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# On the construction of a potential from Cauchy data

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*Abstract*

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We investigate the problem of recovering a potential $q(y)$ in the differential equation:

$$-\Delta u+q(y)u=0,\quad (x, y)\in(0, 1)\times(0, 1),$$
$$u(0, y)=u(1, y)=u(x, 0)=0,$$
$$u(x, 1)=f(x),\qquad u_y(x, 1)=g(x).$$

The method of separation of variables reduces the recovery of $q(y)$ to a nonstandard inverse Sturm–Liouville problem. An asymptotic formula is developed that suggests that under appropriate conditions on the Cauchy pair $(f, g)$, $q(y)$ is uniquely determined up to the mean. Moreover, the recovery of $q(y)$ is comparable to finding a function from its polynomial moments. A reconstruction scheme is suggested and numerical examples are considered.

*Keywords:* Undetermined coefficient; inverse problem; overposed boundary data.

## 1. Introduction

For the domain $\Omega=(0, 1)\times(0, 1)$, consider the problem of determining the potential $q(y)\in L^{\infty}(0, 1)$ in

$$\begin{aligned}&-\Delta u+q(y)u=0,\quad (x, y)\in\Omega,\\&u(0, y)=u(1, y)=u(x, 0)=0,\\&u(x, 1)=f(x),\qquad u_y(x, 1)=g(x).\end{aligned}\tag{1.1}$$

The functions $f$ and $g$ are assumed to be given and $f$ satisfies the condition $f(0)=f(1)=0$. For a general bounded domain $\Omega\subset\mathbb{R}^n$ with smooth boundary, it is known that $q(x)$ in the

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differential operator $-\Delta + q(\boldsymbol{x})$ is uniquely determined by complete knowledge of the Dirichlet to Neumann map

$$\Lambda_q(f) = \left.\frac{\partial V_f}{\partial n}\right|_{\partial\Omega}, \tag{1.2}$$

where $f \in H^{1/2}(\partial\Omega)$ and $V_f \in H^1(\Omega)$ satisfies

$$-\Delta V_f + q(\boldsymbol{x}) V_f = 0, \quad \boldsymbol{x} \in \Omega, \qquad V_f|_{\partial\Omega} = f \tag{1.3}$$

[15]. The map (1.2) is well-defined provided that 0 is not a Dirichlet eigenvalue of $-\Delta + q(\boldsymbol{x})$. More precisely, when $n \geqslant 3$, (1.2) uniquely determines $q(\boldsymbol{x}) \in L^\infty(\Omega)$ [9], and in two dimensions, uniqueness holds, provided that $q(\boldsymbol{x})$ is smooth and sufficiently close to zero [13]. Recently, it has been shown that uniqueness holds in a large class of smooth functions [11].

The inverse potential problem (1.2), (1.3) is intimately related to the Impedance Tomography Problem: to determine the scalar conductivity $\gamma(\boldsymbol{x}) > 0$ in

$$\nabla \cdot \left(\gamma(\boldsymbol{x}) \nabla U_f\right) = 0, \quad \boldsymbol{x} \in \Omega, \qquad U_f|_{\partial\Omega} = f, \tag{1.4}$$

from complete knowledge of the Dirichlet to Neumann map

$$\Lambda_\gamma(f) = \gamma \left.\frac{\partial U_f}{\partial n}\right|_{\partial\Omega}. \tag{1.5}$$

Uniqueness for the Impedance Tomography Problem was first shown in [6] for analytic conductivities and then extended to the case of piecewise analytic conductivities [7]. Uniqueness for $C^\infty(\Omega)$ conductivities in $n \geqslant 3$ was proved in [14]. Subsequent work has shown that this regularity assumption can be relaxed [1,9]. A determination of the information contained in partial knowledge of the Dirichlet to Neumann map has also been analyzed [2–5,10]. For a survey of the Impedance Tomography Problem, the reader is referred to [15].

