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Note

New constructions of divisible designs

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Abstract

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A construction is given for a $(p^{2a}(p+1), p^2, p^{2a+1}(p+1), p^{2a+1}, p^{2a}(p+1))$ (p a prime) divisible difference set in the group $H \times Z_{p^{a+1}}^2$ where H is any abelian group of order $p+1$. This can be used to generate a symmetric semi-regular divisible design; this is a new set of parameters for $\lambda_1 \neq 0$, and those are fairly rare. We also give a construction for a $(p^{a-1}+p^{a-2}+\cdots+p+2, p^{a+2}, p^a(p^a+p^{a-1}+\cdots+p+1), p^a(p^{a-1}+\cdots+p+1), p^{a-1}(p^a+\cdots+p^2+2))$ divisible difference set in the group $H \times Z_{p^2} \times Z_p^a$. This is another new set of parameters, and it corresponds to a symmetric regular divisible design. For $p=2$, these parameters have $\lambda_1 = \lambda_2$, and this corresponds to the parameters for the ordinary Menon difference sets.

1. Introduction

Divisible Designs are combinatorial structures involving points, blocks, and incidence relations that were first studied by Bose and Connor in [3]. The formal difinition can be stated as follows.

Definition 1.1. An incidence structure $\Delta = (\mathscr{P}, \mathscr{B}, \mathscr{I})$ is called a *divisible design* with parameters $m, n, k, \lambda_1, \lambda_2$ if the following conditions are satisfied:

(a) The point set $\mathscr{P}$ is split into m classes of n points. If p and q are points in the same class, then we write $p \sim q$.

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(b) For distinct points p and q, $[p, q]=\lambda_1$ if $p \sim q$, and $[p, q]=\lambda_2$ if $p \nsim q$. ($[p, q]$ is the number of blocks through p and q).

(c) Each block contains exactly k points.

If the number of blocks equals the number of points, then the design is called *square*. If furthermore the dual structure is a divisible design with the same parameters, then the design is called *symmetric* (we are following the terminology of Jungnickel [5]). This paper will only consider square symmetric divisible designs. If we define r to be the number of blocks each point is incident with, then $r=k$ in the square symmetric case, and $k^2-nm\lambda_2=k-\lambda_1+(\lambda_1-\lambda_2)n$.

Definition 1.2. *If* $r>\lambda_1$ and $rk=nm\,\lambda_2$, then call the design *semi-regular*. If $r>\lambda_1$ and $rk>nm\,\lambda_2$, then call the design *regular*.

The remarks prior to the definition imply that the design is semi-regular if $k>\lambda_1$ and $k^2=nm\,\lambda_2$, and it is regular if $k>\lambda_1$ and $k^2>nm\,\lambda_2$.

In this note, we will construct divisible designs with new sets of parameters. One of these designs will be semi-regular, while the other is regular. The constructions make use of a standard technique in design theory (and are similar to the constructions found in [5]), namely to look for a transitive automorphism group of the design; this is called a (divisible) difference set.

Definition 1.3. Let G be a group of order mn and N a subgroup of G of order n. If D is a k-subset of G, then D is called a $(m, n, k, \lambda_1, \lambda_2)$ *divisible difference set* (DDS) provided that the differences dd'^{-1} for $d, d' \in D, d \neq d'$ contain every nonidentity element of N exactly λ_1 times and every element of $G-N$ exactly λ_2 times.

If we can find a group G with a DDS, that is equivalent to a divisible design with a regular automorphism group (see [2]). Thus, we want techniques that will help us find DDS. One helpful way to view DDS is to consider the group ring $Z[G]$. The definition of a DDS immediately yields the group ring equation

$$DD^{(-1)}=k+\lambda\mu_1(N-1)+\lambda_2(G-N)$$

where

$$D=\sum_{d\in D} d \quad \text{and} \quad D^{(-1)}=\sum_{d\in D} d^{-1}.$$

We now restrict our attention to abelian groups; in this case, characters of the group are simply homomorphisms from the group to the complex numbers. Extending this homomorphism to the entire group ring yields a map from the group ring to a number field. The character sum for the character χ on the element D of the group ring yields 3 possible results: $\chi(D)=k$ if χ is the principal (all 1) character,

$$|\chi(D)|=\sqrt{k-\lambda_1}$$

if χ is nonprincipal on N, and

$$|\chi(D)|=\sqrt{k-\lambda_1+(\lambda_1-\lambda_2)n}$$

if χ is principal on N but nonprincipal on G. If we have a subset of the group that satisfies these character sums for every character χ, then that subset will be a DDS (this is because of the orthogonality relations for characters: see [6] for similar arguments). Thus, our strategy will be to come up with a 'candidate' subset of the group, then use character theory to check that all the sums are correct.

