta)(t), \eta) - \psi(\kappa(0, \eta)(t), \eta)$$

$$= \Big(P_0 + (1 - P_0)\Big)\Big(\psi(\kappa(P_0 r_0, \eta)(t), \eta) - \psi(\kappa(0, \eta)(t), \eta)\Big)$$

$$= \kappa(P_0 r_0, \eta)(t) + \frac{1}{2\pi}\int_{\mathbb{T}}\Big(\psi(\kappa(P_0 r_0, \eta)(t, x), \eta) - \psi(\kappa(0, \eta)(t, x), \eta)\Big)dx$$

$$= \kappa(P_0 r_0, \eta)(t) + \frac{1}{2\pi}\int_{\mathbb{T}}\int_0^1 D_1\psi\big(\tau\kappa(P_0 r_0, \eta)(t), \eta\big)\kappa(P_0 r_0, \eta)(t, x)d\tau dx$$

holds for all $t > 0$. Notice that

$$e^{\omega t}t^{1-\mu}\|\kappa(P_0 r_0, \eta)(t)\|_{F_1} \le \|\kappa(P_0 r_0, \eta)\|_{\mathbb{F}_1(\mathbb{R}_+, \omega)}$$

and

$$\sup_{t \in \mathbb{R}_+}\|e^{\omega t}\kappa(P_0 r_0, \eta)(t)\|_{F_\mu}$$

are finite quantities by the fact that $\kappa(P_0 r_0, \eta) \in \mathbb{F}_1(\mathbb{R}_+, \omega)$ and the embedding (4.7). We note that the reference for (4.7) does not explicitly include the unbounded interval $J = \mathbb{R}_+$; however, the methods of the proof extend to this unbounded case with little trouble. Meanwhile, the remaining term in $r(t) - (r_\star + \eta)$ above is scalar-valued, so we bound $D_1\psi(\tau\kappa(P_0 r_0, \eta)(t), \eta)\kappa(P_0 r_0, \eta)(t)$ in the $C(\mathbb{T})$-topology, which is then bounded in the $h^\sigma(\mathbb{T})$-topology for any $\sigma \in \mathbb{R}_+ \setminus \mathbb{Z}$. In particular, observe that, by (4.6),

$$\sup_{\tilde{\rho} \in U_0}\|D_1\psi(\tilde{\rho}, \eta)\kappa(P_0 r_0, \eta)(t)\|_{h^\alpha} \le N\|\kappa(P_0 r_0, \eta)(t)\|_{h_0^\alpha}, \qquad t > 0,$$

and we conclude that the bounds

$$(4.19) \qquad e^{\omega t}t^{1-\mu}\|r(t) - (r_\star + \eta)\|_{E_1} \le \Big(1 + c_1 N\Big)\|\kappa(P_0 r_0, \eta)\|_{\mathbb{F}_1(\mathbb{R}_+, \omega)}$$

and

$$(4.20) \qquad e^{\omega t}\|r(t) - (r_\star + \eta)\|_{E_\mu} \leq \Big(c_2 + c_3 N\Big)\|\kappa(P_0 r_0, \eta)\|_{\mathbb{F}_1(\mathbb{R}_+, \omega)}$$

hold uniformly for $t > 0$. Here the constant c_1 comes from the embedding $F_1 \hookrightarrow F_0$, and the constants c_2 and c_3 come from the embeddings (4.7). Finally, by the regularity of κ, we may assume that $\tilde{U}$ was chosen sufficiently small to ensure that $D_1 \kappa$ is uniformly bounded from $\tilde{U}$ into $\mathbb{F}_1(\mathbb{R}_+, \omega)$. Recalling that $\kappa(0, \eta) = 0$, it follows that

$$(4.21) \qquad \begin{aligned} \|\kappa(P_0 r_0, \eta)\|_{\mathbb{F}_1(\mathbb{R}_+, \omega)} &\leq \int_0^1 \|D_1\kappa(\tau P_0 r_0, \eta) P_0 r_0\|_{\mathbb{F}_1(\mathbb{R}_+, \omega)} \ d\tau \\ &\leq \tilde{M}\|P_0 r_0\|_{F_\mu} \leq M\|r_0 - r_\star\|_{E_\mu}, \end{aligned}$$

where $M := \|P_0\| \sup_{(\tilde{\rho}, \eta) \in \tilde{U}} \|D_1\kappa(\tilde{\rho}, \eta)\|_{\mathcal{L}(F_\mu, \mathbb{F}_1(\mathbb{R}_+, \omega))}$. The claim now follows from (4.21) and the inequalities (4.19)–(4.20). $\quad\square$

Remark 7. With Theorem 4.5 established, we note that the equivolume manifold $\mathcal{M}^{2+\alpha} = \mathcal{M}^{2+\alpha}(r_\star) \subset h^{2+\alpha}(\mathbb{T})$ is a local stable manifold for the cylinder of radius $r_\star > 1$. Moreover, these manifolds foliate the interval $(1, \infty) \subset h^{2+\alpha}(\mathbb{T})$ with the radius $r_\star$ a parameter which separates leaves of the foliation.

5. Instability of cylinders with radius $0 < r_\star < 1$. Taking advantage of the reduced problem (4.5) and the connection with (1.3), we proceed with the following result regarding instability of cylinders with radius $0 < r_\star < 1$, in the setting of F_μ. Differences in volume between the initial data r_0 and the cylinder $r_\star$ are not a factor in the following argument, so we assume that the parameter η, associated with the reduced problem (4.5), is simply taken to be zero for this proof.

