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Determination of a Potential from Cauchy Data: Uniqueness and Distinguishability

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Abstract

The problem of recovering a potential $q(y)$ in the differential equation:

$$
\begin{aligned}
-\Delta u + q(y)\, u = 0 \qquad (x,y) \in (0,1)\times(0,1) \\
u(0,y) = u(1,y) = u(x,0) = 0 \\
u(x,1) = f(x)\ ,\ u_y(x,1) = g(x)
\end{aligned}
$$

is investigated. The method of separation of variables reduces the recovery of $q(y)$ to a non-standard inverse Sturm-Liouville problem. Employing asymptotic techniques and integral operators of Gel'fand-Levitan type, it is shown that, under appropriate conditions on the Cauchy pair (f,g), $q(y)$ is uniquely determined, in a local sense, up to its mean. We characterize the ill-posedness of this inverse problem in terms of the "distinguishability" of potentials. An estimate is derived which indicates the maximum level of measurement error under which two potentials, differing only far away from $y=1$, can be resolved.

KEY WORDS: inverse problem, undetermined coefficient, overposed boundary data

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1. INTRODUCTION.

For the domain $\Omega = (0,1)\times(0,1)$, consider the problem of determining the potential $q(y) \in L^\infty(0,1)$ in

$$-\Delta u + q(y)\,u = 0 \qquad (x,y) \in \Omega$$
$$u(0,y) = u(1,y) = u(x,0) = 0 \tag{1.1}$$
$$u(x,1) = f(x)\ ,\ u_y(x,1) = g(x)$$

The functions f and g are assumed to be given and f satisfies the condition $f(0) = f(1) = 0$. For a general bounded domain $\Omega \subset R^n$ with smooth boundary, it is known that $q(\vec{x})$ in the differential operator $-\Delta + q(\vec{x})$ is uniquely determined by complete knowledge of the Dirichlet to Neumann map[1]

$$\Lambda_q(f) = \frac{\partial V_f}{\partial n}\Big|_{\partial\Omega} \tag{1.2}$$

where $f \in H^{\frac{1}{2}}(\partial\Omega)$ and $V_f \in H^1(\Omega)$ satisfies

$$-\Delta\, V_f + q(\vec{x})\, V_f = 0 \qquad \vec{x} \in \Omega$$
$$V_f\Big|_{\partial\Omega} = f\,. \tag{1.3}$$

The map (1.2) is well defined provided that 0 is not a Dirichlet eigenvalue of $-\Delta + q(\vec{x})$. More precisely, when $n \geq 3$, (1.2) uniquely determines $q(\vec{x}) \in L^\infty(\Omega)$ [2] and in two dimensions, uniqueness holds provided that $q(\vec{x})$ is smooth and sufficiently close to zero[3]. Recently, it has been shown that uniqueness holds in a larger class of smooth functions[4].

The inverse potential problem (1.2, 1.3) is closely related to the Impedance Tomography Problem: To determine the scalar conductivity $\gamma(\vec{x}) > 0$ in

$$\nabla \cdot (\gamma(\vec{x})\nabla U_f) = 0 \qquad \vec{x} \in \Omega$$
$$U_f\Big|_{\partial\Omega} = f \tag{1.4}$$

from complete knowledge of the Dirichlet to Neumann map

$$\Lambda_\gamma(f) = \gamma\frac{\partial U_f}{\partial n}\Big|_{\partial\Omega}\,. \tag{1.5}$$

Uniqueness for the Impedance Tomography Problem was first shown by Kohn and Vogelius for analytic conductivities and then extended to the case of piecewise analytic conductivities[5–6]. Uniqueness for $C^\infty(\Omega)$ conductivities in $n \geq 3$

was proved by Sylvester and Uhlmann[7]. Subsequent work has shown that this regularity assumption can be relaxed[2,8]. A determination of the information contained in partial knowledge of the Dirichlet to Neumann map has also been analyzed[9–13]. For a survey of the Impedance Tomography Problem, the reader is referred to the paper of Sylvester and Uhlmann[1].

The connection between (1.2, 1.3) and (1.4, 1.5) is made by setting $q = \frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}$ and $V_f = \gamma^{-\frac{1}{2}} U_f$, giving

$$\Lambda_q = \gamma^{-\frac{1}{2}} \Lambda_\gamma \gamma^{-\frac{1}{2}} + \frac{1}{2}\gamma^{-1}\frac{\partial\gamma}{\partial n}. \tag{1.6}$$

Consequently, if Λ_q, $\gamma|_{\partial\Omega}$ and $\left.\frac{\partial\gamma}{\partial n}\right|_{\partial\Omega}$ are known, then so is Λ_γ. Conversely, Λ_γ determines Λ_q.