2. Semi-regular design

Let G be the group of the form $H\times Z_{p^{a+1}}^2$ where H is an abelian group of order $p+1$, p a prime. The generators of the p^{a+1} parts are y and z, and we will write the elements of H as $h_0, h_1, \ldots, h_p$. The subgroup N is the group $\langle y^{p^a}, z^{p^a}\rangle\cong Z_p^2$. We also want to label the cyclic subgroups of order p^{a+1} in a careful way. We will use $D_{1,i}=\langle yz^i\rangle$, $i=0, 1, \ldots, p^{a+1}-1$ and $D_{pj,1}=\langle y^{p\cdot j}z\rangle$, $j=0, 1, \ldots, p^a-1$. Consider the set

$$D=\left(\bigcup_{k=0}^{p-1} h_k\left(\bigcup_{i=0}^{p^a-1} z^i D_{1,ip+k}\right)\right)\cup\left(h_p\bigcup_{j=0}^{p^a-1} y^j D_{p\cdot j,1}\right).$$

This is the 'candidate' subset; we claim that this is a DDS with the proper parameters. The proof is broken down into the following sequence of lemmas.

Lemma 2.1. *D has no repeated elements.*

Proof. Suppose there is a repeated element. By the way we have displayed the set, the repeated element must occur within the same coset of $\langle y, z\rangle$, or unless the k's are the same. We will consider the i case; the j case is similar. If there is a repeated element, there must be an i, i', m, m' so that $z^i(yz^{ip+k})^m=z^{i'}(yz^{i'p+k})^{m'}$. In order for this to occur, $m=m'$ (since the same power of y must be present). Considering the powers of z, we get

$$z^{i+mip+mk}=z^{i'+mi'p+mk},$$

or

$$i(1+mp)\equiv i'(1+mp) \pmod{p^{a+1}}.$$

Since $1+mp$ is invertible mod p^{a+1}, we can conclude that $i=i'$, but this says that the elements are the same. □

Lemma 2.2. *If χ is a character of order p^{a+1} on G/H, then χ is nonprincipal on all of the $D_{i,j}$ except one, where it is principal.*

Proof. The kernel of χ has order p^{a+1}, and $(G/H)/\mathrm{Ker}$ is cyclic. Thus, the kernel must also be cyclic in this case. All of the cyclic groups of that order are $D_{i,j}$'s, so χ is principal on that $D_{i,j}$ and nonprincipal on all of the others. □

Lemma 2.3. *If χ is a character of G/H that is nonprincipal but of order less than p^{a+1}, then*

$$\sum_{i=0}^{p^a-1} \chi(z^i D_{1,ip+k}) = \sum_{j=0}^{p^a-1} \chi(y^j D_{j,1}) = 0$$

for every k.

Proof. Let ξ be a primitive p^{a+1} root of unity, and suppose that χ is a character of order less than p^{a+1}. Then

$$\chi(y) = \xi^{sp^v}, \qquad \chi(z) = \xi^{rp^t}, (r, p) = (s, p) = 1,$$

$1 \leqslant t, v \leqslant a+1$, but not both t and v are $a+1$. We are only considering the $D_{1,ip+k}$ case above (the other argument is similar). There are two cases to consider; the first is the $t = a+1$ case. In this case, χ is nonprincipal on each $D_{1,ip+k}$, so the character sums over these pieces is 0. Thus, the sum over all the $D_{1,ip+k}$ will be 0. If $t \leqslant a$, then suppose that $yz^k \in \mathrm{Ker}(\chi)$; we claim that

$$yz^{ip^{a+1-t}+k} \in \mathrm{Ker}(\chi)$$

for every i. This is a straightforward calculation. Using that, we get

$$\sum_{i=0}^{p^t-1} \chi(z^{ip^{a-t}} D_{1,ip^{a+1-t}+k}) = p^{a+1} \sum_{i=0}^{p^t-1} (\xi^{rp^t})^{ip^{a-t}} = 0.$$

A similar argument shows that if

$$yz^{k'} \notin \mathrm{Ker}(\chi),$$

then

$$yz^{ip^{a+1-t}+k'} \notin \mathrm{Ker}(\chi)$$

for any i, so $\chi(D_{1,ip^{a+1-t}+k'}) = 0$. Combining these sums, we see that

$$\sum_{i=0}^{p^a-1} \chi(z^i D_{1,ip+k}) = 0,$$

which proves the lemma. $\square$

Lemma 2.4. *If χ is a nonprincipal character on G that is principal on $\langle y, z\rangle$, then $\chi(D) = 0$.*

Proof. Since $\chi(D_{i,j}) = p^{a+1}$ for every (i,j) pair, and

$$\sum_{i=0}^{p^a-1} \chi(z^i D_{1,ip+k}) = p^a p^{a+1} = p^{2a+1},$$

then this sum reduces to $\chi(D) = \sum_{k=0}^{p} \chi(h_k) = 0$. $\square$

Putting all this together, we get the following.

