1-7-1993

Nonexistence of Certain Perfect Binary Arrays

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Recommended Citation
Jedwab, Jonathan, and James A. Davis. "Nonexistence of Certain Perfect Binary Arrays." *Electronics Letters* 29, no. 1 (January 7, 1993): 99-101. doi: 10.1049/el:19930065.

NONEXISTENCE OF CERTAIN PERFECT BINARY ARRAYS

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Indexing terms: Binary sequences, Information theory, Signal processing

A perfect binary array (PBA) is an r-dimensional matrix with elements ± 1 such that all out-of-phase periodic autocorrelation coefficients are zero. The two smallest sizes for which the existence of a PBA is undecided, $2 \times 2 \times 3 \times 3 \times 9$ and $4 \times 3 \times 3 \times 9$, are ruled out using computer search and a combinatorial argument.

Introduction: An $s_1 \times \ldots \times s_r$ perfect array is a matrix $(a[j_1, \ldots, j_r])$, $0 \leq j_i < s_i$, with integer elements such that the periodic autocorrelation coefficients

$$R(u_1, \ldots, u_r) = \sum_{j_1=0}^{s_1-1} \ldots \sum_{j_r=0}^{s_r-1} a[j_1, \ldots, j_r] \times a[(j_1 + u_1) \bmod s_1, \ldots, (j_r + u_r) \bmod s_r]$$

are zero for all $(u_1, \ldots, u_r) \neq (0, \ldots, 0)$, $0 \leq u_i \leq s_i$. The array is binary if each matrix element is ± 1, and then $\prod_i s_i = 4N^2$ for some integer N. The energy and sum of A are, respectively, $E = \sum_{j_1=0}^{s_1-1} \ldots \sum_{j_r=0}^{s_r-1} (a[j_1, \ldots, j_r])^2$ and $S = \sum_{j_1=0}^{s_1-1} \ldots \sum_{j_r=0}^{s_r-1} a[j_1, \ldots, j_r]$. See Reference 1 for a general background on perfect arrays and Reference 2 for details of their many uses in signal processing applications.

Perfect binary arrays (PBAs) have recently been constructed [3] with size $2^{a_1} \times \ldots \times 2^{a_u} \times 3^{b_1} \times 3^{b_1} \times \ldots \times 3^{b_r} \times 3^{b_r}$, where $\sum_i a_i = 2a + 2$ and the a_i satisfy the Turyn exponent bound $a_i \leq a + 2$ for all i [4, 5]. No other sizes of PBA are known, and there are many nonexistence results [1] when N does not take the form $2^a 3^b$. The smallest sizes for which the existence of a PBA is currently undecided [3], [6, property 3.5.1] are $2 \times 2 \times 3 \times 3 \times 9$ and $4 \times 3 \times 3 \times 9$ (equivalently [2], sizes $6 \times 6 \times 9$ and $3 \times 9 \times 12$).

Given a perfect array we can form another perfect array by summing every tth array element along any dimension whose size is a multiple of t [7, lemma 2.4]:

(i) *Lemma 1:* Let $(a[j, j_1, \ldots, j_r])$ be an $st \times s_1 \times \ldots \times s_r$ array. Define the $t \times s_1 \times \ldots \times s_r$ array $B = (b[j, j_1, \ldots, j_r])$ by $b[j, j_1, \ldots, j_r] = \sum_{l=0}^{s-1} a[j + lt, j_1, \ldots, j_r]$ for all $0 \leq j < t$, $0 \leq j_i < s_i$. If A is perfect with energy E and sum S then so is B.

A perfect array is invariant under the shearing transformation [7, lemma 8.1]:

(ii) *Lemma 2:* Let $A = (a[l, j, j_1, \ldots, j_r])$ be an $s \times t \times s_1 \times \ldots \times s_r$ array and let c be an integer such that $ct \equiv 0 (\bmod s)$. Define the $s \times t \times s_1 \times \ldots \times s_r$ array $B = (b[l, j, j_1, \ldots, j_r])$ by $b[l, j, j_1, \ldots, j_r] = a[(l - cj) \bmod s, j, j_1, \ldots, j_r]$ for all $0 \leq l < s$, $0 \leq j < t$, $0 \leq j_i < s_i$. A is perfect (with energy E and sum S) if and only if B is.

For example, the 3×3 array

$$A = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -4 & -4 \\ 2 & -4 & -4 \end{bmatrix}$$

is perfect and therefore so is

$$B = \begin{bmatrix} -1 & -4 & -4 \\ 2 & 2 & -4 \\ 2 & -4 & 2 \end{bmatrix}$$

In this Letter we use computer search to determine all possible $9 \times 2 \times 2$ perfect arrays that could arise by summing the nine elements in each 3×3 'slice' of a $9 \times 2 \times 2 \times 3 \times 3$ PBA. We then apply various shearing transformations to the $9 \times 2 \times 2$ perfect arrays to show that the underlying PBA cannot exist. A similar procedure rules out the existence of a $9 \times 4 \times 3 \times 3$ PBA. Although search methods of this sort have been used previously for arrays [8] and for difference sets [9], to our knowledge the additional argument involving shearing transformations is new.

