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Rank one perturbations of self-adjoint operators

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Rank One Perturbations of Self-Adjoint Operators

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1. Introduction

A linear operator T on a Hilbert space $\mathcal{H}$, with inner product $\langle \cdot, \cdot \rangle$, is said to be cyclic if there exists a vector $v \in \mathcal{H}$, a cyclic vector for T, so that the linear span of $\{v, Tv, T^2v, T^3v, \cdots\}$ is all of $\mathcal{H}$. The operator T is self-adjoint if $\langle Tx, y \rangle = \langle x, Ty \rangle$ for all $x, y \in \mathcal{H}$. Two examples of cyclic self-adjoint operators are (1) the operator

$$T : \mathbb{C}^n \to \mathbb{C}^n, \quad Tx = Ax,$$

where $A^* = A$ is a self-adjoint $n \times n$ matrix with distinct eigenvalues and (2) the operator

$$T : L^2[0,1] \to L^2[0,1], \quad (Tf)(x) = xf(x).$$

Note that in (1) the inner product on $\mathbb{C}^n$ is

$$\langle \mathbf{v}, \mathbf{w} \rangle = \sum_{i=1}^{n} v_i \overline{w_i},$$

while the inner product for (2) is

$$\langle f, g \rangle = \int_0^1 f(x)\overline{g(x)}dx.$$

The spectral theorem for cyclic-self adjoint operators T says that there is a measure μ_T on $\mathbb{R}$ so that T is unitarily equivalent to the operator

$$M^\mu : L^2(\mu) \to L^2(\mu), \quad (M^\mu f)(x) = xf(x).$$

In this thesis, I will discuss the details of the work of Simon and Wolff [1, 2] which deals with the properties of the spectral measures of rank-one perturbations of operators. In particular, I will deal with the following problem: Given a cyclic self-adjoint operator T on $\mathcal{H}$ with cyclic vector v, form the family of operators

$$T_\lambda = T + \lambda(v \otimes v),$$

where $\lambda \in \mathbb{R}$ and $(v \otimes v)(w) = \langle w, v \rangle v$. These operators turn out to be cyclic and self-adjoint (see the details in the thesis) and so, by the spectral theorem, there is a family of measures $\{\mu_\lambda : \lambda \in \mathbb{R}\}$ associated with the family $\{T_\lambda : \lambda \in \mathbb{R}\}$.

I will focus on this, almost magical, property of these measures:

$$\int_{-\infty}^{\infty} \left(\int f(x) d\mu_\lambda(x) \right) d\lambda = \int f(x)dx.$$

This theorem was shown by Simon[1] but the details in their paper are a bit vague. In this thesis, we will prove this theorem in its full detail. We will also work out some specific examples this theorem in two main cases (1) self-adjoint matrices and (2) multiplication by x on $L^2[0,1]$.

In Section 2 of this thesis, we prove the spectral theorem (as stated above) for cyclic-self adjoint matrices. In Section 3, we prove the Simon-Wolff formula which requires an elaborate approximation argument using harmonic functions and the Hahn-Banach separation theorem. In Section 4 we work out some specific examples of the Simon-Wolff formula for self-adjoint matrices – proving some interesting integration formulas along the way. In section 5, we compute the family of spectral measures for multiplication by x on $L^2[0,1]$.

2. The Spectral Theorem

We will need the spectral theorem stated in terms of $L^2(\mu)$, where μ is a measure on $\mathbb{R}$. But before we discuss the spectral theorem, we would like to review some basic linear algebra.

Definition 2.1.

(i) An $n \times n$ matrix T of complex numbers is *self-adjoint* if $T^* = T$, where T^* is the conjugate transpose of T.
(ii) A matrix T is *cyclic* if there exists a vector $\mathbf{v}$ such that $\text{Span}\{\mathbf{v}, T\mathbf{v}, T^2\mathbf{v}\dots\} = \mathbb{C}^n$.
(iii) A matrix T is *unitary* if $T^*T = I$.

Theorem 2.2 (The Spectral Theorem)**.** *Given any self-adjoint $n \times n$ matrix T, there exists a unitary matrix P such that*

$$T = PDP^*,$$

where $D = diag\{\lambda_1, \dots, \lambda_n\}$ and $\lambda_1, \dots, \lambda_n$ are eigenvalues of T.

Proof. From linear algebra, we know that for a self-adjoint $n \times n$ matrix T, there exists an orthonormal basis for $\mathbb{C}^n$, each vector of which is an eigenvector for T. Let $\{\mathbf{v_1}, \dots, \mathbf{v_n}\}$ be such a basis, and $\{\lambda_1, \dots, \lambda_n\}$ be the corresponding eigenvalues. We then construct

$$P = \begin{bmatrix}\mathbf{v_1}| & \cdots & |\mathbf{v_n}\end{bmatrix},$$

and D a diagonal matrix with $\{\lambda_1, \dots, \lambda_n\}$ as diagonal entries. Given P and D we have

$$\begin{aligned} TP &= \begin{bmatrix}T\mathbf{v_1}| & \cdots & |T\mathbf{v_n}\end{bmatrix} \\ &= \begin{bmatrix}\lambda_1\mathbf{v_1}| & \cdots & |\lambda_n\mathbf{v_n}\end{bmatrix} \\ &= PD. \end{aligned}$$

Since the columns of P form an orthonormal basis for $\mathbb{C}^n$, we get

$$(PP^*)_{ij} = \sum_{k=1}^{n} P_{ik}\overline{P_{jk}} = \langle \mathbf{v_i}\mathbf{v_j}\rangle = 0$$

for $i \neq j$. Thus P is unitary. Therfore we have

$$T = PDP^{-1} = PDP^*.$$

□

Corollary 2.3. *A self-adjoint matrix T can be written in the form*

$$T = \lambda_1 P_1 + \cdots + \lambda_n P_n,$$

where $\{P_i : i = 1, \dots, n\}$ form a set of orthogonal projections onto the eigenspace of T according to the λ_i's.

Proof. Let $P_i = PI_iP^*$, where P is defined as in Theorem 2.2, and I_i is an $n \times n$ matrix with all zero entries except for a 1 at the ith diagonal entry. Then the desired equality comes easily from the equality in Theorem 2.2. Also it is easy to see that $P_iP_j = \delta_{ij}P_i$. □

As an example to the above corollary, consider the self-adjoint matrix

$$T = \begin{bmatrix} 1 & 2i \\ -2i & 1 \end{bmatrix}.$$

It is easy to obtain the eigenvalues $\lambda_1 = 3, \lambda_2 = -1$, and the corresponding normalized eigenvectors

$$\mathbf{v_1} = \frac{1}{\sqrt{2}} \begin{bmatrix} i \\ 1 \end{bmatrix}, \mathbf{v_2} = \frac{1}{\sqrt{2}} \begin{bmatrix} i \\ -1 \end{bmatrix}.$$

Then we have

$$P = \frac{1}{\sqrt{2}} \begin{bmatrix} i & i \\ 1 & -1 \end{bmatrix} \text{and } D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}.$$

Thus

$$\begin{aligned} T &= \lambda_1 P_1 + \lambda_2 P_2 \\ &= \lambda_1 P \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} P^* + \lambda_2 P \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} P^* \\ &= 3 \begin{bmatrix} \frac{1}{2} & \frac{i}{2} \\ -\frac{i}{2} & \frac{1}{2} \end{bmatrix} - 1 \begin{bmatrix} \frac{1}{2} & -\frac{i}{2} \\ \frac{i}{2} & \frac{1}{2} \end{bmatrix} \end{aligned}$$

Corollary 2.4. *A self-adjoint matrix T has only real eigenvalues.*

Proof. From Theorem 2.2, we take conjugate transpose and get

$$T^* = (PDP^*)^* = PD^*P^*.$$

Since $T = T^*$, we have $D = D^*$. Therefore T has only real eigenvalues. □

Theorem 2.5. *A self-adjoint operator $T : \mathbb{C}^n \to \mathbb{C}^n$ is cyclic iff T has n distinct eigenvalues.*

Proof. We will identify T with its matrix representation. If T is self-adjoint and has distinct eigenvalues, then we can write T as

$$T = PDP^{-1},$$

where $P^{-1} = P^*$, and D is a diagonal matrix with entries being the distinct eigenvalues $\{\lambda_1, \dots, \lambda_n\}$ of T. Let

$$\mathbf{v} = P \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix},$$

then we have

$$T^i\mathbf{v} = PD^iP^{-1}\mathbf{v} = P \begin{bmatrix} \lambda_1^i \\ \lambda_2^i \\ \vdots \\ \lambda_n^i \end{bmatrix}.$$

We want to show that $\{T^i\mathbf{v} : i = 0, 1, \ldots, n-1\}$ are linearly independent. Assume that there exists a vector $\mathbf{c} = (c_0, c_1, \ldots, c_{n-1})$ such that

$$\sum_{i=0}^{n-1} c_i T^i \mathbf{v} = 0,$$

which means

$$P \sum_{i=0}^{n-1} \begin{bmatrix} \lambda_1^i \\ \lambda_2^i \\ \vdots \\ \lambda_n^i \end{bmatrix} c_i = 0.$$

