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PROOF OF THE BARKER ARRAY CONJECTURE

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ABSTRACT. Using only elementary methods, we prove Alquaddoomi and Scholtz's conjecture of 1989, that no $s \times t$ Barker array having $s, t > 1$ exists except when $s = t = 2$.

1. INTRODUCTION

Binary sequences and arrays whose out-of-phase aperiodic autocorrelations are collectively small are particularly useful in digital communication systems, especially synchronisation and radar. The search for such sequences and arrays dates from the 1950s [2], [16] and continues to the present day [7], [9], [13], [14]. We define an $s \times t$ *array* to be a two-dimensional array (a_{ij}) of complex-valued elements satisfying

$$a_{ij} = 0 \text{ unless } 0 \le i < s \text{ and } 0 \le j < t.$$

The array is *binary* if all nonzero elements a_{ij} take values in $\{1, -1\}$. The *aperiodic autocorrelation function* of an $s \times t$ array $A = (a_{ij})$ is given by

$$C_A(u, v) = \sum_i \sum_j a_{ij}\overline{a_{i+u,j+v}} \text{ for integer } u,\ v \text{ satisfying } |u| < s \text{ and } |v| < t.$$

We refer to an $s \times 1$ array as a *sequence of length* s, abbreviating the array (a_{i0}) to (a_i) and its aperiodic autocorrelation function $C_A(u, 0)$ to $C_A(u)$.

Alquaddoomi and Scholtz [1] defined an $s \times t$ *Barker array* to be an $s \times t$ binary array A for which

$$|C_A(u, v)| \le 1 \text{ for all } (u, v) \ne (0, 0).$$

This generalises the notion of a *Barker sequence* from one dimension (the case $s = 1$ or $t = 1$) to two dimensions; see [10] and [11] for recent nonexistence results for Barker sequences. The 2×2 array $\left[\begin{smallmatrix} + & + \\ + & - \end{smallmatrix}\right]$ is a Barker array, but it is conjectured that there are no other sizes for a (truly two-dimensional) Barker array.

Conjecture 1.1 (Alquaddoomi and Scholtz [1]). If an $s \times t$ Barker array exists for $s, t > 1$, then $s = t = 2$.

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In this paper we prove Conjecture 1.1 using only elementary methods. We include short proofs of key auxiliary results obtained elsewhere, in order to make the paper self-contained. Theorem 1.2 summarises the previous state of knowledge regarding Conjecture 1.1.

Theorem 1.2 (Jedwab [6], Jedwab, Lloyd and Mowbray [8]). *Let A be an $s \times t$ Barker array with $s, t > 1$. Then*

Case 1. s, t even: $s = t$. *If* $t > 2$, *then* $t \equiv 0 \pmod 4$ *and* $t \geq 12$.
Case 2. s even, $t > 1$ odd: $s > t$. $s = 4S^2$ *and* $t = T^2$ *for integers* S, T. *There exists a Barker sequence of length* s.
Case 3. $s, t > 1$ odd: $st \geq 3^{11}$. *Write* $t = \prod_j p_j^{\alpha_j}$, *where the* $\{p_j\}$ *are distinct primes and* $\alpha_j \geq 1$ *for all* j. *Then* $\alpha_j \geq 2$ *for all* j *and* $\alpha_k > 2$ *for some* k. *If* $st \equiv 1 \pmod 4$, *then* $p_j \equiv 1 \pmod 4$ *for all* j.

Following [1], define the following function for an $s \times t$ array $A = (a_{ij})$:

$$P_A(u, v) = C_A(u, v) + C_A(u, v - t) \quad \text{for } -s < u < s \text{ and } 0 \leq v < t. \tag{1.1}$$

Any expression involving $P_A(u, v)$ or $C_A(u, v)$ will implicitly refer only to values of (u, v) for which the function is defined. In terms of the array elements a_{ij} we have

$$P_A(u, v) = \sum_i \sum_{j=0}^{t-1} a_{ij} \overline{a_{i+u,(j+v) \bmod t}}. \tag{1.2}$$

Alquaddoomi and Scholtz [1] established Lemma 1.3 for binary arrays, and then used it to prove Proposition 1.4 for Barker arrays. This generalised the approach taken by Tuyrn and Storer in their classical paper [15] on the one-dimensional (sequence) case.