The connection between (1.2), (1.3) and (1.4), (1.5) is made by setting $q = \Delta\sqrt{\gamma}\,/\sqrt{\gamma}$ and $V_f = \gamma^{-1/2} U_f$, giving

$$\Lambda_q = \gamma^{-1/2} \Lambda_\gamma \gamma^{-1/2} + \tfrac{1}{2}\gamma^{-1} \frac{\partial\gamma}{\partial n}. \tag{1.6}$$

Consequently, if $\Lambda_q$, $\gamma|_{\partial\Omega}$ and $(\partial\gamma/\partial n)|_{\partial\Omega}$ are known, then so is $\Lambda_\gamma$. Conversely, $\Lambda_\gamma$ determines $\Lambda_q$.

The goal of this paper is two-fold: to quantify the information that is contained in a single Cauchy data pair $(f, g)$ in (1.1) and to provide a reconstruction algorithm to extract this information. There is strong evidence to suggest that under appropriate conditions, a single pair $(f, g)$ determines $q(y)$ up to the mean. In practice, it will be shown that only a limited class of $q(y)$ can be numerically reconstructed, for only a limited class of $f$ and $g$ pairs will give usable numerical information. Of course, since the data is given in a direction that is perpendicular to the dependence of $q$, difficulty is to be expected in this worst-case situation.

Uniqueness results for univariate conductivies have been proved. Kohn and Vogelius [7] have shown that a finite set of Cauchy pairs uniquely determines a layered conductivity $\gamma(x, y) = \gamma(y)$ on an infinite strip $-\infty < x < \infty$, $0 < y < 1$. Their method involves reducing the problem to a classical inverse Sturm–Liouville problem, where uniqueness is known. Recently, Sylvester [12] proved that a radial conductivity $\gamma(r)$ on the unit circle is uniquely determined by the

Dirichlet to Neumann map. The problem of this paper, while similar in many respects, is fundamentally different. Our data provides no information about the mean $\bar{q} = \int_0^1 q(y)\,\mathrm{d}y$ and the question of uniqueness up to the mean reduces to a nonstandard inverse Sturm–Liouville problem.

This paper is organized as follows. In Section 2 the method of separation of variables is used to reduce the uniqueness question to the aforementioned nonstandard inverse Sturm–Liouville problem. The information contained in a Cauchy data pair $(f, g)$ is determined through the derivation of an asymptotic formula. In Section 3 a numerical scheme for the recovery of $q$ is proposed. The numerical examples and the asymptotics show that reconstructing $q$ from such data is similar to finding a function from its moments.

## 2. Reduction to Sturm–Liouville form

Using the following expansions for $U_f$ and the known functions $f$ and $g$:

$$U_f(x, y) = \sum_{n=1}^{\infty} b_n X_n(x) Y_n(y), \tag{2.1a}$$

$$f(x) = \sum_{n=1}^{\infty} f_n \sin(\pi n x), \tag{2.1b}$$

$$g(x) = \sum_{n=1}^{\infty} g_n \sin(\pi n x), \tag{2.1c}$$

it is easily verified that $U_f$ satisfies

$$U_f(x, y) = \sum_{n=1}^{\infty} b_n Y_n(y) \sin(\pi n x), \tag{2.2}$$

where $Y_n$ is the solution of the Sturm–Liouville problem

$$\begin{aligned} &-Y_n''(y) + q(y) Y_n(y) = -n^2\pi^2 Y_n(y), \\ &Y_n(0) = Y_n'(1) - \beta_n Y_n(1) = 0, \end{aligned} \tag{2.3}$$

with $\beta_n = g_n/f_n$ when $f_n \neq 0$. Equation (2.3) is a pathological version of the inverse Sturm–Liouville problem. For each $n$, the boundary condition at $y = 1$ is modified and a single negative eigenvalue of $q$ is being specified. In the classical inverse Sturm–Liouville problem, for a given boundary condition, a complete spectrum is specified. In general, additional data is then specified to uniquely determine $q \in L_2$. In our modified problem, when $q = 0$, it is easily verified that $\beta_n = \pi n \coth(\pi n)$ and the resulting eigenfunctions are $\sinh(\pi n y)$. The ill-posedness in recovering $q(y)$ is manifested in the exponential growth of these eigenfunctions.

