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Lester Caudill
University of Richmond, lcaudill@richmond.edu

Peter A. Perry

Albert W. Schueller

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ISOSPECTRAL SETS FOR FOURTH-ORDER ORDINARY DIFFERENTIAL OPERATORS*

LESTER F. CAUDILL, JR.[†], PETER A. PERRY[‡], AND ALBERT W. SCHUELLER[§]

Abstract. Let $L(p)u = D^4u - (p_1u')' + p_2u$ be a fourth-order differential operator acting on $L^2[0,1]$ with $p \equiv (p_1, p_2)$ belonging to $L^2_{\mathbb{R}}[0,1] \times L^2_{\mathbb{R}}[0,1]$ and boundary conditions $u(0) = u''(0) = u(1) = u''(1) = 0$. We study the isospectral set of $L(p)$ when $L(p)$ has simple spectrum. In particular we show that for such p, the isospectral manifold is a real-analytic submanifold of $L^2_{\mathbb{R}}[0,1] \times L^2_{\mathbb{R}}[0,1]$ which has infinite dimension and codimension. A crucial step in the proof is to show that the gradients of the eigenvalues of $L(p)$ with respect to p are linearly independent: we study them as solutions of a non-self-ajdoint fifth-order system, the *Borg system*, among whose eigenvectors are the gradients.

Key words. inverse spectral problem, ordinary differential equations

AMS subject classifications. 34A55, 34L20

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1. Introduction.

This paper initiates a study of isospectral sets of coefficients for self-adjoint, fourth-order ordinary differential operators, in Liouville–Green normal form, on the finite interval $[0,1]$. Such operators are labelled by a pair of coefficients $p = (p_1, p_2)$. Our motivation is twofold: first, we would like to understand the inverse spectral problem for fourth-order operators such as the Euler–Bernoulli operator of mechanics; second, we would like to develop techniques of analysis which are systematic in nature and are therefore likely to be useful in the study of other singular and higher order ordinary differential operators. Our goal is to understand the set of coefficients isospectral to a given pair $p = (p_1, p_2)$ as a Hilbert submanifold of a suitable Hilbert space of coefficients, in analogy to the analysis of the second-order Sturm–Liouville problem carried out by Trubowitz and his collaborators (see [11, 18, 19, 26, 27], and see [10] for more recent results).

As in the work of Trubowitz et. al., we use methods of global analysis to study the isospectral manifold as a level set of the direct spectral map from coefficients to spectra. For the class of operators we consider, the gradient $g_n(x;p)$ of a given eigenvalue $\lambda_n(p)$ is an ordered pair consisting of an eigenfunction square and the square of its derivative, and so the gradient of the mapping from coefficients to spectra is the infinite sequence of all such ordered pairs. A crucial part of the analysis is to show that these ordered pairs form a linearly independent set.

Our approach differs from the approach to the Sturm–Liouville problem taken in [11, 18, 19, 26, 27] in two respects. First of all, we use resolvent perturbation

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[†]Department of Mathematics and Computer Science, University of Richmond, Richmond, VA 23173 (lcaudill@richmond.edu).

[‡]Department of Mathematics, University of Kentucky, Lexington, KY 40506-0027 (perry@ms.uky.edu).

[§]Department of Mathematics, Whitman College, Walla Walla, WA 99362 (schuellaw@whitman.edu).

techniques rather than integral equations and complex analysis to obtain the necessary eigenvalue and eigenfunction asymptotics: see the thesis of the third author [29], where these techniques are developed at greater length. Secondly, we study orthogonality properties of the gradients, not using special identities, but rather by studying an associated non-self-adjoint, fifth-order system, the *Borg system*, among whose eigenvectors are exactly the gradients $g_n(x,p)$. Our system is the analogue, for fourth-order differential operators, of a third-order non-self-adjoint eigenvalue problem introduced by Borg [9] in his study of completeness of eigenfunction squares in the Sturm–Liouville problem. We believe this technique to be a powerful one which admits generalization to other inverse spectral problems involving ordinary differential operators.

To describe our results in detail, we first specify the class of fourth-order operators which we will study. In order to study the isospectral set as a Hilbert manifold, we wish to study operators $L(p)$ where p ranges over a Hilbert space of coefficients; here $L(p)$ is the operator

$$L(p)u = D^4u - D(p_1Du) + p_2u \tag{1.1}$$

on $L^2[0,1]$, where $p = (p_1, p_2)$. In what follows, we will impose "double Dirichlet" boundary conditions $u(0) = u''(0) = u(1) = u''(1) = 0$, although our methods can be used to treat other separated, self-adjoint boundary conditions.

A natural choice for the Hilbert space of coefficients is $E \equiv L^2_{\mathbb{R}}[0,1] \times L^2_{\mathbb{R}}[0,1]$, where $L^2_{\mathbb{R}}[0,1]$ denotes real-valued, square-integrable, measurable functions on $[0,1]$. For such singular coefficients it is convenient to define the operator $L(p)$ by the method of sesquilinear forms (see, for example, Kato [17, Chapter 6]). Since we wish to study real analyticity of various maps on E, it will also be convenient to introduce $E_{\mathbb{C}} \equiv L^2_{\mathbb{C}}[0,1] \times L^2_{\mathbb{C}}[0,1]$ and define $L(p)$ for $p \in E_{\mathbb{C}}$. To this end, we introduce the sesquilinear form

$$q(u,v) = \int_0^1 u''(x)\overline{v''(x)} + p_1(x)u'(x)\overline{v'(x)} + p_2(x)u(x)\overline{v(x)}\,dx \tag{1.2}$$

with the form domain

$$\mathcal{Q}(q) = \{u \in H^2[0,1] : u(0) = u(1) = 0\}$$

for $p \in E_{\mathbb{C}}$. It is not difficult to see that the form q with $p = 0$ is a closed positive form. Using this fact and simple perturbative estimates, one can show that the form q with $0 \neq p \in E_{\mathbb{C}}$ is also closed and sectorial, i.e., that the set

$$\{q(u,u) : u \in \mathcal{Q}(q), \|u\|_{L^2[0,1]} = 1\}$$

is contained in a sector of the complex plane of the form $\Re(z) \geq -c$, $|\Im(z)| \leq (\Re(z)+c)$. Here c depends only on $\|p\|_E$; a complete proof is given in [29, section 5.2].