Consider the potential problem (1.3) for $n = 2$. In practice, one has access to only a finite number of Cauchy data pairs (points in the graph of Λ_q), so a general $q(x, y)$ cannot be uniquely determined. However, it is reasonable to ask whether, by imposing additional structure on q, unicity can be restored (see, e.g.,[14–18]). In the present paper, we consider the special case in which the potential q varies only in one coordinate direction, that is, $q(x, y) = q(y)$. Such layered structures often arise in applications. For example, in the modelling of steady-state heat distribution, a composite material formed by hot-rolling several metals together exhibits such behavior. Furthermore, the object being studied may be oriented in such a way that only a portion of its boundary may be accessible for measurement. This is reflected in our assumption that the data is given only on the top boundary of the square.

Uniqueness results for univariate conductivies in two dimensions have been proved. Kohn and Vogelius[6] have shown that a finite set of Cauchy pairs uniquely determines a layered conductivity $\gamma(x, y) = \gamma(y)$ on an infinite strip $-\infty < x < \infty$, $0 < y < 1$. Their method involves reducing the problem to a classical inverse Sturm-Liouville problem, where uniqueness is known. Sylvester[19] proved that a radial conductivity $\gamma(r)$ on the unit circle is uniquely determined by the Dirichlet to Neumann map. The problem of this paper, while similar in many respects, is fundamentally different. Our data provides no information about the mean $\bar{q} = \int_0^1 q(y)\,dy$ and the question of uniqueness up to the mean reduces to a non-standard inverse Sturm-Liouville problem. Recently, Fang-Lin and Gilbert-Lin have established conditions under which a positive and radially symmetric q on the unit disk can be identified from a single Cauchy data pair[14,17].

The present paper, based on an asymptotic expansion (equation (2.7) below) derived by Caudill and Lowe[20] to quantify the information content in the data pair (f, g) in (1.1), has been prepared with two goals in mind. The first goal is to show that a near-constant potential is uniquely determined up to its mean by the single Cauchy data pair (f, g). The second goal is to demonstrate the severe ill-posedness of this problem by determining the degree of measurement accuracy required to distinguish potentials which differ only far away from $y = 1$. Of course, since the data is given in a direction that is perpendicular to the dependence of q, difficulty is to be expected in this worst-case situation.

This paper is organized as follows: In section 2 the method of separation of variables is used to reduce the uniqueness question to the aforementioned non-standard inverse Sturm-Liouville problem. In section 3 a uniqueness result is established for potentials q with prescribed mean $\bar{q}$ which are close to the constant potential $q = \bar{q}$. In section 4 the resolution possible from measurements of finite precision is discussed.

2. REDUCTION TO STURM-LIOUVILLE FORM.

Using the following expansions for U_f and the known functions f and g:

$$U_f(x, y) = \sum_{n=1}^{\infty} b_n X_n(x) Y_n(y) \tag{2.1a}$$

$$f(x) = \sum_{n=1}^{\infty} f_n \sin \pi n x \tag{2.1b}$$

$$g(x) = \sum_{n=1}^{\infty} g_n \sin \pi n x \tag{2.1c}$$

it is easily verified that U_f satisfies

$$U_f(x,y) = \sum_{n=1}^{\infty} b_n Y_n(y) \sin \pi n x \tag{2.2}$$

where Y_n satisfies the Sturm-Liouville problem:

$$-Y_n''(y) + q(y) Y_n(y) = -n^2\pi^2 Y_n(y) \tag{2.3a}$$

$$Y_n(0) = Y_n'(1) - \beta_n Y_n(1) = 0 \tag{2.3b}$$

with $\beta_n = \frac{g_n}{f_n}$ when $f_n \neq 0$. When $q = 0$, it is easily verified that $\beta_n = \pi n \coth \pi n$ and the resulting eigenfunctions are $\sinh \pi n y$. The ill-posedness in recovering $q(y)$ is manifested in the exponential growth of these eigenfunctions.

At first glance, it may seem that a single data pair should carry very little information about q. However, when the f_n in (2.1b) are nonzero for all n, the "partial" Dirichlet to Neumann map

$$\Lambda_q(f) = \frac{\partial U_f(x,1)}{\partial y} \tag{2.4}$$

where U_f is the solution of

$$\begin{aligned} -\Delta U_f + q(y)\, U_f = 0 \qquad (x,y) \in \Omega \\ U_f(0,y) = U_f(1,y) = U_f(x,0) = 0\,,\ U_f(x,1) = f(x) \end{aligned} \tag{2.5}$$

is known. The β_n are just the eigenvalues of Λ_q :

$$\Lambda_q(\sin \pi n x) = \beta_n \sin \pi n x\ , \qquad n \geq 1. \tag{2.6}$$

The information carried in the sequence $\{\beta_n\}$ about the unknown function $q(y)$ was quantified by Caudill and Lowe[20], who showed that when $q \in L^\infty(0,1)$, this sequence has the asymptotic expansion

$$\beta_n = \beta_n(q) = n\pi \coth n\pi + \int_0^1 q(y) \left(\frac{\sinh n\pi y}{\sinh n\pi} \right)^2 dy + O\left(\frac{1}{n^2} \right). \tag{2.7}$$

In particular, it was noted that this sequence, and consequently the overposed data, carries no information about $\bar{q}$, the mean of q, defined by

$$\bar{q} \equiv \int_0^1 q(y)\, dy.$$

For our purposes, the potential q will be presumed to belong to the space V, defined as

$$V \equiv \left\{ q \in W^{1,\infty}(0,1) \;:\; q' \, is \, left-continuous \, in \, a \, nbhd \, of \; y = 1 \right\} .$$

Here, $W^{1,\infty}(0,1)$ is the Sobolev space of bounded functions possessing bounded weak derivatives.