At first glance, it may seem that a single data pair should carry very little information about $q$. However, when the $f_n$ in (2.1b) are nonzero for all $n$, the "partial" Dirichlet to Neumann map

$$\Lambda_q(f) = \frac{\partial U_f(x, 1)}{\partial y}, \tag{2.4}$$

where $U_f$ is the solution of

$$\begin{aligned} &-\Delta U_f + q(y)U_f = 0, \quad (x, y) \in \Omega, \\ &U_f(0, y) = U_f(1, y) = U_f(x, 0) = 0, \qquad U_f(x, 1) = f(x), \end{aligned} \tag{2.5}$$

is known. The $\beta_n$ are just the eigenvalues of $\Lambda_q$:

$$\Lambda_q(\sin(\pi nx)) = \beta_n \sin(\pi nx), \quad n \geqslant 1. \tag{2.6}$$

The information carried in the sequence $\{\beta_n\}$ about the unknown function $q(y)$ is quantified in Theorem 4. We begin by proving some preliminary lemmas. In what follows, $C$ denotes a generic constant that is independent of $n$.

**Lemma 1.** *For $q \in L^\infty(0, 1)$, let $w$ be the solution of*

$$-w'' + qw = -\pi^2 n^2 w, \quad w(0) = 0, \quad w'(0) = 1. \tag{2.7}$$

*Then for each $y \in [0, 1]$,*

$$\left| w(y) - \frac{\sinh(\pi ny)}{\pi n} \right| \leqslant \frac{C}{n^2} \sinh(\pi ny).$$

**Proof.** The solution $w$ of (2.7) has the representation

$$w(y) = \frac{\sinh(\pi ny)}{\pi n} + \frac{1}{\pi n} \int_0^y \sinh(\pi n(y-t)) q(t) w(t) \, \mathrm{d}t. \tag{2.8}$$

Letting $\psi(y) \equiv \pi n w(y)/(\sinh(\pi ny))$, it follows that $\psi(y)$ satisfies the integral equation

$$\psi(y) = 1 + \frac{1}{\pi n \sinh(\pi ny)} \int_0^y \sinh(\pi n(y-t)) \sinh(\pi nt) q(t) \psi(t) \, \mathrm{d}t. \tag{2.9}$$

One concludes from (2.9) that

$$|\psi(y) - 1| \leqslant \frac{1}{\pi n} \| q \|_\infty \| \psi \|_\infty F(n, y), \tag{2.10}$$

where

$$F(n, y) \equiv \frac{1}{\sinh(\pi ny)} \int_0^y \sinh(\pi n(y-t)) \sinh(\pi nt) \, \mathrm{d}t. \tag{2.11}$$

In Lemma 2 we show that

$$\sup_{n \in \mathbb{N}, y \in [0,1]} F(n, y) \leqslant C < \infty, \tag{2.12}$$

which, using (2.10) and (2.12), gives

$$\| \psi \|_\infty < \frac{1}{1 - C \| q \|_\infty / (\pi n)} \leqslant 2, \tag{2.13}$$

for all $n$ sufficiently large. The desired result follows from (2.10), (2.12) and (2.13). □

**Lemma 2.** *The function* (2.11) *satisfies estimate* (2.12).

**Proof.** A direct integration gives

$$F(n, y) = \tfrac{1}{2}\left[y \coth(\pi n y) - \frac{1}{\pi n}\right].$$

When $0 \leqslant y \leqslant 1/(\pi n)$, then

$$F(n, y) \leqslant \frac{1}{2\pi n} \sup_{z \in [0,1]} |z \coth z - 1| \leqslant C,$$

and if $1/(\pi n) < y \leqslant 1$, then $\lim_{n \to \infty} \coth(\pi n) = 1$ gives

$$F(n, y) \leqslant \tfrac{1}{2}\left(\coth(\pi n) - \frac{1}{\pi n}\right) \leqslant C.$$

The result follows immediately. □

**Lemma 3.** *If $w$ solves* (2.7), *then*

$$\left|\int_0^1 \frac{\sinh(\pi n(1-t))}{\sinh(\pi n)} q(t) w(t)\, \mathrm{d}t\right| \leqslant \frac{C}{n}.$$