Lemma 1.3 (Alquaddoomi and Scholtz [1]). *Let A be an $s \times t$ binary array. Then*

$$P_A(u, v) \equiv P_A(u, v') \pmod 4 \quad \textit{for all } (u, v, v').$$

Proof. Let u, v, v' satisfy $-s < u < s$ and $0 \leq v, v' < t$. From (1.2), $P_A(u, v)$ is the sum of $(s - |u|)t$ nonzero terms, of which exactly $[(s - |u|)t - P_A(u, v)]/2$ are -1 and $[(s - |u|)t + P_A(u, v)]/2$ are $+1$. But from (1.2), the product of these nonzero terms is independent of v. Therefore

$$(-1)^{[(s-|u|)t - P_A(u,v)]/2}$$

is independent of v, which implies $P_A(u, v) \equiv P_A(u, v') \pmod 4$. □

Proposition 1.4 (Alquaddoomi and Scholtz [1]). *Let A be an $s \times t$ Barker array with $st > 2$. Then*

Case 1. s, t even:

$$P_A(u, v) = 0 \quad \textit{for } (u, v) \neq (0, 0).$$

Case 2. s even and t odd:

$$\begin{aligned} P_{A^T}(v, u) &= 0 \quad \textit{for } (u, v) \neq (0, 0), \\ P_A(u, v) &= \begin{cases} 0 & \textit{for } u \textit{ even and } (u, v) \neq (0, 0), \\ k(u) & \textit{for } u \textit{ odd,} \end{cases} \end{aligned}$$

where $k(u) = 1$ *or* -1.

Case 3. s, t odd:

$$P_A(u,v) = \begin{cases} k & \textit{for } u \textit{ even and } (u,v) \neq (0,0), \\ 0 & \textit{for } u \textit{ odd,} \end{cases}$$

where $k = 1$ *or* -1.

Proof. For all u, v satisfying $|u| < s$ and $|v| < t$, $C_A(u,v)$ is the sum of $(s - |u|) \times (t - |v|)$ nonzero terms, each of which is ± 1. Therefore $C_A(u,v) \equiv (s+u)(t+v) \pmod 2$. The Barker array property then implies

(1.3) $$C_A(u,v) = \pm(((s+u)(t+v)) \bmod 2) \quad \text{for } (u,v) \neq (0,0).$$

Case 1. *s, t even:* From (1.3) we have

$$C_A(u,v) = 0 \quad \text{for } u \text{ or } v \text{ even and } (u,v) \neq (0,0).$$

Then by (1.1),

$$P_A(u,v) = 0 \quad \text{for } u \text{ or } v \text{ even and } (u,v) \neq (0,0).$$

Lemma 1.3 then implies that

$$P_A(u,v) = 0 \quad \text{for } (u,v) \neq (0,0).$$

Case 2. *s even, t odd:* From (1.3) we have

(1.4) $$C_A(u,v) = \pm((u(1+v)) \bmod 2) \quad \text{for } (u,v) \neq (0,0).$$

It follows from (1.1) that

$$P_A(u,v) = \begin{cases} 0 & \text{for } u \text{ even and } (u,v) \neq (0,0), \\ \pm 1 & \text{for } u \text{ odd.} \end{cases}$$

Lemma 1.3 then implies that

(1.5) $$P_A(u,v) = \begin{cases} 0 & \text{for } u \text{ even and } (u,v) \neq (0,0), \\ k(u) & \text{for } u \text{ odd,} \end{cases}$$

where $k(u) = 1$ or -1, as required.