It follows from the form representation theorem (see, for example, Theorem VI.2.1 of [17]) that there is a unique sectorial operator $L(p)$, i.e., a unique closed operator with numerical range in a sector, associated with the sesquilinear form q. It follows from the same theorem that for all $p \in E_{\mathbb{C}}$, the domain of $L(p)$ is contained in the $H^3[0,1]$ functions with $u(0) = u''(0) = u(1) = u''(1) = 0$. Thus $L(p)$ is an operator with compact resolvent, and its spectrum consists of an infinite sequence, $\{\lambda_n(p)\}$, of discrete eigenvalues. Using the form representation theorem, one can also show that if $p \in C^1([0,1];\mathbb{C}^2)$, then $L(p)$ is the operator (1.1). For more singular p the action of $L(p)$ may be understood in terms of the quasi-derivatives associated with the operator $L(p)$: see Naimark [25] for the general theory and Schueller [29, section 5.3] for its application to fourth-order operators.

We wish to study isospectral sets of $L(p)$ for $p = (p_1, p_2) \in E$. In order to apply techniques of global analysis, we need to realize the direct spectral map from coefficients to spectral data as a map between Hilbert spaces. To this end, we set

$$\mu_0(p) = \overline{p_1} = \int_0^1 p_1(x)\,dx$$

and

$$\mu_n(p) = \frac{\lambda_n(p) - \lambda_n(0) - n^2\pi^2\overline{p_1}}{n^2\pi^2}.$$

We will show that the sequence $\{\mu_n(p)\}_{n=0}^{\infty}$ belongs to the Hilbert space $F \equiv \ell^2(0 \cup \mathbb{N})$. The *direct spectral map* is the mapping $\mu : E \to F$ defined by $\mu(p) = \{\mu_n(p)\}$. The *isospectral set* $M(p)$ of a given $p \in E$ is the set of all $q \in E$ with $\mu(q) = \mu(p)$. We will say that $p \in E$ has *simple spectrum* if the sectorial operator $L(p)$ has only simple eigenvalues. First of all, we will prove the following theorem.

THEOREM 1.1. *The set of $p \in E$ with simple spectrum is open and dense in E.*

Denote this set by $\mathcal{E}$. There are physically relevant families of fourth-order problems, such as the Liouville–Green normal forms of the Euler–Bernoulli equation, which are known to have simple spectrum (see, e.g., [14]). Thus, restricting attention to $p \in \mathcal{E}$ is not unreasonable for many problems of physical interest. Our main result is as follows.

THEOREM 1.2. *For each $p \in \mathcal{E}$, $M(p) \cap \mathcal{E}$ is a real-analytic submanifold of E of infinite dimension and infinite codimension.*

We can quantify the "size" of $M(p)$ more precisely by introducing some auxiliary boundary value problems associated with the formal differential operator $L(p)$. To define these auxiliary boundary value problems, we introduce the closed sesquilinear forms q_1 and q_2 which are given by the expression (1.2) defined on the respective domains

$$\mathcal{Q}_1 = \{u \in H^2[0,1] : u(0) = 0, u'(1) = 0\}$$

and

$$\mathcal{Q}_2 = \{u \in H^2[0,1] : u(0) = 0, u(1) = u'(1) = 0\}.$$

We denote the associated sectorial operators by $L_1(p)$ and $L_2(p)$, and their corresponding eigenvalues by $\sigma_n(p)$ and $\tau_n(p)$, respectively. For $p \in C_0^\infty([0,1];\mathbb{C}^2)$, these operators carry the following boundary conditions:

$$L(p) : u(0) = u''(0) = u(1) = u''(1) = 0, \tag{1.3}$$

$$L_1(p) : u(0) = u''(0) = u'(1) = u'''(1) = 0, \tag{1.4}$$

$$L_2(p) : u(0) = u''(0) = u(1) = u'(1) = 0. \tag{1.5}$$

We conjecture that for $p \in \mathcal{E}$ and q in a dense and open subset of $M(p)$, the three sets of eigenvalues $\{\lambda_n(q)\}, \{\sigma_n(q)\}, \{\tau_n(q)\}$ give local coordinates for $M(p)$. We expect to prove this in a subsequent paper.

Theorem 1.2 involves a study of the differential of the direct spectral map μ. We will first study μ on a dense subset $\mathcal{D}$ of $\mathcal{E}$ consisting of functions $p \in C_0^\infty((0,1);\mathbb{R}^2)$ such that the spectra of each of the three boundary value problems is simple and the intersection of the sets $\{\lambda_n(p)\}$, $\{\sigma_n(p)\}$, and $\{\tau_n(p)\}$ is empty. We show that the set $\mathcal{D}$ is dense in $\mathcal{E}$ in Theorem 3.1. We will also show that eigenvalues and eigenfunctions associated with operators $L(q)$ with $q \in \mathcal{E}$ can be well approximated by those associated with operators $L(p)$ with $p \in \mathcal{D}$.

To show that the isospectral manifold is real-analytic, we wish to apply the real-analytic implicit function theorem (see, for example, [26, p. 154]). Theorem 1.2 follows from Theorem 1.3.

THEOREM 1.3. *The direct spectral map μ is a real-analytic mapping from $\mathcal{E}$ to F. For $p \in \mathcal{E}$ and each $q \in M(p)$, there is an orthogonal decomposition $T_q\mathcal{E} = \mathcal{E}_v(q) \oplus \mathcal{E}_h(q)$ such that $d\mu(p)$ is a linear isomorphism of $\mathcal{E}_v(q)$ onto F and $\mathcal{E}_h(q)$ has infinite dimension.*

We will prove Theorem 1.3 by showing that (1) the map μ is real-analytic as a map from $\mathcal{E}$ into F, (2) the differential $d\mu(q)$ for an arbitrary $q \in \mathcal{E}$ is well approximated by the differential $d\mu(p)$ of a "nearby" $p \in \mathcal{D}$, and (3) the differential $d\mu(p)$ has the required mapping properties for $p \in \mathcal{D}$.

