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A historical survey of methods of solving cubic equations

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A HISTORICAL SURVEY OF METHODS OF SOLVING CUBIC EQUATIONS

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INTRODUCTION

It has been said that the labor-saving devices of this modern age have been made possible by the untiring efforts of lazy men. While working with cubic equations, solving them according to the standard methods appearing in modern text-books on the theory of equations, it became apparent, that in many cases, the finding of solutions was a long and tedious process involving numerical calculations into which numerous errors could creep. Confessing to laziness, and having been told at an impressionable age that "any fool can do it the hard way but it takes a genius to find the easy way", it became of interest to find a simpler method of solution. It eventually became clear that it is necessary to find out what has been done in the past to accomplish this.

The information is found to be interesting and varied, but scattered among many sources. These sources are brought together here, not only in the hope that this history will be helpful in learning about the development of cubic equations, but also to challenge the reader to find solutions of his own, which will not only reduce the labor and errors involved, but also will explain the true mathematical meaning of a cubic equation.

An altar used for prayer to angry gods is the legendary beginning for the many attempts to find solutions to cubic equations. The altar was in the shape of a cube. Kneeling before this cube, ancient Greeks had prayed successfully to their powerful gods for many years. However, when pestilence came to the land and prayers to the gods went unanswered, legend has it that the oracle - he who passes as the local politician of today - sought to appease the people with an explanation for the lands depression.

"Ah", said the oracle, "The gods do not answer our prayer because he is insulted by our meager altar. It must be doubled in size."

Work began at once to double the size of the cube. When finished, bad times remained. Most conveniently, the oracle discovered that an error had been made. The stupid slaves had simply constructed a cube with edges twice as long! Work such as that would not pacify the gods. The oracle, stalling for time in hopes that good fortune would gradually return, consulted Plato, a renowned philosopher and a man interested in all problems. Could he and his disciples find a solution? Plato's mathematicians applied their skill.

An important contribution to this problem was made by Hippocrates of Chios around 430 B. C. He showed that the problem of doubling the cube could be reduced to finding two mean proportionals between a given line and another twice as long. He failed, however, to find the two mean proportionals by geometric construction with ruler and compasses, the so-called Euclidian tools.

Having been defrauded of his property Hippocrates was considered slow and stupid by his contemporaries. It was also said that he had actually accepted pay for the teaching of mathematics! However, he was a talented mathematician and his work on the doubling of the cube presented a challenge to others. Who could solve this problem depending upon cubic equations and appease the angry gods?

Archimedes, legend has it, was an inventor of war machines and mirrors to reflect the sun's rays to destroy the enemy by fire. His mechanical inventions won for him the admiration of his fellow citizens. He himself took more pride in his accomplishments in the field of pure science. He is known to have said, "Every kind of art which is connected with daily needs is ignoble and vulgar." Archimedes, during the second century before

Christ, gave a geometric solution to cubic equations with the help of spheric sections.

Commercialism - the inate desire of all mankind to make money - was the incentive for even earlier attempts to solve problems involving the cubic. The Babylonians, situated as they were on great caravan routes, consulted their mathematicians in order to obtain tables of squares and cubes so that they could have solutions to equations involving lengths, breadths and volumes.

The love of beauty, which, regardless of plague, famine and rumors of war, remains through history with all peoples, was an added incentive to the Babylonians. Their erratic Tigris and Euphrates rivers must be controlled. Dams must be constructed and canals built to irrigate the fields so that the gardens of Babylonia would be recorded in history for their loveliness. The skillful mathematicians must find solutions to equations concerning volumes.

Known to the West as a great Persian poet and philosopher, Omar Khayyam is known to the East primarily as a great astronomer and mathematician. Omar Khayyam's verse was regarded by orthodox Mohammedans as heretical, materialistic and even atheistic. He was under constant observation because it was suspicioned that his poetry contained political and anti-religious meanings. But in spite of the general unpopularity of his poetic works, it is found that the Arab historians and biographers treat him with the highest esteem for his scientific work. The works of the Greeks in finding solutions to cubic equations was built up into a general method by Omar Khayyam in the eleventh century. He not only classified cubic equations but also found many solutions by means of the intersection of various conic sections.

The dark ages throughout the world brought a halt to the progress and dissemination of knowledge. It was not until the Renaissance (1450-1630) that the Italian mathematicians succeeded in getting algebraic solutions to cubic equations.

In 1505 Scipione del Ferro, a professor of mathematics, solved the equation $x^3 + mx = n$. He did not publish his solution. It was the practice in those days to keep discoveries secret so that rivals could not have the advantage in solving publicly proposed problems. This led to many disputes over priority.

Nicolo of Brescia, known as Tartaglia, the stammerer, was involved in a serious dispute over priority. The product of a poor family, self-taught, and having the disadvantage of a speech impediment, Tartaglia had worked hard to excel in mathematics. After much labor, he succeeded in finding solutions to cubics of the form $x^3 + mx = n$, $x^3 = mx + n$, and $x^3 + px^2 = q$. He was entreated by friends to make known his solutions immediately but, thinking he would soon publish an algebra in which he would make known his solutions, he refused. He did divulge his secret, however, to one he thought a scholar and a gentleman, Hieronimo Cardan. He was betrayed when Cardan published the method as his own.

Cardan was characterized as a man of genius, folly, self-conceit and mysticism. He was already recognized as an outstanding mathematician. However, his desire to excel at any cost caused him to break his vow to Tartaglia. When Tartaglia accused Cardan publicly, he found that Cardan had powerful friends both politically and socially and barely escaped with his life.

Since the solutions given by Tartaglia, the aim has been to introduce refinements and to simplify the method. There have been attempts to solve the so-called "irreducible" case not possible by the Cardan-Tartaglia method. Trigonometric solutions have been developed and there are methods using rapidly converging series. In all of these the goal has been simplicity, accuracy and a clear definition of the problem.

Mathematics, throughout history, has been used as a tool in religion, politics, economics, and in furthering the ambition of selfish men. It is used today, with more refinement, for the same reasons. It is also used for the betterment and progress of man in the hands of chemists, physicists and engineers. But mathematics reaches the pinnacle of truth when it is used as a tool of the mathematician. Mathematical truth for its own sake is unblemished.

Therefore, the consideration of only what is important from a purely mathematical point of view is desirable for this history. The political, social, and economic settings in which these developments have taken place will not be included. This information can be found in many of the references. Since in some cases there is more than one mathematician who solved a problem, it has been necessary to be selective in what is presented here in order to avoid unnecessary duplication as far as the understanding of the historical development of cubic equations is concerned. In this case the names of other "solvers" have been mentioned. More information concerning them can also be found in many of the references.

The numbers between the slant lines refer to the references in the bibliography. For example, /4;37/ refers to reference number 4 page 37. These will represent the main sources of information and the sources to which the reader may go in order to more fully inform himself concerning matters discussed.

This history brings together the most important aspects in the development of solutions to the cubic equation and presents a selected list of literature and notes.

CHAPTER I

THE BABYLONIANS

Babylonian mathematicians were makers of mathematical tables and computers of great ability. Their aptitude in these fields was probably due to their advanced economic development.

Arithmetic, in Babylonia, had become a well-developed algebra by 2000 B.C.. Babylonian cuneiform texts which are perhaps the oldest used texts for quadratic equations (around 1800 B. C.) also give exercises using cubic equations. These are separated from their geometric and surveying problems and show purely algebraic character, although some show a geometric origin /2/.

The problems on cubic equations in the text are numbered 1,2,3,12,14,15. They are classified as (1) pure equation (number 14), (2) normal form (numbers 1, 12 and 15) and (3) general form (numbers 2 and 3) /1;119/, /3/.

The pure equation in modern notation is as follows:

$$V = xyz = 1\ 30/60, \quad y = x, \quad z = ux \qquad (u = 12)$$

which gives

$$x = \sqrt[3]{V/u} = \sqrt[3]{V/12} = \sqrt[3]{1/8} = 1/2 = y, \quad z = 6$$

The terms used for the unknowns are length.

breadth and depth for x,y, and cross-section for xyz.

To show an example of the normal form number 1 is given:

$$xyz + xy = 1\ 10/60,\ y = 40/60\ x,\ z = 12x$$

from which is obtained

$$(12x)^3 + (12x)^2 = 4 \cdot 60 + 12\ (= 252)$$

whereby the solution (12x) = 6 will follow. The ancient text does not show how this is achieved. However, included in the text is a table which contains the sum of the cube and square numbers of the form $n^3 + n^2$ for n = 1 to 30. Its use is evident from the above example.

In problem number 12 the procedure is as follows:

$$V = xyz = 3/60 + 20/60^2\ (= 1/18),\ y = x,\ z = ux + 7,$$
$$(u = 12)$$

from which

$$V = x^3 + 7x^2$$

This is of the form $n^3 + n^2$. The answer obtained is

$$ux = 1 \text{ or } x = 1/12.$$

As examples of the general form problems 2 and 3 are as follows:

$$xyz + xy = 1\ 10/60,\ z = ux,\ x + y = 50/60 \quad \text{number 2}$$
$$(u = 12)$$
$$xyz + xy = 1\ 10/60,\ z = ux,\ x - y = 10/60 \quad \text{number 3}$$

The solution x = 1/2 is stated after short computation, probably through interpolation using different values of x.