**Proof.** Writing

$$\int_0^1 \frac{\sinh(\pi n(1-t))}{\sinh(\pi n)} q(t) w(t)\, \mathrm{d}t = \int_0^1 \frac{\sinh(\pi n(1-t))}{\sinh(\pi n)} \left(w(t) - \frac{\sinh(\pi n t)}{\pi n}\right) q(t)\, \mathrm{d}t$$
$$+ \int_0^1 \frac{\sinh(\pi n(1-t))}{\sinh(\pi n)} \frac{\sinh(\pi n t)}{\pi n} q(t)\, \mathrm{d}t$$

and using Lemmas 1 and 2, we obtain

$$\left|\int_0^1 \frac{\sinh(\pi n(1-t))}{\sinh(\pi n)} q(t) w(t)\, \mathrm{d}t\right| \leqslant C \| q \|_\infty \left[\frac{1}{n^2} + \frac{1}{n}\right] F(n, 1) \leqslant \frac{C}{n}. \qquad \square$$

**Theorem 4.** *For $q \in L^\infty(0, 1)$,*

$$\beta_n(q) = \pi n \coth(\pi n) + \int_0^1 \left(\frac{\sinh(\pi n y)}{\sinh(\pi n)}\right)^2 q(y) dy + \mathrm{O}\left(\frac{1}{n^2}\right). \tag{2.14}$$

**Proof.** Using the integral representation (2.8), we have

$$\frac{\beta_n(q)}{\pi n} - \coth(\pi n) = \frac{\int_0^1 \sinh(\pi n t) q(t) w(t)\, \mathrm{d}t}{\sinh^2(\pi n)\left(1 + [\sinh(\pi n)]^{-1} \int_0^1 \sinh(\pi n(1-t)) q(t) w(t)\, \mathrm{d}t\right)}.$$

Consequently,

$$\frac{\beta_n(q)}{\pi n} - \coth(\pi n) - \frac{1}{\pi n}\int_0^1 \left(\frac{\sinh(\pi n t)}{\sinh(\pi n)}\right)^2 q(t)\,\mathrm{d}t$$
$$= \frac{A-B}{1+[\sinh(\pi n)]^{-1}\int_0^1 \sinh(\pi n(1-t))q(t)w(t)\,\mathrm{d}t}, \tag{2.15}$$

where

$$A \equiv \frac{1}{\sinh^2(\pi n)}\int_0^1 \sinh(\pi n t)q(t)\left[w(t) - \frac{\sinh(\pi n t)}{\pi n}\right]\mathrm{d}t,$$
$$B \equiv \frac{1}{\pi n \sinh^3(\pi n)}\left(\int_0^1 \sinh^2(\pi n t)q(t)\,\mathrm{d}t\right)\left(\int_0^1 \sinh(\pi n(1-t))q(t)w(t)\,\mathrm{d}t\right).$$

Observing that by Lemma 3, the denominator of (2.15) remains bounded away from zero as $n \to \infty$, Lemmas 1 and 3 give

$$\left|\frac{\beta_n(q)}{\pi n} - \coth(\pi n) - \frac{1}{\pi n}\int_0^1 \left(\frac{\sinh(\pi n t)}{\sinh(\pi n)}\right)^2 q(t)\,\mathrm{d}t\right| \leqslant \frac{C}{n^3}.$$

This concludes the proof of Theorem 4. □

For each real number $r$, the set

$$\left\{\psi_{n_k}(y) \equiv \left(\frac{\sinh(\pi n_k y)}{\sinh(\pi n_k)}\right)^2\right\}_{k=1}^{\infty} \tag{2.16}$$

spans $L_2^r(0, 1) \equiv \{q \in L_2(0, 1): \bar{q} \equiv \int_0^1 q(y)\,\mathrm{d}y = r\}$, whenever

$$\sum_{k=1}^{\infty} \frac{1}{n_k} = \infty. \tag{2.17}$$

To see this, extend $q$ to be even on $[-1, 1]$. Then

$$\alpha_{n_k} \equiv \int_0^1 q(y)\sinh^2(\pi n_k y)\,\mathrm{d}y = \frac{1}{4}\int_{-1}^1 q(y)\,\mathrm{e}^{2\pi n_k y}\,\mathrm{d}y - \tfrac{1}{2}r.$$