In what follows, the term *Müntz-Szász set* will refer to any subsequence $\{n_k\}_{k=1}^{\infty}$ of the natural numbers N possessing the property

$$\sum_{k=1}^{\infty} \frac{1}{n_k} = \infty \, .$$

3. UNIQUENESS.

We now state the main result of this section.

Theorem 1. *Let q_c be a constant, and denote $\gamma_n \equiv \sqrt{n^2\pi^2 + q_c}$. If $\beta_n(q) = \beta_n(q_c) = \gamma_n \coth \gamma_n$ on a Müntz-Szász set for some potential $q \in V$ with $\bar{q} = q_c$, then $q \equiv q_c$.*

The corresponding result for the two-dimensional problem (1.1) can be stated as follows.

Theorem 1′. *Let $q \in V$, and denote by f_n^c, g_n^c the Fourier sine coefficients for a Cauchy data pair corresponding to $q = q_c$. If*

$$\frac{g_n}{f_n} = \frac{g_n^c}{f_n^c}$$

on a Müntz-Szász set, and if $\bar{q} = q_c$*, then* $q \equiv q_c$.

The proof of Theorem 1 is presented through a sequence of lemmas, and employs an integral operator of Gel'fand-Levitan[21] type to reduce the problem to one involving the moments of a particular kernel function. A similar approach has been applied to the two-spectrum problem of classical inverse Sturm-Liouville theory[22–23].

Following Gel'fand-Levitan, the solution $Y_n(y;q)$ of (1) can be represented as

$$Y_n(y;q) = \frac{\sinh \gamma_n y}{\gamma_n} + \frac{1}{\gamma_n}\int_0^y M(y,t)\sinh \gamma_n t \, dt \,, \tag{3.1}$$

where $M(y,t)$ solves the Goursat problem

$$M_{yy}(y,t) - M_{tt}(y,t) + (q(y) - q_c)M(y,t) = 0, \quad 0 < t \le y < 1 \tag{3.2a}$$

$$M(y,0) = 0 \tag{3.2b}$$

$$M(y,y) = \frac{1}{2}\int_0^y (q(s) - q_c)\, ds \,. \tag{3.2c}$$

It is important to note that $M(y,t)$ is independent of n. Combining representation (3.1) with condition (2.3b) yields the additional condition

$$\int_0^1 M_y(1,t)\sinh \gamma_n t \, dt = \beta_n \int_0^1 M(1,t)\sinh \gamma_n t\, dt, \quad \forall n \in N \,. \tag{3.3}$$

In terms of M, the hypothesis $\bar{q} = q_c$ of Theorem 1 is equivalent to the condition

$$M(1,1) = 0. \tag{3.4}$$

We also remark that condition (3.2b) implies the condition

$$M_y(1,0) = 0. \tag{3.5}$$

Using the overposed condition (3.3), integration by parts and conditions (3.4) and (3.5) yield

$$\begin{aligned}\int_0^1 \{M_{yt}(1,t) + M_{tt}(1,t)\} \cosh \gamma_n t \, dt \\ = \frac{q(1) - q_c}{2}\cosh \gamma_n - M_t(1,0) \\ + \frac{\gamma_n^2}{\sinh \gamma_n}\int_0^1 M(1,t)\sinh \gamma_n(1-t)\, dt,\end{aligned} \tag{3.6}$$

where the boundary condition (3.2c) is used to derive the equality

$$M_y(1,1) + M_t(1,1) = \frac{q(1) - q_c}{2}. \tag{3.7}$$

We now consider the behavior of the components of (3.6) for large n.

Lemma 2. *For $q \in V$,*

$$\|M_t(1,t)\|_\infty\,,\ \|M_{yt}(1,t) + M_{tt}(1,t)\|_\infty < \infty\,.$$

Proof. Under the characteristic change of variables $\xi = y + t$, $\eta = y - t$, the solution $M(y,t)$ of (3.2) can be represented as

$$M^*(\xi,\eta) = -\frac{1}{4}\int_0^\xi \int_0^\eta q\left(\frac{\sigma+\tau}{2}\right) M^*(\sigma,\tau)\,d\tau\,d\sigma\,,$$

where

$$M^*(\xi,\eta) \equiv M\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right) = M(y,t)\,.$$

From this, one can compute

$$M_t(y,t) = q(y) + \frac{1}{4}\int_0^\xi q\left(\frac{\sigma+\eta}{2}\right) M^*(\sigma,\eta)\,d\sigma - \frac{1}{4}\int_0^\eta q\left(\frac{\xi+\tau}{2}\right) M^*(\xi,\tau)\,d\tau$$

and

$$\begin{aligned} M_{yt}(y,t) + M_{tt}(y,t) &= \frac{1}{4}q'(y) - \frac{1}{4}q(y)M(y,t) \\ &\quad - \frac{1}{4}\int_0^\eta \left\{\frac{1}{2}q'\left(\frac{\xi+\tau}{2}\right) M^*(\xi,\tau) + q\left(\frac{\xi+\tau}{2}\right) M^*_\xi(\xi,\tau)\right\} d\tau\,. \end{aligned}$$

The result follows upon letting $y \to 1$ and noting that $W^{1,\infty}(0,1) \subseteq C([0,1])$. □

The next two results are immediate consequences of Hölder's inequality.