One can bring every cubic equation of the form $x^3 + ax^2 + bx + c = 0$ into the form $n^3 + n^2 = p$. The transformation to $u^3 + qu^2 = r$ comes from the substitution $x = u + s$ and s can be determined from a quadratic equation. If $u^3 + qu^2 = r$ is divided by q^3, then

$$(u/q)^3 + (u/q)^2 = r/q^3$$

which is again the form $n^3 + n^2 = p$ and is found in the tables.

For each of the procedures the old Babylonian methods appear sufficient. Otto Neugebauer, to whom we owe most of our present knowledge of the Babylonian achievements, believes that they were quite capable of reducing the general cubic equation, although he has, as yet, no evidence that they actually did do it.

CHAPTER II

THE GREEKS

The Greek concern with cubic equations grew out of their determination to solve the two problems

(1) the doubling of the cube and

(2) the trisection of any angle.

The first real progress in the doubling of the cube was the reduction of the problem by Hippocrates of Chios (about 440 B. C.) to the construction of two mean proportionals between two given line segments s and 2s/4;82/, /5;38/. If the two mean proportionals are denoted by x and y , then

$$s : x = x : y = y : 2s$$

From these proportions one obtains $x^2 = sy$ and $y^2 = 2sx$. Eliminating y , it is found that $x^3 = 2s^2$. Thus x is the edge of a cube having twice the volume of the cube of edge s.

He failed to find the two mean proportionals by geometric construction with ruler and compass.

Archytas (400 B. C.) was also one of the first to give a solution to the problem of duplicating the cube. His solution rests on finding a point of intersection of a right circular cylinder, a torus of zero inner diameter, and a right circular cone /5;28/, /4;83/,/7a;84/.

The solution by Menaechmus (375 B. C.) a pupil of Plato, was given in two ways /5;44/. He showed that two parabolas having a common vertex, axes at right angles, and such that the latus rectum of one is double that of the other will intersect in another point whose abscissa (or ordinate) will give a solution. If the equations of the parabolas are $y^2 = 2ax$ and $x^2 = ay$, they intersect in a point whose abscissa is given by $x^3 = 2a^3$ /1;126/. He also showed that the same point could be determined by the intersection of the parabola $y^2 = 2ax$ and the hyperbola $xy = a^2$. The first method was probably suggested by the form in which Hippocrates had presented his problem, i.e., to find x and y so that $a:x = x:y = y:2a$, which gives $x^2 = ay$ and $y^2 = 2ax$.

Thus the finding of two mean proportionals gives the solution of any pure cubic equation, or the equivalent of extracting the cube root.

In the two propositions, On the Sphere and Cylinder II, 1,5, Archimedes (240 B. C.) uses the two mean proportionals when it is required to find x where

$$a^2:x^2 = x:b$$

which today would be stated $x^3 = a^2b$.

In another problem (On the Sphere and Cylinder II,4) he reduces the problem to dividing a sphere by a plane into two segments whose volumes are in a given ratio /6;cxxvi,62-72/, /1;127/,/5;65/, /32;128-159/. Since the geometrical form of the proof is intricate, it will not be given here. The procedure is clear if stated as follows:

Problem:

To cut a given sphere by a plane so that the segments shall have a given ratio.

Stages of the proof:

(a) Archimedes says that if the problem is propounded in the general form, it requires a "diorismos", (that is, it is necessary to investigate the limits of possibility), but, if there be added the conditions existing in a particular case, it does not require a "diorismos". Therefore, in considering a particular case, the problem becomes as follows: Given two straight lines a and b and an area c^2, to divide a at x so that

$$\frac{a - x}{b} = \frac{c^2}{x^2} \qquad \text{or} \qquad x^2(a - x) = bc^2$$

(b) Analysis of this general problem, in which it is shown that the required point can be found as the intersection of a parabola whose equation is $ax^2 = c^2y$ and a hyperbola whose equation is $(a - x)y = ab$.

(c) Synthesis of this general problem according as bc^2 is greater than, equal to or less than $4a^3/27$. (If greater, there is no real solution; if equal, there is one real solution; if less, there are two real solutions.)

(d) Proof that $x^2(a - x)$ is greatest when $x = 2a/3$. This is done in two parts: (1) if x has any value less than $2a/3$, (2) if x has any value greater $2a/3$, then $x^2(a - x)$ has a smaller value than when $x = 2a/3$.

(e) Proof that, if bc^2 is less than $4a^3/27$, there are always two real solutions.

(f) Proof that, in the particular case of the general problem to which Archimedes has reduced his original problem, there is always a real solution.

(g) Synthesis of the original problem.

Of these stages, (a) and (g) are found in the Archimedian texts. Eustocius, the historian, found stages (b) through (d) in an old book which he claims is the work of Archimedes. Eustocius added stages (e) and (f) himself.

In the technical language of Greek mathematics, the general problem requires a "diorismos". In modern language there must be limiting conditions if the equation $x^2(a - x) = bc^2$ is to have a real root lying between zero and a.

In our algebraic notation, $x^2(a - x)$ is a maximum when $x = 2a/3$. This can be easily proved by the calculus, By differentiating and equating the result to zero, it is found that

$$2ax - 3x^2 = 0 \quad \text{and} \quad x(2 - 3x) = 0$$

from which is obtained $x = 0$ (minimum value) and $x = 2a/3$ (maximum value). This method, of course, was not used by Archimedes.

In showing that the required point can be found as the intersection of the two conics, Archimedes proved that if $4a^3/27 = bc^2$ then the parabola $x^2 = c^2y/a$ touches the hyperbola $(a - x)y = ab$ at the point $(2a/3, 3b)$ because they both touch, at this point the same straight line $(9bx - ay - 3ab = 0)$. This may be proved in the following manner.

The points of intersection of the parabola and the hyperbola are given by the equation

$$x^2(a - x) = bc^2$$

which may be written

$$x^3 - ax^2 + 4a^3/27 = 4a^3/27 - bc^2,$$

length, FE is a second ruler at right angles to the first with C a fixed peg in it. (Fig. I). This peg moves in a slot made in a third ruler parallel to its length, while this ruler has a fixed peg on it, D, in a straight line with the slot in which C moves; and the peg D can move along the slot in AB.

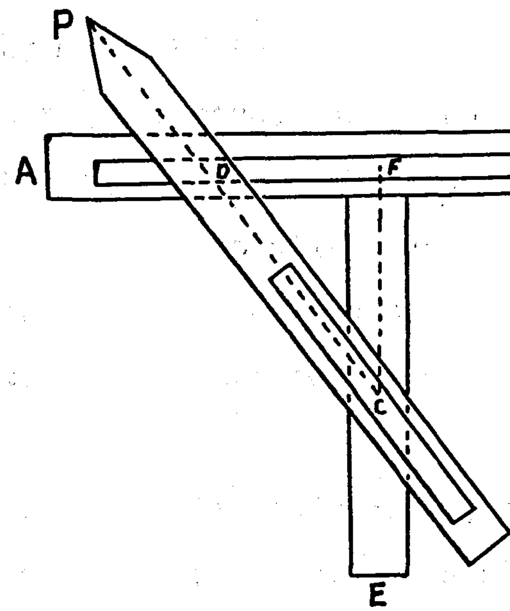

Fig. I

If the ruler PD moves so that the peg D describes the length of the slot in AB on each side of F, the extremity of the ruler, P, describes the curve which is the conchoid. Nicomedes called the straight line AB the "ruler", the fixed point C the "pole", and the length PD the "distance", and the fundamental property of the curve, which in polar coordinates would now be denoted by the equation $r = a \neq b\sec\theta$, is that, if any radius vector be drawn from C to the curve, as CP, the length intercepted on the radius vector between the curve and the straight line AB is constant. Thus any problem in which one of the two given lines is a straight line can be solved by means of the intersection of the other line with a certain conchoid whose pole is a fixed point to which the required straight line must verge. In practice the conchoid was not always actually drawn, but for greater convenience, the ruler was moved about the fixed point until by trial the intercept was made equal to the given length.

Hippias of Elis (420 B.C.), better known as a statesman and a philosopher, made his single contribution to mathematics by the invention of a simple device for trisecting an angle. This curve was called the quadratrix /5;32/,/1;125/.

If the radius of a circle (Fig.II) rotates uniformly around the center O from the position OA through a right angle to OB, and at the same time a straight line, which is drawn perpendicular to OB, moves parallel to itself from the position OA to BC, the locus of their intersection will be the quadratrix.

Let OR and MQ be the position of these lines at any time, and let them cut in P, a point on the curve. Then

OM:OB = arc AR: arc AB = angle AOP: angle AOB.

Similarly, if OR be another position of the radius,

OM':OB = angle AOP': angle AOB.

Therefore,

OM:OM' = angle AOP: angle AOP';

therefore,

angle AOP': angle P'OP = OM':M'M.

Hence, if the angle AOP is given, and it is required to divide it in any given ratio, it is sufficient to divide OM in that ratio at M', and draw the line M'P'. Then OP' will divide AOP in the required ratio. If OA is taken as the initial line, OP = r, the angle OP = θ, and OA = a, then θ :1/2π = r sinθ:a and the equation of the curve is πr = 2aθ cosecθ.