Theorem 2.1. *D is a* $(p^{2a}(p+1), p^2, p^{2a+1}(p+1), p^{2a+1}, p^{2a}(p+1))$ *DDS in G.*

Proof. Consider the different character sums. If χ is principal, then the character sum is simply the number of elements in D, and this is obviously $p^{a+1}(p^{a+1}+p^a)=p^{2a+1}(p+1)=k$ since there are no repeated elements (Lemma 2.1). If χ is nonprincipal on N, then the character sum will be 0 on all the D_{ij} except one, where the sum has modulus

$$p^{a+1}=\sqrt{p^{2a+1}(p+1)-p^{2a+1}}=\sqrt{k-\lambda_1}$$

(by Lemma 2.2). Finally, if χ is principal on N but nonprincipal on G, then the sum is

$$\begin{aligned}0&=\sqrt{p^{2a+1}(p+1)-p^{2a+1}-(p^{2a+1}-p^{2a+1}-p^{2a})p^2}\\&=\sqrt{k-\lambda_1+(\lambda_1-\lambda_2)n}\end{aligned}$$

by Lemmas 2.3 and 2.4. Thus, D is a DDS in G. $\square$

This DDS generates a divisible design with the same parameters. The design is semi-regular since

$$k=p^{2a+1}(p+1)>\lambda_1=p^{2a+1},$$

and

$$k^2=p^{4a+2}(p+1)^2=nm\lambda_2.$$

The author has not found this set of parameters in the literature, so this seems to be a new set of parameters for a semi-regular design.

It is worth making two comments about this construction. First, we could have proved this using a group ring argument. The group ring argument is more difficult, but it has the advantage of allowing the groups to be nonabelian. In the nonabelian case, the h_k must be carefully chosen to match a condition much like the condition found in Dillon's paper [4]. The second comment involves a construction found in a paper by Arasu and Pott [1]. They have a recursive construction involving the same parameters as the above for the $p=2$ case, but their construction is in a different group. The $p=2$ case can be thought of as a divisible difference set analog of the Menon difference sets, and that is the basis of their construction.

3. Regular design

We will use the same pattern that we used in Section 2 to establish another new divisible design. We will consider the group $G=H\times Z_{p^2}\times Z_p^a$, where H is an abelian group of order $p^{a-1}+\cdots+p+2$. The subgroup N is the group $Z_{p^2}\times Z_p^a$. The elementary abelian subgroup of rank $a+1$ has

$$\frac{p^{a+1}-1}{p-1}=p^a+\cdots+p+1$$

hyperplanes; label these hyperplanes $H_0, H_1, \ldots, H_{p^a+p^{a-1}+\cdots+p}$. Label the elements of H by $h_0, h_1, \ldots, h_{p^{a-1}+p^{a-2}+\cdots+p+1}$. If y is an element of order p^2, then the 'candidate' DDS is

$$D=\left(\bigcup_{k=0}^{p^{a-1}+p^{a-2}+\cdots+p} h_k \bigcup_{i=p}^{p-1} y^i H_{i+pk}\right) \cup (h_{p^{a-1}+p^{a-2}+\cdots+p+1} H_{p^a+p^{a-1}+\cdots+p}).$$

This DDS will have the parameters $(p^{a-1}+p^{a-2}+\cdots+p+2, p^{a+2}, p^a(p^a+p^{a-1}+\cdots+p+1), p^a(p^{a-1}+\cdots+p+1), p^{a-1}(p^a+\cdots+p^2+2))$. We do the same four lemmas as in Section 2.

Lemma 3.1. *D has no repeated elements.*

Proof. If there were a repeated element, it would occur in one of the 'h_k' cosets, but the different powers of y prevent any repetition. □

Lemma 3.2. *If χ is any character that is nonprincipal on the elementary abelian subgroup of rank $a+1$, then $|\chi(D)|=\sqrt{k-\lambda_1}$.*

Proof. Every nonprincipal character is nonprincipal on all the hyperplanes except 1, so the character sum is

$$\begin{aligned}|\chi(D)| &= |H_{i'}| \\ &= p^a \\ &= \sqrt{p^a(p^a+p^{a-1}+\cdots+p+1)-p^a(p^{a-1}+\cdots+p+1)} \\ &= \sqrt{k-\lambda_1}. \quad \square\end{aligned}$$

Lemma 3.3. *If χ is any character that is principal on the elementary abelian subgroup of rank $a+1$ but nonprincipal on y, then $|\chi(D)|=\sqrt{k-\lambda_1}$.*