Lemma 3. *For continuous q, there exists a finite constant A for which*

$$\left|\int_0^1 \{M_{yt}(1,t) + M_{tt}(1,t)\} \cosh \gamma_n t\,dt\right| \le A\frac{e^{\gamma_n}}{n}\,.$$

Lemma 4. *For $f \in L^\infty(0,1)$, there exists a finite constant K for which*

$$\left|\int_0^1 f(t) \cosh \gamma_n(1-t)\,dt\right| \le K\frac{e^{\gamma_n}}{n}\,.$$

Lemma 5. *There exists a finite constant B for which*

$$\left|\frac{\gamma_n^2}{\sinh\gamma_n}\int_0^1 M(1,t)\sinh\gamma_n(1-t)\,dt\right| \le B\,.$$

Proof. We have

$$\int_0^1 M(1,t)\sinh\gamma_n(1-t)\,dt = \frac{1}{\gamma_n}\int_0^1 M_t(1,t)\cosh\gamma_n(1-t)\,dt\,.$$

Lemmas 2 and 4 then imply

$$\begin{aligned}\left|\frac{\gamma_n^2}{\sinh\gamma_n}\int_0^1 M(1,t)\sinh\gamma_n(1-t)\,dt\right| \\ &= \left|\frac{\gamma_n}{\sinh\gamma_n}\int_0^1 M_t(1,t)\cosh\gamma_n(1-t)\,dt\right| \\ &\le \frac{\gamma_n}{\sinh\gamma_n}K\frac{\mathrm{e}^{\gamma_n}}{n} \le B\,,\end{aligned}$$

as desired. □

Setting

$$\begin{aligned} A_n &= n\mathrm{e}^{-\gamma_n}\int_0^1 \{M_{yt}(1,t)+M_{tt}(1,t)\}\cosh\gamma_n t\,dt\,,\\ C_n &= \left(\frac{q(1)-q_c}{2}\right)\mathrm{e}^{-\gamma_n}\cosh\gamma_n\,,\\ B_n &= -M_t(1,0)+\frac{{\gamma_n}^2}{\sinh\gamma_n}\int_0^1 M(1,t)\sinh\gamma_n(1-t)\,dt\,,\end{aligned}$$

equation (3.6) can be rewritten as

$$A_n\frac{\mathrm{e}^{\gamma_n}}{n} = C_n\mathrm{e}^{\gamma_n} + B_n\,,$$

or

$$C_n = \frac{A_n}{n} - B_n\mathrm{e}^{-\gamma_n}\,. \tag{3.8}$$

Letting $n\to\infty$ in (3.8), we conclude that $q(1)=q_c$. Hence, $C_n = 0\,\forall\, n$. Consequently, we conclude

$$A_n = O\left(n\mathrm{e}^{-\gamma_n}\right)\,,$$

i.e. the sequence of real numbers

$$\int_0^1 \{M_{yt}(1,t) + M_{tt}(1,t)\} \cosh \gamma_n t \, dt \tag{3.9}$$

remains bounded as $n \to \infty$. In view of the exponential growth of the sequence

$$\{\cosh \gamma_n t\}_{n=1}^{\infty} ,$$

we can establish the following.

Lemma 6. *Let $\phi \in L^\infty(0,1)$. If meas(suppϕ) $\neq 0$, then*

$$\left| \int_0^1 \phi(t) \mathrm{e}^{\gamma_n t} dt \right| \longrightarrow \infty, \ n \to \infty .$$

Proof. We prove the result for the case $\phi(t)$ a step function, the generalization being easily made. For a partition $\{x_j\}_{j=0}^N$ of $[0,1]$ with $x_{j-1} < x_j$, $\forall j$, write

$$\phi(t) = \sum_{j=1}^N \phi_j \chi_j(t) ,$$

where $\chi_j(t)$ is the indicator function of the interval $(x_{j-1}, x_j]$, and assume $\phi \not\equiv 0$. Denote by J the largest value of the index j for which $\phi_j \neq 0$. Then,

$$\begin{aligned} \int_0^1 \phi(t)\mathrm{e}^{\gamma_n t} dt = \sum_{j=1}^J \phi_j \int_{x_{j-1}}^{x_j} \mathrm{e}^{\gamma_n t} dt &= \sum_{j=1}^J \phi_j \left(\frac{\mathrm{e}^{\gamma_n x_j} - \mathrm{e}^{\gamma_n x_{j-1}}}{\gamma_n} \right) \\ &= \frac{\mathrm{e}^{\gamma_n x_J}}{\gamma_n} \Bigg\{ \phi_J \left(1 - \mathrm{e}^{-\gamma_n (x_J - x_{J-1})} \right) \\ &\quad + \sum_{j=1}^{J-1} \phi_j \left(\mathrm{e}^{-\gamma_n (x_J - x_j)} - \mathrm{e}^{-\gamma_n (x_J - x_{j-1})} \right) \Bigg\} . \end{aligned}$$