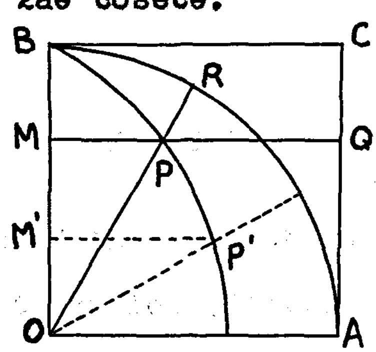

Fig. II

The quadratrix is an example of a transcendental (nonalgebraic) curve which will not only trisect a given angle but will multisect it into any number of equal parts. Another example of this type of curve is the spiral of Archimedes. Archimedes revealed his solution in his Property 8 of the "Liber Assumptorum" /1;121/,/6;cxi/.

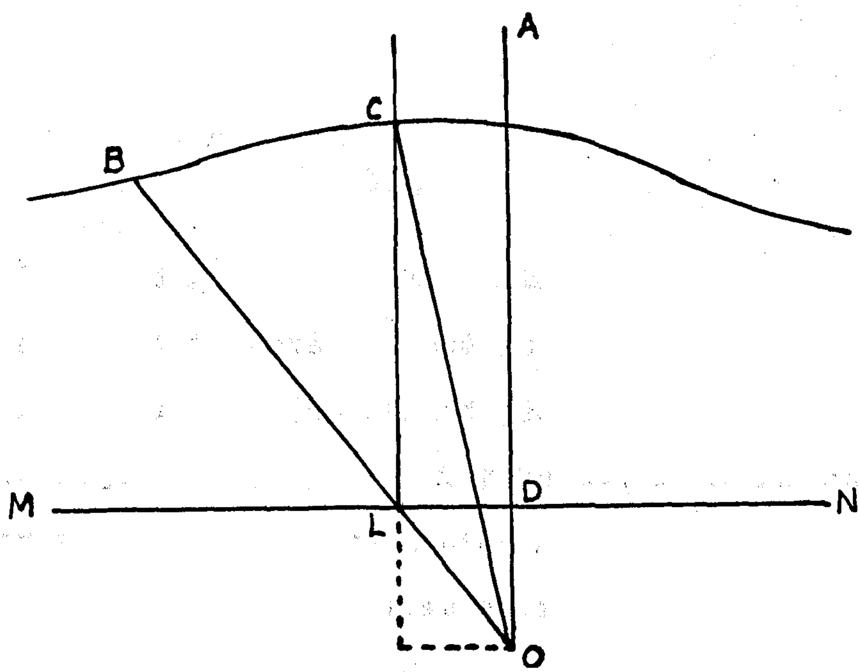

Fig.III

As an example of the trisection of an acute angle, let AOB be any acute angle, (Fig.III). Draw line MN perpendicular to OA, cutting OA and OB in D and L. Now draw the conchoid of MN for pole O and constant 2(OL). At L draw the parallel to OA to cut the conchoid in C. Then OC trisects angle AOB.

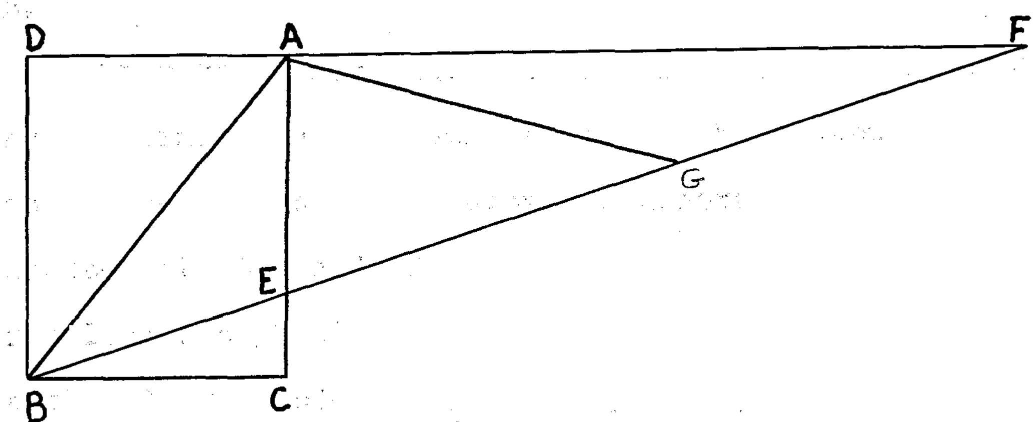

Fig. IV

In dealing with the trisection problem the Greeks appear first to have reduced it to what they called a "verging" problem (so-called geometry of motion). Any acute angle ABC (Fig. IV) may be taken as the angle between a diagonal BA and a side BC of a rectangle BCAD /4;85/, /6;cxii/. Consider a line through B cutting CA in E and DA produced in F, and such that EF = 2(BA). Let G be the midpoint of EF. Then

$$EG = GF = GA = BA$$

whence

$$\angle ABG = \angle AGB = \angle GAF + \angle GFA = 2\angle GFA = 2\angle GBC,$$

and BEF trisects angle ABC. Thus the problem is reduced to that of constructing a straight-line segment EF of given length 2(BA) between AC and AF so that FE "verges" toward B.

Over the years many mechanical contrivances, linkage machines, and compound compasses have been devised to solve the trisection problem. A general angle may be trisected with the aid of a conic. The early Greeks were not familiar enough with conics to accomplish this and the earliest proof of the type was given by Pappus (300 A.D.) /6;cxi/.

It was not until the nineteenth century that it was shown that the duplication of the cube and the trisection of an angle could not be accomplished by means of rulers and compasses. Pierre Laurent Wantzel (1814-1848) gave the first rigorous proofs of this. /7;350/.

The following theorem was established to show the impossibility of solving these two problems with Euclidian tools. /4;96/.

From a given unit length it is impossible to construct with Euclidian tools a segment in the magnitude of whose length is a root of a cubic equation with rational coefficients but with no rational roots.

In the duplication problem, take for the unit of length the edge of the given cube and let x denote the edge of the cube to be found. Then one must have $x^3=2$. If the problem is solvable with Euclidian tools one could construct from the unit segment another segment of length x. But this is impossible since $x^3 = 2$ is a cubic equation with rational coefficients but without rational roots.

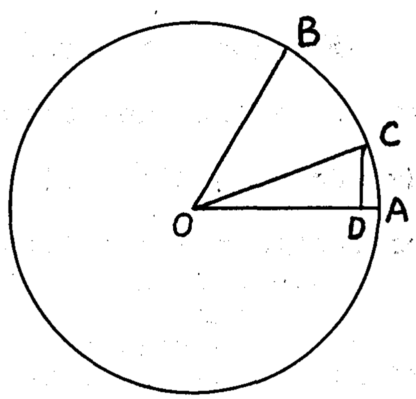

Fig. V

In showing that the general angle cannot be trisected with Euclidian Tools, it is only necessary to show that some particular angle cannot be trisected /4;97/. From trigonometry,

$$\cos\theta = 4\cos^3(\theta/3) - 3\cos(\theta/3).$$

Taking $\theta = 60^\circ$ and setting $x = \cos(\theta/3)$ this becomes

$$8x^3 - 6x - 1 = 0.$$

Let OA be a given unit segment. Describe the circle with center O and radius OA, and with A as a center and AO as radius draw an arc to cut the circle in B (Fig. V). Then angle BOA = 60°. Let trisector OC, which makes angle COA = 20°, cut the circle in C, and let D be the foot of the perpendicular from C on OA. Then OD = $\cos 20^{\circ}$ which is also equal to x. It follows that if a 60° angle can be trisected with Euclidian tools, in other words if OC can be drawn with these tools, then we can construct from a unit segment OA another segment of length x. But this is impossible by the theorem, since the above cubic equation has rational coefficients but no rational roots.

Of course some angles can be trisected with Euclidian tools. What has been shown is that not all angles can be trisected with straightedge and compasses.

Lorenzo Mascheroni (1750-1800), an Italian, proved that all constructions possible with ruler and compasses are possible with compasses alone. Jean Victor Poncelet (1788-1867), a Russian, proved that all such constructions are possible with ruler alone, if a fixed circle with its center in the plane of construction is given.

Francois Vieta (1540-1603) gave a proof that each of the two famous problems depend upon the solution of a cubic equation /5;208/.

Diophantus of Alexander (about 75 A.D.) presented an interesting problem involving cubics. This is stated in the following manner /32;539/:

To find a right-angled triangle such that its area, added to one of the perpendiculars, makes a square, while its perimeter is a cube.

He begins his proof by letting the area of the triangle be equal to x and the hypotenuse be some square number minus x, say $16 - x$.

Since the area is equal to x, then the product of the sides about the right angle is equal to 2x. However, 2x can be factored into x and 2 so that we can make one of the sides of the right angle equal to 2 and the other equal to x.

The perimeter is $16 - x + 2 + x$, or 18, which is not a cube, but is made up of a square (16) + 2. It is required, therefore, to find a square number which, when 2 is added, makes a cube. In other words, the cube must exceed the square by 2.

Let the side of the square equal $m + 1$ and the side of the cube equal $m - 1$. Then the square equals $m^2 + 2m + 1$ and the cube equals $m^3 + 3m - 3m^2 - 1$.