To explain steps (2) and (3) more fully, let $z_n(\cdot;p)$ be the normalized eigenfunction corresponding to eigenvalue $\lambda_n(p)$. (Note that since $p \in \mathcal{E}$, this eigenfunction is unique up to a phase.) Let $\langle \cdot\,,\,\cdot\rangle_E$ denote the inner product on E. A short calculation shows that for $p \in \mathcal{D}$, the differential $d\mu(p)$ is given by

$$d\mu(p)(v_1, v_2) = \{\langle g_n(\ \cdot\ ; p), v\rangle_E\}_{n=0}^\infty\,, \tag{1.6}$$

where the gradients g_n are given by

$$g_0(x;p) = (1,0)$$

and

$$g_n(x;p) = \left(\frac{z_n'(x;p)^2}{n^2\pi^2} - 1, \frac{z_n(x;p)^2}{n^2\pi^2}\right), \quad n \geq 1\,. \tag{1.7}$$

We wish to take $\mathcal{E}_v$ to be the span of the g_n and $\mathcal{E}_h$ to be its orthogonal complement, the kernel of $d\mu$. If $\zeta = \sum_j c_j g_j$ and c denotes the sequence $\{c_j\}$, then $\|\zeta\|_E^2 = \langle c, A(q)c\rangle$, where A is the operator on F with matrix $\langle g_i, g_j\rangle_E$. If A is a bounded invertible operator, then the g_j form a *Riesz basis* [8] for $\mathcal{E}_v$, and the operator $T : \mathcal{E}_v \to F$ defined by $T\zeta = c$, is boundedly invertible. Moreover, $S(q) = d\mu(q) \circ T^{-1} \in \mathcal{B}(F,F)$ has matrix $A(q)$ and so is a linear isomorphism. In step (2), we show that for any $\epsilon > 0$ and each $q \in \mathcal{E}$, there is a $p \in \mathcal{D}$ such that $\|S(q) - S(p)\|_{\mathcal{B}(F,F)} < \epsilon$. In step (3) we show that $S(p)$ is boundedly invertible for each $p \in \mathcal{D}$ by proving that $A(p)$ has the same property. In order to do so we show that $A(0)$ is boundedly invertible, and, by perturbation estimates and linear independence of the g_n, that the same holds true for $A(p)$ if $p \in \mathcal{D}$. Since the boundedly invertible operators are open in $\mathcal{B}(F,F)$, this shows that $A(q)$, and hence $S(q)$, is boundedly invertible for any $q \in \mathcal{E}$.

The required perturbation estimates on the g_n show that

$$\sum_n \|g_n(\ \cdot\ ;p) - g_n(\ \cdot\ ;0)\|_E^2$$

is finite. In order to obtain these estimates, we exploit the observation that the functions z_n^2 and $(z_n')^2$ can be recovered from the respective residues of the operators $(L(p)-z)^{-1}$ and $D(L(p)-z)^{-1}D$ at $z=\lambda_n(p)$. We use resolvent perturbation theory to estimate the differences $z_n(\,\cdot\,;p)^2 - z_n(\,\cdot\,;0)^2$ and $z_n'(\,\cdot\,;p)^2 - z_n'(\,\cdot\,;0)^2$.

To prove that the g_n are linearly independent for any $p \in \mathcal{D}$, we introduce an auxiliary fifth-order differential system, the *Borg system*, satisfied by the $g_n(p)$. An analogous third-order equation was used by Borg [9] in his study of eigenfunction-squares in the Sturm–Liouville problem. The Borg system for a fourth-order operator takes the form

$$\mathcal{M}(p)g = \lambda \mathcal{B}(p)g$$

for matrix-valued differential operators $\mathcal{M}(p)$ of fifth order and $\mathcal{B}(p)$ of third order. For $p \in \mathcal{D}$, the generalized resolvent

$$R(\lambda) = (\mathcal{M}(p) - \lambda\mathcal{B}(p))^{-1}$$

has simple poles, among which are the eigenvalues $\lambda_n(p)$, with corresponding generalized eigenfunctions

$$\widehat{g}_n(x;p) = \left(z_n(x;p)^2, z_n'(x;p)^2\right)$$

The remaining poles are associated with the two auxiliary boundary value problems; since $p \in \mathcal{D}$, these are distinct from the poles $\lambda_n(p)$, and, as we shall see, all of the poles of the generalized resolvent are simple. Using the simplicity of poles, we can then construct a biorthogonal set from the rank-one residues of the generalized resolvent $R(\lambda)$; this proves linear independence of the $\widehat{g}_n(p)$. The linear independence of the gradients $g_n(p)$ is an easy consequence. The residues of the Borg operator furnish tangent vectorfields to the isospectral manifold; we expect, but have not yet proved, that they are a basis for its tangent space.

We note that the eigenvalue equation

$$D^2\left(r(x)D^2y\right) = \mu\rho(x)y$$

for the Euler–Bernoulli beam can be transformed, by means of a Liouville transform, into Liouville–Green normal form for smooth coefficients (see, e.g., [5]). Thus our results apply to Euler–Bernoulli problems with suitable boundary conditions.

A number of results exist in the literature regarding the inverse spectral problem for fourth-order differential operators. Barcilon [1, 2, 3, 4, 5, 6, 7] proved that the density and bending stiffness of an Euler–Bernoulli beam can be recovered from three sets of spectra, showed that fewer than three spectra do not uniquely determine these coefficients, and also proved some general results on inverse spectral problems for differential equations of nth order in Liouville normal form. He also showed that three sequences of eigenvalues corresponding to certain distinct boundary conditions contain the same information as one sequence of eigenvalues together with two sets of norming constants [3]. McLaughlin developed a Gel'fand–Levitan-type reconstruction algorithm for smooth coefficients from one spectrum and two sequences of norming constants [21, 22, 23, 24]. In McLaughlin's papers [20, 21] it is shown that the isospectral set for the operator $L(0)$ with boundary conditions $u(0)=u'(0)=u(1)=u'(1)=0$ is infinite-dimensional with infinite codimension and that the isospectral set for the operator $L(p)$ with the same boundary conditions is also infinite dimensional, with infinite codimension, provided that the eigenvalues satisfy certain asymptotic forms. Gladwell gave necessary and sufficient conditions on spectral data to produce an Euler–Bernoulli beam with strictly positive (i.e., physical) density and bending stiffness [13] and carried out numerical reconstructions of Euler–Bernoulli beams from finite spectral data [15, 16].