Using the change of variables $y \to (2\pi)^{-1}\ln x$, one obtains

$$\alpha_{n_k} + \tfrac{1}{2}r = \frac{1}{8\pi}\int_{\mathrm{e}^{-2\pi}}^{\mathrm{e}^{2\pi}} q\left(\frac{1}{2\pi}\ln x\right)x^{n_k-1}\,\mathrm{d}x. \tag{2.18}$$

By the Müntz–Szász Theorem on the interval $[\mathrm{e}^{-2\pi}, \mathrm{e}^{2\pi}]$ and the monotonicity of $\ln x$, $q$ is uniquely determined in $L_2^r[0, 1]$ by the polynomial moments (2.18).

Theorem 4 suggests that $q \in L_2^r(0, 1)$ is uniquely determined by $\{\beta_{n_k}\}$, whenever $\{n_k\}$ satisfies (2.17). The sequence $\{\beta_{n_k}\}$ provides information on the Fourier coefficients of $q$ in the spanning set (2.16). Equation (2.18) shows that the reconstruction of $q$ in the expansion set (2.16) is comparable to finding a function from its polynomial moments. The graphs of

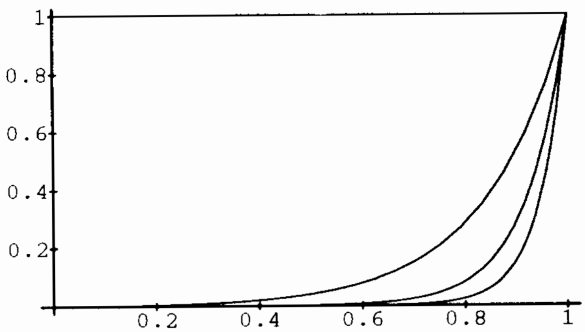

Fig. 1.

Table 1

| $N$ | Condition number |
|---|---|
| 2 | 40 |
| 3 | 1422 |
| 4 | 49 128 |
| 5 | 1 672 642 |

$\{\psi_n(y)\}_{n=1}^3$ are shown in Fig. 1. For the case $n_k = k$, the condition number of the $N \times N$ matrix $[A]_{i,j} = \int_0^1 \psi_i(y)\psi_j(y)\,\mathrm{d}y$ is given in Table 1.

Consequently, measurement errors in $\beta_n$ are profoundly magnified, and, realistically, only $\{\beta_n\}_{n=1}^3$ provide usable numerical information for the reconstruction of $q(y)$. Of course, information given at $y = 1$ should only provide information about $q(y)$ for $y$ near 1. Consequently, any approach that is used to determine $q$ on (0, 1) must contend with the exponential propagation of errors into the interior.

## 3. Reconstruction algorithm and numerical examples

The reconstruction of $q(y)$ from finite data is accomplished by a shooting method. This method was previously applied for the reconstruction of Sturm–Liouville potentials from finite spectral data [8]. Given the data $\{\beta_n\}_{n=1}^N$, a potential $q(y) = \sum_{k=1}^N q_k \psi_k(y)$ in the span of a basis set $\{\psi_k(y) \mid k = 1, \ldots, N\}$ is constructed that is compatible with the data $\{\beta_n\}_{n=1}^N$. This is accomplished by finding a zero of the map $F : \mathbb{R}^N \to \mathbb{R}^N$ given by

$$F(\boldsymbol{q}) \equiv \begin{pmatrix} Y_1'(1; \boldsymbol{q}) - \beta_1 Y_1(1; \boldsymbol{q}) \\ Y_2'(1; \boldsymbol{q}) - \beta_2 Y_2(1; \boldsymbol{q}) \\ \vdots \\ Y_N'(1; \boldsymbol{q}) - \beta_N Y_N(1; \boldsymbol{q}) \end{pmatrix}, \tag{3.1}$$