Since it is required to have the cube exceed the square by 2, 2 is added to the square

$$m^2 + 2m + 3 = m^3 + 3m - 3m^2 - 1$$

from which $m = 4$.

Therefore, the side of the square is equal to 5 and that of the cube is equal to three; and hence, the square is 25 and the cube is 27.

The right-angled triangle is transformed to meet the new conditions. The area is still x but the hypotenuse is now 25 - x; the base remains 2 and the perpendicular equals x.

The condition is still left that the square of the hypotenuse is equal to the sum of the squares of the two sides. Therefore,

$$x^2 + 625 - 50x = x^2 + 4$$

from which

$$x = 621/50$$

and the conditions are satisfied.

multiplied by the square of the arbitrary number, and the cube of the arbitrary number, give the cube (of the given number)."

Expressions for n^3 involving series were given by Sridhara (750), Mahavira and Narayana (1356). The formula

$$n^3 = \sum_{1}^{n} \left\{3r(r - 1) + 1\right\}$$

was given by Sridhara in these words:

"The cube (of a given number) is equal to the series whose terms are formed by applying the rule, 'the last term multiplied by thrice the preceding term plus one', to the terms of the series whose first term is zero, the common difference is one and the last term is the given number."

Mahavira gave the above in the form

$$n^3 = 3\sum_{2}^{n} r(r - 1) + n \text{ .}$$

He said,

"In the series, wherein one is the first term as well as the common difference and the number of terms is equal to the given number (n), multiply the preceding term by the immediately following one. The sum of the products so obtained, when multiplied by three and added to the last term (i.e., n) becomes the cube (of n)."

Narayana stated his series in this way:

"From the series whose first term and common difference are each one, (the last term being the given number) the sum of the series formed by the last term multiplied by three and the preceding added to one, gives the cube (of the last term)."

Mahavira also mentioned the results

$$x^3 = x + 3x + 5x + \ldots\ldots \text{ to } x \text{ terms}$$

$$x^3 = x^2 + (x - 1)\{1 + 3 + \ldots\ldots + (2x - 1)\},$$

in these words:

"The cube (of a given number) is equal to the sum of the series whose first term is the given number, the common difference is twice that number, and the number of terms is (equal to) that number."

or

"The square of the given number when added to the product of that number minus one (and) the sum of the series in which the first term is one, the common difference two and the number of terms (is equal to) that number, gives the cube."

The Hindu terms for cube-root are "ghana-mula" and "ghana-pada". The first description of the operation of the cube-root is found in work entitled "Aryabhatiya", which was written by Aryabhata /31;175/.

The method is described as follows:

Divide the second "aghana" (hundreds) place by thrice the square of the cube-root; substract from the first "aghana" (tens) place the square of the quotient multiplied by thrice the preceding (cube-root), and (substract) the cube (of the quotient) from the "ghana" (units) place; (the quotient put down at the next place (in the line of the root) gives the root)."

The present method of extracting the cube-root is a contraction of Aryabhata's method.

Bhaskara gave as a numerical example of a cubic equation

$$x^3 + 12x = 6x^2 + 35$$

which gives the root $x = 5$ after conversion to

$$(x - 2)^3 = 3^3.$$

CHAPTER IV

THE CHINESE, JAPANESE AND ARABS

Little is known about the early mathematical works of the Chinese and Japanese. This is because both countries, for centuries, enjoyed almost complete isolation from the rest of the world. As a result, their early mathematical achievements did not affect or contribute to the progress of mathematics in the west.

In the first half of the seventh century, Wang Hs' Iao-T'ung published a work entitled "Ch'i-ku Suan-ching", in which numerical cubic equations appear for the first time in Chinese mathematics. He gave several problems leading to cubics. One of them is as follows /8;74/:

"There is a right triangle, the product of whose two sides is 706 1/50, and whose hypotenuse is greater than the first side by 30 9/60. It is required to know the lengths of the three sides."

He gave the answer as 14 7/10, 49 1/5, 51 1/4. He also gave the following rule:

"The product, P, being squared and being divided by twice the surplus, S, make the result "shih" or the constant class. Halve the surplus and make it the "lien-fa" or the second degree class. And carry out the operation of evolution according to the extraction of the cube root. The result gives the first side. Adding the surplus to it, one gets the hypotenuse. Divide the product with the first side and the quotient is the second side."

This rule leads to the cubic equation

$$x^3 + S/2x^2 - P^2/28 = 0.$$

The method of solution is similar to the process of extracting the cube-root, but Wang Hs' Iao-T'ung did not give the details.

Horner's method of approximating to the roots of a numerical equation was known to the Chinese in the thirteenth century. This method was later adopted by the Japanese and published in the eighth book of the "Tegen Shinan" of Sato Moshun in 1698 /26;115/.

The first solution of the problem of trisecting an angle by the Arabs is found in the geometry of the "Three Brothers", Muhammed, Ahmed and Alhasan, sons of Musa ibn Shakir (about 875) /1;124/,/7;171/,/5;104/. They depended heavily on the Greeks, using the conchoid in the trisection problem.

The first to state the Archimedian problem of dividing a sphere by a plane so that the two segments should be in a certain ratio and stating this in the form of a cubic equation was Al-Mahani of Bagdad (about 860) /8;107/; /7;171/; while Abu Ja'far Alchazin was the first Arab to solve the problem by conic sections. Solutions were given also by Al-Kuhi, Al-Hasas ibn Al-Haitam, and others. Another difficult problem, to determine the side of a regular heptagon, required the construction of the side from the equation

$$x^3 - x^2 - 2x + 1 = 0.$$

It was attempted by many and finally solved by Abu'l Jud.

A noteworthy work was produced by Al-Biruni (about 1048). He knew a rigorous approximation procedure in order to figure out the roots of cubic equations. His method used a polygon of seven or nine sides /1;130/. A similar method was figured out by Gijat Eddin Alkasi (about 1435).

The one who did most to elevate to a method the solution of algebraic equations was the poet Omar Khayyam (about 1045-1123) /8;107/; /7a;286/. He divided cubics into two classes, the trinomial and the quadrinomial, and each class into families and species. Each species was treated separately but according to a general plan.

He believed that cubics could not be solved by calculation. He rejected negative roots and sometimes failed to find the positive ones.

Omar gave cubic equations reducible to quadratic forms as containing three species /9;64/:

(1) A cube and squares are equal to roots.

$(x^3 + cx^2 = bx)$

(2) A cube and two roots equal three squares,

$(x^3 + bx = cx^2)$ (general case)

or a square plus two equal to three roots,

$(x^3 + 2x = 3x^2)$ (particular case).

(3) A cube is equal to a square and three roots,

$(cx^2 + bx = x^3)$ (general case)

or a square equal to a root plus the number three,

$(x^3 = 1.x^2 + 3x)$ (particular case).

To give an example of the proofs, (2) is considered

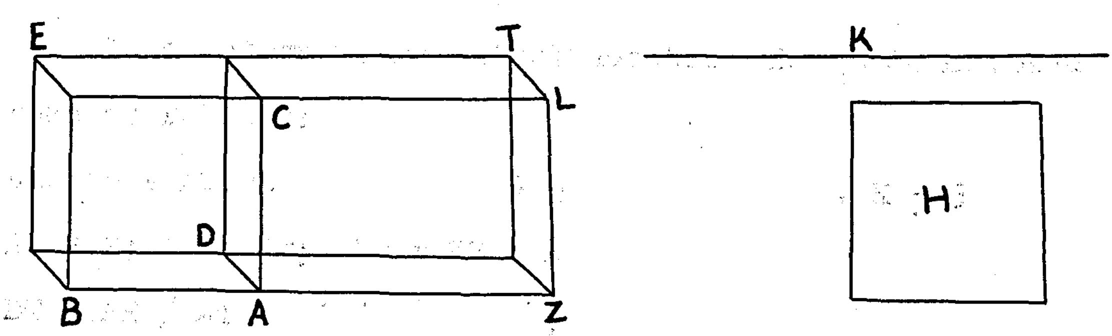

Fig. VI

Let the cube ABCDE with its two roots be equal to three squares, and let the square H equal CB and also let K be equal to the number three. Then the product of H by K will be equal to three times the square of the cube AE. Construct on AC a rectangle equal to the number two and complete the solid AZCTD. It will be equal, then, to the number of roots. But the line ZB multiplied by the square AC gives the solid BT, and the solid AT is equal to the number of sides. Consequently, the solid BT will be equal to the cube plus a quantity equal to the number of its sides. Hence, solid BT is equal to the number of squares. Consequently, the line ZB is equal to three and the rectangle BL is equal to a square plus two. Then a square plus two is equal to three roots because the rectangle BL is equal to a square plus two. Then a square plus two is equal to three roots because the rectangle BL is formed by multiplying AB by three.

In a modern mathematical notation the problem would read as follows:

the cube ABCDE $= x^3$, $H = CB = x^2$; $K = 3$, $H \cdot K = 3x^2$;

$AL = 2$; $AT = 2x$; $BT = ZB \cdot \overline{AC}^2 = ZB \cdot x^2$;

$BT = AE + AT = x^3 + 2x = 3x^2$; then $ZB = 3$;

$BL = BC + AL = x^2 + 2$; $BL = ZB \cdot AB = 3x$; then $x^2 + 2 = 3x$.