The term in brackets remains nonzero and bounded as $n \to \infty$. Noting that $\gamma_n \approx n\pi$ for n large, the conclusion of the lemma follows immediately. □

Combining (3.9) with Lemma 6, we immediately deduce

Lemma 7.

$$q(1) - q_c = 0\,;$$

$$M_{yt}(1,t) + M_{tt}(1,t) = 0\,, \quad 0 \le t \le 1\,.$$

Writing

$$M_y(1,t) + M_t(1,t) = M_y(1,1) + M_t(1,1) - \int_t^1 \{M_{yt}(1,s) + M_{tt}(1,s)\}\, ds$$

and using Lemma 7 and condition (3.7), one can establish the following key lemma.

Lemma 8.

$$M_y(1,t) + M_t(1,t) = 0\,, \quad 0 \le t \le 1\,.$$

We may now prove the following.

Lemma 9.

$$M(1,t) = M_y(1,t) = 0\,, \quad 0 \le t \le 1\,.$$

Proof. From condition (3.3) and Lemma 8, we have

$$\begin{aligned}\int_0^1 M_y(1,t) \sinh \gamma_n t\, dt &= \beta_n \int_0^1 M(1,t) \sinh \gamma_n t\, dt \\ &= -\coth \gamma_n \int_0^1 M_t(1,t) \cosh \gamma_n t\, dt \\ &= \coth \gamma_n \int_0^1 M_y(1,t) \cosh \gamma_n t\, dt\end{aligned}$$

This can be rewritten as

$$\begin{aligned}0 &= \int_0^1 M_y(1,t)\{\coth \gamma_n \cosh \gamma_n t - \sinh \gamma_n t\}\, dt \\ &= \frac{1}{\sinh \gamma_n} \int_0^1 M_y(1,t) \cosh \gamma_n (1-t)\, dt\,.\end{aligned}$$

So,

$$\int_0^1 M_y(1,t) \cosh \gamma_n (1-t)\, dt = 0 \quad \forall n\,. \tag{3.10}$$

In [20], it is shown that the sequence

$$\{(\sinh n_k \pi t)^2\}_{k=1}^{\infty}$$

is complete up to the mean in $L^2(0,1)$ whenever $\{n_k\}_{k=1}^{\infty}$ is a Müntz-Szász set. A similar argument shows that the sequence

$$\{\cosh \gamma_n (1-t)\}_{k=1}^{\infty}$$

is complete in $L^2(0,1)$ whenever $\{n_k\}_{k=1}^{\infty}$ is a Müntz-Szász set. The assertion of this lemma now follows from (3.10). □

Proof of Theorem 1. The Goursat problem (3.2), combined with the condition

$$M(1,t) = M_y(1,t) = 0$$

has been shown by Suzuki[23] to imply $M(y,t) \equiv 0$. The result $q \equiv q_c$ follows immediately. □

4. DISTINGUISHABILITY OF POTENTIALS.

In the work[11], D. Isaacson introduced the notion of "distinguishability" for the Impedance Tomography problem (1.4)-(1.5) as a means of characterizing unrecoverable information in the presence of measurement errors. In the context of the present problem, one is led to the following notion:

We say that two potentials $q_1(y)$ and $q_2(y)$ are *distinguishable by measurements of precision* ϵ if

$$|\beta_n(q_1) - \beta_n(q_2)| > \epsilon$$

for some $n \in \mathbf{N}$. Likewise, the potentials q_1 and q_2 are *not* distinguishable by measurements of precision ϵ if for all $n \in \mathbf{N}$ we have

$$|\beta_n(q_1) - \beta_n(q_2)| \leq \epsilon\,.$$

In other words, q_1 and q_2 are indistinguishable if, up to the accuracy of the measuring instruments, their β_n's are the same.

It is indicated by the asymptotic formula (2.7) that the Cauchy data pair (f, g) of problem (1.1) provides information about the potential $q(y)$ in the form of its moments with respect to the set

$$\{\rho_n(y)\}_{n=1}^{\infty} = \left\{ \left(\frac{\sinh \gamma_n y}{\sinh \gamma_n} \right)^2 \right\}_{n=1}^{\infty} .$$

The graphs of the first three of these functions for the case $q_c = 0$ are shown in Figure 1.