THEOREM 5.1. *Let $r_\star \in (0, 1)$ and $\mu \in [1/2, 1]$ such that $4\mu + \alpha \notin \mathbb{Z}$. Then the equilibrium $r_\star$ of (1.3) is unstable in the topology of $h^{4\mu + \alpha}(\mathbb{T})$ for initial values in $h^{4\mu + \alpha}(\mathbb{T})$. More precisely, there exist $\varepsilon > 0$ and a sequence of initial values $(r_n) \subset h^{4\mu + \alpha}(\mathbb{T})$ such that*

- $\lim_{n \to \infty} \|r_n - r_\star\|_{h^{4\mu + \alpha}} = 0$, *and*
- *for each n, there exists $t_n \in J(r_n)$ so that $\|r(t_n, r_n) - r_\star\|_{h^{4\mu + \alpha}} \geq \varepsilon$.*

Proof. (i) We begin by showing that 0 is an unstable equilibrium for the reduced problem (4.5) centered at $r_\star$. Let $L := D_1\mathcal{G}_\star(0, 0)$ be the linearization of $\mathcal{G}_\star$ at $\tilde{\rho} = 0$. We can restate the evolution equation (4.5) in the equivalent form

$$(5.1) \qquad \begin{cases} \tilde{\rho}_t - L\tilde{\rho} = g(\tilde{\rho}), & t > 0, \\ \tilde{\rho}(0) = \tilde{\rho}_0, \end{cases}$$

where $g(\tilde{\rho}) := \mathcal{G}_\star(\tilde{\rho}, 0) - L\tilde{\rho}$. Using the quasilinear structure of $[\tilde{\rho} \mapsto \mathcal{G}_\star(\tilde{\rho}, 0)]$ it is not difficult to see that for every $\beta > 0$ there exists a number $\varepsilon_0 = \varepsilon_0(\beta) > 0$ such that

$$(5.2) \qquad \|g(\tilde{\rho})\|_{F_0} \leq \beta\|\tilde{\rho}\|_{F_1}, \quad \tilde{\rho} \in \mathbb{B}_{F_\mu}(0, \varepsilon_0) \cap F_1,$$

where we will be assuming throughout that $\tilde{\rho} \in U_0$, to guarantee that $\mathcal{G}_\star(\tilde{\rho}, 0)$, and subsequently $g(\tilde{\rho})$, is defined. It follows from (4.17) that

$$\sigma(L) \cap [\operatorname{Re} z > 0] \neq \emptyset,$$

and we may choose numbers $\omega, \gamma > 0$ such that

$$[\omega - \gamma \leq \operatorname{Re} z \leq \omega + \gamma] \cap \sigma(L) = \emptyset \quad \text{and} \quad \sigma_+ := [\operatorname{Re} z > \omega + \gamma] \cap \sigma(L) \neq \emptyset,$$

i.e., the strip $[\omega - \gamma \leq \mathrm{Re}\, z \leq \omega + \gamma]$ does not intersect $\sigma(L)$ and there is at least one point of $\sigma(L)$ to the right of the line $[\mathrm{Re}\, z = \omega + \gamma]$.

We define P_+ to be the spectral projection, in F_0, with respect to the spectral set σ_+, and let $P_- := 1 - P_+$. Then $P_+(F_0)$ is finite-dimensional and the topological decomposition

$$F_0 = P_+(F_0) \oplus P_-(F_0)$$

reduces L, so that $L = L_+ \oplus L_-$, where $L_\pm$ is the part of L in $P_\pm(F_0)$, respectively, with the domains $D(L_\pm) = P_\pm(F_1)$. Moreover, $P_\pm$ decomposes F_1 by the embedding $F_1 \hookrightarrow F_0$, and, without loss of generality, we can take the norm on F_1 so that

$$\|v\|_{F_1} = \|P_+ v\|_{F_1} + \|P_- v\|_{F_1}.$$

We note that

$$\sigma(L_-) \subset [\mathrm{Re}\, z < \omega - \gamma], \qquad \sigma(L_+) = \sigma^+ \subset [\mathrm{Re}\, z > \omega + \gamma].$$

This implies that there is a constant $M_0 \geq 1$ such that

(5.3)
$$\begin{aligned}
\|e^{L_- t} P_-\|_{\mathcal{L}(F_0)} &\leq M_0 e^{(\omega - \gamma) t}, \\
\|e^{-L_+ t} P_+\|_{\mathcal{L}(F_0)} &\leq M_0 e^{-(\omega + \gamma) t}, \qquad t \geq 0,
\end{aligned}$$

where $\{e^{L_- t} : t \geq 0\}$ is the analytic semigroup in $P_-(F_0)$ generated by L_- and $\{e^{L_+ t} : t \in \mathbb{R}\}$ is the group in $P_+(F_0)$ generated by the bounded operator L_+.

From (4.14)–(4.15) and [46, Theorem 5.2] one sees that $(\mathbb{F}_0(J), \mathbb{F}_1(J))$ is a pair of maximal regularity for $-L$, and it is easy to see that $-L_-$ inherits the property of maximal regularity. In particular, the pair $(P_-(\mathbb{F}_0(J)), P_-(\mathbb{F}_1(J)))$ is a pair of maximal regularity for $-L_-$. In fact, since $type(-\omega + L_-) < -\gamma < 0$ we see that $(P_-(\mathbb{F}_0(\mathbb{R}_+)), P_-(\mathbb{F}_1(\mathbb{R}_+)))$ is a pair of maximal regularity for $(\omega - L_-)$. This, in turn, implies the a priori estimate

(5.4)
$$\|e^{-\omega t} w\|_{\mathbb{F}_1(J_T)} \leq M_1 \Big(\|w_0\|_{F_\mu} + \|e^{-\omega t} f\|_{\mathbb{F}_0(J_T)} \Big)$$

for $J_T := [0, T]$, any $T \in (0, \infty)$ (or $J_T = \mathbb{R}_+$ for $T = \infty$), with a universal constant $M_1 > 0$, where w is a solution of the linear Cauchy problem

$$\begin{cases} \dot{w} - L_- w = f, \\ w(0) = w_0, \end{cases}$$

with $(f, w_0) \in \big(C\big((0,T), P_- F_0\big), P_- U_0 \big)$.