The species which Omar Khayyam claims could not be proved except by the properties of conics include fourteen: one simple equation (that in which a number is equal to a cube); six trinominal equations; and seven tetranomial equations.

The six species of trinomial equations are (as taken from a translation of his work /9;64-86/):

(1) A cube and sides are equal to a number.

$(x^3 \neq bx = a)$

This species does not present varieties of cases or impossible problems. It is solved by means of the properties of the circle combined with those of the parabola.

(2) A cube and a number are equal to sides.

$(x^3 \neq a = bx)$

This species includes different cases and some impossible problems. The species is solved by means of the properties of the parabola and the hyperbola.

(3) A cube is equal to sides plus a number.

$(x^3 = bx \neq z)$

This species has no variety of cases and no impossible solutions. It is solved by means of the properties of the parabola and hyperbola.

(4) A cube and squares are equal to a number.

$(x^3 + cx^2 = a)$

This species has no variety of cases and no impossible problems. It is solved by means of the properties of the parabola and hyperbola combined.

(5) A cube and a number are equal to a square.

$(x^3 + a = cx^2)$

This species has a variety of cases some of which are impossible. It is solved by the properties of the hyperbola and parabola.

(6) A cube is equal to squares and numbers.

$(x^3 = cx^2 + a)$

This species has neither a variety of cases nor impossible solutions. It is solved by means of the properties of the hyperbola and parabola.

The tetranomial equations are as follows:

(1) A cube, squares and sides are equal to numbers.

$(x^3 + cx^2 + bx = a)$

This species has no varieties of cases and no impossible problems. It is solved by means of the properties of a hyperbola combined with those of a circle.

(2) A cube, squares, and numbers are equal to sides.

$$x^3 + cx^2 + a = bx$$

There are a variety of cases in this species and some may be impossible. This species is solved by means of the properties of two hyperbolas.

(3) A cube, sides and numbers are equal to squares.

$$x^3 + bx + a = cx^2$$

There are a variety of cases, some impossible. It is solved by means of a circle and hyperbola.

(4) Numbers, sides and squares are equal to a cube.

$$cx^2 + bx + a = x^3$$

This species has no variety of cases. It is solved by using the properties of two hyperbolas.

(5) A cube and squares are equal to sides and a number.

$$x^3 + cx^2 = bx + a$$

This species has a variety of cases but no impossible cases. It is solved by using the properties of two hyperbolas.

(6) A cube and sides are equal to squares and numbers.

$(x^3 + bx = cx^2 + a)$

This species has a variety of cases and forms. It has no impossible problems. It is solved by means of the properties of the circle and hyperbola.

(7) A cube and numbers are equal to sides and squares.

$(x^3 + a = bx + cx^2)$

This species has different cases, some impossible. It is solved by means of the properties of two hyperbolas.

As mentioned previously, Omar Khayyam did not concern himself with negative or imaginary roots. This might have been due to the fact that he rarely completed construction of his curves, using semi-circles, semi-parabolas and only one branch of the hyperbola. His procedures were always logical.

Although a manuscript of Omar's work on algebra was noticed in 1742, his work was not made generally available to European scholars until 1851.

CHAPTER V

THE RENAISSANCE

After the work done by Omar Khayyam, nothing of real importance was accomplished in the field of cubic equations until the middle ages. One of the most important contributions was made by Leonardo of Pisa (Fibonacci).
His greatest work, the "Liber Abaci" was published in 1202. To him is owed the first renaissance of mathematics on Christian soil. His work contains the knowledge the Arabs had in arithmetic and algebra. He advocated the use of the Arab notation. His concern with cubic equations was confined to the following problem:

To find by the methods used in the tenth book of Euclid a line whose length x should satisfy the equation $x^3 + 2x^2 + 10x = 20$.

Leonardo showed by geometry that the problem was impossible, but he gave an approximate value of the root /8;120/, /5;159/, /7b;457/, /57/, /59/, /60/.

Scipione Del Ferro, a professor of mathematics at the University of Bologna, solved the equation $x^3 + mx = n$ in 1515. He did not say how he arrived at the solution /7b;459/.

The first really great algebra to be printed was the *Ars Magna* of Girolamo Cardan (1501-1576) which was published in 1545. Among its contents was a solution of cubic equations. A contemporary of his was Niccola Fontana (Tartaglia, the stammerer) (1500-1559). Cardano, in his *Ars Magna*, stated that an equation of the type $x^3 + px = q$ was solved by a method discovered by Scipio del Ferro. Tartaglia claimed priority for the method of solving equations of the type $x^3 + px^2 = q$ and also the method claimed for del Ferro. A discussion of these claims is to be found in all of the general histories of mathematics. In modern publications the solution is usually referred to as "Cardan's Solution" although some use "Cardan-Tartaglia Solution".

A translation of Cardan's solution in modern symbols is given here /11;203/.

Given $x^3 + 6x = 20$

let $u^3 - v^3 = 20$ and $u^3v^3 = (1/3 \cdot 6)^3 = 8$.

Then $(u - v)^3 + 6(u - v) = u^3 - v^3$,

for $u^3 - 3u^2v + 3uv^2 - v^3 + 6u - 6v = u^3 - v^3$,

whence $3uv(v - u) = 6(v - u)$

and $uv = 2$.

Hence $x = u - v$.

But $u^3 = 20 + v^3 = 20 + 8/u^3$,

whence $u^6 = 20u^3 + 8$,

which is a quadratic in u^3. Hence u^3 can be found, and therefore v^3, and therefore $u - v$. A "geometric" demonstration was also given by Cardan /11;204/.

Cardan discussed negative roots and proved that imaginary roots occur in pairs. He showed that if the three roots of a cubic equation were real, his solution gave them in a form which involved imaginary quantities. Cardan also noted the difficulty in the irreducible case in cubics.

Rafael Bombelli of Bologna published a noteworthy algebra in 1572. In this work he showed that in the irreducible case of a cubic equation, the roots are all real. In textbooks it is shown that if $(n/2)^2 + (m/3)^3$ is negative, then the cubic equation $x^3 + mx = n$ has three real roots. In this case, however, the Cardan-Tartaglia solution expresses these roots as the difference of two cube roots of complex imaginary numbers. Bombelli pointed out the reality of these apparently imaginary roots /4;221/, /5;203/ , /8;135/, /1;139/.
In his publication he also remarks that the problem to trisect a given angle is the same as that of the solution

of a cubic equation.

Michael Stifel (1486 - 1567) was the greatest German algebraist in the sixteenth century. In 1553 he published an improved edition of Christoff Rudolf's book on algebra entitled <u>Die Coss</u> /1;139/. Rudolf gave three numerical cubic equations. One of them gives an interesting method as follows in modern notation:

Given $x^3 = 10x^2 + 20x + 48$

by adding 8 to both sides one obtains

$$x^3 + 8 = 10x^2 + 20x + 56.$$

Dividing by $x + 2$

$$x^2 - 2x + 4 = 10x + 56/x + 2,$$

Now assuming that the two members may be split into

$$x^2 - 2x = 10x$$

and $4 = 56/x + 2$

then both of these equations are satisfied by $x = 12$. However, the method is not general.

Similar solutions using special cases were worked out by Nicolas Petri around 1567 and are found in a subdivision on "Cubica Coss". He gave eight cubic equations. For example:

$$x^3 = 9x + 28, \quad 23x^3 + 32x = 905\ 5/9, \quad x^3 = 3x^2 + 5x + 16.$$

He solved these using Cardan's method /7b;465/.

After the ground work had been laid by Cardan and Tartaglia, it was Francois Vieta (1540-1603), one of the greatest mathematicians of the sixteenth century, who generalized the method. Vieta began with the form

$$x^3 + px^2 + qx + r = 0$$

and using the substitution

$$x = y - p/3$$

reduced the equation to the form

$$y^3 + 3by = 2c.$$

He then made the substitution

$$z^3 + yz = b, \text{ or } y = b - z^2/z,$$

which gave

$$z^6 + 2cz^3 = b^3$$

which he solved as a quadratic /4;222/, /7b;465/, /5;205/, /8;137/.

He made an outstanding contribution to the symbolism of algebra and to the development of trigondmetry. Vieta showed that both the trisection and the duplication problems depend upon the solution of cubic equations. Work on these problems led him to the discovery of a trigonometrical solution of Cardan's irreducible case in cubics. He applied the equation:

$$(2 \cos 1/3\ \phi)^3 - 3(2 \cos 1/3\ \phi) = 2 \cos \phi$$

to the solution of $x^3 - 3a^2x = a^2b$, when a 1/2b, by placing $x = 2a \cos 1/3\ \phi$, and determining ϕ from $b = 2a \cos \phi$ /1;147-150/.

Vieta clearly established the fact that although he did not recognize negative roots of cubic equations, he understood the relationship between the positive roots and its coefficients. In his treatise <u>De Emendatione Aequationum</u> (published by Alexander Anderson in 1615) he states that the equation whose roots are x = a, b, and c is /10;149/.

$$x^3 - (a + b + c)x^2 + (ab + bc + ca)x = abc.$$

This relation can be used to solve problems of the following type:

Given $3x^3 - 16^3x^2 + 23x - 6 = 0$ and the product of two roots is 1.