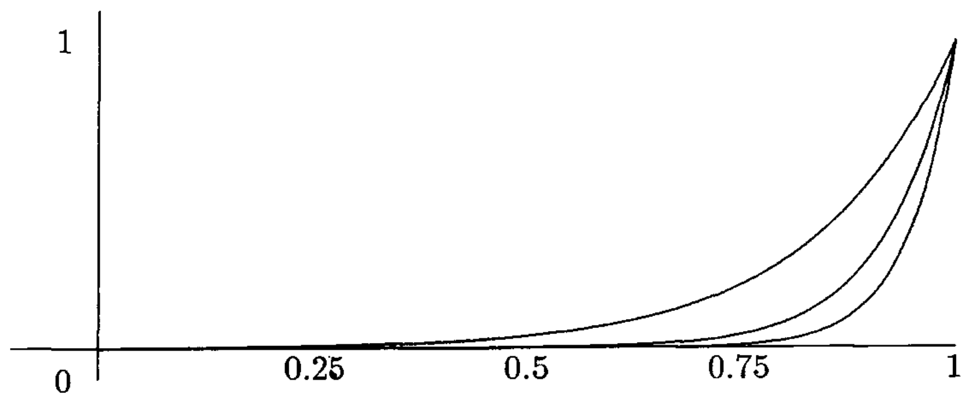

FIGURE 1. The graphs of $\rho_n(y)$, $n = 1, 2, 3$.

We note that the lack of separation in these functions away from $y = 1$ reflects the fact that the Cauchy data (f, g) is being given at $y = 1$. In view of Figure 1, it is to be expected that, up to some tolerance level, the data sequence $\{\beta_n(q)\}$ cannot "detect" a change in $q(y)$ away from $y = 1$. To illustrate this point, and to provide an idea of the precision necessary to distinguish between two such potentials, we consider the following example:

Define the potential q_1 on the interval $[0, 1]$ by

$$q_1(y) = \begin{cases} \alpha, & 0 \le y \le \delta, \\ 0, & \delta < y \le 1, \end{cases}$$

where $\alpha > 0$ and $\delta \in (0, \frac{1}{2})$. We determine those values of α and δ for which q_1 is indistinguishable from the zero potential by measurements of precision ϵ.

Theorem 10. *For all n, $0 < \alpha \le \alpha_M$ and $0 < \delta \le \frac{1}{2}$,*

$$\beta_n(q_1) - \beta_n(0) \le 0.0041\, \delta^2 \alpha \cosh \delta \sqrt{\pi^2 + \alpha_M} \cosh \pi\delta .$$

Corollary 11. *For a given measurement precision ϵ, q_1 is indistinguishable from the zero potential for each $\alpha \in (0, \alpha_M)$ and $\delta \in (0, \frac{1}{2})$ obeying*

$$\alpha \leq \frac{\epsilon}{0.0041\,\delta^2 \cosh \delta\sqrt{\pi^2 + \alpha_M} \cosh \pi\delta}.$$

The proofs of these results are presented through a sequence of lemmas. But first, the solution of the forward problem gives

$$\beta_n(q_1) = n\pi \left(\frac{A_n \sinh n\pi + B_n \cosh n\pi}{A_n \cosh n\pi + B_n \sinh n\pi} \right),$$

where

$$A_n = \frac{f_n\{n\pi \sinh(\gamma_n - n\pi)\delta - (\gamma_n - n\pi) \cosh \gamma_n\delta \sinh n\pi\delta\}}{\gamma_n \cosh \gamma_n\delta \sinh n\pi(1-\delta) + n\pi \sinh \gamma_n\delta \cosh n\pi(1-\delta)} \tag{4.1}$$

$$B_n = \frac{f_n - A_n \cosh n\pi}{\sinh n\pi} \tag{4.2}$$

and

$$\gamma_n \equiv \sqrt{n^2\pi^2 + \alpha}.$$

Using $\beta_n(0) = n\pi \coth n\pi$,

$$\beta_n(q_1) - \beta_n(0) = \frac{n\pi}{\sinh n\pi} \left(\frac{-A_n}{A_n \cosh n\pi + B_n \sinh n\pi} \right).$$

It follows from (4.1) and (4.2) that

$$\begin{aligned}
\beta_n(q_1) - \beta_n(0) &= \frac{n\pi}{\sinh n\pi} \left(\frac{-A_n}{f_n} \right) \\
&= \frac{n\pi}{\sinh n\pi} \left\{ \frac{(\gamma_n - n\pi) \cosh \gamma_n\delta \sinh n\pi\delta - n\pi \sinh(\gamma_n - n\pi)\delta}{\gamma_n \cosh \gamma_n\delta \sinh n\pi(1-\delta) + n\pi \sinh \gamma_n\delta \cosh n\pi(1-\delta)} \right\} \\
&= \frac{\gamma_n n^2\pi^2}{D \sinh n\pi} \left\{ \cosh \gamma_n\delta \frac{\sinh n\pi\delta}{n\pi} - \cosh n\pi\delta \frac{\sinh \gamma_n\delta}{\gamma_n} \right\} \\
&= \frac{\gamma_n n^2\pi^2 \cosh \gamma_n\delta \cosh n\pi\delta}{D \sinh n\pi} \left(\frac{\tanh n\pi\delta}{n\pi} - \frac{\tanh \gamma_n\delta}{\gamma_n} \right),
\end{aligned} \tag{4.3}$$

where

$$D \equiv \gamma_n \cosh \gamma_n\delta \sinh n\pi(1-\delta) + n\pi \sinh \gamma_n\delta \cosh n\pi(1-\delta).$$