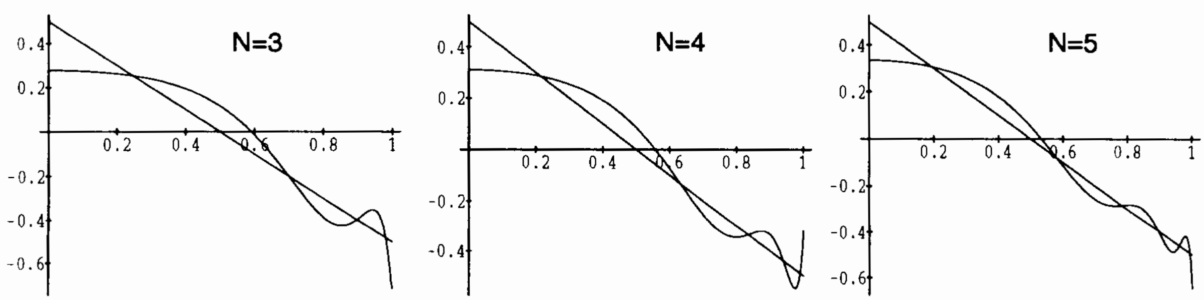

Fig. 2.

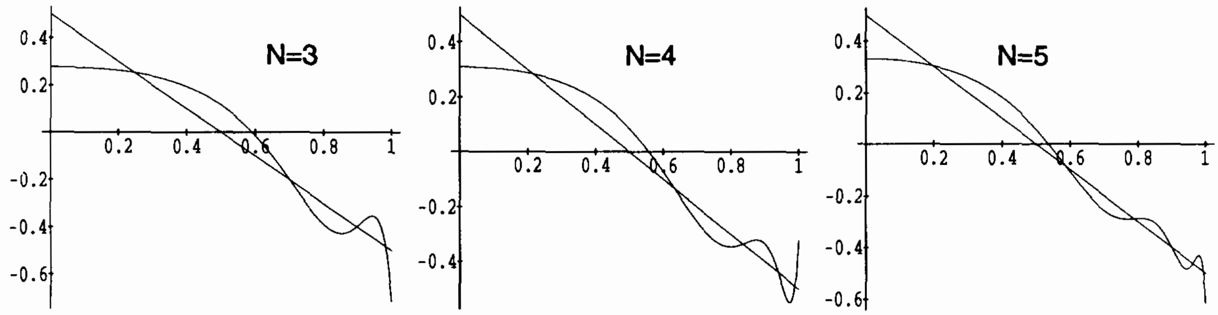


Fig. 3.

where $Y_n(y) \equiv Y_n(y; q)$ is the solution of

$$-Y_n''(y) + q(y)Y_n(y) = -n^2\pi^2 Y_n(y), \quad Y_n(0) = 0, \quad Y_n'(0) = 1, \tag{3.2}$$

for $n = 1, \ldots, N$. The zero is found by the Newton scheme

$$F_q(q^{(\nu)})\delta q^{(\nu)} = -F(q^{(\nu)}), \qquad q^{(\nu+1)} = q^{(\nu)} + \delta q^{(\nu)}, \tag{3.3}$$

where the $nm$th entry of the Jacobian matrix $F_q(q^{(\nu)})$ is the solution of the differential equation

$$\begin{aligned} &-w''(y) + q(y)w(y) = -n^2\pi^2 w(y) - \psi_m(y)Y_n(y), \\ &w(0) = 0, \qquad w'(0) = 0. \end{aligned} \tag{3.4}$$

The following numerical examples illustrate the inherent ill-posedness of this problem. For convenience, we consider $q(y)$ with zero mean and the reconstructions from the data $\{\beta_n\}_{n=1}^N$ employ either the set

$$\left\{ \left( \frac{\sinh(\pi n y)}{\sinh(\pi n)} \right)^2 - \int_0^1 \left( \frac{\sinh(\pi n y)}{\sinh(\pi n)} \right)^2 \mathrm{d}y \right\}_{n=1}^N \tag{3.5}$$

or a set consisting of cosine functions.