(ii) By way of contradiction, suppose that the equilibrium 0 is stable for (4.5). Then for every $\varepsilon > 0$ there exists a number $\delta > 0$ such that (5.1) admits for each $\tilde{\rho}_0 \in \mathbb{B}_{F_\mu}(0, \delta)$ a global solution

$$\tilde{\rho} = \tilde{\rho}(\cdot, \tilde{\rho}_0) \in C^1_{1-\mu}(\mathbb{R}_+, F_0) \cap C_{1-\mu}(\mathbb{R}_+, F_1) \cap C(\mathbb{R}_+, U_0),$$

which satisfies

(5.5)
$$\|\tilde{\rho}(t)\|_{F_\mu} < \varepsilon, \qquad t \geq 0.$$

We can assume without loss of generality that β and ε are chosen such that

$$(5.6) \qquad 2C_0(M_0 + M_1\gamma)\beta \leq \gamma \qquad \text{and} \qquad \varepsilon \leq \varepsilon_0(\beta),$$

where $C_0 := \max\{\|P_-\|_{\mathcal{L}(F_0)}, \|P_+\|_{\mathcal{L}(F_0)}\}$. As $P_+(F_0)$ is finite-dimensional, we may also assume that

$$\|P_+v\|_{F_\nu} = \|P_+v\|_{F_0}, \qquad v \in F_0, \quad \nu \in \{\mu, 1\},$$

where we also use the fact that $P_+F_0 \subset D(L^n)$ for every $n \in \mathbb{N}$; cf. [48, Proposition A.1.2].

Claim 1. For any initial value $\tilde\rho_0 \in \mathbb{B}_{F_\mu}(0,\delta)$, $P_+\tilde\rho$ admits the representation

$$(5.7) \qquad P_+\tilde\rho(t) = -\int_t^\infty e^{L_+(t-s)} P_+ g(\tilde\rho(s))\,ds, \qquad t \geq 0.$$

For this we first establish that, for any $\tilde\rho_0 \in \mathbb{B}_{F_\mu}(0,\delta)$,

$$e^{-\omega t}\tilde\rho \in BC_{1-\mu}(\mathbb{R}_+, F_1) := \left\{ u \in C((0,\infty), F_1) : \sup_{t \in \mathbb{R}_+} t^{1-\mu}\|u(t)\|_{F_1} < \infty \right\}.$$

First notice that the mapping property

$$g : \mathbb{F}_1(J_T) \cap C(J_T, U_0) \to \mathbb{F}_0(J_T), \qquad 0 < T < \infty,$$

which follows in the same way as the mapping property derived for $\mathcal{G}_\star$ in the proof of Theorem 4.5 above, together with the inequalities (5.2) and (5.4) yields

$$(5.8) \qquad \begin{aligned} &\|e^{-\omega t}P_-\tilde\rho\|_{B_{1-\mu}(J_T, F_1)} \\ &\leq M_1\left(\|P_-\tilde\rho_0\|_{F_\mu} + C_0\beta\|e^{-\omega t}P_+\tilde\rho\|_{B_{1-\mu}(J_T, F_1)} + C_0\beta\|e^{-\omega t}P_-\tilde\rho\|_{B_{1-\mu}(J_T, F_1)}\right) \end{aligned}$$

for any $0 < T < \infty$. Due to (5.6), we have $M_1 C_0\beta \leq 1/2$ and can further conclude

$$(5.9) \qquad \|e^{-\omega t}P_-\tilde\rho\|_{B_{1-\mu}(J_T, F_1)} \leq 2M_1\left(\|P_-\tilde\rho_0\|_{F_\mu} + C_0\beta\|e^{-\omega t}P_+\tilde\rho\|_{B_{1-\mu}(J_T, F_1)}\right).$$

It follows from (5.5) that

$$t^{1-\mu}\|e^{-\omega t}P_+\tilde\rho(t)\|_{F_1} \leq t^{1-\mu}e^{-\omega t}C_0\|\tilde\rho(t)\|_{F_\mu} \leq C_0 C_1\varepsilon,$$

where $C_1 := \sup\{t^{1-\mu}e^{-\omega t} : t \geq 0\} < \infty$. Inserting this result into (5.9) yields

$$(5.10) \qquad \|e^{-\omega t}\tilde\rho\|_{B_{1-\mu}(J_T, F_1)} \leq 2M_1\|P_-\tilde\rho_0\|_{F_\mu} + (2M_1 C_0\beta + 1)C_0 C_1\varepsilon \leq C_2$$

for any $0 < T < \infty$. However, since T is arbitrary and (5.10) is independent of T we conclude that $e^{-\omega t}\tilde\rho \in BC_{1-\mu}(\mathbb{R}_+, F_1)$ for any initial value $\tilde\rho_0 \in \mathbb{B}_{F_\mu}(0,\delta)$. Next we note that, for $s \geq t$, by (5.3)

$$(5.11) \qquad \begin{aligned} \|e^{L_+(t-s)}P_+ g(\tilde\rho(s))\|_{F_0} &\leq M_0 C_0\beta e^{(\omega+\gamma)(t-s)}\|\tilde\rho(s)\|_{F_1} \\ &\leq M_0 C_0\beta e^{\omega t}e^{\gamma(t-s)}s^{\mu-1}\|e^{-\omega s}\tilde\rho\|_{B_{1-\mu}(\mathbb{R}_+, F_1)}, \end{aligned}$$

which shows that the integral in (5.7) exists for any $t \geq 0$, with convergence in F_1. Moreover,

$$(5.12) \qquad \left\|\int_t^\infty e^{L_+(t-s)}P_+ g(\tilde\rho(s))\,ds\right\|_{F_0} \leq e^{\omega t}M_0 C_0 C_3\beta\|e^{-\omega t}\tilde\rho\|_{B_{1-\mu}(\mathbb{R}_+, F_1)},$$

where $C_3 := \sup\left\{ \int_t^\infty e^{\gamma(t-s)} s^{\mu-1}\, ds : t \geq 0 \right\} < \infty$. Noting that $w = P_+\tilde{\rho}$ solves the Cauchy problem

$$\begin{cases} \dot{w} - L_+ w = P_+ g(\tilde{\rho}), \\ w(0) = P_+\tilde{\rho}_0, \end{cases}$$

it follows from the variation of parameters formula that, for $t \geq 0$ and $\tau > 0$,

$$P_+\tilde{\rho}(t) = e^{L_+(t-\tau)} P_+\tilde{\rho}(\tau) + \int_\tau^t e^{L_+(t-s)} P_+ g(\tilde{\rho}(s))\, ds.$$

Since this representation holds for any $\tau > 0$, the claim follows from (5.3) and (5.5) by sending τ to ∞.