Now, since P^{-1} exists, we have

$$\sum_{i=0}^{n-1} \begin{bmatrix} \lambda_1^i \\ \lambda_2^i \\ \vdots \\ \lambda_n^i \end{bmatrix} c_i = 0.$$

Notice that the above is equivalent to

$$\begin{bmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \cdots & \lambda_n^{n-1} \end{bmatrix} \mathbf{c} = 0,$$

and that the Vandermonde matrix has

$$\det \begin{bmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \cdots & \lambda_n^{n-1} \end{bmatrix} = \prod_{1 \le i \le j \le n} (\lambda_j - \lambda_i) \neq 0.$$

Thus $\mathbf{c} = \mathbf{0}$ and $\{T^i\mathbf{v} : i = 0, 1, \ldots, n-1\}$ are linearly independent. Therefore T is cyclic with cyclic vector

$$\mathbf{v} = P \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

To prove the other direction, we assume for the sake of contradiction that T is cyclic and does not have distinct eigenvalues. Without loss of generality, we assume that $\lambda_1 = \lambda_2 = \lambda$, so

$$T = P \begin{bmatrix} \lambda & & & \\ & \lambda & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} P^*,$$

and

$$T^i = P \begin{bmatrix} \lambda^i & & & \\ & \lambda^i & & \\ & & \ddots & \\ & & & \lambda_n^i \end{bmatrix} P^*, i \in \mathbb{N}.$$

Let $\mathbf{v}$ be a cyclic vector of T and $\mathbf{w} = P^*\mathbf{v}$, then any vector in $\mathrm{Span}\{\mathbf{v}, T\mathbf{v}, T^2\mathbf{v}, \ldots\}$ will be of the form $q(T)\mathbf{v}$ where q is a polynomial, and hence of the form

$$q(T)\mathbf{v} = P \begin{bmatrix} w_1 q(\lambda) \\ w_2 q(\lambda) \\ \vdots \\ w_n q(\lambda_n) \end{bmatrix}.$$

Let $\mathbf{x} = (-w_2, w_1, 0, \ldots, 0)$. It is obvious that $\mathbf{x} \perp \mathrm{Span}\{\mathbf{v}, T\mathbf{v}, T^2\mathbf{v}, \ldots\}$, so $\mathrm{Span}\{\mathbf{v}, T\mathbf{v}, T^2\mathbf{v}, \ldots\} \neq \mathbb{C}^n$. This contradicts the fact that $\mathbf{v}$ is a cyclic vector for T.

□

Definition 2.6. We define $L^2(\mu) = \{f : \mathbb{R} \to \mathbb{C}, \int |f(x)|^2 d\mu(x) < \infty\}$, which is a Hilbert space with inner product

$$\langle f, g \rangle = \int f\bar{g} d\mu.$$

Theorem 2.7. *Given any cyclic self-adjoint operator $T : \mathbb{C}^n \to \mathbb{C}^n$, there exists a measure μ on $\mathbb{R}$ and a unitary operator $U : \mathbb{C}^n \to L^2(\mu)$ such that*

$$UTU^{-1} = M,$$

where $(Mf)(x) = xf(x)$ on $L^2(\mu)$.

Proof. By Corollary 2.3 we can write T in terms of its distinct eigenvalues λ_i and orthogonal projections P_i:

$$T = \sum_{i=1}^{n} \lambda_i P_i.$$

Let $\mathbf{v}$ be a cyclic vector of T, and we define a discrete measure

$$\mu = \sum_{i=1}^{n} \|P_i \mathbf{v}\|^2 \delta_{\lambda_i}$$

on $\mathbb{R}$ and the resulting $L^2(\mu) = \{f : \{\lambda_i, i = 1, \ldots, n\} \to \mathbb{C}\}$.

Now we want to show that there exists a unitary operator $U : \mathbb{C}^n \to L^2(\mu)$, such that $UTU^* = M$. Since $\{P_i v : i = 1...n\}$ forms a basis for $\mathbb{C}^n$, for any $\mathbf{w} \in \mathbb{C}^n$ we have $\mathbf{w} = \sum_{i=1}^n c_i P_i \mathbf{v}$ for some c_i's. We then define $U : \mathbb{C}^n \to L^2(\mu)$ by

$$U\mathbf{w} = \sum_{i=1}^{n} c_i \chi_{\{\lambda_i\}},$$

where for a set A we define $\chi_A(x)$ as

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Then we have

$$\begin{aligned}UT\mathbf{w} &= U\sum_{i=1}^{n}\lambda_i P_i\left(\sum_{j=1}^{n}c_jP_j\mathbf{v}\right)\\ &= U\sum_{i=1}^{n}\lambda_i c_i P_i\mathbf{v}\\ &= \sum_{i=1}^{n}\lambda_i c_i \chi_{\{\lambda_i\}}.\end{aligned}$$

On the other hand,

$$MU\mathbf{w} = M\sum_{i=1}^{n}c_i\chi_{\{\lambda_i\}} = \sum_{i=1}^{n}\lambda_i c_i\chi_{\{\lambda_i\}},$$

so we have $UT = MU$, and we want to show U is unitary, meaning that U is norm preserving and onto.

For norm preserving, given any arbitrary $\mathbf{w} \in \mathbb{C}^n$,

$$\begin{aligned}\|U\mathbf{w}\|^2 &= \int \sum_{i=1}^{n}c_i\chi_{\{\lambda_i\}}\overline{\sum_{j=1}^{n}c_j\chi_{\{\lambda_j\}}}d\mu(x)\\ &= \sum_{i=1}^{n}|c_i|^2\|P_i\mathbf{v}\|^2\\ &= \|\mathbf{w}\|^2.\end{aligned}$$

To show onto, we need to show that for every element $f \in L^2(\mu)$, there exists a $\mathbf{w} \in \mathbb{C}^n$ such that $U\mathbf{w} = f$. Since any $f \in L^2(\mu)$ can be written in the form $\sum_{i=1}^{n} c_i\chi_{\{\lambda_i\}}$, we can always find the desired $\mathbf{w} = \sum_{i=1}^{n} c_iP_i\mathbf{v}$.

□

3. The Disintegration Theorem

As we have shown in Section 2, for each cyclic, self-adjoint $T : \mathbb{C}^n \to \mathbb{C}^n$, there is a corresponding measure μ as prescribed in Therorem 2.7. Now we would like to describe one-dimentional perturbations to T as the following:

$$T_\lambda = T + \lambda\mathbf{v}\otimes\mathbf{v},$$

where $\lambda \in \mathbb{R}$ and $\mathbf{v}$ is a cyclic vector for T, and $\mathbf{v}\otimes\mathbf{v}$ is defined as the following:

Definition 3.1. We define the operation $\otimes$ that maps an ordered pair of n-dimentional vectors $\{\mathbf{v},\mathbf{w}\}$ to an $n\times n$ operator as

$$(\mathbf{v}\otimes\mathbf{w})\,\mathbf{u} = \langle\mathbf{u},\mathbf{w}\rangle\mathbf{v},$$

where $\mathbf{u}, \mathbf{v},$ and $\mathbf{w}$ are any n-dimentional vectors.

Lemma 3.2. $T_\lambda = T_\lambda^*$.

Proof. We can show that T_λ is also self-adjoint by showing that $\mathbf{v}\otimes\mathbf{v}$ is self-adjoint. For any $\mathbf{w} \in \mathbb{C}^n$,

$$\langle(\mathbf{v}\otimes\mathbf{v})\mathbf{u},\mathbf{w}\rangle = \langle\langle\mathbf{u},\mathbf{v}\rangle\mathbf{v},\mathbf{w}\rangle$$

$$= \sum_i \left(\sum_k u_k \overline{v_k} \right) v_i \overline{w_i}$$
$$= \sum_i \sum_k u_k \overline{v_k} v_i \overline{w_i}.$$

Similarly,

$$\begin{aligned}
\langle \mathbf{u}, (\mathbf{v} \otimes \mathbf{v})\mathbf{w} \rangle &= \langle \mathbf{u}, \langle \mathbf{w}, \mathbf{v} \rangle \mathbf{v} \rangle \\
&= \sum_i u_i \overline{\left(\sum_k w_k \overline{v_k} \right) v_i} \\
&= \sum_i \sum_k u_i \overline{w_k} v_k \overline{v_i} \\
&= \sum_i \sum_k u_k \overline{w_i} v_i \overline{v_k} \quad \text{(switched dummy indices } i \text{ and } k) \\
&= \langle (\mathbf{v} \otimes \mathbf{v})\mathbf{u}, \mathbf{w} \rangle.
\end{aligned}$$

Therefore T_λ is self-adjoint. □

Lemma 3.3. *T_λ is cyclic with the same cyclic vector $\mathbf{v}$ as T.*

Proof. We will show that $\text{Span}\{\mathbf{v}, T_\lambda \mathbf{v}, \dots, {T_\lambda}^n \mathbf{v}\} = \mathbb{C}^n$. First notice that

$$\begin{aligned}
T_\lambda \mathbf{v} &= T\mathbf{v} + \lambda(\mathbf{v} \otimes \mathbf{v})\mathbf{v} \\
&= T\mathbf{v} + \lambda \|\mathbf{v}\| \mathbf{v}.
\end{aligned}$$

Since $\lambda \|\mathbf{v}\| \in \mathbb{R}$, $T\mathbf{v} \in \text{Span}\{\mathbf{v}, T_\lambda \mathbf{v}\}$ and $T_\lambda \mathbf{v} = q_1(T)\mathbf{v}$, where q_i is a polynomial of order $i \in \mathbb{N}$. Now we will proceed to prove the induction statement: for all $k > 1, T^k \mathbf{v} \in \text{Span}\{\mathbf{v}, T_\lambda \mathbf{v}, \dots, {T_\lambda}^k \mathbf{v}\}$ and $T_\lambda^k \mathbf{v} = q_k(T)\mathbf{v}$ if $\mathbf{v}, T\mathbf{v}, \dots, T^{k-1}\mathbf{v} \in \text{Span}\{\mathbf{v}, T_\lambda \mathbf{v}, \dots, {T_\lambda}^{k-1} \mathbf{v}\}$ and $T_\lambda^{k-1} \mathbf{v} = q_{k-1}(T)\mathbf{v}$.