Our results appear to be the first systematic study of the isospectral manifold for fourth-order differential operators. It should be noted that an operator very similar to our "Borg operator" in the constant coefficient case appears in Barcilon's analysis [3] of the inverse spectral problem.

The plan of this paper is as follows. In section 2 we prove some basic results about the spectra of $L(p)$ and the two associated boundary value problems. In section 3 we prove Theorem 1.1. In section 4 we show that the sequence of functions $\{g_n(q)\}$ is stable, in $\ell^2(\mathbb{N}; E)$-sense, under small perturbations of $q \in \mathcal{E}$ and that the map μ has its range in F. In section 5, we prove that the map μ is an analytic mapping from $\mathcal{E}$ into F. In section 6, we introduce and analyze the Borg system and use it to prove linear independence of the vectors $g_n(p)$ for each fixed $p \in \mathcal{D}$. Finally, in section 7 we give the proofs of Theorems 1.2 and 1.3. In Appendix A, we collect some important estimates on the integral kernels of the resolvents, at $p = 0$, of each of the three boundary value problems considered. In Appendix B, we discuss the boundary conditions on the Borg system and prove some technical domain results needed for section 6.

The results in sections 2 and 4 are proved for a number of separated self-adjoint boundary conditions in the Ph.D. thesis of the third author [29].

2. Spectra. In this section we prove some basic results about the spectra of the operators $L(p)$, $L_1(p)$, and $L_2(p)$ for $p \in E$. The symbol $L_\#(p)$ will denote one of the operators $L(p)$, $L_1(p)$, or $L_2(p)$, and $\lambda_n^\#(p)$ will denote the nth eigenvalue of $L_\#(p)$. Similarly, $q_\#$ denotes one of the three sesquilinear forms q, q_1, or q_2.

The spectra of $L(0)$, $L_1(0)$, and $L_2(0)$ are given by explicit transcendental equations (see, for example, [29]). From these, we easily deduce that $\lambda_n(0) = n^4\pi^4$, $\sigma_n(0) = (n+\frac{1}{2})^4\pi^4$, and $|\tau_n(0)^{1/4} - (n+\frac{1}{4})\pi| \le 4e^{-n\pi}$. We expect the eigenvalues of $L_\#(p)$ for $p \ne 0$ to approach these values asymptotically so that the three sets of spectra "separate" for n large. We will use resolvent perturbation theory to show this is the case.

The following technical lemma will enable us to prove certain resolvent estimates for coefficients $p \in C_0^\infty((0,1), \mathbb{C}^2)$ and extend them by continuity to $p \in E_{\mathbb{C}}$. Recall that a mapping f from an open subset of a Banach space E into a Banach space F is called *compact* if $f(p_n)$ converges strongly to $f(p)$ in F whenever p_n converges weakly to p in E.

LEMMA 2.1. *For any fixed z with $\Re(z)$ sufficiently negative, the mapping $p \mapsto (L(p) - z)^{-1}$ is a compact mapping from E into the bounded operators on $L^2[0,1]$.*

Proof. Suppose that $p_n \to p$ weakly in E. It is easy to verify that the sesquilinear forms $q_\#^n$ associated with p_n converge to the sesquilinear form $q_\#$ associated with p so that $L_\#(p_n)$ converges to $L_\#(p)$ in the strong resolvent sense (see Kato [17, Theorem VIII.3.6]). It is also easy to check that there is a fixed c, depending only on $\sup \|p_n\|_E$, such that $\Re q_\#^n(u,u) \ge -c+1$ and that the operators $(L_\#(0)+1)^{1/2}(L_\#(p_n)+c)^{-1}$ are bounded uniformly in n. Let $R_n = (L_\#(p_n)+c)^{-1} - (L_\#(p)+c)^{-1}$ and let P_N project onto the first N eigenvectors of $L_\#(0)$. We may estimate

$$\|R_n\| \le \|P_N R_n\| + \|(I - P_N)R_n\|$$

$$\leq \|P_N R_n\| + \|(I - P_N)(L_\#(0) + c)^{-1/2}\| \, \|(L_\#(0) + c)^{1/2} R_n\|.$$

The second right-hand term goes to zero as $N \to \infty$ uniformly in n, and the first right-hand term goes to zero as $n \to \infty$ for each fixed N by the compactness of P_N and the fact that R_n converges strongly to zero. ∎

Note that the same proof works if z is only required to lie in the common resolvent set of the operators $L(p_n)$ and $L(p)$.

Let $\rho(L_\#(p))$ denote the resolvent set of the operator $L_\#(p)$. The remarks in the proof of Lemma 2.1 show that there is a fixed half-plane $\Re(z) < -c$ so that $(L_\#(p) - z)^{-1}$ exists for any z in this half-plane and any $p \in E_{\mathbb{C}}$ with $\|p\|_{E_{\mathbb{C}}} \leq M$. Thus the set

$$S_M = \cap \{\rho(L_\#(p)) : \|p\|_{E_{\mathbb{C}}} \leq M\}$$

has nonempty interior; we will shortly show that it includes the complement of a countable union of discs whose size depends on M and which are centered at the eigenvalues of $L_\#(0)$. In what follows, denote by $B_M(0)$ the set $\{p \in E : \|p\|_E < M\}$.