We next consider the function

(1.6) $$P_{A^T}(v,u) = C_A(u,v) + C_A(u-s,v).$$

From (1.4),

(1.7) $$P_{A^T}(v,u) = 0 \quad \text{for } u \text{ even and } (u,v) \neq (0,0).$$

Lemma 1.3 applied to A^T states that

$$P_{A^T}(v,u) \equiv P_{A^T}(v,u') \pmod 4 \quad \text{for all } (u,u',v),$$

giving

(1.8) $$P_{A^T}(v,u) = 0 \quad \text{for } (u,v) \neq (0,0), \text{ except when } s = 2 \text{ and } (u,v) = (1,0)$$

(since, when $s = 2$ and $v = 0$, there is no value of u satisfying the conditions of (1.7)).

To complete the proof of Case 2, we now derive a contradiction for the case $s = 2$, so that (1.8) holds without exception. By assumption $st > 2$ and $s = 2$, so $t > 1$ and we can choose an even value of v satisfying $0 < v < t$. From (1.5),

$$k(1) = P_A(1,v) = P_A(1,t-v),$$

and so from (1.1) and (1.4),

$$\pm 1 = C_A(1, v) = C_A(1, -v). \tag{1.9}$$

But by (1.8), $P_{A^T}(v, 1) = 0$, and so from (1.6) we get

$$\begin{aligned} 0 &= C_A(1, v) + C_A(-1, v) \\ &= C_A(1, v) + C_A(1, -v) \end{aligned}$$

since $C_A(u, v) = C_A(-u, -v)$ for all u, v. This contradicts (1.9).
Case 3. *s, t odd:* From (1.3) we have

$$C_A(u, v) = \pm(((1 + u)(1 + v)) \bmod 2) \quad \text{for } (u, v) \neq (0, 0).$$

Then by (1.1),

$$P_A(u, v) = \begin{cases} \pm 1 & \text{for } u \text{ even and } (u, v) \neq (0, 0), \\ 0 & \text{for } u \text{ odd.} \end{cases}$$

Lemma 1.3 then implies that

$$P_A(u, v) = \begin{cases} k(u) & \text{for } u \text{ even and } (u, v) \neq (0, 0), \\ 0 & \text{for } u \text{ odd,} \end{cases}$$

where $k(u) = 1$ or -1. By symmetry in s and t we also obtain

$$P_{A^T}(v, u) = \begin{cases} k'(v) & \text{for } v \text{ even and } (u, v) \neq (0, 0), \\ 0 & \text{for } v \text{ odd,} \end{cases}$$

where $k'(v) = 1$ or -1. But, for u, v even and $(u, v) \neq (0, 0)$, by (1.3) the single nonzero contribution to $P_A(u, v) = C_A(u, v) + C_A(u, v - t)$ and to $P_{A^T}(v, u) = C_A(u, v) + C_A(u - s, v)$ is the same term $C(u, v)$, and so $k(u) = k'(v) = k$. □

Proposition 1.4 is implied by Theorem 2 and (21)–(23) of [1]. Lemma 3.5 of [6] shows that an $s \times t$ binary array A having $P_A(u, v) = 0$ for all $(u, v) \neq (0, 0)$ is equivalent to A being simultaneously a perfect binary array and a "quasiperfect" binary array. This in turn is equivalent to the -1 elements of A corresponding to a $(4N^2, 2N^2 - N, N^2 - N)$-difference set in $\mathbb{Z}_s \times \mathbb{Z}_t$, where $st = 4N^2$ (see [4], for example), and the -1 elements of $\left[\begin{smallmatrix} A \\ -A \end{smallmatrix}\right]$ corresponding to an $(st, 2, st, st/2)$ relative difference set in $\mathbb{Z}_{2s} \times \mathbb{Z}_t = \langle x \rangle \times \langle y \rangle$, where $x^{2s} = y^t = 1$, relative to $\langle x^s \rangle$ (see [17]). See [3] or [12] for a background on difference sets and relative difference sets.

2. Proof of the conjecture

We begin with two lemmas.