Since

$$T_\lambda^{k-1} \mathbf{v} = q_{k-1}(T)\mathbf{v},$$
$${T_\lambda}^{k-1} \mathbf{v} = \sum_{i=0}^{k-1} a_i T^i \mathbf{v}, \text{ for some } a_i \in \mathbb{C}, a_{k-1} \neq 0.$$

Then

$$\begin{aligned}
{T_\lambda}^k \mathbf{v} &= T_\lambda \sum_{i=0}^{k-1} a_i T^i \mathbf{v} \\
&= (T + \lambda \mathbf{v} \otimes \mathbf{v}) \sum_{i=0}^{k-1} a_i T^i \mathbf{v} \\
&= \sum_{i=1}^{k} a_{i-1} T^i \mathbf{v} + \lambda \langle \sum_{i=0}^{k-1} a_i T^i \mathbf{v}, \mathbf{v} \rangle \mathbf{v} \\
&= q_k(T)\mathbf{v}.
\end{aligned}$$

Since $\mathbf{v}, T\mathbf{v}, \dots, T^{k-1}\mathbf{v} \in \text{Span}\{\mathbf{v}, T_\lambda \mathbf{v}, \dots, {T_\lambda}^{k-1} \mathbf{v}\}$, we have

$$T^k \mathbf{v} \in \text{Span}\{\mathbf{v}, T_\lambda \mathbf{v}, \dots, {T_\lambda}^k \mathbf{v}\}.$$

Induction complete. Thus, $\text{Span}\{\mathbf{v}, T\mathbf{v}, \ldots, T^k - \mathbf{v}\} = \mathbb{C}^n \subseteq \text{Span}\{\mathbf{v}, T_\lambda \mathbf{v}, \ldots, {T_\lambda}^k \mathbf{v}\}$, which implies that

$$\mathbb{C}^n = \text{Span}\{\mathbf{v}, T_\lambda \mathbf{v}, \ldots, {T_\lambda}^k \mathbf{v}\}.$$

Therefore T_λ is cyclic with cyclic vector $\mathbf{v}$. □

With the above lemma, we can assign each T_λ its spectral measure μ_λ in a similar fashion as we did for T. Now we are ready to present the following disintegration theorem of Simon [1]. For the rest of this section, μ_λ is spectral measure for T_λ. Note that each μ_λ is of the form

$$\mu_\lambda = \sum_{i=1}^{n} c_j^{(\lambda)} \delta_{\lambda_j(\lambda)}$$

Theorem 3.4. *For $f \in C(\mathbb{R}) \ni f(x) \in \mathcal{O}(\frac{1}{x^2})$ as $x \to \pm\infty$,*

$$\int_{-\infty}^{\infty} \left(\int f(t) d\mu_\lambda(t) \right) d\lambda = \int_{-\infty}^{\infty} f(t) dt$$

We first show two lemmas that prove the above equality for a special family of functions:

Lemma 3.5. *Let*

$$F(z) = \int \frac{d\mu(t)}{t - z}$$

and

$$F_\lambda(z) = \int \frac{d\mu_\lambda(t)}{t - z}.$$

Then

$$F_\lambda(z) = \frac{1}{F(z)^{-1} + \lambda}.$$

Proof. For any self-adjoint operator $T : \mathbb{C}^N \to \mathbb{C}^N$ we have

$$T = \sum_{j=1}^{N} \lambda_j P_j,$$

and for any polynomial $q(x)$, we have

$$q(T) = \sum_{j=1}^{N} q(\lambda_j) P_j.$$

Thus

$$(T - zI)^{-1} = \sum_{j=1}^{N} \frac{1}{\lambda_j - z} P_j,$$

and so

$$\langle (T - zI)^{-1} \mathbf{v}, \mathbf{v} \rangle = \sum_{j=1}^{N} \frac{1}{\lambda_j - z} \langle P_j \mathbf{v}, \mathbf{v} \rangle,$$

where

$$\langle P_j \mathbf{v}, \mathbf{v} \rangle = \langle {P_j}^2 \mathbf{v}, \mathbf{v} \rangle = \langle P_j \mathbf{v}, P_j \mathbf{v} \rangle = \|P_j \mathbf{v}\|^2.$$

Hence

$$\langle (T - zI)^{-1} \mathbf{v}, \mathbf{v} \rangle = \sum_{j=1}^{N} \frac{1}{\lambda_j - z} \|P_j \mathbf{v}\|^2 = \int \frac{d\mu_T(t)}{t - z},$$

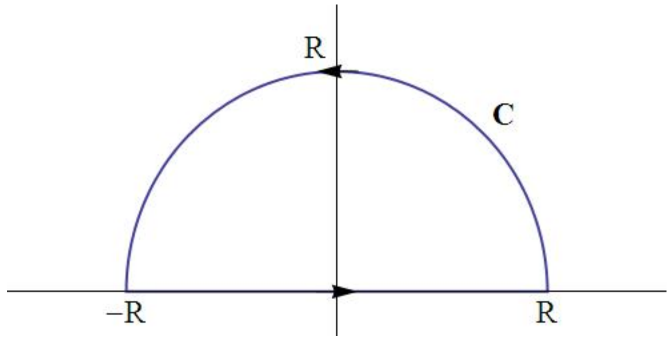

FIGURE 1. The upper hemisphere C_R and the closed path D_R

where $d\mu_T$ is the spectral measure for T. Thus

$$F_\lambda(z) = \langle (T_\lambda - zI)^{-1}\mathbf{v}, \mathbf{v}\rangle.$$

On the other hand, for any $\mathbf{w} \in \mathbb{C}^n$,

$$\begin{aligned}((T_\lambda - zI)^{-1} - (T - zI)^{-1})\mathbf{w} &= (T - zI)^{-1}(T - zI - (T_\lambda - zI))(T_\lambda - zI)^{-1}\mathbf{w} \\ &= -(T - zI)^{-1}(\lambda\mathbf{v} \otimes \mathbf{v})(T_\lambda - zI)^{-1}\mathbf{w} \\ &= -\lambda(T - zI)^{-1}\langle (T_\lambda - zI)^{-1}\mathbf{w}, \mathbf{v}\rangle\mathbf{v} \\ &= -\lambda\langle \mathbf{w}, (T_\lambda - \bar{z}I)^{-1}\mathbf{v}\rangle((T - zI)^{-1}\mathbf{v}) \\ &= -\lambda((T - zI)^{-1}\mathbf{v}) \otimes ((T_\lambda - \bar{z}I)^{-1}\mathbf{v})\mathbf{w}.\end{aligned}$$

Thus

$$\begin{aligned}F_\lambda(z) - F(z) &= \langle (((T_\lambda - zI)^{-1} - (T - zI)^{-1})\mathbf{v}, \mathbf{v}\rangle \\ &= -\lambda\langle ((T - zI)^{-1}\mathbf{v}) \otimes ((T_\lambda - \bar{z}I)^{-1}\mathbf{v})\mathbf{v}, \mathbf{v}\rangle \\ &= -\lambda\langle (T_\lambda - zI)^{-1}\mathbf{v}, \mathbf{v}\rangle\langle (T - zI)^{-1}\mathbf{v}, \mathbf{v}\rangle \\ &= -\lambda F_\lambda(z)F(z).\end{aligned}$$

Therefore

$$F_\lambda(z) = \frac{1}{F(z)^{-1} + \lambda}. \quad \square$$

The first class of functions that we will prove Theorem 3.4 for is the following:

Lemma 3.6. *For* $f_z(t) = (t - z)^{-1} - (t + i)^{-1}, z \in \mathbb{C} \setminus \mathbb{R}$,

$$\int \left(\int f_z(t) d\mu_\lambda(t) \right) d\lambda = \int f_z(t) dt.$$

Proof. For RHS, we want to show:

$$\int_{-\infty}^{\infty} f_z(t) dt = \begin{cases} 2\pi i & \text{if } \Im z > 0 \\ 0 & \text{if } \Im z < 0 \end{cases}$$