LEMMA 2.2. *Fix $M > 0$ and let U be the interior of the set S_M. The mapping $\Psi(z, p) = (L_\#(p) - z)^{-1}$ is a compact analytic mapping from $U \times B_M(0)$ to the bounded operators on $L^2[0, 1]$.*

Proof. Compactness is an immediate consequence of Lemma 2.1 and the first resolvent formula. The resolvent identity

$$\begin{aligned}(L_\#(p) - z)^{-1} &- (L_\#(q) - z)^{-1} \\ &= (L_\#(q) - z)^{-1}(D(q_1 - p_1)D + (q_2 - p_2))(L_\#(p) - z)^{-1}\end{aligned}$$

holds, where $D(q_1 - p_1)D + (q_2 - p_2)$ is understood as a sesquilinear form on the form domain of $L_\#(0)$. This shows that Ψ is norm continuous. A short calculation with difference quotients shows that $(L_\#(p) - z)^{-1}$ is differentiable in the complex sense and that

$$d\Psi_{z,p}(w, h) = w(L_\#(p) - z)^{-2} + (L_\#(p) - z)^{-1}(Dh_1 D + h_2)(L_\#(p) - z)^{-1}. \qquad ∎$$

For numbers $R > 3$ and $\alpha \in (2, 3)$, we define a region $C^\#_{R,\alpha}$ of $\mathbb{C}$ as follows. Let N be an integer obeying the bounds

$$(8R)^{\frac{1}{3-\alpha}} < N < (16R)^{\frac{1}{3-\alpha}}, \tag{2.1}$$

let

$$D^\#_N = \{z \in \mathbb{C} : |z| < \lambda^\#_N(0) + RN^\alpha\}$$

be a disc containing the first N eigenvalues of $L_\#(0)$, and let

$$E^\#_n = \{z \in \mathbb{C} : |z - \lambda^\#_n(0)| < Rn^\alpha\},$$

a disc containing the nth eigenvalue of $L_\#(0)$. We set

$$C^\#_{R,\alpha} = D_N \cup \left(\cup_{n=N+1}^\infty E_n\right).$$

Thus, the set $C^\#_{R,\alpha}$ is the union of a large disc containing the first N eigenvalues of $L_\#(0)$ and infinitely many small discs each containing exactly one of the remaining eigenvalues (see Figure 2.1). We will show that this region still contains the spectrum of $L_\#(p)$ for R sufficiently large, depending on $\|p\|_E$.

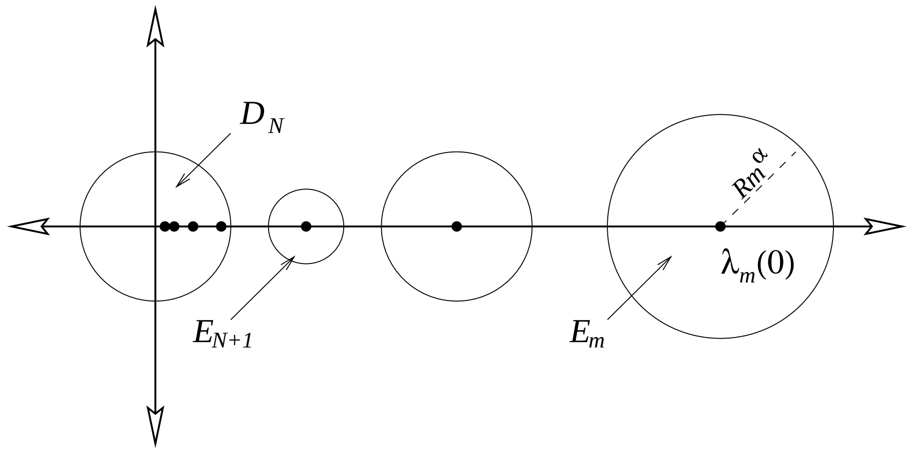

FIG. 2.1. *The set $C_{R,\alpha}$ (not drawn to scale).*

It is easy to check the following purely geometric properties of $C^{\#}_{R,\alpha}$.

LEMMA 2.3.

(i) *Fix one set of boundary conditions. For any $R > 3$, $\alpha \in (2, 3)$, and $m > N$, the regions $D^{\#}_N$ and $E^{\#}_m$ are mutually disjoint.*

(ii) *Let $E^{(1)}_n$ and $E^{(2)}_n$ be the regions E_n associated with two distinct sets of boundary conditions. Then $E^{(1)}_n$ and $E^{(2)}_m$ are disjoint for $n \geq N+1$ and $m \geq N+1$.*

Proof. (i) The discs E_m have radii Rm^α, while $\lambda^{\#}_{m+1}(0) - \lambda^{\#}_m(0) \geq m^3$ if $m \geq 2$. Thus the discs will be separate if $2R(m+1)^\alpha < m^3$. This is true if $m > N$. The regions $D^{\#}_N$ and $E^{\#}_{N+1}$ will be disjoint so long as $\lambda^{\#}_{N+1}(0) - R(N+1)^\alpha - (\lambda^{\#}_N(0) + RN^\alpha) > 0$ or $N^3 > RN^\alpha + R(N+1)^\alpha$. This is guaranteed by the choice of N. (ii) Note that, by the mean value theorem, $(x + \frac{1}{4})^4 - x^4 \geq x^3$. Since the fourth roots of any two eigenvalues of the three operators are separated by a distance of at least $1/4$, it follows that the eigenvalue $\lambda^{\#}_m(0)$ associated with any one of the boundary conditions is separated from the closest eigenvalue associated with any of the three boundary conditions by at least $(m-1)^3$. Thus, it suffices to show that $(m-1)^3 > Rm^\alpha + R(m-1)^\alpha$. This will be true if $\left(\frac{m-1}{m}\right)^3 m^{3-\alpha} > R(1 + (\frac{m-1}{m})^\alpha)$. But $m > 1$ and $m^{3-\alpha} > N^{3-\alpha} > 8R$ by the choice of N. $\square$

We can now state our rough bounds on the location of $\lambda^{\#}_n(p)$.