Search for 9 × 2 × 2 perfect arrays: Suppose there exists a 9 × 2 × 2 × 3 × 3 PBA, say $A = (a[i, j, k, l, m])$; then A has energy 324 and (changing the sign of all array elements if necessary) sum 18. Define the arrays $B = (b[i, j, k])$, $C = (c[i, j])$, $C' = (c'[i, k])$ and $D = (d[i])$ of sizes 9 × 2 × 2, 9 × 2, 9 × 2 and 9, respectively by

$$b[i, j, k] = \sum_{l=0}^{2} \sum_{m=0}^{2} a[i, j, k, l, m] \quad (1)$$

$$c[i, j] = b[i, j, 0] + b[i, j, 1] \quad (2)$$

$$c'[i, k] = b[i, 0, k] + b[i, 1, k] \quad (3)$$

$$d[i] = c[i, 0] + c[i, 1] = c'[i, 0] + c'[i, 1] \quad (4)$$

By Lemma 1, each of B, C, C' and D is a perfect array with energy 324. This energy constraint and eqns. 1–4 imply that

$$b[i, j, k] \in \{-9, -7, \ldots, 7, 9\}$$
$$c[i, j], c'[i, k], d[i] \in \{-18, -16, \ldots, 16, 18\} \quad (5)$$

for all i, j, k. We may assume, by translation if necessary, that

$$|d[0]| = \max_i |d[i]| \quad (6)$$

The first stage of the search is to find all possible arrays D satisfying the above conditions (it is computationally infeasible to determine the arrays B directly). The algorithm recursively fixes the value of $d[i]$ for successive values of i, taking all possibilities for $d[i]$ satisfying eqns. 5 and 6 and backtracking if the cumulative sum of squares exceeds 324. Because the array sum is 18, when $i = 8$ we have the additional constraint $d[8] = 18 - \sum_{i=0}^{7} d[i]$. Of those arrays surviving this process, only those that are perfect are retained. After exclusion of translations and reflections, there remain eight arrays D.

The second stage is to search in a similar way for all pos-·· e arrays C which satisfy eqn. 4 for some array D from the st stage. Taking each array D in turn, we recursively fix the alue of $c[i, 0]$ and $c[i, 1]$ subject to eqns. 4 and 5. It is easy to see that a perfect array of size 2 with sum 18 must consist of elements 18 and 0, and so by lemma 1 we also have (after translation if necessary) $c[8, 0] = 18 - \sum_{i=0}^{7} c[i, 0]$ and $c[8, 1] = -\sum_{i=0}^{7} c[i, 1]$. Each of the arrays D from the first stage gives rise to a group of four 9 × 2 perfect arrays C.

The third and final stage is to find all possible arrays B which satisfy eqns. 2 and 3 simultaneously for a pair of arrays C, C' belonging to the same group from the second stage. Now a perfect array of size 2 × 2 with energy 324 and sum 18, none of whose elements is zero, must consist of elements 9, 9, 9 and −9. Therefore we may also impose $b[8, 0, 0] = 9 - \sum_{i=0}^{7} b[i, 0, 0]$, $b[8, 0, 1] = 9 - \sum_{i=0}^{7} b[i, 0, 1]$, $b[8, 1, 0] = 9 - \sum_{i=0}^{7} b[i, 1, 0]$, and $b[8, 1, 1] = -9 - \sum_{i=0}^{7} b[i, 1, 1]$. The results of this search are:

(iii) *Proposition 1:* Let A be a 9 × 2 × 2 × 3 × 3 PBA and let $B = (b[i, j, k])$ be the 9 × 2 × 2 array given by $b[i, j, k] = \sum_{l=0}^{2} \sum_{m=0}^{2} a[i, j, k, l, m]$; then, up to translation, the array B is in magnitude

9 3 3 9 3 3 9 3 3	1 1 1 1 1 1 1 1 1
1 1 1 1 1 1 1 1 1	1 1 1 1 1 1 1 1 1

and the three elements of magnitude 9 do not all have the same sign.