Let the roots be a, b, c. Their sum is 16/3; their sum taken two at a time is 23/3; their product is 2. From this last relationship, c is found immediately to be 2. The other roots may be found either by solving the remaining equations for a and b or by reducing the original equation to a quadratic by dividing it by x - c.

As was recognized by Vieta, Rene Descarte (1596-1650) proved that every geometric problem giving rise to a cubic equation can be reduced either to the duplication of the cube or to the trisection of an angle. In the third book of his Géomètrie he pointed out that if a cubic equation (with rational coefficients) has a rational root, then it can be factored and the cubic can be solved geometrically by the use of ruler and compasses. He derived the cubic $z^3 = 3z - q$ as the equation upon which the trisection of an angle depends and effected the trisection with the aid of a parabola and circle /8;173-180/. He also gave the rule for determining a limit to the number of positive and negative roots of an algebraic equation and introduced the method of indeterminate coefficients for the solution of equations.

CHAPTER VI

THE SEVENTEENTH AND EIGHTEENTH CENTURIES

The latter half of the seventeenth century saw the beginning of many attempts to refine and simplify solutions to cubic equations. There was increased activity in the search for new solutions.

Johann Hudde (1633-1704) was the author of an ingenious method for finding equal roots. As an illustration let the equation be

$$x^3 - x^2 - 8x + 12 = 0.$$

Taking an arithmetical progression 3,2,1,0,of which the highest term is equal to the degree of the equation, multiply each term of the given equation by the corresponding term of the progression obtaining

$$3x^3 - 2x^2 - 8x + 0 = 0$$

or $$3x^2 - 2x - 8 = 0.$$

The last equation is one degree lower than the original one. The next step in the procedure is to find the greatest common divisor of the two equations. This is $x - 2$, so 2 is one of the two equal roots. If there had been no common divisor then the original equation would not have had equal roots /8;180/.

This method will be recognized as being analagous to the modern method of determining when a cubic equation has equal roots. Taking the derivative of the original cubic, the equation

$$3x^2 - 2x - 8 = 0$$

is immediately obtained. The rest of the procedure is the same.

Hudde simplified the work of Vieta. He also realized that a letter in an equation might stand for either a positive or negative number.

His method of solving the cubic equation is as follows /7b;466/:

Given $x^3 = qx + r$

let $x = y + z$

Substituting this in the original equation one obtains

$$y^3 + 3y^2z + 3yz^2 + z^3 = qx + r.$$

Now let $y^3 + z^3 = r$

and $3zy^2 + 3z^2y = qx$

which gives $y = q/3z$.

Therefore,

$$y^3 = r - z^3 = q^3/27z^3,$$

and $z^3 = r/2 \pm \sqrt{r^2/4 - q^3/27} = A$

and $y^3 = r/2 \mp \sqrt{r^2/4 - q^3/27} = B.$

Hence,

$$x = \sqrt[3]{A} + \sqrt[3]{B}.$$

Christian Huygens (1629-1695) used a method similar to Hudde's /1;145/. He began with

$$x^3 + px - q = 0$$

and let $x = y - z$ and $3yz = p$.

Through these substitutions is obtained

$$y^3 - z^3 - q = 0\ , \quad y^3 - p^3/27y^3 - q = 0\ ,$$

$$y^6 = qy^3 + p^3/27\ , \quad y = \sqrt[3]{q/2 + \sqrt{q^2/4 + p^3/27}}\ .$$

From this value of y one can obtain z and find

$$x = \sqrt[3]{q/2 + \sqrt{q^2/4 + p^3/27}} - p/3/\sqrt[3]{q/2 + \sqrt{q^2/4 + p^3/27}}\ .$$

In tracing back a geometrical solution of the trisection of an angle, Albert Girard in 1629 gave his solution in algebraic terms as follows /1;150/:

Given

$$x^3 = px + q$$

let

$$x_1 = 2r \cos \phi$$

$$x_2 = - 2r \cos (60^\circ + \phi)$$

$$x_3 = - 2r \cos (60^\circ - \phi)$$

where

$$r = \sqrt{p/3} \quad \text{and} \quad \cos 3\phi = 3q/2pr.$$

A. Cagnoli in 1786 gave a trigonometric solution using as his given equation:

$$x^3 - px \pm q = 0 \quad \text{where} \quad 4p^3 \geqq 27\,q^2 .$$

He uses the relationship

$$\sin 3\phi = 3q/p \cdot 1/2\sqrt{p/3}$$

and with this help determines that

$$x_1 = \pm \sin \phi \cdot 2\ p/3$$
$$x_2 = \pm \sin (60^\circ - \phi) \cdot 2\sqrt{p/3}$$
$$x_3 = \mp \sin (60^\circ + \phi) \cdot 2\sqrt{p/3} .$$

For the form $x^3 + px \pm q = 0$ he uses the tangents of two angles to obtain his solution /1;151/.

Ehrenfried W. Tschirnhausen (1631-1708) endeavored to solve equations of any degree by removing all terms except the first and last. This method bears his name. The equation $x^3 + qx + r = 0$ is combined with the equation $x^2 + vx + w = y$ and x is eliminated between them leaving a cubic equation in y. Through an appropiate selection of v and w, one obtains

$$y^3 = \text{a constant}$$

and by substitution in the quadratic in x, a solution is found /1;146/.

Isaac Newton (1642-1727) published in 1707 a treatise restating Descartes' rule of signs in accurate form and gave formulae expressing the sum of the powers of roots up to the sixth power /8;191-205/. He used his formulae for fixing an upper limit of real roots by showing that the sum of any even power of all the roots must exceed the same even power of any one of the roots. He showed that in equations with real coefficients, imaginary roots always occur in pairs. He also developed a rule for determining the inferior limit of the number of imaginary roots, and the superior limits for the number of positive and negative roots. He did not prove this rule.

The treatise on "Method of Fluxions" contains Newton's method of approximating the roots of numerical equations. The earliest printed account of this appeared in Wallis' Algebra in 1685. He explained it by working one example /8;202/, /7b;473/. He assumes an approximate value is known which differs from the true value by less than one-tenth of that value.

Given

$$y^3 - 2y - 5 = 0$$

He takes $2<y<3$ and substitutes $y = 2 + p$ in the equation, which becomes $p^3 + 6p^2 + 10p - 1 = 0$. Neglecting the higher powers of p, he gets $10p - 1 = 0$. Taking $p = .1 + q$, he gets $q^3 + 6.3q^2 + 11.23q + .061 = 0$. From $11.23q + .061 = 0$ he gets $q = -.0054 + r$, and by the same process, $r = -.00004853$. Finally $y = 2 + .1 - .0054 - .00004853 = 2.09455147$. Newton arranges his work in a paradigm /64/. If there is doubt whether $p = .1$ is sufficiently close, he suggests finding p from $6p^2 + 10p - 1 = 0$. He does not show that even the last will give a close solution. He finds by the same type of procedure - by a rapidly converging series - the value of y in terms of x in the equation $y^3 + axy + a^2y - x^3 - 2a^3 = 0$.

Joseph Raphson 91648-1715) in 1690 gave a method closely resembling Newton's. The only difference is that Newton derived each successive step, p,q,r, of approach to the root, from a new equation, while Raphson found it each time by substitution in the original equation. In Newton's cubic, Raphson would not find the second correction by the use of $x^3 + 6x^2 + 10x - 1 = 0$ but would substitute $2.1 + q$ in the original equation, finding $q = -.0054$. He then would substitute $2.o946 + r$ in the original equation, finding $r = .0004853$, and so on. The method used in modern textbooks, therefore, should properly be called the Newton-Raphson method /8;202/.

The defects in the process (successive corrections not always yielding results converging to the true value of the root sought) were removed by J. Raymond Mourraille in 1768 and a half century later by Fourier /8;247/. Mourraille and Fourier introduced geometrical considerations. Mourraille concluded that security is insured if the first approximation is selected so that the curve is convex toward the axis of x for the interval between the approximation and the root. He shows that this condition is sufficient, but not necessary.

A noteworthy algebra by John Wallis (1616-1703) was published in 1685. In this work Wallis makes the first recorded attempt to give a graphical representation of complex roots. This book was the first serious attempt to give a history of mathematics in England. His account of the history of mathematics in antiquity was very comprehensive. This book was used as a standard textbook for many years. The algebra of Thomas Harriot (1560-1621) and Newton were discussed in detail. In many instances, work which should have been credited to Vieta, Girard, and Descartes was credited to Harriot by Wallis. In the <u>Treatise of Algebra</u> it is difficult to distinguish between what was done by others and the work actually done by Wallis. He gives twenty-five improvements in algebra.

In Section V of the Algebra, the following propositions are given /10;141/:

(1) An equation $a^3 - 3b^2a = 2c^3$ is satisfied by one root if b is less than c; as a matter of fact, in that case there is only one positive root.

(2) The equation $a^3 - 3b^2 = 2c^3$ is satisfied by one root if $c = b$.

(3) The equation $a^3 - 3b^2a = -2c^3$ is satisfied by two roots when c is less than b, whereas, of course, in that particular case the equation has two positive roots and one negative.