Claim 2. For any $\tilde{\rho}_0 \in \mathbb{B}_{F_\mu}(0, \delta)$ it must hold that

$$\|P_+\tilde{\rho}_0\|_{F_\mu} \leq 2M_0 M_1 C_3 \|P_-\tilde{\rho}_0\|_{F_\mu}.$$

From (5.7) and (5.11) follows

$$\begin{aligned} (5.13) \qquad & \|e^{-\omega t} P_+\tilde{\rho}\|_{B_{1-\mu}(\mathbb{R}_+, F_0)} \\ & \leq \frac{M_0 C_0 \beta}{\gamma} \left(\|e^{-\omega t} P_+\tilde{\rho}\|_{B_{1-\mu}(\mathbb{R}_+, F_1)} + \|e^{-\omega t} P_-\tilde{\rho}\|_{B_{1-\mu}(\mathbb{R}_+, F_1)} \right), \end{aligned}$$

where we have used the fact that $\sup_{t \geq 0}\left\{ t^{1-\mu} \int_t^\infty e^{\gamma(t-s)} s^{\mu-1}\, ds \right\} \leq 1/\gamma$. Adding the estimates in (5.8) and (5.13) and employing (5.6) yields

$$(5.14) \qquad \|e^{-\omega t}\tilde{\rho}\|_{B_{1-\mu}(\mathbb{R}_+, F_1)} \leq 2M_1 \|P_-\tilde{\rho}_0\|_{F_\mu}.$$

The representation (5.7) in conjunction with (5.12) and (5.14) then implies

$$(5.15) \qquad \|P_+\tilde{\rho}_0\|_{F_\mu} \leq M_0 C_0 C_3 \beta \|e^{-\omega t}\tilde{\rho}\|_{B_{1-\mu}(\mathbb{R}_+, F_1)} \leq M_0 C_3 \|P_-\tilde{\rho}_0\|_{F_\mu},$$

where the last inequality follows from the fact that $2C_0 M_1 \beta \leq 1$. We have thus demonstrated the claim.

Notice that the preceding claim contradicts the stability assumption. In particular, if $\tilde{\rho}_0 \in \mathbb{B}_{F_\mu}(0, \delta)$, $\tilde{\rho}_0 \neq 0$, is chosen such that $P_-\tilde{\rho}_0 = 0$, then it must hold that $P_+\tilde{\rho}_0 = 0$, and hence $\tilde{\rho}_0 = 0$, leading to a clear contradiction. In particular, we conclude that there exists $\tilde{\varepsilon} > 0$ and a sequence $(\tilde{\rho}_n) \subset F_\mu$ such that $\tilde{\rho}_n \to 0$ and the solution $\tilde{\rho}(\cdot, \tilde{\rho}_0)$ satisfies $\|\tilde{\rho}(t_n, \tilde{\rho}_n)\|_{F_\mu} \geq \tilde{\varepsilon}$ for some $t_n \in J(\tilde{\rho}_n)$.

(iii) The result now follows by application of the projection P_0 to perturbations of the cylinder $r_\star$. In particular, stability of the $r_\star$ for (1.3) would necessarily imply stability of 0 for (4.5), which contradicts the conclusion of parts (i) and (ii). □

Remark 8. With explicit knowledge of eigenvalues for the linearization, we can also conclude existence of stable, unstable, and center manifolds for (1.3) about cylinders $r_\star \in (0, 1)$. See [18, 64] for existence of such manifolds for nonlinear parabolic problems with continuous maximal regularity. The characterization of the unstable manifolds is a very interesting open question, especially regarding investigation of pinch-off behavior.

6. Bifurcation results. In this section we turn our attention to interactions between the family of cylinders and the family of unduloids. We have already seen that the radius $r_\star = 1$ plays a critical role in the dynamics of the cylinders. The change of stability for cylinders above and below this critical radius suggests that there is a bifurcation at $r_\star = 1$. Indeed, we will confirm this bifurcation, using results of Crandall and Rabinowitz [16], and investigate properties of the bifurcation. Herein we take the parameter $\lambda := 1/r_\star$ as our bifurcation parameter, $r_\star > 0$.

From the reductions developed in section 4, it suffices to study the bifurcation equation

$$(6.1) \qquad \bar{\mathcal{G}}(\tilde{\rho}, \lambda) := \mathcal{G}_\star(\tilde{\rho}, 0) = P_0 G(\psi(\tilde{\rho}) + r_\star) = 0, \qquad \lambda = 1/r_\star,$$

in the setting of $(\tilde{\rho}, \lambda) \in F_1 \times (0, \infty)$, where we use $\psi(\tilde{\rho}) := \psi(\tilde{\rho}, 0)$ to economize notation. Recalling the explicit characterization (4.17), we note that the eigenvalues of $D_1 \mathcal{G}_\star(0,0)$ all have multiplicity two in the setting of F_1, regardless of the value of the parameter $r_\star$. The techniques of [16], where the authors derive results for operators with simple eigenvalues, are not directly applicable in this setting. We may choose at this point to employ more general bifurcation results for high-dimensional kernels, such as the results contained in [40, section I.19], or we can simplify our setting to make accessible the results of [16].

Whether we choose to simplify our setting or use higher-dimensional bifurcation results, we can make good use of the following observation. Due to the periodicity enforced in the problem, the set of equilibria of (1.3) is invariant under shifts along the axis of rotation. More precisely, considering the translation operators T_a, discussed in the proof of Proposition 2.4 in the appendix, one can easily verify that $G(T_a \bar{r}) = 0$ if and only if $G(\bar{r}) = 0$, $a \in \mathbb{R}$. Obviously, this invariance carries over to the reduced problem (4.5) and subsequently to the bifurcation equation (6.1).

One can take advantage of this shift invariance of equilibria in the context of bifurcation with high-dimensional kernels by constructing a two-dimensional bifurcation parameter $\tilde{\lambda} = (1/r_\star, a)$ and eventually observes two-dimensional bifurcating *surfaces* of equilibria in F_1; cf. [40, Theorem I.19.2 and Remark I.19.7]. On the other hand, we will make use of this invariance to simplify the setting in which we are looking for equilibria and make accessible the methods of Crandall and Rabinowitz for operators with simple eigenvalues. The specific simplification that we apply to our setting has also been employed by Escher and Matioc [25] and is supported by the following proposition, which allows us to consider the class

$$F_{1,\text{even}} := h_{0,\text{even}}^{4+\alpha}(\mathbb{T})$$

of functions which are even, i.e., symmetric about $[x = 0]$, and $h_0^{4+\alpha}$ regular.