Nonexistence of 9 × 2 × 2 × 3 × 3 PBA: Suppose that A is a 9 × 2 × 2 × 3 × 3 PBA. We now use proposition 1 and lemma 2 to obtain a contradiction by progressively constraining the elements of A. Define the arrays $A'_c = (a'_c[i, j, k, l, m])$ and $A''_c = (a''_c[i, j, k, l, m])$ by $a'_c[i, j, k, l, m] = a[(i - cl) \bmod 9, j, k, l, m]$ and $a''_c(i, j, k, l, m] = a[(i - cm) \bmod 9, j, k, l, m]$; then by lemma 2, A'_c and A''_c are each a 9 × 2 × 2 × 3 × 3 PBA, provided $c \equiv 0 (\bmod 3)$. Define the 9 × 2 × 2 array $B = (b[i, j, k])$ by $b[i, j, k] = \sum_{l=0}^{2} \sum_{m=0}^{2} a[i, j, k, l, m]$, and define B'_c and B''_c similarly from A'_c and A''_c.

Translate A if necessary so that B has the form given in proposition 1; then for $(i, j, k) = (0, 0, 0)$, $(3, 0, 0)$ and $(6, 0, 0)$ the nine ±1 elements of A summing to $b[i, j, k]$ must be equal, so

$$a[3i, 0, 0, l, m] = a[3i, 0, 0, 0, 0] \quad \text{for all } 0 \le i, l, m < 3 \quad (7)$$

Because the three elements $a[3i, 0, 0, 0, 0]$ $(0 \le i < 3)$ do not all have the same sign, we also have

$$\sum_{i=0}^{2} a[3i, 0, 0, 0, 0] = \pm 1 \quad (8)$$

Then $b'_6[0, 0, 0] = \sum_{l=0}^{2} \sum_{m=0}^{2} a[3l, 0, 0, l, m] = \sum_{m=0}^{2} (\pm 1) = \pm 3$. By proposition 1, the elements of magnitude 9 in B'_6 must therefore occur when $(j, k) = (0, 0)$, say at $i = 1, 4$ and 7; then $a'_6[3i + 1, 0, 0, l, m] = a'_6[3i + 1, 0, 0, 0, 0]$ for all $0 \le i, l, m < 3$, and $\sum_{i=0}^{2} a'_6[3i + 1, 0, 0, 0, 0] = \pm 1$, or equivalently

$$a[3i + 1, 0, 0, l, m] = a\{[3(i - 1) + 1] \bmod 9, 0, 0, 0, 0] \quad \text{for all } 0 \le i, l, m < 3 \quad (9)$$

$$\sum_{i=0}^{2} a[3i + 1, 0, 0, 0, 0] = \pm 1 \quad (10)$$

It follows from eqns. 7 and 8 and 9 and 10, respectively, that $b'_3[0, 0, 0] = \pm 3$ and $b'_3[1, 0, 0] = \pm 3$, which by proposition 1 forces $a'_3[3i + 2, 0, 0, l, m] = a'_3[3i + 2, 0, 0, 0, 0]$ for all $0 \le i, l, m < 3$, and $\sum_{i=0}^{2} a'_3[3i + 2, 0, 0, 0, 0] = \pm 1$, and so

$$a[3i + 2, 0, 0, l, m] = a\{[3(i + 1) + 2] \bmod 9, 0, 0, 0, 0\} \quad \text{for all } 0 \le i, l, m < 3 \quad (11)$$

$$\sum_{i=0}^{2} a[3i + 2, 0, 0, 0, 0] = \pm 1 \quad (12)$$

Finally from eqns. 7–12 we have $b''_6[0, 0, 0] = \pm 3$, $b''_6[1, 0, 0] = \pm 3$, and $b''_6[2, 0, 0] = \pm 3$, which contradicts proposition 1.

Nonexistence of 9 × 4 × 3 × 3 PBA: The nonexistence of a 9 × 4 × 3 × 3 PBA follows similar lines, with the rows of the possible 9 × 4 perfect arrays being the same as those in proposition 1. The difference in that lemma 1 only provides one way to sum to a 9 × 2 perfect array (using $t = 2$ on the dimension of size 4), rather than two. This means there are fewer constraints on the possible 9 × 4 arrays and so the search takes considerably longer.

Summary: By computer search and combinatorial argument we have established that there is no 2 × 2 × 3 × 3 × 9 or 4 × 3 × 3 × 9 perfect binary array. To our knowledge this is the first nonexistence result for a PBA with $N = 2^a 3^b$ which improves on the Turyn exponent bound for N of this form [4, 10]. There remain eleven non-equivalent values $\{s_1, \ldots, s_r\}$ for which the existence of an $s_1 \times \ldots \times s_r$ PBA with $N < 20$ is undecided [3, 6], namely: {2, 2, 4, 5, 5}, {2, 8, 5, 5}, {4, 4, 5, 5}, {2, 2, 16, 9}, {4, 16, 9}, {8, 8, 9}, {4, 3, 3, 5, 5}, {2, 2, 2, 2, 3, 3, 9}, {2, 2, 4, 3, 3, 9}, {2, 8, 3, 3, 9} and {4, 4, 3, 3, 9}.

Acknowledgments: The authors thank C. Mitchell for independently verifying the results of the computer searches.

13th September 1992

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