Let C_R be an open path of the upper hemisphere of radius R, and D_R the closed path of C_R and the diameter, as shown in Figure 1, then

$$\begin{aligned}
\lim_{R\to\infty}\left|\int_{C_R} f_z(t)dt\right| &= \lim_{R\to\infty}\left|\int_{C_R}\left((t-z)^{-1}-(t+i)^{-1}\right)dt\right| \\
&= \lim_{R\to\infty}\left|\int_{C_R}\frac{z+i}{(t-z)(t+i)}dt\right| \\
&= |z+i|\lim_{R\to\infty}\int_{C_R}\frac{1}{|t-z||t+i|}|dt| \\
&\le |z+i|\lim_{R\to\infty}\int_{C_R}\frac{1}{(|t|-|z|)(|t|-|i|)}|dt| \\
&= |z+i|\lim_{R\to\infty}\frac{1}{(R-|z|)(R-|i|)}\int_{C_R}|dt| \\
&= |z+i|\lim_{R\to\infty}\frac{2\pi R}{(R-|z|)(R-|i|)} \\
&= 0.
\end{aligned}$$

Thus

$$\lim_{R\to\infty}\int_{C_R} f_z(t)dt = 0,$$

and therefore

$$\begin{aligned}
\int_{-\infty}^{\infty} f_z(t)dt &= \lim_{R\to\infty}\int_{-R}^{R} f_z(t)dt \\
&= \lim_{R\to\infty}\int_{-R}^{R} f_z(t)dt + \lim_{R\to\infty}\int_{C_R} f_z(t)dt \\
&= \lim_{R\to\infty}\oint_{D_R} f_z(t)dt \\
&= \lim_{R\to\infty}\left(\oint_{D_R}(t-z)^{-1}dt - \oint_{D_R}(t+i)^{-1}dt\right).
\end{aligned}$$

Now, we will show that for each R large enough,

$$\oint_{D_R}(t-c)^{-1}dt = \begin{cases} 2\pi i & \text{if } \Im c > 0 \\ 0 & \text{if } \Im c < 0, \end{cases} \tag{3.7}$$

so for RHS

$$\begin{aligned}
\int_{-\infty}^{\infty} f_z(t)dt &= \lim_{R\to\infty}\left(\oint_{D_R}(t-z)^{-1}dt - \oint_{D_R}(t+i)^{-1}dt\right) \\
&= \begin{cases} 2\pi i & \text{if } \Im z > 0 \\ 0 & \text{if } \Im z < 0. \end{cases}
\end{aligned}$$

Definition 3.8. A function is analytic on an open set $D \subseteq \mathbb{C}$ if for all $x_0 \in D$, $f(x)$ is infinitely differentiable at x_0, and the Taylor series of f at x in a neighborhood of x_0 converges to $f(x)$.

To show Equation (3.7), we first consider the case $\Im c > 0$. We deform the contour D to a circle of radius $r = \Im c/2$. Clearly $(t-c)^{-1}$ is analytic in the region between D and the circle. By the Cauchy Deformation Theorem,

$$\begin{aligned}\lim_{R\to\infty}\oint_{D_R}(t-c)^{-1}dt &= \oint_{\odot(r)}(t-c)^{-1}dt \\ &= \int_0^{2\pi}(c+re^{it'}-c)^{-1}ire^{it'}dt' \quad (t=c+re^{it'}) \\ &= 2\pi i.\end{aligned}$$

In the case $\Im c < 0$, c is outside the contour D, so $(t-c)^{-1}$ is clearly analytic in D.By Green's Theorem,

$$\lim_{R\to\infty}\oint_{D_R}(t-c)^{-1}dt = 0.$$

This establishes the RHS of (3.4) for $f_z(t)$. Now given Lemma 3.5, the LHS of (3.4) with $f_z(t)$then becomes

$$\begin{aligned}\int\left(\int f_z(t)d\mu_\lambda(t)\right)d\lambda &= \int\left(\int(t-z)^{-1}d\mu_\lambda(t) - \int(t+i)^{-1}d\mu_\lambda(t)\right)d\lambda \\ &= \int(F_\lambda(z)-F_\lambda(-i))d\lambda \\ &= \int\left((\lambda-(-F(z)^{-1}))^{-1}-(\lambda-(-F(-i)^{-1}))^{-1}\right)d\lambda.\end{aligned}$$

Due to Equation (3.7), if we can show that $\Im z\cdot\Im(-F(z)^{-1})\geq 0$, then similar to that on RHS, we have on LHS

$$\int\int f_z(t)d\mu_\lambda(t)d\lambda = \begin{cases}2\pi i & \text{if } \Im z > 0 \\ 0 & \text{if } \Im z < 0\end{cases}.$$

To show $\Im z\cdot\Im(-F(z)^{-1})\geq 0$, first we show $\Im F(z)\cdot\Im(-F(z)^{-1})\geq 0$: let $F(z) = x+iy, x, y\in\mathbb{R}$, then

$$\Im(-F(z)^{-1}) = \Im\left(\frac{iy-x}{x^2+y^2}\right) = \frac{y}{x^2+y^2},$$

which has the same sign as $y = \Im F(z)$. Now recall that

$$F(z) = \int\frac{d\mu(t)}{t-z} = \sum_j c_j\frac{1}{t_j-z},$$

where $c_i\in\mathbb{R}^+$. Similar to what we just showed for $\Im F(z)$, $\Im\left(\frac{1}{t_i-z}\right)$ shares the same sign as $\Im(z-t_i) = \Im z$ for all i. Thus

$$\Im(F(z)) = \sum_j c_j\Im\left(\frac{1}{t_j-z}\right)$$

shares the same sign as $\Im z$. Therefore $\Im z\cdot\Im(-F(z)^{-1})\geq 0$. □

Thus, Theorem 3.4 is proved for $f_z(t)$ as a lemma. Now we want to show that the theorem works for all functions $f\in C(\mathbb{R})$ and $f\in\mathcal{O}(\frac{1}{1+x^2})$. To do this, we need a few tools.

Definition 3.9. We define the Poisson kernel:

$$P_{x+iy}(t) = \frac{1}{\pi}\frac{y}{(x-t)^2+y^2}, x \in \mathbb{R}, y \in \mathbb{R}^+.$$

It is easy to show that $\int_{-\infty}^{\infty} P_{x+iy}(t)f(t)dt$ is harmonic on the upper-half plane, and that

$$\lim_{y\to 0}\int_{-\infty}^{\infty} P_{x+iy}(t)f(t)dt = f(x), \tag{3.10}$$

for suitably smooth functions f. Now let μ be a measure on $\mathbb{R}$ with $\int_{-\infty}^{\infty}\frac{d\mu(t)}{1+t^2} < \infty$, then similarly $\int_{-\infty}^{\infty} P_{x+iy}(t)d\mu(t)$ is harmonic on $\mathbb{C}^+$.

Theorem 3.11. *Let $g \in C_c(\mathbb{R})$ and $d\mu = \sum_{j=1}^{n} c_j\delta_{\lambda_j}$, then*

$$\lim_{y\to 0^+}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} P_{x+iy}(t)d\mu(t)\right)g(x)dx \to \int_{-\infty}^{\infty} g(t)d\mu(t).$$

Proof. Since we have integration over $d\mu(t)$ on both sides, due to linearity of the discrete measure, it suffices to show that the result holds for $d\mu(t) = \delta_c(x)$ for some $c \in \mathbb{R}$.

RHS is obviously $g(c)$. Since

$$\int_{-\infty}^{\infty}\frac{y}{(x-c)^2+y^2} = \pi, \text{ for } y > 0,$$

we write RHS as

$$\int_{-\infty}^{\infty}\left(\frac{1}{\pi}\frac{y}{(x-c)^2+y^2}\right)g(c)dx,$$

and need to show that RHS = LHS. Since

$$\lim_{y\to 0^+}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} P_{x+iy}(t)d\mu(t)\right)g(x)dx = \lim_{y\to 0^+}\int_{-\infty}^{\infty}\left(\frac{1}{\pi}\frac{y}{(x-c)^2+y^2}\right)g(x)dx,$$

we have

$$LHS - RHS = \lim_{y\to 0^+}\int_{-\infty}^{\infty}\left(\frac{1}{\pi}\frac{y}{(x-c)^2+y^2}\right)(g(x)-g(c))dx.$$