PROPOSITION 6.1. *For every equilibrium $\bar{\rho}$ of (4.5), there exists $x_0 = x_0(\bar{\rho}) \in \mathbb{T}$ for which the translation $T_{x_0}\bar{\rho}$ is in the space $F_{1,\text{even}} := h_{0,\text{even}}^{4+\alpha}(\mathbb{T})$ of even functions on $\mathbb{T}$ in the class F_1. That is, up to translations on $\mathbb{T}$, all equilibria of (4.5) are even.*

Proof. From Remark 2 and Proposition 4.3, we know that $\bar{\rho}$ must correspond to the projection of an undulary curve $R(\mathcal{H}, B)$, modulo translations along the x-axis. Choosing $x_0 \in \mathbb{T}$ so that $T_{x_0}\bar{\rho} = P_0 R(\cdot; \mathcal{H}, B)$ readily verifies that $R(\cdot; \mathcal{H}, B)$ is symmetric about $s = \pi/2\mathcal{H}$. The claim follows from $x(\pi/2\mathcal{H}) = 0$. $\square$

From this observation, we see that there is no loss of generality if we focus our bifurcation analysis on the setting of $\tilde{\rho} \in F_{1,\text{even}}$. One benefit of working in this setting is that we have the Fourier series representation

$$\tilde{\rho}(x) = \sum_{k \geq 1} a_k \cos(kx), \quad \{a_k\} \subset \mathbb{R}, \qquad \text{for all} \quad \tilde{\rho} \in F_{1,\text{even}}.$$

We are now prepared to prove our first bifurcation result.

THEOREM 6.2 (bifurcation of reduced problem). *For every $\ell \in \mathbb{N}$, $(0, \ell) \in h_{0,\text{even}}^{4+\alpha}(\mathbb{T}) \times (0, \infty)$ is a bifurcation point for (6.1). In particular, there exist a positive constant $\delta_\ell > 0$ and a nontrivial analytic curve*

$$(6.2) \qquad \left\{ (\tilde{\rho}_\ell(s), \lambda_\ell(s)) \in h_{0,\text{even}}^{4+\alpha} \times \mathbb{R} : s \in (-\delta_\ell, \delta_\ell), (\tilde{\rho}_\ell(0), \lambda_\ell(0)) = (0, \ell) \right\},$$

such that

$$\bar{\mathcal{G}}(\tilde{\rho}_\ell(s), \lambda_\ell(s)) = 0 \qquad \text{for all} \quad s \in (-\delta_\ell, \delta_\ell),$$

and all solutions of (6.1) in a neighborhood of $(0, \ell)$ are either a trivial solution $(0, \lambda)$ or an element of the nontrivial curve (6.2). Moreover, if $\lambda \in (0, \infty) \setminus \mathbb{N}$, then $(0, \lambda)$ is not a bifurcation point for (6.1).

Proof. We first note that bifurcation can only occur at points $(0, \lambda)$ for which $D_1\bar{\mathcal{G}}(0, \lambda)$ is not bijective. We can see from (4.14)–(4.15) that

$$(6.3) \qquad D_1\bar{\mathcal{G}}(0, \lambda) = -\partial_x^2 \left(\lambda^2 + \partial_x^2 \right) \Big|_{F_{1,\text{even}}},$$

which is realized as a Fourier multiplier with the symbol

$$\left(M_k \right)_{k \in \mathbb{N}} = \left(k^2(\lambda^2 - k^2) \right)_{k \in \mathbb{N}},$$

and we see that the operator is bijective whenever $\lambda \in (0, \infty) \setminus \mathbb{N}$. Hence, it follows that bifurcation can occur only at points of the form $(0, \ell)$, $\ell \in \mathbb{N}$.

We now fix $\ell \in \mathbb{N}$ and proceed to verify that $(0, \ell)$ is indeed a bifurcation point for (6.1). By compactness of the resolvent $R(\lambda) := (\lambda - DG_\star(0))^{-1}$, $\lambda \in \rho(DG_\star(0))$, it follows that $D_1\bar{\mathcal{G}}(0, \ell)$ is a Fredholm operator of index zero. Further, we see that

$$N_\ell := N(D_1\bar{\mathcal{G}}(0, \ell)) = \text{span}\{\cos(\ell x)\},$$
$$R_\ell := R(D_1\bar{\mathcal{G}}(0, \ell)) = \overline{\text{span}}\{\cos(kx) : k \geq 1, k \neq \ell\},$$

where $N(B)$ and $R(B)$ denote the kernel and the range, respectively, of the operator B. Since $h^\sigma(\mathbb{T}) \hookrightarrow L_2(\mathbb{T})$, we can borrow the L_2-inner product to realize N_ℓ as a topological complement to R_ℓ as subspaces of $F_{1,\text{even}}$. Meanwhile, following from (6.3), we compute the mixed derivative

$$(6.4) \qquad D_2 D_1\bar{\mathcal{G}}(0, \ell) = -2\ell \, \partial_x^2 \Big|_{F_{1,\text{even}}}.$$

Now take $\hat{v}_0 := \cos(\ell \cdot) \in N_\ell$ and observe that

$$D_2 D_1\bar{\mathcal{G}}(0, \ell)\hat{v}_0 = 2\ell^3 \, \cos(\ell \cdot) \notin R_\ell,$$

from which the result follows by [16, Theorem 1.7] or [40, Theorem I.5.1]. □

Remark 9. Following from the previous result, we are able to track the behavior of the so-called *critical eigenvalue* $\mu_\ell(\lambda)$ of the linearization $D_1\bar{\mathcal{G}}(0, \lambda)$ about the trivial equilibria $(0, \lambda)$. In particular, we choose $\mu_\ell(\lambda)$ to be the eigenvalue of $D_1\bar{\mathcal{G}}(0, \lambda)$ which passes through 0 with nonvanishing speed at $\lambda = \ell$; the existence of $\mu_\ell(\lambda)$ is guaranteed by the bifurcation observed above; cf. [40, sections I.6 and I.7]. Moreover, employing eigenvalue perturbation techniques, we can also track the associated perturbed eigenvalue $\hat{\mu}_\ell(s)$ of the linearization $D_1\bar{\mathcal{G}}(\tilde{\rho}_\ell(s), \lambda_\ell(s))$ about the nontrivial

equilibria. These eigenvalues will play a crucial role in the following instability results for the branches of bifurcating equilibria.