For $\delta > 0$, $\psi(x) = \tanh \delta x / x$ is a decreasing function of x. Consequently, (4.3) gives

$$\beta_n(q_1) - \beta_n(0) \geq 0, \qquad \forall n,$$

so that

$$|\beta_n(q_1) - \beta_n(0)| = \beta_n(q_1) - \beta_n(0).$$

Lemma 12. *The expression D defined above satisfies the bound*

$$D \geq \gamma_n \sinh n\pi \,,$$

yielding

$$\beta_n(q_1) - \beta_n(0) \leq \frac{n^2\pi^2}{\sinh^2 n\pi} \cosh \gamma_n\delta \cosh n\pi\delta \left\{ \frac{\tanh n\pi\delta}{n\pi} - \frac{\tanh \gamma_n\delta}{\gamma_n} \right\}. \quad (4.4)$$

Proof. Set

$$g(\delta) \equiv \gamma_n \cosh \gamma_n\delta \sinh n\pi(1-\delta) + n\pi \sinh \gamma_n\delta \cosh n\pi(1-\delta) > 0 \qquad \text{on} \quad [0,1].$$

Then,

$$\begin{aligned} g'(\delta) &= (\gamma_n^2 - n^2\pi^2) \sinh \gamma_n\delta \sinh n\pi(1-\delta) \\ &= \alpha \sinh \gamma_n\delta \sinh n\pi(1-\delta) \ \geq 0. \end{aligned}$$

which gives

$$g(\delta) \geq g(0) = \gamma_n \sinh n\pi \,,$$

and the estimate (4.4) follows. ☐

Next, we derive a bound on the difference term in (4.4).

Lemma 13. *For each n, $0 < \alpha \leq \alpha_M$ and $0 < \delta < 1$,*

$$\frac{\tanh n\pi\delta}{n\pi} - \frac{\tanh \gamma_n\delta}{\gamma_n} \leq 0.34342\, \delta^2 \frac{\alpha}{2n\pi} \,,$$

yielding

$$\beta_n(q_1) - \beta_n(0) \leq \frac{0.17171 n\pi\delta^2\alpha \cosh \delta\sqrt{n^2\pi^2 + \alpha_M} \cosh n\pi\delta}{\sinh^2 n\pi}. \qquad (4.5)$$

Proof. An application of the Mean Value Theorem to the decreasing function $\psi(x) \equiv \dfrac{\tanh \delta x}{x}$ gives

$$\frac{\tanh n\pi\delta}{n\pi} - \frac{\tanh \gamma_n \delta}{\gamma_n} \leq \sup_{\sigma \in (n\pi, \gamma_n)} (-\psi'(\sigma))(\gamma_n - n\pi), \tag{4.6}$$

where

$$-\psi'(x) = \frac{\sinh \delta x \cosh \delta x - \delta x}{x^2 \cosh^2 \delta x}.$$

The change of variables $y = \delta x$ gives

$$\phi(y) \equiv -\psi'\left(\frac{y}{\delta}\right) = \delta^2 \left(\frac{\sinh y \cosh y - y}{y^2 \cosh^2 y}\right).$$

We have

$$\phi'(y) = 2\delta^2 \left(\frac{y^2 \mathrm{sech}^2 y \tanh y - \tanh y + y \mathrm{sech}^2 y}{y^3}\right),$$

so the critical values of $\phi(y)$ on the interval $(\delta n\pi, \delta\gamma_n)$ occur when

$$y^2 \mathrm{sech}^2 y \tanh y - \tanh y + y \mathrm{sech}^2 y = 0,$$

which has the solution $y_0 \approx 0.91993767$. Now,

$$\begin{aligned} \phi'(y) &> 0, \qquad 0 < y < y_0 \\ \phi'(y) &< 0, \qquad y_0 < y, \end{aligned}$$

so

$$\sup_{(\delta n\pi, \delta\gamma_n)} \phi(y) \leq \phi(y_0) \approx 0.34342\,\delta^2.$$

Hence,

$$\begin{aligned} \frac{\tanh n\pi\delta}{n\pi} - \frac{\tanh \gamma_n \delta}{\gamma_n} &\leq 0.34342\delta^2(\gamma_n - n\pi) \\ &\leq 0.34342\delta^2 \frac{\alpha}{2n\pi}, \end{aligned} \tag{4.7}$$

where the last inequality follows from the Mean Value Theorem.