Figure 2 shows the reconstruction of $q(y) = \frac{1}{2} - y$ using the data $\{\beta_n\}_{n=1}^N$ and the set (3.5). Essentially identical results are achieved in Fig. 3 by using a Fourier reconstruction of $q$ in this same set.

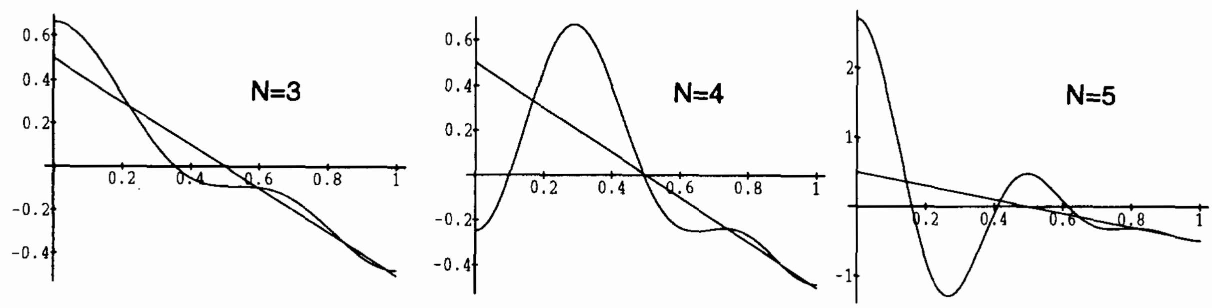


Fig. 4.

Table 2
Comparison between $\beta_n$ for actual and reconstructed $q$; absolute deviation

| $n$ | $N=3$ | $N=4$ | $N=5$ |
|---|---|---|---|
| 1 | 0 | 0 | 0 |
| 2 | 0 | 0 | 0 |
| 3 | 0 | 0 | 0 |
| 4 | $5.9 \cdot 10^{-5}$ | 0 | 0 |
| 5 | $1.1 \cdot 10^{-5}$ | $1.5 \cdot 10^{-5}$ | 0 |
| 6 | $1.4 \cdot 10^{-5}$ | $3.2 \cdot 10^{-5}$ | $4.0 \cdot 10^{-6}$ |

Table 3
Comparison between $\beta_n$ for actual and reconstructed $q$; absolute deviation

| $n$ | $N=3$ | $N=4$ |
|---|---|---|
| 1 | 0 | 0 |
| 2 | 0 | 0 |
| 3 | 0 | 0 |
| 4 | $3.2 \cdot 10^{-4}$ | 0 |
| 5 | $5.7 \cdot 10^{-4}$ | $8.1 \cdot 10^{-5}$ |
| 6 | $7.3 \cdot 10^{-4}$ | $1.7 \cdot 10^{-4}$ |

When $N = 6$, the reconstructions degrade significantly, with the introduction of large oscillations. The condition numbers encountered in Table 1 explain this phenomenon. Reconstructions using the set $\{\cos(\pi n y)\}_{n=1}^{N}$ are shown in Fig. 4. Predictably, the reconstructions degrade when moving from $y = 1$ to $y = 0$. Table 2 shows a comparison between $\{\beta_n\}_{n=1}^{6}$ of the actual $q$ and the reconstructions in Fig. 4. It is observed that there is excellent agreement.

When reconstructing $q(y) = e^{y^2} - \int_0^1 e^{y^2}\, dy$, similar behavior is encountered. The seemingly accurate reconstructions in Fig. 5 result because $q$ and the elements in the set (3.5) have the same general form. The oscillations encountered in Fig. 6 when using a set of cosine functions are quite pronounced. However, Table 3 shows that $\{\beta_n\}_{n=1}^{6}$ for the actual $q$ and the reconstructions in Fig. 6 are very close.