THEOREM 6.3. *Each of the bifurcations established in Theorem* 6.2 *is a subcritical pitchfork-type bifurcation. More precisely, for all $\ell \in \mathbb{N}$, we have*

$$\dot{\lambda}_\ell(0) = 0 \quad and \quad \ddot{\lambda}_\ell(0) < 0,$$

where "$\cdot$" denotes the derivative with respect to the parameter s. Moreover, it holds that the perturbed eigenvalues $\hat{\mu}_\ell(s)$ are strictly positive for $|s| > 0$ chosen sufficiently small.

Proof. Utilizing the methods of [40, sections I.6 and I.7], and the techniques developed in the previous sections of the paper, one can explicitly verify that the bifurcations observed above are indeed subcritical pitchfork bifurcations. The result for the perturbed eigenvalues now follows from the eigenvalue perturbation techniques in [40, section I.7]; see also Amann [1, section 27].   □

With these bifurcation results established in the setting of the reduced problem, we will now go about deriving results for the original problem (1.3). Recalling the definition of the operator $G_\star$ from section 4.1, we introduce the notation

$$G(\rho, \lambda) := G(\rho + 1/\lambda) = G_\star(\rho) \qquad for \quad \lambda = 1/r_\star.$$

We are now interested in finding solutions to the bifurcation equation

(6.5) $$G(\rho, \lambda) = 0, \qquad (\rho, \lambda) \in h^{4+\alpha}(\mathbb{T}) \times (0, \infty),$$

associated with the full problem (1.3).

We begin analyzing (6.5) by *lifting* the bifurcation results already established for the reduced problem. We make use of the connections established in section 4.2, and we also establish the following connection between the eigenvalues of $D_1\bar{\mathcal{G}}$ and $DG(\cdot, \lambda)$ at equilibria.

PROPOSITION 6.4. *Suppose $\bar{\mathcal{G}}(\tilde{\rho}, \lambda) = 0$ and $\mu \neq 0$. Then*

μ *is an eigenvalue for* $D_1\bar{\mathcal{G}}(\tilde{\rho}, \lambda)$ $\Longleftrightarrow$ μ *is an eigenvalue for* $D_1G(\psi(\tilde{\rho}), \lambda)$.

Proof. (i) First, suppose that $D_1\bar{\mathcal{G}}(\tilde{\rho}, \lambda)\tilde{h} = \mu\tilde{h}$ for some $\tilde{h} \in F_1 \setminus \{0\}$, and let $h := D\psi(\tilde{\rho})\tilde{h}$. Then $h \in E_1 \setminus \{0\}$, by injectivity of $D\psi(\tilde{\rho})$, and it follows from (4.13) that

$$D_1G(\psi(\tilde{\rho}), \lambda)h = \mu h.$$

We also observe that this assertion is true in case $\mu = 0$.

(ii) Now suppose that $D_1G(\psi(\tilde{\rho}), \lambda)h = \mu h$ for some $h \in E_1 \setminus \{0\}$. We conclude from (4.12) that $h \in T_{\psi(\tilde{\rho})}\mathcal{M}_0$, so that there exists a unique $\tilde{h} \in F_1 \setminus \{0\}$ for which $h = D\psi(\tilde{\rho})\tilde{h}$. Then (4.13) shows that

$$\mu D\psi(\tilde{\rho})\tilde{h} = D\psi(\tilde{\rho})D_1\bar{\mathcal{G}}(\tilde{\rho}, \lambda)\tilde{h},$$

and finally, by injectivity of $D\psi(\tilde{\rho})$, we conclude that $\mu\tilde{h} = \bar{\mathcal{G}}(\tilde{\rho}, \lambda)\tilde{h}$, as desired.   □

We are now prepared to prove the main result regarding bifurcation of the original problem (1.3) in the setting of $h^{4+\alpha}(\mathbb{T})$, and instability of the bifurcating unduloids.

THEOREM 6.5 (bifurcation of full problem). *Fix $\ell \in \mathbb{N}$. Then*
(a) *the set*

$$(6.6) \qquad \left\{ \psi(\tilde{\rho}_\ell(s)) + 1/\lambda_\ell(s) : s \in (-\delta_\ell, \delta_\ell) \right\} \subset h^{4+\alpha}(\mathbb{T})$$

is an analytic curve of equilibria for the problem (1.3) which bifurcates subcritically from the family of cylinders $r_\star \in (0, \infty)$ at the cylinder $r_\star = 1/\ell$;
(b) *there exists some $\varepsilon_\ell > 0$ so that for every $s \in (-\delta_\ell, \delta_\ell)$*

$$\psi(\tilde{\rho}_\ell(s)) + 1/\lambda_\ell(s) = R(B, \ell) \qquad \text{for some} \quad B \in (-\varepsilon_\ell, \varepsilon_\ell),$$

i.e., the family (6.6) of equilibria is exactly the even presentations of $2\pi/\ell$-periodic undulary curves in some neighborhood of the cylinder $r_\star = 1/\ell$;
(c) *the undulary curves $R(B, \ell)$ are unstable for $|B| > 0$ chosen sufficiently small.*
Proof. (a) It follows from Proposition 4.3 and Theorem 6.2 that the family

$$\left\{ (\psi(\tilde{\rho}_\ell(s)), \lambda_\ell(s)) : s \in (-\delta_\ell, \delta_\ell) \right\} \subset E_1 \times (0, \infty)$$

consists of solutions to the bifurcation equation (6.5). The regularity of the curve follows from the regularity of the bifurcating branch in Theorem 6.2 and regularity of the mapping ψ. By definition of the bifurcation function $G(\cdot, \lambda)$, it follows that the family (6.6) is indeed equilibria of the original equation (1.3) which intersect the family of cylinders at $r_\star = 1/\ell$, when $s = 0$. Meanwhile, the bifurcation parameter λ remains unchanged in lifting from the reduced problem to the full problem, and hence we see that