Proof. Because χ is principal on the elementary abelian subgroup, it is principal on every hyperplane. Thus, the sum becomes

$$\begin{aligned}\chi(D) &= \sum_{k=0}^{p^{a-1}+\cdots+p} \chi(h_k) \sum_{i=0}^{p-1} \chi(y^i H_{i+pk}) + \chi(h_{p^{a-1}+\cdots+1} H_{p^a+\cdots+p}) \\ &= p^a \left\{ \sum_{k=0}^{p^{a-1}+\cdots+p} \chi(h_k) \sum_{i=0}^{p-1} (\chi(y))^i + \chi(h_{p^{a-1}+\cdots+1}) \right\} \\ &= p^a \chi(h_{p^{a-1}+\cdots+1}).\end{aligned}$$

Thus, $\chi(D)$ has modulus $p^a=\sqrt{k-\lambda_1}$. □

Lemma 3.4. *If χ is principal on N but nonprincipal on G, then*

$$|\chi(D)|=(p-1)p^a=\sqrt{k-\lambda_1+(\lambda_1-\lambda_2)n}.$$

Proof. As in the above proof, χ is principal on all the hyperplanes, so the sum reduces quickly. Also, χ is principal on y. Thus, the sum is

$$\chi(D)=p(p^a)\left\{\sum_{k=0}^{p^{a-1}+\cdots+p}\chi(h_k)\right\}+p^a(\chi(h_{p^{a-1}+\cdots+1}))$$

$$=-(p-1)p^a(\chi(h_{p^{a-1}+\cdots+1})).$$

Thus, $\chi(D)$ has character sum of modulus $(p-1)p^a=\sqrt{k-\lambda_1+(\lambda_1-\lambda_2)n}$ as claimed. □

Putting all of this together, we get the following.

Theorem 3.1. *The set D is a $(p^{a-1}+p^{a-2}+\cdots+p+2, p^{a+2}, p^a(p^a+p^{a-1}+\cdots+p+1), p^a(p^{a-1}+\cdots+p+1), p^a+\cdots+p^2+2))$ DDS in G.*

Proof. Once again, we simply need to organize the lemmas to show that the character sum match what they should. Lemma 3.1 shows that the character sum for the principal character is equal to k. Lemmas 3.2 and 3.3 show that if the character is nonprincipal on N, then the sum will have modulus $\sqrt{k-\lambda_1}$. Finally, Lemma 3.4 shows that if the character is principal on N but nonprincipal on G, then the character sum will have modulus $\sqrt{k-\lambda_1+(\lambda_1-\lambda_2)n}$, and this shows that D is a DDS with the correct parameters. □

We can use the fact that $k>\lambda_1$ and $\lambda_1>\lambda_2$ to show that the design associated to this divisible difference set is a regular design (see [5] for similar arguments). Two other comments seem appropriate at this point. First, we should examine what happens with the prime $p=2$. In this case

$$m=\frac{2^n-1}{2-1}+1;\qquad n=2^{a+2};\qquad k=2^a\left(\frac{2^{a+1}-1}{2-1}\right);$$

$$\lambda_1=2^a\left(\frac{2^a-1}{2-1}\right);\qquad \lambda_2=2^{a-1}\left(\frac{2^{a+1}-1}{2-1}-2+1\right).$$

Thus, $\lambda_1=\lambda_2=2^{2a}-2^a$, and this means that the DDS is actually an ordinary difference set. As a matter of fact, the parameters of the ordinary difference set. As a matter of fact, the parameters of the ordinary difference set are Menon ($v=2^{2n+2}$, $k=2^{2n+1}-2^n$, $\lambda=2^{2n}-2^n$), so these parameters can be thought of as a prime generalization of the ordinary Menon difference sets (see [2] for further details on Menon difference sets).

Second, as with the construction in Section 2, this proof could have been done using group rings, and that would have included some nonabelian examples.

References

[1] K.T. Arasu and A. Pott, Some constructions of group divisible designs with Singer groups, Discrete Math., to appear.

[2] T. Beth, D. Jungnickel and H. Lenz, Design Theory (Cambridge Univ. Press, Cambridge, 1986).

[3] R.C. Bose and W.S. Connor, Combinatorial properties of group divisible incomplete block designs, Ann. Math. Stat. 23 (1952) 367–383.

[4] J.F. Dillon, Variations on a scheme by McFarland for noncyclic difference sets, J. Combin. Theory Ser. A 40 (1985) 9–21.

[5] D. Jungnickel, On automorphism groups of divisible designs, Canad. J. Math. 34 (1982) 257–297.

[6] R.J. Turyn, Character sums and difference sets, Pac. J. Math. 15 (1965) 319–346.