Because $g(x)$ is continuous at c, there exists a $\delta > 0$ for each $\epsilon > 0$ such that for all $|x-c| < \delta, |g(x)-g(c)| < \epsilon$. Thus

$$\begin{aligned}
&\lim_{y\to 0^+}\int_{-\infty}^{\infty}\left(\frac{1}{\pi}\frac{y}{(x-c)^2+y^2}\right)(g(x)-g(c))dx \\
&= \lim_{y\to 0^+}\int_{|x-c|>\delta}\left(\frac{1}{\pi}\frac{y}{(x-c)^2+y^2}\right)(g(x)-g(c))dx \\
&\quad + \lim_{y\to 0^+}\int_{|x-c|<\delta}\left(\frac{1}{\pi}\frac{y}{(x-c)^2+y^2}\right)(g(x)-g(c))dx \\
&= 0 + \lim_{y\to 0^+}\int_{|x-c|<\delta}\left(\frac{1}{\pi}\frac{y}{(x-c)^2+y^2}\right)(g(x)-g(c))dx \\
&\leq \lim_{y\to 0^+}\int_{|x-c|<\delta}\left(\frac{1}{\pi}\frac{y}{(x-c)^2+y^2}\right)|g(x)-g(c)|dx \\
&\leq \epsilon\lim_{y\to 0^+}\int_{|x-c|<\delta}\left(\frac{1}{\pi}\frac{y}{(x-c)^2+y^2}\right)dx
\end{aligned}$$

$$= \epsilon \lim_{y\to 0^+} \frac{2}{\pi}\tan^{-1}\left(\frac{\delta}{y}\right) dx$$
$$= \epsilon.$$

Therefore $LHS = RHS$. ☐

Corollary 3.12. *If $\int_{-\infty}^{\infty} P_{x+iy}(t)d\mu(t) = 0$ for all $x, y \in \mathbb{R}$, then $\mu \equiv 0$*

Proof. From Theorem 3.11, if $\int_{-\infty}^{\infty} P_{x+iy}(t)d\mu(t) = 0$ for all $x, y \in \mathbb{R}$, then

$$\lim_{y\to 0^+}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} P_{x+iy}(t)d\mu(t)\right) g(x)dx \to 0, \text{ for all } g,$$

which means that

$$\int_{-\infty}^{\infty} g(t)d\mu(t) = 0, \text{ for all } g.$$

This can only be true if $\mu \equiv 0$. ☐

Let $\widehat{\mathbb{R}} = \mathbb{R}\cup\{\infty\}$ and define a norm in $C(\widehat{\mathbb{R}})$ by

$$\|f\|_{C(\widehat{\mathbb{R}})} = sup\{|f(x)|, x \in C(\widehat{\mathbb{R}})\}.$$

One can show that $C(\widehat{\mathbb{R}})$, with this norm, is a Banach space (a complete normed linear space).

Let ν be a finite measure on $\widehat{\mathbb{R}} = \mathbb{R}\cup\{\infty\}$, and let $\ell : C(\widehat{\mathbb{R}}) \to \mathbb{C}$ be defined by

$$\ell(f) = \int f(t)d\nu(t).$$

Then ℓ is clearly a linear transformation. We know that

$$|\ell(f)| = \left|\int f(t)d\nu(t)\right| \le \int |f(t)||d\nu(t)| \le \|f\|_{C(\widehat{\mathbb{R}})}\|\nu(\widehat{\mathbb{R}})\|.$$

This says that ℓ is continuous.

Definition 3.13. Given a Banach space $\mathcal{X}$, the dual space $\mathcal{X}^*$ is the space of all $\ell : \mathcal{X} \to \mathbb{C}$, where ℓ is linear and continuous.

Following the definition, $C(\widehat{\mathbb{R}})^*$ is the set of all continuous functions from $C(\widehat{\mathbb{R}})^*$ to $\mathbb{C}$. We know the following theorem:

Theorem 3.14 (Riesz representation theorem)**.** *For any $\ell \in C(\widehat{\mathbb{R}})^*$, there exists a measure ν on $\widehat{\mathbb{R}}$ such that*

$$\ell(f) = \int f(t)d\nu(t).$$

Theorem 3.15 (Hahn-Banach seperation theorem)**.** *Let M be a closed subspace of a Banach space $\mathcal{X}$, $M \subsetneq \mathcal{X}$, and $f_0 \notin M$. Then there exists a function $\ell \in \mathcal{X}^*$ such that*

$$\ell(f) = 0\,\forall f \in M,$$
$$\ell(f_0) = 1.$$

Combining Theorem 3.14 and Theorem 3.15, the following corollary is immediate:

Corollary 3.16. *Let M be a closed subspace of $C(\widehat{\mathbb{R}})$, $M \subsetneq C(\widehat{\mathbb{R}})$, and $f_0 \notin M$. Then there exists a finite measure ν on $\widehat{\mathbb{R}}$ such that*

$$\int f(t)d\nu(t) = 0 \, \forall f \in M,$$

$$\int f_0(t)d\nu(t) = 1.$$

A lemma then follows:

Lemma 3.17.

$$M = \mathit{Clos}(\mathit{Span}\{(1+t^2)P_{x+iy}(t) : x \in \mathbb{R}, y > 0\}) = C(\widehat{\mathbb{R}}).$$

Proof. Suppose that there exists $f_0 \in C(\widehat{\mathbb{R}}) \setminus M$, then by Corollary 3.16, there exists a measure ν such that $\int f_0(t)d\nu(t) = 1$, and that $\int f(t)d\nu(t) = 0$ for all $f \in M$. This implies that

$$\int (1+t^2)P_{x+iy}(t)d\nu(t) = 0.$$

Let $d\mu(t) = (1+t^2)d\nu(t)$ and apply Corollary 3.12, we get $\mu = 0$, and thus $\nu = 0$. This contradicts with the fact that $\int f_0(t)d\nu(t) = 1$. Therefore $C(\widehat{\mathbb{R}}) \setminus M = \emptyset$. □

Now, back to the proof that Theorem 3.4 works for all f such that $f \in C(\mathbb{R})$ and $f \in \mathcal{O}\left(\frac{1}{x^2}\right)$. Since

$$\left(\frac{1}{t-z} + \frac{1}{t+i}\right) - \left(\frac{1}{t-\bar{z}} + \frac{1}{t+i}\right) = 2iP_{x+iy}(t),$$

the theorem works for all $P_{x+iy}(t)$ with $x \in \mathbb{R}, y > 0$, i.e.

$$\int \left(\int P_{x+iy}(t)d\mu_\lambda(t)\right) d\lambda = \int P_{x+iy}(t)dt.$$

According to Lemma 3.17, for all f such that $f \in C(\mathbb{R})$ and $f \in \mathcal{O}\left(\frac{1}{x^2}\right)$, and any $\epsilon > 0$, there exists a $g(t) \in \mathrm{Span}\{P_{x+iy}(t) : x \in \mathbb{R}, y > 0\}$ such that

$$|g(t)(1+t^2) - f(t)(1+t^2)| \leq \frac{\epsilon}{2\pi}.$$

Then

$$\begin{aligned}
&\left|\int \left(\int f(t)d\mu_\lambda(t)\right) d\lambda - \int f(\lambda)d\lambda\right| \\
=&\left|\int \left(\int (f(t)-g(t))d\mu_\lambda(t)\right) d\lambda + \int \left(\int g(t)d\mu_\lambda(t)\right) d\lambda \right.\\
&\left. - \int (f(\lambda)-g(\lambda))d\lambda - \int g(\lambda)d\lambda\right|.
\end{aligned}$$

Since $g \in \mathrm{Span}\{P_{x+iy}(t) : x \in \mathbb{R}, y > 0\}$, and we have proved Theorem 3.4 for the Poisson kernels, we know that

$$\int \left(\int g(t)d\mu_\lambda(t)\right) d\lambda = \int g(\lambda)d\lambda.$$

Thus the above equation reduces to

$$\left|\int \left(\int (f(t)-g(t))d\mu_\lambda(t)\right) d\lambda - \int (f(\lambda)-g(\lambda))d\lambda\right|$$

$$\leq \int \left(\int |f(t) - g(t)| d\mu_\lambda(t) \right) d\lambda + \int |f(\lambda) - g(\lambda)| d\lambda$$

Now, since

$$\begin{aligned} \int \left(\int |f(t) - g(t)| d\mu_\lambda(t) \right) d\lambda &\leq \frac{\epsilon}{2\pi} \int \left(\int \frac{1}{1+t^2} d\mu_\lambda(t) \right) d\lambda \\ &= \frac{\epsilon}{2\pi} \int \left(\int P_{0+1i}(t) d\mu_\lambda(t) \right) d\lambda \\ &= \frac{\epsilon}{2\pi} \int P_{0+1i}(\lambda) d\lambda \\ &= \frac{\epsilon}{2\pi} \int \frac{1}{1+\lambda^2} d\lambda \\ &= \frac{\epsilon}{2\pi} \pi \\ &= \frac{\epsilon}{2}, \end{aligned}$$

and

$$\int |f(\lambda) - g(\lambda)| d\lambda \leq \frac{\epsilon}{2\pi} \int \frac{1}{1+\lambda^2} d\lambda = \frac{\epsilon}{2},$$

we have

$$\begin{aligned} &\left| \int \left(\int f(t) d\mu_\lambda(t) \right) d\lambda - \int f(\lambda) d\lambda \right| \\ &\leq \int \left(\int |f(t) - g(t)| d\mu_\lambda(t) \right) d\lambda + \int |f(\lambda) - g(\lambda)| d\lambda \\ &\leq \epsilon, \end{aligned}$$

for any $\epsilon > 0$. Therefore Theorem 3.4 is proved.