$$\dot{\lambda}_\ell(0) = 0 \quad \text{and} \quad \ddot{\lambda}_\ell(0) < 0$$

from Theorem 6.3, and so we conclude that the given curve bifurcates subcritically.
(b) By Remark 3(f) it follows that ψ preserves the symmetry of even functions on $\mathbb{T}$, and since $\tilde{\rho}_\ell(s) \in F_{1,\text{even}}$, it follows that the functions in the family (6.6) are even on $\mathbb{T}$. Meanwhile, by the characterization of equilibria established in section 3, and the fact that

$$\psi(\tilde{\rho}_\ell(0)) + 1/\lambda_\ell(0) = 1/\ell = R(0, \ell),$$

it follows that the family (6.6) must coincide with the family of $2\pi/\ell$-periodic undulary curves $R(B, \ell)$ for some continuum of values $B \in (-\varepsilon_\ell, \varepsilon_\ell)$.
(c) To prove that the unduloids (6.6) are unstable, we mimic the proof of Theorem 5.1 in the current setting. In particular, define

$$G_\ell(\rho, s) := G(\rho + \psi(\tilde{\rho}_\ell(s)), \lambda_\ell(s)), \qquad \text{and}$$
$$L_\ell(s) := D_1 G_\ell(0, s) = D_1 G(\psi(\tilde{\rho}_\ell(s)), \lambda_\ell(s)),$$

acting on functions $\rho \in E_1$. It follows by Theorem 6.3 and Proposition 6.4 that

$$\sigma(L_\ell(s)) \cap [\text{Re } z > 0] \neq \emptyset,$$

provided $|s| > 0$ is chosen sufficiently small. Meanwhile, the operator $G_\ell(\cdot, s)$ has a similar quasilinear structure as $\mathcal{G}_\star$ and so the analogue to inequality (5.2) is also derived for

$$g_\ell(\rho, s) := G_\ell(\rho, s) - L_\ell(s)\rho.$$

Utilizing [40, Proposition I.7.2] and the explicit characterization (4.16) of the spectra $\sigma(DG_\star(\eta))$, we can control the eigenvalues of the perturbed linearization $L_\ell(s)$, so that, for sufficiently small values of $|s| > 0$, we can derive the necessary *spectral gap* condition

$$[\omega - \gamma \le \operatorname{Re} z \le \omega + \gamma] \cap \sigma(L_\ell(s)) = \emptyset \quad \text{and} \quad \sigma_+ := [\operatorname{Re} z > \omega + \gamma] \cap \sigma(L_\ell(s)) \neq \emptyset$$

for some $\gamma, \omega > 0$. The remainder of the proof now follows as in the proof of Theorem 5.1 with the observation that $-L_\ell(s)$ satisfies maximal regularity properties, which follows by uniform ellipticity of $L_\ell(s)$ and an argument similar to the proof of the stated claim in the proof of Lemma 2.1 in the appendix. $\square$

Remark 10. Note that we only prove nonlinear instability for unduloids with sufficiently small parameter values $|B| > 0$. Relying on previous results in [9, 66, 67] for the stationary *trapped drop* capillary problem, the authors of [10] observe that the linearized problem at nontrivial unduloids always has an unstable eigenvalue. Applying this observation in our setting, we can in fact conclude nonlinear instability of the entire family of nontrivial unduloids $R(B, k), |B| \in (0, 1), k \in \mathbb{N}$.

7. Appendix. In this section we outline the proofs of Lemma 2.1 and Propositions 2.2 and 2.4; see [47] for more details.

Proof of Lemma 2.1. Fix $\mu \in [1/2, 1]$ as indicated.

Claim. $\mathcal{A}(\rho) \in \mathcal{MR}_\nu(E_1, E_0)$ for $\rho \in V_\mu$, $\nu \in (0, 1]$. This claim will follow from [46], though the setting of that paper differs slightly from the current setting and warrants a brief discussion. First, for $\rho \in V_\mu$ define the coefficients

$$b_4(\rho) := \frac{1}{(1 + \rho_x^2)^2} \quad \text{and} \quad b_3(\rho) := \frac{2\rho_x(1 + \rho_x^2 - 3\rho\rho_{xx})}{\rho(1 + \rho_x^2)^3},$$

so that $\mathcal{A}(\rho) = b_4(\rho)\,\partial_x^4 + b_3(\rho)\,\partial_x^3$. By our choice of μ, it follows that $V_\mu \subset h^{2+\alpha}(\mathbb{T}, \mathbb{R})$, so that $b_4, b_3 \in E_0$ and $\mathcal{A}(\rho)$ is a uniformly elliptic differential operator. By [46, Theorem 5.2] we conclude that

$$\mathcal{A}(\rho) \in \mathcal{MR}_\nu\big(h^{4+\alpha}(\mathbb{T}, \mathbb{C}), h^\alpha(\mathbb{T}, \mathbb{C})\big), \qquad \nu \in (0, 1],$$

where we utilize the notation $h^{k+\alpha}(\mathbb{T}, \mathbb{C})$ to be clear that the space consists of $\mathbb{C}$-valued functions over $\mathbb{T}$ and does not coincide with the spaces E_μ being considered herein. However, $h^{k+\alpha}(\mathbb{T}, \mathbb{C})$ does coincide with the complexification of $h^{k+\alpha}(\mathbb{T}, \mathbb{R})$ (up to equivalent norms), and it is a straightforward exercise to see that the property of maximal regularity continues to hold under restriction to the subspaces $h^\sigma(\mathbb{T}, \mathbb{R})$.