THEOREM 2.4. *Fix $M > 0$ and let $p \in E_{\mathbb{C}}$ with $\|p\|_{E_{\mathbb{C}}} < M$. There is a number $R > 3$ depending only on M so that:*

(i) *The spectrum of $L_{\#}(p)$ is contained in $C^{\#}_{R,\alpha}$.*

(ii) *The operators $L_{\#}(p)$ have exactly N eigenvalues in the region D_N.*

(iii) *The eigenvalues of $L_{\#}(p)$ with index $n \geq N+1$ are all simple.*

(iv) *The sets $\{\lambda_n(p)\}^\infty_{n=N+1}$, $\{\sigma_n(p)\}^\infty_{n=N+1}$, and $\{\tau_n(p)\}^\infty_{n=N+1}$ have empty intersection.*

Proof. The operators $L_{\#}(p)$ are sectorial with spectrum contained in a half-plane $\Re(\lambda) > -C(M)$, where $C(M)$ is a positive constant depending only on M [29]. We will construct the resolvents $(L_{\#}(p) - z)^{-1}$ perturbatively from $(L_{\#}(0) - z)^{-1}$ and estimate $\|(L_{\#}(p) - z)^{-1}\|$ uniformly in p with $\|p\|_E < M$ and $z \notin C^{\#}_{R,\alpha}$ for sufficiently large R. This will give (i); (iv) will then follow from Lemma 2.3(ii) and the following argument. Observe that for $p = 0$, the region D_N contains exactly N eigenvalues, and the regions E_m each contain one eigenvalue. Analyticity of the resolvents as operator-valued analytic functions of $p \in E$ with $\|p\|_E < M$ will imply analyticity of the projections onto the eigenspaces of eigenvalues contained in $D_N^{\#}$ and $E_n^{\#}$; by standard perturbation theoretic arguments (see, for example, [28, section XII.2], and especially the lemma following Theorem XII.7), the corresponding spectral multiplicities must be stable under perturbation from 0 to p. This gives (ii) and (iii).

We now turn to the perturbative estimates that prove (i). We will use strong estimates on the integral kernel $G_0^{\#}(x, y; z)$ for the operator $(L_{\#}(0) - z)^{-1}$ in order to estimate $(L_{\#}(p) - z)^{-1}$ in operator norm. By Lemma 2.1, it suffices to prove these operator norm estimates for $p \in C_0^{\infty}((0, 1), \mathbb{C}^2)$. This assumption allows us to bypass domain questions which might arise if the coefficients were more singular.

We begin by noting the resolvent equations, true for $\Re(z)$ sufficiently negative,

$$(L_{\#}(p) - z)^{-1} = (L_{\#}(0) - z)^{-1} - (L_{\#}(0) - z)^{-1}(Dp_1D + p_2)(L_{\#}(p) - z)^{-1}.$$

Since $(L_{\#}(p) - z)^{-1}$ maps into $H^3[0, 1]$ and p_1 and p_2 are smooth, the composition of $Dp_1D + p_2$ with $(L_{\#}(p) - z)^{-1}$ is well defined. From this equation it is easy to derive the useful identity

$$(L_{\#}(p) - z)^{-1} = A(z) + B(z)(L_{\#}(p) - z)^{-1}\,, \tag{2.2}$$

where

$$\begin{aligned} A(z) &= (L_{\#}(0) - z)^{-1} \\ &\quad - (L_{\#}(0) - z)^{-1}Dp_1(I + C(z))^{-1}D(L_{\#}(0) - z)^{-1}, \end{aligned} \tag{2.3}$$

$$\begin{aligned} B(z) &= (L_{\#}(0) - z)^{-1}Dp_1(I + C(z)^{-1}D(L_{\#}(0) - z)^{-1}p_2 \\ &\quad - (L_{\#}(0) - z)^{-1}p_2, \end{aligned} \tag{2.4}$$

and

$$C(z) = D(L_{\#}(0) - z)^{-1}Dp_1. \tag{2.5}$$

These equations are valid whenever $\|C(z)\| < 1/2$. Using Lemma A.4(e), we see that this holds for $z \notin C_{R,\alpha}^{\#}$ so long as $CR^{-1/2}(\ln R)^{1/2}\|p_1\|_{L^2} < 1/2$, where C is a numerical constant; we can ensure this by choosing a sufficiently large R depending only on M. We wish to show that, by increasing R if necessary, we can make $\|B(z)\| < 1/2$ for $z \notin C_{R,\alpha}^{\#}$ and then show that $\|A(z)\| \le CR^{-1}$ for a constant C. We can then use standard analytic continuation arguments to conclude that $\|(L_{\#}(p) - z)^{-1}\|$ is bounded for $z \notin C_{R,\alpha}^{\#}$ for sufficiently large R depending on M.

First, we show how to choose R depending on M so that $\|B(z)\| < 1/2$ for $z \notin C_{R,\alpha}^{\#}$. From Lemma A.4(b), (d), and (e), we see that $\|B(z)\| \le CR^{-1}$ for a constant C depending on M so that $\|B(z)\| < 1$ for R sufficiently large. The estimates in Lemma A.4 also show that

$$\|A(z)\| \le \frac{C}{R} + \frac{C}{R^2}(1 + \|p_1\|_{L^2}),$$

which gives an estimate of the desired form. ◻

3. Approximation. First of all, we give the proof of Theorem 1.1. It is not difficult to see that the set $\mathcal{E}$ is open, since all eigenvalues $\lambda_n(p)$ with $n > N$ are simple, and for each of the eigenvalues $\lambda_n(p)$ with $1 \leq n \leq N$, there is a neighborhood U_n in E of any $q \in \mathcal{E}$ such that $\lambda_n(p)$ is simple for all $p \in U_n$. Taking $U = \cap_{n=1}^N U_n$ we obtain an open neighborhood of q contained in $\mathcal{E}$.