Combining (4.7) with (4.4) yields

$$\begin{aligned}\beta_n(q_1)-\beta_n(0) &\le \frac{n^2\pi^2}{\sinh^2 n\pi}\cosh\gamma_n\delta\cosh n\pi\delta\left\{0.343428^2\frac{\alpha}{2n\pi}\right\}\\ &= \frac{0.17171n\pi\delta^2\alpha\cosh\gamma_n\delta\cosh n\pi\delta}{\sinh^2 n\pi}\\ &\le \frac{0.17171n\pi\delta^2\alpha\cosh\delta\sqrt{n^2\pi^2+\alpha_M}\cosh n\pi\delta}{\sinh^2 n\pi}, \quad \text{for} \quad \alpha\le\alpha_M.\end{aligned}$$

This completes the proof of Lemma 13. □

Proof of Theorem 10. For $0<\delta\le\frac{1}{2}$, the right-hand side of (4.5) is decreasing in n. Indeed, for

$$H(x)\equiv\frac{x\cosh\delta\sqrt{x^2+\alpha_M}\cosh\delta x}{\sinh^2 x}$$

on the interval $[\pi,\infty)$,

$$\begin{aligned}&H'(x)\sinh^3 x\\ &=\cosh\delta\sqrt{x^2+\alpha}\cosh\delta x[\sinh x - x\cosh x]\\ &+\delta x\cosh\delta\sqrt{x^2+\alpha}[\sinh\delta x\sinh x-\cosh\delta x\cosh x]\\ &+x\cosh\delta x\left[\frac{\delta x}{\sqrt{x^2+\alpha}}\sinh\delta\sqrt{x^2+\alpha}\sinh x-(1-\delta)\cosh\delta\sqrt{x^2+\alpha}\cosh x\right].\end{aligned}$$

The first two terms in the sum are negative for $x\ge\pi$. Writing the term $\frac{\delta x}{\sqrt{x^2+\alpha}}\sinh\delta\sqrt{x^2+\alpha}\sinh x-(1-\delta)\cosh\delta\sqrt{x^2+\alpha}\cosh x$ as

$$\begin{aligned}&=\left(\frac{\delta x}{\sqrt{x^2+\alpha}}-(1-\delta)\right)\sinh\delta\sqrt{x^2+\alpha}\sinh x\\ &+(1-\delta)\left(\sinh\delta\sqrt{x^2+\alpha}\sinh x-\cosh\delta\sqrt{x^2+\alpha}\cosh x\right), \qquad (4.8)\end{aligned}$$

the second term on the right-hand side of (4.8) is seen to be negative, as is the first term when $\delta\le\frac{1}{2}$, and consequently, $H'(x)\le 0$ on $[\pi,\infty)$. Thus, the right-hand side of (4.5) is a decreasing function of n, implying

$$\begin{aligned}\beta_n(q_1)-\beta_n(0)&\le\frac{0.17171\,\pi\delta^2\alpha\cosh\delta\sqrt{\pi^2+\alpha_M}\cosh\pi\delta}{\sinh^2\pi}\\ &\le 0.0041\,\delta^2\alpha\cosh\delta\sqrt{\pi^2+\alpha_M}\cosh\pi\delta, \quad \forall n\in\mathbf{N},\end{aligned}$$

as desired. □

In view of Corollary 11, a unit jump on the interval $[0, \delta]$ cannot be detected by measurements of precision ϵ if

$$\epsilon \geq Q(\delta),$$

where

$$Q(\delta) \equiv 0.0041\, \delta^2 \cosh \delta \sqrt{\pi^2 + 1} \cosh \pi\delta.$$

The graph of $Q(\delta)$ is shown in Figure 2. Selected values of $Q(\delta)$ are listed in Table I.

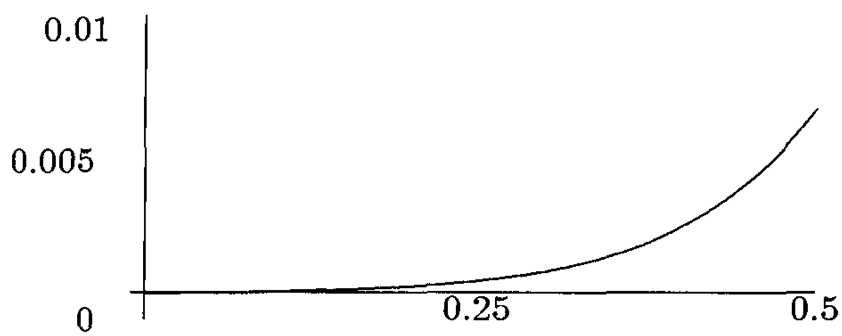

FIGURE 2. The graph of $Q(\delta)$, for $0 \leq \delta \leq \frac{1}{2}$.

TABLE I. Values of $Q(\delta)$ for small δ.

δ	$Q(\delta)$
.5	6.9×10^{-4}
.2	2.4×10^{-4}
.1	4.5×10^{-5}
.01	4.1×10^{-7}
.001	4.1×10^{-9}
.0001	4.1×10^{-11}

The high degree of ill-posedness is borne out by the exponential behavior of $Q(\delta)$. For example, Table I shows that measurements of precision $\epsilon = 10^{-6}$ cannot "sense" a unit jump in potential on the interval $[0, 0.01]$.

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