The regularity assertion for $(\mathcal{A}, f)$ follows from the fact that the mappings

$$[r \mapsto 1/r] : V_0 \to E_0, \ [r \mapsto r_x] : h^{\sigma+1}(\mathbb{T}) \to h^\sigma(\mathbb{T}), \ [(r, s) \mapsto rs] : E_0 \times E_0 \to E_0$$

are real analytic, the additional observation that the mapping $\mathcal{A} : V_\mu \to \mathcal{L}(E_1, E_0)$ inherits the regularity of the coefficients $b_3, b_4 : V_\mu \to E_0$, and the fact that $\mathcal{MR}_\mu(E_1, E_0)$ is an open subset of $\mathcal{L}(E_0, E_1)$; cf. [12, Lemma 2.5(a)]. $\square$

Proof of Proposition 2.2. In case $\mu \in [1/2, 1)$, the result follows from Lemma 2.1 and [12, Theorems 4.1, 5.1, and 6.1]. When $\mu = 1$ we note that the existence and uniqueness of a maximal solution

$$r(\cdot, r_0) \in C^1(J(r_0), E_0) \cap C(J(r_0), E_1)$$

follows from [12, Theorem 4.1(b)]. However, for the semiflow properties, we will consider (1.3) as a fully nonlinear equation and apply results of Angenent [5]. In particular, for $r \in V_1$ we use the representation $G(r) = -\mathcal{A}(r)r + f(r)$ and (2.2)–(2.3) to see that the Fréchet derivative DG has the structure

$$DG(r) = -\frac{1}{(1 + r_x^2)^2} \partial_x^4 + \sum_{k=0}^{3} B_k(r) \partial_x^k,$$

where the coefficients $B_k(r) \in E_0$ for every $r \in V_1$, $k = 0, \ldots, 3$. From this computation it follows that $-DG(r)$ is a uniformly elliptic operator from E_1 to E_0, and so, using the results of [46] as in the highlighted claim in the proof of Lemma 2.1 above, we see that $-DG(r) \in \mathcal{MR}_1(E_1, E_0)$ for all $r \in V_1$. Now the fact that (1.3) generates an analytic semiflow on V_1 follows from [5, Corollary 2.9]. $\quad\square$

Proof of Proposition 2.4. For $a \in \mathbb{R}$ let $T_a : \mathbb{T} \to \mathbb{T}$ be the translation operator, where $T_a(x)$ denotes the unique element in $\mathbb{T}$ that is in the coset $[x + a] \in \mathbb{R}/2\pi\mathbb{Z}$ of $(x + a)$. T_a naturally acts on functions $u \in C(\mathbb{T}, \mathbb{R})$ by virtue of $(T_a u)(x) := u(T_a(x))$. As in [28] one shows that, for $a \in \mathbb{R}$, the family of translations $\{T_{ta} : t \in \mathbb{R}\}$ induces a strongly continuous group of contractions on any of the spaces E_μ, with infinitesimal generator A_a given by

$$D(A_a) = h^{1+4\mu+\alpha}(\mathbb{T}, \mathbb{R}), \qquad A_a = a\partial_x.$$

Let $r_0 \in V_\mu$ be fixed, and let

$$r = r(\cdot, r_0) \in C_{1-\mu}^1(J(r_0), E_0) \cap C_{1-\mu}(J(r_0), E_1)$$

be the unique solution to (2.1) on the maximal interval of existence $J(r_0) = [0, t^+(r_0))$. Let $t_1 \in (0, t^+(r_0))$ be fixed and set $I := [0, t_1]$. Then there exists $\delta > 0$ such that $(1 + \lambda)t \in J(r_0)$ for all $(t, \lambda) \in I \times (-\delta, \delta)$. Finally, for $(\lambda, a) \in W := (-\delta, \delta)^2$ we set

$$r_{\lambda,a}(t) := T_{ta} r((1 + \lambda)t), \qquad t \in I;$$

i.e., $r_{\lambda,a}(t, x) = r((1 + \lambda)t, T_{ta}(x))$ for $(t, x) \in I \times \mathbb{T}$. One verifies that

$$r_{\lambda,a} \in \mathbb{E}_1(I) := BUC_{1-\mu}^1(I, E_0) \cap BUC_{1-\mu}(I, E_1).$$

Moreover, since the nonlinear mapping $[r \mapsto G(r)]$ is equivariant with respect to translations, i.e., $T_b G(r) = G(T_b r)$ for any $b \in \mathbb{R}$, we obtain that $r_{\lambda,a}$ is a solution of the parameter-dependent equation

(7.1)
$$\begin{cases} \partial_t v = (1 + \lambda)G(v) + a\partial_x v, & t > 0, \\ v(0) = r_0, \end{cases}$$

on the time interval I.

Now, for $\mathbb{U}(I) := \mathbb{E}_1(I) \cap C(I, V)$ we define

$$\Phi : \mathbb{U}(I) \times W \to \mathbb{E}_0(I) \times E_\mu, \quad \Phi(v, (\lambda, a)) = \big(\partial_t v - (1 + \lambda)G(v) - a\partial_x v, \, \gamma v - r_0\big),$$

where $\mathbb{E}_0(I) := BUC_{1-\mu}(I, E_0)$, and we note that $\Phi(r_{\lambda,a}, (\lambda, a)) = (0, 0)$. Moreover,

$$\Phi \in C^\omega\Big(\mathbb{U}(I) \times W, \mathbb{E}_0(I) \times E_\mu\Big), \quad D_1\Phi(r, (0, 0)) = \Big(\frac{d}{dt} - DG(r), \gamma\Big),$$

where we use the same notation for $r = r(\cdot, r_0)$ and its restriction to the time interval I. Exactly as in the proof of [12, Theorem 6.1] one shows that

$$D_1\Phi(r, (0,0)) \in \mathcal{L}_{isom}(\mathbb{E}_1(I), \mathbb{E}_0(I) \times E_\mu).$$

Finally, according to the implicit function theorem (cf. [20, Theorem 15.3] or [22, (10.2.1)]), there exist a neighborhood of r in $\mathbb{E}_1(I)$ and a neighborhood of $(0,0)$ in $\mathbb{R}^2$, which we will again denote by $\mathbb{U}(I)$ and W, respectively, and a mapping $g \in C^\omega(W, \mathbb{E}_1(I))$ such that

$$\Phi(v, (\lambda, a)) = (0,0) \quad \text{if and only if} \quad v = g(\lambda, a)$$

whenever $(v, (\lambda, a)) \in \mathbb{U}(I) \times W$. We conclude that $g(\lambda, a) = r_{\lambda, a}$ and

$$(7.2) \qquad\qquad [(\lambda, a) \mapsto r_{\lambda, a}] \in C^\omega(W, \mathbb{U}(I)).$$

For $t_0 \in (0, t_1)$ and $x_0 \in \mathbb{T}$ fixed, we see that

$$(7.3) \qquad\qquad [(\lambda, a) \mapsto r((1+\lambda)t_0, T_{t_0 a}(x_0))] \in C^\omega(W, \mathbb{R}),$$

and the assertion follows since (t_0, x_0) can be chosen arbitrarily. $\qquad\square$

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