Lemma 2.1. *Let $A = (a_{ij})$ be an $s \times t$ binary array and let ζ be a (not necessarily primitive) t^{th} root of unity. Let $X = (x_i)$ be the complex-valued sequence of length s given by*

$$x_i = \sum_j a_{ij} \zeta^j. \tag{2.1}$$

Then

$$C_X(u) = \sum_{v=0}^{t-1} P_A(u, v) \zeta^{-v} \quad \textit{for all } u.$$

Proof. From (1.2), for all u,

$$\begin{aligned}\sum_{v=0}^{t-1} P_A(u,v)\zeta^{-v} &= \sum_{v=0}^{t-1}\sum_i\sum_j a_{ij}\overline{a_{i+u,(j+v) \bmod t}}\zeta^{-v} \\ &= \sum_i\sum_j a_{ij}\sum_{k=0}^{t-1}\overline{a_{i+u,k}}\zeta^{j-k},\end{aligned}$$

writing $k = (j+v) \bmod t$ and using $\zeta^t = 1$. Hence, for all u,

$$\begin{aligned}\sum_{v=0}^{t-1} P_A(u,v)\zeta^{-v} &= \sum_i\sum_j a_{ij}\zeta^j\overline{\sum_k a_{i+u,k}\zeta^k} \\ &= \sum_i x_i\overline{x_{i+u}} \\ &= C_X(u),\end{aligned}$$

as required. □

Lemma 2.2. *Let $X = (x_i)$ be a complex-valued sequence of length s for which*

$$C_X(u) = 0 \quad \textit{for } u \neq 0.$$

Then, for some I satisfying $0 \le I < s$,

$$|x_i|^2 = \begin{cases} 0 & \textit{for } i \neq I, \\ C_X(0) & \textit{for } i = I. \end{cases}$$

Proof. By the definition of aperiodic autocorrelation, we are given that

$$\sum_i x_i\overline{x_{i+u}} = 0 \quad \text{for } 0 < u < s. \tag{2.2}$$

We prove by induction on s that, for some I satisfying $0 \le I < s$,

$$|x_i|^2 = 0 \quad \text{for } i \neq I.$$

The case $s = 1$ is immediate (take $I = 0$). Assume case $s-1$ to be true. Put $u = s-1$ in (2.2) to give $x_0\overline{x_{s-1}} = 0$. This implies, without loss of generality, that $x_{s-1} = 0$. Then from (2.2) we have

$$\sum_{i=0}^{s-u-2} x_i\overline{x_{i+u}} = 0 \quad \text{for } 0 < u < s-1.$$

By the inductive hypothesis it follows that, for some I satisfying $0 \le I < s-1$, $|x_i|^2 = 0$ for $i \neq I$. Combining this with $x_{s-1} = 0$ gives the case s, completing the induction.

Furthermore, by the definition of aperiodic autocorrelation, $C_X(0) = \sum_i |x_i|^2$, and so $C_X(0) = |x_I|^2$, as required. □

The case $\zeta = 1$ of Lemma 2.1 was used as a starting point in [5], [6] and [8] to derive equations in the row sums $\sum_j a_{ij}$ of an $s \times t$ Barker array from Proposition 1.4, eventually leading to Theorem 1.2. We will now use the case where ζ is a primitive t^{th} root of unity to prove Conjecture 1.1.

Theorem 2.3. *If an $s \times t$ Barker array $A = (a_{ij})$ exists for $s, t > 1$, then $s = t = 2$.*

Proof. Let ζ be a primitive t^{th} root of unity and define $X = (x_i)$ as in (2.1). We will show that the case s, t even forces the result $s = t = 2$, whereas the case s even, t odd and the case s, t odd both result in a contradiction. These three cases are exhaustive, because the transpose of a Barker array is also a Barker array.

Case 1. *s, t even:* Proposition 1.4 and Lemma 2.1 together give

$$C_X(u) = \begin{cases} 0 & \text{for } u \neq 0, \\ st & \text{for } u = 0, \end{cases}$$

using $P_A(0,0) = C(0,0) = st$. Then by Lemma 2.2 there is some I satisfying $0 \le I < s$ for which

$$|x_I|^2 = st. \tag{2.3}$$

But by (2.1),

$$\begin{aligned} |x_I|^2 &= \left| \sum_{j=0}^{t-1} a_{Ij}\zeta^j \right|^2 \\ &\le \left(\sum_{j=0}^{t-1} |a_{Ij}\zeta^j| \right)^2 \\ &= t^2. \end{aligned}$$

It follows from (2.3) that

$$s \le t, \text{ with equality} \Leftrightarrow \arg(a_{Ij}\zeta^j) \text{ is constant for all } j \text{ satisfying } 0 \le j < t. \tag{2.4}$$

Since s is even, by symmetry in s and t (or equivalently by applying the same procedure to A^T) we have $t \le s$, forcing equality. Therefore $s = t$ and, since $t > 1$, by (2.4) we have $t = 2$.