To see that $\mathcal{E}$ is dense, we will exploit analytic perturbation theory. Let $p \in E$ with $\|p\| < M$, and suppose that one or more of the eigenvalues of $L(p)$ are degenerate. Consider the family of operators $M(t) = L(tp)$ where $|t| < 2$. By changing the values of R and N as defined in section 2 if necessary, we may assume that the conclusions of Theorem 2.4 hold for all tp with $|t| < 2$. Let

$$P(t) = \frac{1}{2\pi i} \int_{\partial D_N} (M(t) - z)^{-1} \, dz \,,$$

where D_N is as defined in the previous section. For t real, $P(t)$ is an orthogonal projection onto the first N eigenvalues of $M(t)$, counted with multiplicity. By Theorem XII.12 of [28], we can find an analytic family of holomorphically invertible operators $U(t)$ defined for $|t| < 2$ so that $U(t)$ is unitary for t real and $U(t)P(0)U(t)^{-1} = P(t)$. The operator $m(t) = U(t)^{-1}M(t)U(t)$ commutes with $P(0)$ and may be regarded, for t real, as a Hermitian matrix acting on $\mathbb{C}^N$; the first N eigenvalues of $L(tp)$ are simple if and only if the eigenvalues of $m(t)$ are simple. Observe that at $t = 0$, the first N eigenvalues of $L(0)$ are simple by explicit calculation, so the same holds for $|t|$ small. Moreover, the eigenvalues of $m(t)$ are analytic functions of t ([17, Theorem II.6.1]). Thus a given pair of eigenvalues of $m(t)$ can be degenerate for at most finitely many t between 0 and 1. Hence the same holds true of $L(tp)$, so for each $\epsilon > 0$ there is a $t \in (1-\epsilon, 1)$ so that $tp \in \mathcal{E}$. This shows that $\mathcal{E}$ is dense in E, and completes the proof of Theorem 1.1.

Next, we prove Theorem 3.1.

THEOREM 3.1. *Let $\mathcal{D}$ be the set of $p \in C_0^\infty((0,1);\mathbb{R}^2)$ such that $L_\#(p)$ has simple spectrum and the spectra of $L(p)$, $L_1(p)$, and $L_2(p)$ have empty intersection. Then $\mathcal{D}$ is dense in $\mathcal{E}$.*

Proof. First we show how small perturbations may be used to make the spectra of $L_1(p)$ and $L_2(p)$ simple, and the spectra of the three operators nonintersecting. By Theorem 2.4, we need only show that the first N eigenvalues of each operator are simple and that the union of the intersection of the three sets $\{\lambda_n\}_{n=1}^N$, $\{\sigma_n\}_{n=1}^N$, and $\{\tau_n\}_{n=1}^N$ is empty. Let $M(t) = L(tp)$, $M_1(t) = L_1(tp)$, and $M_2(t) = L_2(tp)$. By the technique used above we may associate with these operators analytic, $N \times N$ matrix-valued functions $m(t)$, $m_1(t)$, $m_2(t)$ whose eigenvalues depend holomorphically on t with $|t| < 2$. By explicit calculation, the matrices $m(0)$, $m_1(0)$, and $m_2(0)$ have simple spectra with empty intersection, so the same is true for $|t|$ small by analytic perturbation theory. Analyticity of the eigenvalues implies that the spectra of $m(t)$, $m_1(t)$, and $m_2(t)$ must be simple and have empty intersection for all but a countable set of t with no accumulation point in the region $\{t \in \mathbb{C} : |t| < 2\}$. Thus, given p and ϵ we can find a q with $\|p - q\| < \epsilon$ so that $L(q)$, $L_1(q)$, and $L_2(q)$ have simple spectrum and the three spectra have empty intersection. Since $C_0^\infty((0,1);\mathbb{R}^2)$ is norm dense in E and the simplicity and empty intersection properties involve only finitely many eigenvalues, there is an r with $\|r - q\| < \epsilon$ and $r \in \mathcal{E} \cap C_0^\infty((0,1);\mathbb{R}^2)$ so that $L(r)$, $L_1(r)$, and $L_2(r)$ have simple spectra and their spectra have empty intersection. It follows that $\mathcal{D}$ is dense in $\mathcal{E}$, as asserted. $\square$

4. Stability estimates. For $m > N$, let $z_m(\ \cdot\ ; p)$ denote the normalized eigenfunction of $L(p)$ corresponding to the eigenvalue $\lambda_m(p)$, and let $u_m(\cdot)$ denote the corresponding eigenfunction of $L(0)$. We wish to derive $C[0,1]$-norm estimates on the differences $z_m^2 - u_m^2$ and $(z_m')^2 - (u_m')^2$ and also on the differences $z_m^2(\ \cdot\ ; p) - z_m^2(\ \cdot\ ; q)$ and $(z_m')^2(\ \cdot\ ; p) - (z_m')^2(\ \cdot\ ; q)$. These estimates will be used to analyze the operator $A(p)$ discussed in the introduction.

Let $G(x, y; z)$ denote the integral kernel of the operator $(L(p) - z)^{-1}$. In order to estimate the above quantities, we observe that we can recover $z_m(x, q)^2$ from the diagonal of the residue of $G(x, y; z)$ at $z = \lambda_m(p)$, and $z_m'(x, q)^2$ from the diagonal of the residue of $G_{xy}(x, y; z)$. We will exploit resolvent perturbation theory to prove the following estimates.

THEOREM 4.1. *Let $M > 0$ and let p and q belong to E with $\|p\|_E$, $\|q\|_E < M$. For $\alpha \in (2, 3)$ choose R and N as in Theorem 2.4. Then there are constants C_1 and C_2 depending on M so that for any $m > N$,*

(a) $\sup_{x\in[0,1]} |z_m(x; p)^2 - z_m(x; q)^2| \le C_1 m^{2-\alpha}$ *and*

(b) $\sup_{x\in[0,1]} |z_m'(x; p)^2 - z_m'(x; q)^2| \le C_2 m^{4-\alpha}$

hold.

As an immediate corollary, setting $q = 0$, we have the following theorem.

THEOREM 4.2. *Let $M > 0$ and $p \in E$ with $\|p\|_E < M$. For $\alpha \in (2, 3)$ choose R and N as in Theorem 2.4. Then there are constants C_1 and C_2 depending on M so that for any $m > N$, the estimates*

(a) $\sup_{x\in[0,1]} |z_m(x; p)^2 - z_m(x; 0)^2| \le C_1 m^{2-\alpha}$ *and*

(b) $\sup_{x\in[0,1]} |z_m'(x; p)^2 - z_m'(x; 0)^2| \le C_2 m^{4-\alpha}$ *hold.*

To prove these results, we first note the following lemma.