Wallis was familiar with imaginaries, and he knew that all such roots occured in pairs. Moreover he would not allow the use of the word "impossible" as applied to an equation with imaginary roots. In his own words, "These are not impossible equations, and are not altogether useless but may be made use of to very good purposes. For they serve not only to show that the case proposed (which resolves itself into such an impossible equation) is an Impossible case, and cannot be so performed as was supposed, but it also shows the measure of that impossibility, how far it is impossible, and what alteration in the case proposed would make it possible." /10;156/.

Harriot had shown how, by multiplication, compound equations could be derived from laterals. Wallis carried the investigation much further, by illustrating how these compound equations might by division be reduced to much simpler equations. Wallis' own example is as follows /10;157/:

The equation

$$\begin{aligned} aaa - baa &+ bca - bcd = 0 \\ - caa &+ bda \\ - daa &+ cda \end{aligned}$$

is composed of three laterals,

$$a - b = 0,$$
$$a - c = 0,$$
$$a - d = 0.$$

Now suppose the compound equation be divided by one of the simple, say by

$$a - d = 0,$$

the result is a quadratic equation containing the other two roots

$$\begin{aligned} aa - ba& \\ - ca &+ bc = 0. \end{aligned}$$

Hence to solve an equation such as

$$aaa - 10aa + 31a - 30 = 0$$

Wallis says, "If by any means I have discovered the value of one root, (suppose $a = 2$) I may (dividing the original equation by $a - 2$) depress that cubic into a quadratic, namely,

$$aa - 8a + 15 = 0,$$

which can easily be solved ".

Harriot investigated the equation

$$aaa + 3bba = 2ccc.$$

His method was essentially the same as Vieta's. Putting $a = ee - bb/e$, one obtains (in modern notation)

$$\frac{e^6 - 3b^2e^4 + 3b^4e^2 - b^6 + 3b^2e^4 - 3b^4e^2}{e^3} = 2c^3,$$

whence

$$e^6 = b^6 + 2c^3e^3,$$

which was solved by completing the square.

Wallis gave a new method of solving the cubic which he said he discovered about 1647. This method is substantially an application of Cardan's, though Wallis claimed he knew nothing of it. This investigation includes a method of extracting the cube root of a binomial which he claimed as his own. This method had previously been given by Girard. The method is illustrated by Wallis in the following manner /10;159/.

Given	$aaa - 6a = 40$
Put	$a = \frac{e^2 + 2}{e},$
the equation becomes	$e^2 - 40e^3 + 8 = 0,$
whence	$e^3 = 20 \pm \sqrt{392},$
This becomes	$e^3 = 20 \pm 14\sqrt{2}$
from which	$e = 2 \pm \sqrt{2}.$
so that	$a = \frac{e^2 + 2}{e} = \frac{8 \pm 4\sqrt{2}}{2 \pm \sqrt{2}} = 4.$

To this Wallis says, "Therefore, those equations which have been reputed desparate are truly solved as the others, and thus by casting out the second term, the cubic may be reduced to one of the following forms, of which one root at least is Real, Affirmative or Negative; (the others being sometimes Real and sometimes Imaginary)" /10;160/.

Equation	Root
$aaa + 3ba - 2d = 0.$	$+\sqrt[3]{d + \sqrt{dd + bbb}} - \sqrt[3]{-d + \sqrt{dd + bbb}} = a.$
$aaa + 3ba + 2d = 0.$	$-\sqrt[3]{d + \sqrt{dd + bbb}} + \sqrt[3]{-d + \sqrt{dd + bbb}} = a.$
$aaa - 3ba - 2d = 0.$	$+\sqrt[3]{d + \sqrt{dd - bbb}} + \sqrt[3]{+d - \sqrt{dd - bbb}} = a.$

$aaa - 3ba + 2d = 0. \quad -\sqrt[3]{d + \sqrt{dd - bbb}} - \sqrt[3]{+ d - \sqrt{dd - bbb}} = a.$

Having given the solution of each of these types, Wallis had given a solution of all cubic equations, at least, as far as one of its roots is concerned. He had previously shown that once one root is discovered, the cubic can always be reduced to a quadratic by division.

Joseph Louis Lagrange (1736-1813) gave a solution of cubic equations by a method of combinations. Previous solutions, as has been seen, were made by a substitution method. In the substitution method the original forms are transformed so that the determination of the roots is made to depend upon simpler functions (resolvents). In the method of combination auxiliary quantities are substituted for certain simple combinations (types) of the unknown roots of the equation, and auxiliary equations (resolvents) are obtained for these quantities with the aid of the coefficients of the given equation. "Réflexions sur la résolution algébrique des équations" (published in Memoirs of the Berlin Academy for the years 1770 and 1771) contains all known algebraic solutions of equations of lower degree whose roots are linear functions of the required roots, and of the roots of unity. In this study Lagrange considered the number of values a rational function can assume when its variables are permuted in every possible way /8;253/.

Langrange's solution of the cubic equation must be introduced by a brief discussion of Newton's formulas and the fundamental theorem on symmetric functions upon which it depends. Newton's formulas are a particular case of the general theorem of symmetric functions.

Symmetric functions of the roots of an equation are those which are not altered if any two of the roots are interchanged. For example, if x_1, x_2, x_3 are the roots of a cubic equation,
$x_1 + x_2 + x_3$, $x_1x_2 + x_1x_3 + x_2x_3$, $x_1x_2x_3$ are symmetric functions, for all the roots are involved alike. The functions p_1, q and r of

$$x^3 + p_1x^2 + qx + r = 0$$

are the simplest symmetric functions of the roots, each root entering in the first degree only in any one of them. One can often obtain a great variety of symmetric functions in terms of the coefficients of the equation whose roots are being considered. A symmetric function is usually represented by the Greek letter Σ attached to one term of it, from which the entire expression may be written /25;134/.

Lagrange considered a rational integral function in three variables x_1, x_2, x_3. If they are permuted in six possible manners, the function, in general, will acquire six distinct values. In particular cases, however, it may happen that the number of distinct values will be less than six, then it will be either one (for symmetric functions) or two or three. Lagrange showed that it is possible to find a linear function whose cube has only two different values. His method is as follows /24;273/. Let ω be an imaginary cube root of unity and consider the linear function

$$x_1 + \omega x_2 + \omega^2 x_3.$$

To every even permutation of the indices 123, 231, 312 there correspond three values of this function:

$$y_1 = x_1 + \omega x_2 + \omega^2 x_3 ,$$
$$y_2 = x_2 + \omega x_3 + \omega^2 x_1 ,$$
$$y_3 = x_3 + \omega x_1 + \omega^2 x_2 ,$$

and to every odd permutation 132, 213, 321 there correspond three more:

$$y_4 = x_1 + \omega x_3 + \omega^2 x_2 ,$$
$$y_5 = x_2 + \omega x_1 + \omega^2 x_3 ,$$
$$y_6 = x_3 + \omega x_2 + \omega^2 x_1 .$$

Observe that

$$y_2 = \omega^2 y_1 ,$$
$$y_3 = \omega y_1 ,$$
$$y_5 = \omega y_4 ,$$
$$y_6 = \omega^2 y_4 ,$$

so that

$$y_1^3 = y_2^3 = y_3^3 ,$$

$$y_4^3 = y_5^3 = y_6^3 .$$

Hence,

$$(x_1 + \omega x_2 + \omega^2 x_3)^3$$

has only two distinct values

$$t_1 = (x_1 + \omega x_2 + \omega^2 x_3)^3 , \quad t_2 = (x_1 + \omega^2 x_2 + \omega x_3)^3,$$

and the combinations $t_1 + t_2$ and $t_1 t_2$ are symmetric functions of x_1, x_2, x_3.

Suppose that x_1, x_2, x_3 are the roots of a cubic equation

$$x^3 + px^2 + qx + r = 0,$$

then

$$t_1 + t_2 = 2\Sigma x_1^3 - 3\Sigma x_1^2 x_2 + 12 x_1 x_2 x_3$$

in which

$$\Sigma x_1^3 = -p^3 + 3pq - 3r, \Sigma x_1^2 x_2 = -pq + 3r, \Sigma x_1 x_2 x_3 = -r$$

and on substituting

$$t_1 + t_2 = -2p^3 + 9pq - 27r.$$

Also in finding t_1t_2 :

$$\sqrt[3]{t_1t_2} = \Sigma x_1^2 - \Sigma x_1x_2.$$

where

$$\Sigma x_1^2 = p^2 - 3q \quad \text{and} \quad \Sigma x_1x_2 = q$$

from which

$$t_1t_2 = (p^2 - 3q)^3 .$$

Consequently, t_1 and t_2 are the roots of the quadratic equation

$$t^2 + (2p^3 - 9pq + 27r)\, t + (p^3 - 3q)^3 = 0$$

and can be found algebraically. Having found t_1 and t_2, on extracting cube roots, one obtains

$$x_1 + \omega x_2 + \omega^2 x_3 = \sqrt[3]{t_1} \text{, and } x_1 + \omega^2 x_2 + \omega x_3 = \sqrt[3]{t_2} ,$$

and also

$$x_1 + x_2 + x_3 + = -p.$$

By solving these equations one obtains the roots

$$x_1 = 1/3(-p + \sqrt[3]{t_1} + \sqrt[3]{t_2}),$$

$$x_2 = 1/3(-p + \omega^2 \sqrt[3]{t_1} + \omega \sqrt[3]{t_2})$$

$$x_3 = 1/3(-p + \omega \sqrt[3]{t_1} + \omega^2 \sqrt[3]{t_2})$$

of the cubic equation. Between the cube roots there exists the relation

$$\sqrt[3]{t_1} \cdot \sqrt[3]{t_2} = p^2 - 3q .$$

CHAPTER VII

THE NINETEENTH AND TWENTIETH CENTURIES

The search for new methods of solutions to cubic equations continues into modern times. Some of the methods consist of different approaches to known procedures /75/, /79/. Solutions using mechanical apparatus and hydraulic apparatus have been suggested /70/, /73/. Work continues to be done on the nature of the roots and on the approximation to the roots /71/, /77/, /79/. Several different procedures will be presented here.