4. Some Matrix Examples

We will now compute some specific examples of the disintegration formula for $A_\lambda = A + \lambda \mathbf{v} \otimes \mathbf{v}$, where

$$A = \begin{bmatrix} a & c \\ c & b \end{bmatrix},$$

$a, b, c \in \mathbb{R}, a \neq b$. Note that for any vector

$$\mathbf{v} = \begin{bmatrix} d \\ 1 \end{bmatrix},$$

$$A\mathbf{v} = \begin{bmatrix} ad + c \\ cd + b \end{bmatrix},$$

then

$$\det \big[\mathbf{v} | A\mathbf{v} \big] = cd^2 + bd - ad - c.$$

Let $\delta = b - a$, then the roots for

$$\det \big[\mathbf{v} | A\mathbf{v} \big] = cd^2 + \delta d - c$$

would be

$$d = \begin{cases} \frac{-\delta \pm \sqrt{\delta^2 + 4c^2}}{2c} & c \neq 0 \\ 0 & c = 0. \end{cases}$$

Thus, for values of d that does not meet the above roots, $\mathbf{v}$ would be an cyclic vector for A, as well as for $A+\lambda\mathbf{v}\otimes\mathbf{v}$.

We first investigate a specific cyclic vector

$$\mathbf{v}=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}.$$

It is easy to verify that $\mathbf{v}$ will never make $\det\left[\mathbf{v}|A\mathbf{v}\right]=0$, so it is always a cyclic vector for A. A standard matrix calculation shows that the eigenvalues for A_λ are

$$\lambda_1=\frac{1}{2}\left(a+b+\lambda-\sqrt{(a-b)^2+(2c+\lambda)^2}\right),$$

$$\lambda_2=\frac{1}{2}\left(a+b+\lambda+\sqrt{(a-b)^2+(2c+\lambda)^2}\right),$$

and following the procedure described in Thereom 2.7, we have the spectral measure for A_λ:

$$\mu_\lambda=\frac{-2c-\lambda+\sqrt{(a-b)^2+(2c+\lambda)^2}}{2\sqrt{(a-b)^2+(2c+\lambda)^2}}\delta_{\lambda_1}+\frac{2c+\lambda+\sqrt{(a-b)^2+(2c+\lambda)^2}}{2\sqrt{(a-b)^2+(2c+\lambda)^2}}\delta_{\lambda_2}.$$

Then by Theorem 3.4, for $f\in C(\mathbb{R})\ni f(x)\in\mathcal{O}(\frac{1}{x^2})$ as $x\to\pm\infty$,

$$\begin{aligned}&\int_{-\infty}^{\infty}\left(\int f(t)d\mu_\lambda(t)\right)d\lambda\\ =&\int_{-\infty}^{\infty}\left(\frac{-2c-\lambda+\sqrt{(a-b)^2+(2c+\lambda)^2}}{2\sqrt{(a-b)^2+(2c+\lambda)^2}}f(\lambda_1)+\frac{2c+\lambda+\sqrt{(a-b)^2+(2c+\lambda)^2}}{2\sqrt{(a-b)^2+(2c+\lambda)^2}}f(\lambda_2)\right)d\lambda\\ =&\int_{-\infty}^{\infty}f(t)dt\end{aligned}$$

Example 4.1. Let $f(t)=e^{-t^2}$, then we have

$$\begin{aligned}&\int_{-\infty}^{\infty}\frac{1}{2\sqrt{(a-b)^2+(2c+\lambda)^2}}\times\\ &\left((\sqrt{(a-b)^2+(2c+\lambda)^2}-2c-\lambda)\exp\Big(-\frac{1}{4}(-\sqrt{(a-b)^2+(2c+\lambda)^2}+a+b+\lambda)^2\Big)\right.\\ &\left.+(\sqrt{(a-b)^2+(2c+\lambda)^2}+2c+\lambda)\exp\Big(-\frac{1}{4}(\sqrt{(a-b)^2+(2c+\lambda)^2}+a+b+\lambda)^2\Big)\right)d\lambda\\ &=\sqrt{\pi}.\end{aligned}$$

Example 4.2. Let $f(t)=\frac{1}{1+x^2}$, then we have

$$\int_{-\infty}^{\infty}\left(\frac{1}{A\lambda^2+B\lambda+C}\right)d\lambda=\pi,$$

where A,B,C are constants independent of λ:

$$A=\frac{a^2-4c(a+b)+2ab+b^2+4c^2+4}{2\left(a^2+b^2+2\right)-4c(a+b)+4c^2},$$

$$B=\frac{-2c^2(a+b)+c(4-4ab)+2(a(b(a+b)+1)+b)+4c^3}{a^2-2c(a+b)+b^2+2c^2+2},$$

$$C = \frac{2\left(a^2+1\right)\left(b^2+1\right)+c^2(4-4ab)+2c^4}{a^2-2c(a+b)+b^2+2c^2+2}.$$

If we now set $b=0, c=0$, and $a \neq 0$, we have

$$\int_{-\infty}^{\infty}\left(\frac{a^2+2}{\left(\frac{a^2}{2}+2\right)\lambda^2+2a\lambda+2(a^2+1)}\right)d\lambda = \pi.$$

We can take derivatives with respect to a on both sides, and obtain

$$\int_{-\infty}^{\infty}\left(\frac{(a+\lambda)(a\lambda-2)}{\left(a^2\left(\lambda^2+4\right)+4a\lambda+4\lambda^2+4\right)^2}\right)d\lambda = 0.$$

Example 4.3. Let $a=1, b=0, c=0$, and $f(x)=\frac{1}{1+x^p}$, where p is a positive even number. Then

$$\int_{-\infty}^{\infty}\left(\frac{1-\lambda/\sqrt{\lambda^2+1}}{\left(-\sqrt{\lambda^2+1}+\lambda+1\right)^p+2^p}+\frac{1+\lambda/\sqrt{\lambda^2+1}}{\left(\sqrt{\lambda^2+1}+\lambda+1\right)^p+2^p}\right)d\lambda = \frac{\pi\csc\left(\frac{\pi}{p}\right)}{2^{p-2}p}.$$

In addition to functions that are nonzero on $(-\infty,\infty)$, we would like to study step functions of the form

$$f(x) = g(x)(\theta_{m_1}(x)-\theta_{m_2}(x)),$$

where g is integrable on (m_1,m_2) and $\theta_m(x)$ is the Heaviside function:

Definition 4.4. For $m\in\mathbb{R}$, the function $\theta_m:\mathbb{R}\to\mathbb{R}$ is defined as

$$\theta_m(x) = \begin{cases} 0 & \text{if } x\le m \\ 1 & \text{if } x>m \end{cases}.$$

Applying Theorem 3.4 to these step functions then yields

$$\begin{aligned}
&\int_{m_1\le\lambda_1\le m_2}\left(\frac{-2c-\lambda+\sqrt{(a-b)^2+(2c+\lambda)^2}}{2\sqrt{(a-b)^2+(2c+\lambda)^2}}f(\lambda_1)\right)d\lambda \\
&+\int_{m_1\le\lambda_2\le m_2}\left(\frac{2c+\lambda+\sqrt{(a-b)^2+(2c+\lambda)^2}}{2\sqrt{(a-b)^2+(2c+\lambda)^2}}f(\lambda_2)\right)d\lambda \\
&=\int_{m_1\le\lambda_1\le m_2}f_1(\lambda)d\lambda+\int_{m_1\le\lambda_2\le m_2}f_2(\lambda)d\lambda \\
&\quad (f_1 \text{ and } f_2 \text{ are just the terms above in parentheses as a function of } \lambda) \\
&=\int_{m_1}^{m_2}g(t)dt.
\end{aligned}$$

To simplify the above equation, we would like to find out the ranges for λ corresponding to $m_1\le\lambda_1\le m_2$ and $m_1\le\lambda_2\le m_2$. We take derivative of λ_1 with respect to λ:

$$\frac{d\lambda_1}{d\lambda} = \frac{d}{d\lambda}\left(\frac{1}{2}\left(a+b+\lambda-\sqrt{(a-b)^2+(2c+\lambda)^2}\right)\right)$$

$$= \frac{1}{2}\left(1 - \frac{2c+\lambda}{\sqrt{(a-b)^2+(2c+\lambda)^2}}\right)$$
$$\geq 0.$$

Since λ_1 is monotonically increasing with respect to λ, we calculate the limits as λ approaches $\pm\infty$:

$$\lim_{\lambda\to-\infty} \frac{1}{2}\left(a+b+\lambda-\sqrt{(a-b)^2+(2c+\lambda)^2}\right) = -\infty$$
$$\lim_{\lambda\to\infty} \frac{1}{2}\left(a+b+\lambda-\sqrt{(a-b)^2+(2c+\lambda)^2}\right) = \frac{1}{2}(a+b-2c).$$

Thus, we only need to solve for λ from $\lambda_1(\lambda) = m$ for $m = m_1$ and $m = m_2$ respectively. The solution only exists for $m < \frac{1}{2}(a+b-2c)$ and is calculated to be

$$\Lambda(m) = -\frac{2\left(ab-c^2\right)+2m(-a-b)+2m^2}{a+b-2c-2m}.$$

Similarly,

$$\frac{d\lambda_2}{d\lambda} \geq 0,$$
$$\lambda_2 \in \left(\frac{1}{2}(a+b-2c), \infty\right),$$

and the solution only exists for $m > \frac{1}{2}(a+b-2c)$ in the same form $\Lambda(m)$.