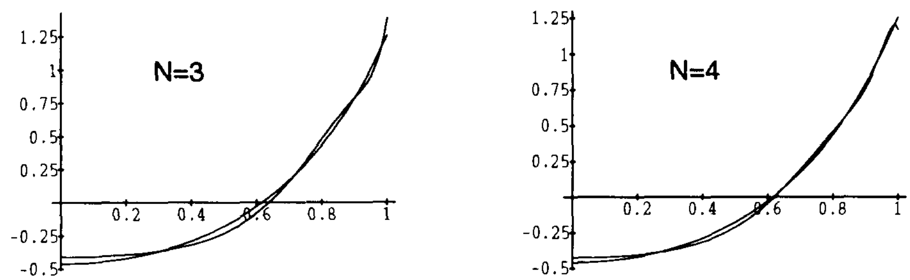


Fig. 5.

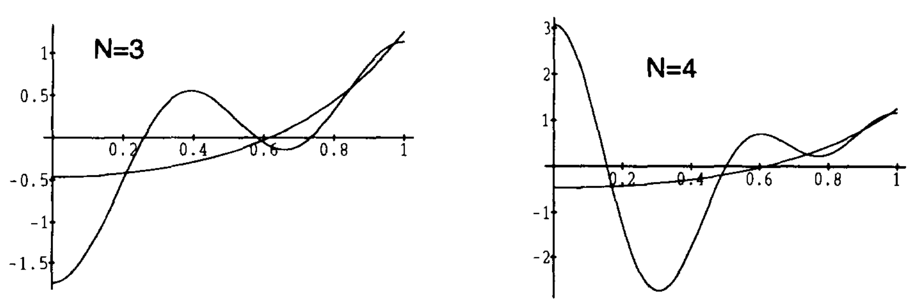


Fig. 6.

## 4. Conclusions

We have investigated the problem of recovering a potential $q(y)$ in (1.1) on the unit square $(0, 1) \times (0, 1)$. The forward solution $u$ in (1.1) satisfies homogenous Dirichlet conditions on three sides and a single Cauchy data pair $(f, g)$ is prescribed along $y = 1$. This represents a worst-case situation, for the overposed data is given in a direction that is perpendicular to the dependence of $q$. Employing separation of variables, the recovery of $q$ reduces to solving the nonstandard inverse Sturm–Liouville problem (2.3). The decomposition of $f$ and $g$ in (2.1b), (2.1c) transforms the overposed data $(f, g)$ into a data sequence $\{\beta_n\}$. When $q$ is a perturbation of 0, Lemma 1 shows that for all $n$ a solution of (2.3) behaves like $\sinh(\pi n y)$. These exponential solutions are a consequence of the negative term on the right-hand side of the differential equation and cause a great deal of numerical difficulty. This is contrary to the situation which is encountered in a classical inverse Sturm–Liouville problem. For example, given Dirichlet eigenvalues and endpoint data for $q$ near 0, the corresponding forward solutions are of the form $\sin(\pi n y)$. These functions are considerably more tractable in the context of numerics. The information that is contained in the sequence $\{\beta_n\}$ is quantified in Theorem 4. Formula (2.14) shows that $\{\beta_n\}$ does not contain any information about the mean of $q$. However, it does give a strong indication that when the mean of $q$ is known and (2.17) is satisfied, then $q$ is uniquely determined by $\{\beta_{n_k}\}$ or equivalently $(f, g)$. The sequence $\{\beta_{n_k}\}$ provides information on the Fourier coefficients of $q$ in the spanning set (2.16) of $L_2^{\bar{q}}(0, 1)$.

Equation (2.18), Fig. 1 and Table 1 emphasize that reconstructing $q$ in the set (2.16) is comparable to the difficult numerical problem of finding a function from a Müntz–Szász set of moments. The condition numbers encountered in Table 1 show that in the presence of noise, only $\{\beta_n\}_{n=1}^{3}$ realistically provide usable numerical data. Equivalently, only $f$ in the span of $\{\sin(\pi n x)\}_{n=1}^{3}$ provide usable data. The numerical experiments illustrate a degradation in the reconstructions as one moves from the boundary $y = 1$ to the interior $y = 0$. This is clearly consistent with the overposed data being given at $y = 1$.

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