The value of x in any algebraic equation may be expressed as an infinite series. Let the equation be of any degree, and by dividing by the coefficient of the first power of x let it be placed in the form

$$a = x + bx^2 + cx^3 + dx^4 + ex^5 + fx^6 + \dots\dots .$$

Now let it be assumed that x can be expressed by the series

$$x = a + ma^2 + na^3 + pa^4 + qa^5 + \dots\dots .$$

By inserting this value of x in the equation and equating the coefficients of like powers of a, the values of m, n, etc. are found, and then the following is an expression of one of the roots of the equation:

$$x = a - ba^2 + (2b^2 - c)a^3 - (5b^3 - 5bc + d)a^4$$
$$+ (14b^4 - 21b^2c + 6bd + 3c^2 - e)a^5 \ldots\ldots$$

In order for this series to converge rapidly, a must be a small fraction.

To apply this to a cubic equation, the coefficients d, e, etc. are made zero.

Example:

$$x^3 - 3x + 0.6 = 0$$

Divide by 3, the coefficient of the x term and obtain

$$0.2 = x - 1/3x^3.$$

Then

$$a = 0.2,\ b = 0,\ c = -1/3,$$

and

$$x = 0.2, + 0.2^3/3 + 0.2^5/3 + \ldots. = 0.20277$$

which is the value of one of the roots correct to the 4th decimal place /33;27/.

When a cubic equation has three real roots, a convenient solution is by trigonometry.

Given

$$y^3 + 3By + 2C = 0$$

let

$$y = 2r \sin\theta$$

then

$$8 \sin^3 \theta + 6 B \sin\theta / r^2 + 2C/r^3 = 0.$$

By comparison with the known identity

$$8\sin 3\theta - 6\sin\theta + 2 \sin 3\theta = 0$$

$r = -B \quad \sin 3\theta = c/\sqrt{-B^3}$, in which B is always negative for the case of three real roots.

3θ is found in a table and then θ is known. Therefore,

$$y_1 = 2r\sin\theta$$

$$y_2 = -2r\sin(240^\circ + \theta)$$

$$y_3 = 2r\sin(120^\circ + \theta)$$

are the real roots.

When B^3 is negative and is less than C^2, and when B^3 is positive, the solution fails since one root is real and the others imaginary. In this case, a similar solution is obtained by means of hyperbolic sines /69/.

D. B. Steinman in 1950 gave a shortcut method for solving cubics /53/. The solution is explained by use of numerical examples. Two of them will be given here.

(1) $x^3 = 5x^2 + 2x + 3$

Write the coefficients and computations in the following manner.

	5	25	135	740 ....
		2	10	54 ....
			3	15 ....
$r =$	5	27	148	809 ....

$x = 809/148 = 5.466$

Explanation:

The coefficient of x^3 is unity. The other given numerical coefficients (5, 2, 3) are written diagonally across the rows and columns. The rows are written by multiplying the coefficients heading each row by the sequence of values of r, as these values become available. The values of r are written by adding the terms in each respective column. The values of r_n/r_{n-1}, the successive approximations to x, are given by the converging ratios:

$x = 5/1,\ 27/5,\ 148/27,\ 809/148,\ \ldots.$

or $x = 5,\ 5.4,\ 5.48,\ 5.466,\ \ldots.$

Stopping the computation at this point gives the answer correct to four significant figures.

(2) $x^3 = 2x^2 - 5x + 1$

Let $x = 1/y$

Then $y^3 = 5y^2 - 2y + 1$

Write in the following manner.

	5	25	115
		-2	-10
			1
r =	5	23	106

$x = 1/y = 0.217$

In example (1), the absolute value of x was greater than unity which is a necessary condition for convergence. If the absolute value of x is less than unity, then by substituting the reciprocal one obtains an equation which has a root whose absolute value is greater than one. The transformed equation has the coefficients of the squared term and the first degree term interchanged, with the signs changed. Since the solution requires 1/y, the inverse ratios are used to give the reciprocal.

The application of this method can be further simplified in special cases. It can be used when the coefficient of x^3 is not unity. It can be used to find the cube root of a number.

Consider the equation

$$x^3 + x^2 - 2 = 0$$

By transforming this equation by the substitution

$$x = y - 1/3$$

to

$$y^3 - 1/3y - 52/27 = 0$$

and using the Cardan-Tartaglia Formula, one obtains the following real root for the original equation:

$$x_1 = 1/3 \left(\sqrt[3]{26 + 15\sqrt{3}} + \sqrt[3]{26 - 15\sqrt{3}} - 1\right)$$

Using Steinman's method:

$$x^3 = -x^2 + 2$$

$$\begin{array}{rrr} -1 & 1 & -1 \\ & 0 & 0 \\ & & 2 \\ \hline -1 & 1 & 1 \end{array}$$

$r = -1 \quad 1 \quad 1$

$x = 1/1; 1/1$

$x = 1$

Which is exact and the only real root of the equation. However, this method has its weaknesses.

CHAPTER VIII

THE CONCLUSION, BIBLIOGRAPHY AND NOTES

This history has presented a survey of the many attempts to interpret and obtain solutions to cubic equations - - a survey covering almost 4000 years. It has been seen that there are many different methods of solution and a number of different approaches to solutions. They are varied and interesting. All of them contribute to a more clear definition of the problems involved.

Some of the procedures are short, but are not always accurate or are only approximations. There are some which are exact but are long and tedious. There are methods which are long and not always accurate. Improvements are needed.

By now it is hoped that there are many ideas for new solutions in the mind of the reader. Perhaps also he is asking himself questions.

It is sincerely hoped, however, he is not saying to himself what Omar Khayyam said in his Rubaiyat as translated by Edward Fitzgerald:

"Myself when young did eagerly frequent
Doctor and Saint, and heard great argument
About it and about; but evermore
Came out by the same door where in I went."

One of the most important parts of any history is the source material. The material presented here has been selected carefully in order that it may be of maximum use to the reader. A number of references have been discarded as not containing sufficient material on the subject to warrent the time it takes to obtain a reference work. Some have not been included because of their poorly presented contents.

References in several foreigh languages are here since the scholar has at least an acquaintance with some language other than his own.

Summaries are given of many of the articles. Notes are included when it appeared they would be helpful. The numbers appearing in parenthesis following the reference work are the Library of Congress card catalogue numbers.

It is hoped that this selected listing of source material will be of maximum benefit to the reader.

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49. Hogan, Joseph T., "Simple Chart Solves Cubic Equations" Chemical Engineering v.62, Dec. 1955, p. 222. (TN 1M45)

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The derivation of this method is not presented in this article. The steps of the method may be constructed from the following:

(1). With the cubic equation stated in the form $x^3 + ax^2 + bx + c = 0$ determine the coordinates of the point of inflection.

(2). Restate the cubic equation in terms of the new coordinates in reference to the new axis that intersect at the point of inflection of the cubic curve.

(3). Three distinct cases arise. Each is identified by the sign of the slope of the cubic curve at the point of inflection.

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VITAE

Minna Newbold Burgess Connor was born in Petersburg, Virginia on August 21, 1922. She was the fourth child of a family of three girls and two boys. Her elementary education was received in North Carolina and West Virginia. She was graduated from Eastern High School in Baltimore, Md. in 1939. She attended Concord College, Athens, W. Va. and was graduated with a Bachelor of Science degree in January, 1946. While here she assisted in the art and physics departments. The years between 1939 and 1946 were war years. Consequently her activities were varied in this interval. She worked for Western Electric Company, Baltimore, Md. as an electrical equipment inspector in 1942. She taught high school in West Virginia in 1943. In 1944 she attended Purdue University, Lafayette, Ind., receiving a diploma after completing a special wartime course in Aeronautical engineering. After leaving Purdue she was employed by the Curtis-Wright Corporation, Columbus, Ohio as a flight-test engineer's assistant until the time of her marriage to John Samuel Connor of Allentown, Penna. in April, 1945. Mr. Connor is now the division engineer of Allied Products of Reynolds Metals Company. In Januray, 1946 she began her duties as a graduate research fellow in Psychology at Lehigh University, Bethlehem, Penna. She continued in this capacity until June, 1948. Since then the Connors have lived in several cities in Pennsylvania, in St. Louis, Mo. and for their second time in Richmond. In 1953 she taught school in Chesterfield County. She began her graduate work in mathematics at the University of Richmond in September, 1954. She receives her Master of Science degree in August, 1956.