Therefore, our integration formula for a step function then becomes

$$\int_{m_1}^{m_2} g(t)dt = \begin{cases} \int_{\Lambda(m_1)}^{\Lambda(m_2)} f_1(\lambda)d\lambda & \text{if } m_2 \leq \frac{1}{2}(a+b-2c) \\ \int_{\Lambda(m_1)}^{\infty} f_1(\lambda)d\lambda + \int_{-\infty}^{\Lambda(m_2)} f_2(\lambda)d\lambda & \text{if } m_1 < \frac{1}{2}(a+b-2c) < m_2 \\ \int_{\Lambda(m_1)}^{\Lambda(m_2)} f_2(\lambda)d\lambda & \text{if } \frac{1}{2}(a+b-2c) \leq m_1 \end{cases}$$

Example 4.5. Let $a = 1, b = 0, c = 0$, and $f(x) = \theta_0(x) - \theta_1(x)$. Since $\frac{1}{2}(a+b-2c) = \frac{1}{2} \in (m1, m2) = (0,1)$, we have

$$\int_{m_1}^{m_2} g(t)dt = 1$$
$$= \int_0^\infty \left(\frac{-\lambda+\sqrt{1+\lambda^2}}{2\sqrt{1+\lambda^2}}\right)d\lambda + \int_{-\infty}^0 \left(\frac{\lambda+\sqrt{1+\lambda^2}}{2\sqrt{1+\lambda^2}}\right)d\lambda$$
$$= \int_0^\infty \left(\frac{-\lambda+\sqrt{1+\lambda^2}}{\sqrt{1+\lambda^2}}\right)d\lambda \quad \text{(change of variable for the second integral)}$$

This is verifiable through classical calculation.

Example 4.6. Let $a = 1, b = -1, c = 0$, and $f(x) = x^p(\theta_0(x)-\theta_1(x)), p \in \mathbb{N}$. Since $\frac{1}{2}(a+b-2c) = 0 = m_1$, we have

$$\int_{m_1}^{m_2} g(t)dt = \frac{1}{p+1}$$
$$= \int_{-\infty}^0 \left(\frac{(\lambda+\sqrt{4+\lambda^2})^{p+1}}{2^{p+1}\sqrt{4+\lambda^2}}\right)d\lambda$$

$$= \int_0^\infty \left(\frac{(-\lambda + \sqrt{4+\lambda^2})^{p+1}}{2^{p+1}\sqrt{4+\lambda^2}} \right) d\lambda$$

5. Computing the Point Masses

In the work of Simon and Wolff [2], they present a direct way to compute the μ_λ measure in terms of the spectral measure μ_0 for a given T via the following theorem:

Definition 5.1. We define function

$$B(x) = \left(\int (x-y)^{-2} d\mu_0(y) \right)^{-1}$$

with the convention that $\infty^{-1} = 0$.

Theorem 5.2 (Simon-Wolff)**.** *Fix $\lambda \neq 0$. Then $d\mu_\lambda$ has an atom at $x_0 \in \mathbb{R}$, i.e. $\mu_\lambda(\{x_0\}) > 0$ iff*

$$\lim_{\epsilon \to 0^+} F_0(x_0 + i\epsilon) = -\lambda^{-1} \tag{5.3}$$

and

$$B(x_0) \neq 0. \tag{5.4}$$

Moreover, $\lambda^{-2}B(x_0)$ is precisely the μ_λ measure of $\{x_0\}$.

Proof. From

$$\begin{aligned} F_\lambda(x_0 + i\epsilon) &= \int \frac{d\mu_\lambda(t)}{t - (x_0 + i\epsilon)} \\ &= \int \frac{(t - x_0 + i\epsilon) d\mu_\lambda(t)}{(t-x_0)^2 + \epsilon^2}, \end{aligned}$$

we have

$$\begin{aligned} \Im F_\lambda(x_0 + i\epsilon) &= \epsilon \int \frac{d\mu_\lambda(t)}{(t-x_0)^2 + \epsilon^2}, \\ \Re F_\lambda(x_0 + i\epsilon) &= \int \frac{(t-x_0) d\mu_\lambda(t)}{(t-x_0)^2 + \epsilon^2}. \end{aligned}$$

Then, since

$$\lim_{\epsilon \to 0^+} \frac{\epsilon^2}{(t-x_0)^2 + \epsilon^2} = \begin{cases} 1 & \text{if } t = x_0 \\ 0 & \text{if } t \neq x_0 \end{cases},$$

by dominated convergence theorem, which allows us to push the limit through the integral, we have

$$\lim_{\epsilon \to 0^+} \epsilon \Im F_\lambda(x_0 + i\epsilon) = \lim_{\epsilon \to 0^+} \int \frac{\epsilon^2}{(t-x_0)^2 + \epsilon^2} d\mu_\lambda(t) = \mu_\lambda(\{x_0\}),$$

and similarly

$$\lim_{\epsilon \to 0^+} \epsilon \Re F_\lambda(x_0 + i\epsilon) = \lim_{\epsilon \to 0^+} \int \frac{\epsilon(t-x_0)}{(t-x_0)^2 + \epsilon^2} d\mu_\lambda(t) = 0.$$

Therefore,

$$\lim_{\epsilon \to 0^+} \epsilon F_\lambda(x_0 + i\epsilon) = \mu_\lambda(\{x_0\}) i.$$

Now, if $\mu_\lambda(\{x_0\}) \neq 0$, then $|F_\lambda(x_0 + i\epsilon)| \to \infty$. Together with the fact from Lemma 3.5 that

$$F_0(z) = \frac{1}{\frac{1}{F_\lambda(z)} - \lambda},$$

we have

$$\lim_{\epsilon \to 0^+} F_0(x_0 + i\epsilon) = -\lambda^{-1}.$$

This proves condition (5.3).

Moreover, by the monotone convergence theorem, which again allows us to push the limit through the integral,

$$\lim_{\epsilon \to 0^+} \epsilon^{-1} \Im F_0(x_0 + i\epsilon) = \lim_{\epsilon \to 0^+} \int \frac{d\mu_\lambda(t)}{(t - x_0)^2 + \epsilon^2} = B(x_0)^{-1}.$$

Now, if

$$\lim_{\epsilon \to 0^+} F_0(x_0 + i\epsilon) = -\lambda^{-1},$$

then

$$\lim_{\epsilon \to 0^+} \Im\left(\frac{F_0(x_0 + i\epsilon)}{\epsilon F_\lambda(x_0 + i\epsilon)} \right) = \Im\left(-\frac{\lambda^{-1}}{\mu_\lambda(\{x_0\}) i} \right) = (\lambda \mu_\lambda(\{x_0\}))^{-1}.$$

On the other hand, due to Lemma 3.5,

$$\lim_{\epsilon \to 0^+} \Im\left(\frac{F_0(x_0 + i\epsilon)}{\epsilon F_\lambda(x_0 + i\epsilon)} \right) = \lim_{\epsilon \to 0^+} \Im\big(\epsilon^{-1}(\lambda F_0 + 1)\big) = \lambda \lim_{\epsilon \to 0^+} \Im \epsilon^{-1} F_0 = \lambda B(x_0)^{-1}.$$

Thus, we have

$$\lambda^2 \mu_\lambda(\{x_0\}) = B(x_0),$$

which also proves condition (5.4).

Conversely, if conditions (5.3) and (5.4) hold, then in particular, the above discussion shows that condition (5.3) implies

$$\lambda^2 \mu_\lambda(\{x_0\}) = B(x_0).$$

Thus if $B(x_0) \neq 0$, i.e., condition (5.4), then $\mu_\lambda(\{x_0\}) \neq 0$. $\square$

Example 5.5. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, then we can easily calculate

$$\begin{aligned}
\lim_{\epsilon \to 0^+} F_0(x + i\epsilon) &= \lim_{\epsilon \to 0^+} \int \frac{d\mu_0(t)}{t - (x_0 + i\epsilon)} \\
&= \lim_{\epsilon \to 0^+} \left(\frac{1}{3} \frac{1}{1 - (x_0 + i\epsilon)} + \frac{1}{3} \frac{1}{2 - (x_0 + i\epsilon)} + \frac{1}{3} \frac{1}{3 - (x_0 + i\epsilon)} \right) \\
&= \frac{1}{3} \left(\frac{1}{1 - x_0} + \frac{1}{2 - x_0} + \frac{1}{3 - x_0} \right).
\end{aligned}$$

Thus the atoms for μ_λ would be the x_0's that satisfy the equation

$$\frac{1}{3} \left(\frac{1}{1 - x_0} + \frac{1}{2 - x_0} + \frac{1}{3 - x_0} \right) = -\frac{1}{\lambda},$$

and the corresponding weight for each x_0 is

$$\lambda^{-2} B(x_0) = \lambda^{-2} \left(\int (x_0 - y)^{-2} d\mu_0(y) \right)^{-1}$$

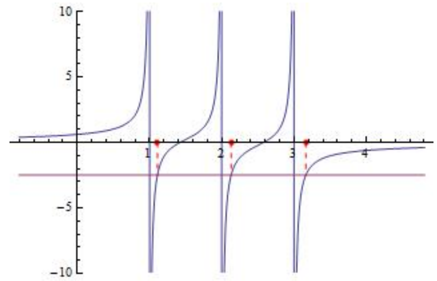

FIGURE 2. The blue curve is the graph of $F_0(x)$. The purple curve is a fixed value of $\lambda = 0.4$. The red dots, point masses of μ_λ, are the solutions to $F_0(x) = -\frac{1}{\lambda}$.

$$= \lambda^{-2}\left(\frac{1}{3(x_0-1)^2} + \frac{1}{3(x_0-2)^2} + \frac{1}{3(x_0-3)^2}\right)^{-1}.$$

See Figure 2 for a drawing which helps explain the computation.