Case 2. *s even, $t > 1$ odd:* By Proposition 1.4, the $t \times s$ array A^T satisfies

$$P_{A^T}(v, u) = 0 \ \text{ for } (u, v) \neq (0, 0).$$

The argument of Case 1 that led to (2.4), when applied to A^T, gives $t \le s$. Furthermore the expression for P_A in Proposition 1.4, together with Lemma 2.1, gives

$$\begin{aligned} C_X(u) &= \begin{cases} 0 & \text{for } u \text{ even and } u \neq 0, \\ k(u)\sum_{v=0}^{t-1}\zeta^{-v} & \text{for } u \text{ odd}, \\ st & \text{for } u = 0 \end{cases} \\ &= \begin{cases} 0 & \text{for } u \neq 0, \\ st & \text{for } u = 0, \end{cases} \end{aligned}$$

since ζ^{-1} is a primitive t^{th} root of unity and $t > 1$. By Lemma 2.2 we then obtain $s \le t$, by the same argument as in Case 1. Since we already have $t \le s$ this implies $s = t$, which contradicts the assumption that s is even and t is odd.

Case 3. $s, t > 1$ *odd:* Proposition 1.4 and Lemma 2.1 together give

$$\begin{aligned} C_X(u) &= \begin{cases} k\sum_{v=0}^{t-1}\zeta^{-v} & \text{for } u \text{ even and } u \neq 0, \\ 0 & \text{for } u \text{ odd}, \\ st + k\sum_{v=1}^{t-1}\zeta^{-v} & \text{for } u = 0 \end{cases} \\ &= \begin{cases} 0 & \text{for } u \neq 0, \\ st - k & \text{for } u = 0, \end{cases} \end{aligned}$$

where $k = 1$ or -1. Then by Lemma 2.2 there is some I satisfying $0 \leq I < s$ for which

$$|x_I|^2 = st - k. \tag{2.5}$$

But, as in Case 1, $|x_I|^2 \leq t^2$ and so

$$st - k \leq t^2.$$

By symmetry in s and t we then have

$$st - k \leq \min\{s^2, t^2\}. \tag{2.6}$$

Suppose, for a contradiction, that $s \neq t$ and without loss of generality that $s \geq t + 1$. Then $st - k \geq t(t+1) - k > t^2$, since $k = 1$ or -1 and $t > 1$. This contradicts (2.6), and so $s = t$.

Then (2.6) forces $k = 1$, and from (2.1) and (2.5) we have

$$\left|\sum_{j=0}^{t-1} a_{Ij}\zeta^j\right|^2 = t^2 - 1. \tag{2.7}$$

Since t is odd, one of the sets $\{j : a_{Ij} = 1\}$ and $\{j : a_{Ij} = -1\}$ contains at most $(t-1)/2$ elements; without loss of generality, suppose it is the former. This implies that

$$\begin{aligned} \left|\sum_{j=0}^{t-1} a_{Ij}\zeta^j\right|^2 &= \left|\sum_{j=0}^{t-1} a_{Ij}\zeta^j + \sum_{j=0}^{t-1}\zeta^j\right|^2 \\ &= \left|2\sum_{j:\, a_{Ij}=1}\zeta^j\right|^2 \\ &\leq 4\left(\sum_{j:\, a_{Ij}=1}|\zeta^j|\right)^2 \\ &\leq 4\left(\frac{t-1}{2}\right)^2 \\ &< t^2 - 1, \end{aligned}$$

since $t > 1$. This contradicts (2.7).

□

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