6. A multiplication operator

So far we have considered only matrix representations of self-adjoint operators for Theorem 3.4. Now we would like to consider operators on $L^2[0,1]$. Let

$$A = M,$$

where M is defined in Theorem 2.7, then

$$d\mu_0(t) = dt.$$

Obviously $\mathbf{1}$ is a cyclic vector for M, so we have

$$A_\lambda = M + \lambda \mathbf{1} \otimes \mathbf{1}$$

as a measure on $[0,1]$, and we would like to know what the corresponding μ_λ is. Due to some technical details in Simon's paper [1], μ_λ has no continuous singular component. Thus we can write $d\mu_\lambda$ in the form

$$d\mu_\lambda(t) = g_\lambda(t)dt + \sum_i c_i^{(\lambda)} \delta_{y_i^{(\lambda)}},$$

and our goal is to find out what $g_\lambda(t)$ and $y_i^{(\lambda)}$'s are. Note that the λ in $c_i^{(\lambda)}\delta_{y_i^{(\lambda)}}$ is to denote their dependence on λ, not an exponent.

From Lemma 3.5, we know that for $y \neq 0$,

$$\begin{aligned} F_0(x+iy) &= \int \frac{d\mu_0(t)}{t-x-iy} \\ &= \int_0^1 \frac{dt}{t-x-iy} \\ &= \int_0^1 \frac{(t-x+iy)dt}{(t-x)^2+y^2} \\ &= \int_{-x}^{1-x} \frac{(t+iy)dt}{t^2+y^2} \end{aligned}$$

$$= \frac{1}{2}\log\left|\frac{(1-x)^2+y^2}{x^2+y^2}\right| + i\left(\arctan\left(\frac{1-x}{y}\right) - \arctan\left(-\frac{x}{y}\right)\right),$$

and we are able to calculate

$$F_\lambda(x+iy) = \frac{F_0(x+iy)}{1+\lambda F_0(x+iy)}. \tag{6.1}$$

On the other hand,

$$F_\lambda(x+iy) = \int \frac{d\mu_\lambda(t)}{t-x-iy} = \int_{-\infty}^{\infty} \frac{g_\lambda(t)dt}{t-x-iy} + \sum_i \frac{c_i^{(\lambda)}}{y_i^{(\lambda)}-x-iy},$$

and

$$\begin{aligned}
&\lim_{y\to 0^+}\left(F_\lambda(x+iy) - F_\lambda(x-iy)\right)\\
=&\int_{-\infty}^{\infty}\left(\frac{1}{t-(x+iy)} - \frac{1}{t-(x-iy)}\right) g_\lambda(t)dt\\
=&\int_{-\infty}^{\infty}\left(\frac{2yi}{(t-x)^2+y^2)}\right) g_\lambda(t)dt\\
=&2\pi i\int_{-\infty}^{\infty} P_{x+iy}(t) g_\lambda(t)dt \quad \text{(from Definition 3.9)}.
\end{aligned}$$

Then, by Equation 3.10, we have

$$\begin{aligned}
g_\lambda(x) &= \lim_{y\to 0^+}\int_{-\infty}^{\infty} P_{x+iy}(t) g_\lambda(t)dt\\
&=\frac{1}{2\pi i}\lim_{y\to 0^+}\left(F_\lambda(x+iy) - F_\lambda(x-iy)\right).
\end{aligned}$$

Before we plug in Equation 6.1, we would like to simplify it:

$$\begin{aligned}
F_\lambda(x+iy) - F_\lambda(x-iy) =& \frac{F_0(x+iy)}{1+\lambda F_0(x+iy)} - \frac{F_0(x-iy)}{1+\lambda F_0(x-iy)}\\
=&\frac{F_0(x+iy) - F_0(x-iy)}{(1+\lambda F_0(x+iy))(1+\lambda F_0(x-iy))}.
\end{aligned}$$

Now plug in

$$F_0(x+iy) = \frac{1}{2}\log\left|\frac{(1-x)^2+y^2}{x^2+y^2}\right| + i\left(\arctan\left(\frac{1-x}{y}\right) - \arctan\left(-\frac{x}{y}\right)\right)$$

and simplify the equation, we get

$$\begin{aligned}
&F_\lambda(x+iy) - F_\lambda(x-iy)\\
=&\frac{2i\left(\arctan\left(\frac{1-x}{y}\right) + \arctan\left(\frac{x}{y}\right)\right)}{\left(1+\frac{1}{2}\lambda\log\left|\frac{(1-x)^2+y^2}{x^2+y^2}\right|\right)^2 + \lambda^2\left(\arctan\left(\frac{1-x}{y}\right) + \arctan\left(\frac{x}{y}\right)\right)^2}.
\end{aligned}$$

Now consider the limit $y \to 0^+$,

$$\lim_{y\to 0^+}\arctan\left(\frac{1-x}{y}\right) = \begin{cases} \frac{\pi}{2} & \text{if } 1-x>0\\ 0 & \text{if } 1-x=0\\ -\frac{\pi}{2} & \text{if } 1-x<0 \end{cases}$$

$$\lim_{y\to 0^+} \arctan\left(\frac{x}{y}\right) = \begin{cases} \frac{\pi}{2} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -\frac{\pi}{2} & \text{if } x < 0 \end{cases},$$

then

$$\lim_{y\to 0^+} \left(\arctan\left(\frac{1-x}{y}\right) + \arctan\left(\frac{x}{y}\right)\right) = \begin{cases} 0 & \text{if } x > 1 \\ \frac{\pi}{2} & \text{if } x = 1 \\ \pi & \text{if } 0 < x < 1 \\ \frac{\pi}{2} & \text{if } x = 0 \\ 0 & \text{if } x < 0 \end{cases}.$$

Therefore we have

$$g_\lambda(x) = \frac{\chi_{(0,1)}(x)}{\left(1 + \lambda \log\left(\frac{1-x}{x}\right)\right)^2 + \lambda^2\pi^2}.$$

Now we consider the point mass. According to Theorem 5.2, to have a point mass at $y^{(\lambda)}$,

$$B(y^{(\lambda)}) = \left(\int_0^1 \frac{dt}{(t - y^{(\lambda)})^2}\right)^{-1} = y^{(\lambda)}(y^{(\lambda)} - 1) \neq 0.$$

The above integral is only defined on $\mathbb{R} \setminus (0,1)$, so the above condition is only satisfied when $y^{(\lambda)} \notin [0,1]$. In addition, it must satisfy the condition that

$$\lim_{\epsilon\to 0^+} F_0(y^{(\lambda)} + i\epsilon) = -\lambda^{-1},$$

so

$$\log\left(1 - \frac{1}{y^{(\lambda)}}\right) = -\lambda^{-1},$$

$$y^{(\lambda)} = \left(1 - e^{-\frac{1}{\lambda}}\right)^{-1}.$$

Note that here the $g_\lambda(t)$ part covers $[0,1]$ while the $y^{(\lambda)}$ covers the complement $\mathbb{R} \setminus [0,1]$. Putting the pieces together, we have

$$d\mu_\lambda(x) = \frac{\chi_{(0,1)}(x)dx}{\left(1 + \lambda\log\left(\frac{1-x}{x}\right)\right)^2 + \lambda^2\pi^2} + \frac{e^{-\frac{1}{\lambda}}}{\lambda^2\left(1 - e^{-\frac{1}{\lambda}}\right)^2}\delta_{\left(1-e^{-\frac{1}{\lambda}}\right)^{-1}}(x).$$

7. Schrödinger Operators

Of great interest in physics, the Schrödinger operator

$$T = -\frac{d^2}{dx^2} + V(x)$$

on $L^2(-\infty,\infty)$, where $V(x)$ is a real-valued function, is self-adjoint. The perturbation

$$T_\lambda = T + \lambda\delta_0$$

is particularly interesting. Simon [1] worked out the spectral theory for these rank one perturbations and, in particular, computed the spectral measures μ_λ for these perturbations. Unlike in our previous examples where the support of μ_λ was a finite set for the self-adjoint matrices and the support was a bounded set for multiplication by x on $L^2(0,1)$, the supports of μ_λ in the Schrödinger case are unbounded sets. Although some of the technical details are somewhat beyond what we are trying to accomplish here, we mention the Schrödinger operator as another example of self-adjoint operator one can consider here.

References

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