LEMMA 4.3. *Let $M > 0$ and $p \in E$ with $\|p\|_E < M$, let $\alpha \in (2, 3)$, and choose R and N as in Theorem 2.4. Then for any $m > N$, the maps $p \mapsto z_m(x; p)$ and $p \mapsto z_m'(x, p)$ are continuous as maps from E to $C[0, 1]$ with the sup norm.*

We do not give the full proof of Lemma 4.3 here but refer the reader to [29, section 5.5]. One first shows the existence of fundamental solutions with the required norm continuity using a Volterra series construction. One then uses the continuity of the eigenvalue map $p \mapsto \lambda_n(p)$, together with explicit formulas for the eigenfunctions in terms of the fundamental solutions, to obtain the required continuity.

By the lemma, it is enough to prove the estimates in Theorem 4.1 for p and q belonging to $C_0^\infty((0, 1); \mathbb{R}^2)$. This restriction facilitates calculations which we will carry out in what follows.

To prove Theorem 4.1 for such smooth coefficients, we exploit the fact that the difference of eigenfunction squares and derivatives can be recovered from the residues of the respective operators

$$\mathbb{A}(p, q; z) = (L(p) - z)^{-1} - (L(q) - z)^{-1}$$

and

$$\mathbb{B}(p, q; z) = D(\mathbb{A}(p, q; z))D,$$

at the appropriate eigenvalue. Here D denotes differentiation with respect to x.

We begin with the resolvent formula

$$(L(p) - z)^{-1} = (L(0) - z)^{-1} - (L(0) - z)^{-1} V_p (L(0) - z)^{-1}$$
$$+ (L(0) - z)^{-1} V_p (L(p) - z)^{-1} V_p (L(0) - z)^{-1}$$

where

$$V_p = -Dp_1D + p_2.$$

From this formula it follows that

$$\begin{aligned}
\mathbb{A}(p,q;z) &= (L(p)-z)^{-1} - (L(q)-z)^{-1} \\
&= (L(0)-z)^{-1}(-V_{p-q})(L(0)-z)^{-1} \\
&+ (L(0)-z)^{-1}V_{p-q}(L(p)-z)^{-1}V_p(L(0)-z)^{-1} \qquad (4.1)\\
&+ (L(0)-z)^{-1}V_q(L(p)-z)^{-1}V_{p-q}(L(q)-z)^{-1}V_p(L(0)-z)^{-1} \\
&+ (L(0)-z)^{-1}V_q(L(q)-z)^{-1}V_{p-q}(L(0)-z)^{-1}
\end{aligned}$$

with an analogous identity for the operator $\mathbb{B}(p,q;z)$. The operator $\mathbb{B}(p,q;z)$ is initially defined on $C_0^\infty(0,1)$ and extended by density to a bounded operator from $L^2[0,1]$ to itself. Let $K_\mathbb{A}$ and $K_\mathbb{B}$ denote the respective integral kernels of $\mathbb{A}(p,q;z)$ and $\mathbb{B}(p,q;z)$. The kernels $K_\mathbb{A}$ and $K_\mathbb{B}$ can be expressed in terms of the integral kernels of $(L(p)-z)^{-1}$ and $(L(q)-z)^{-1}$, which are continuously differentiable in x and y; thus $\mathbb{A}(p,q;z)$ and $\mathbb{B}(p,q;z)$ have continuous kernels. Moreover, the formulas

$$z_m^2(x;p) - z_m^2(x;q) = \frac{1}{2\pi i}\int_{\gamma_m} K_\mathbb{A}(x,x;z)\,dz$$

and

$$z_m'(x;p)^2 - z_m'(x;q)^2 = \frac{1}{2\pi i}\int_{\gamma_m} K_\mathbb{B}(x,x;z)\,dz$$

hold, where γ_m is the contour $\{z : |z-\lambda_m(0)| = Rm^\alpha\}$.

Thus, Theorem 4.1 will follow if we can show that

$$\sup_{z\in\gamma_m}\ \sup_{(x,y)\in[0,1]} |K_\mathbb{A}(x,y;z)| \le C_1\|p-q\|_E m^{2-2\alpha} \qquad (4.2)$$

and

$$\sup_{z\in\gamma_m}\ \sup_{(x,y)\in[0,1]} |K_\mathbb{B}(x,y;z)| \le C_2\|p-q\|_E m^{4-2\alpha}, \qquad (4.3)$$

where C_1 and C_2 depend only on α and M. If T is an integral operator on $L^2[0,1]$ with continuous kernel $K(x,y)$, the sup norm of K is dominated by the $L^1[0,1]\to L^\infty[0,1]$ norm of the operator T. Thus it suffices to estimate the $L^1\to L^\infty$ operator norm of each of the terms in (4.1) and the corresponding identity for $\mathbb{B}(p,q;z)$. Note that each term in (4.1) contains at least one factor involving $p-q$.

Roughly speaking, each resolvent contributes a factor $m^{-\alpha}$, each derivative that occurs contributes a factor m, and each factor of p, q, or $p-q$ contributes a factor $\|p\|_E$, $\|q\|_E$, or $\|p-q\|_E$ to the estimates. This "naive power counting" gives estimates of the desired form. The power counting is justified by the following two results, which themselves depend on Lemma A.5 in Appendix A.

Lemma 4.4. *Let $z \in \gamma_m$, let $0 \leq i, j \leq 1$, and let $M > 0$. Let $r \in C_0^\infty(0,1)$. Then for any p and q with $1 \leq p < \infty$ and $1 \leq q \leq \infty$, the estimate*

$$\|D^i(L(0)-z)^{-1}D^j r\|_{p,q} \leq C_{p,q} m^{i+j-\alpha}\|r\|_2$$

holds.

This lemma is a straightforward consequence of Lemma A.5; the following perturbative argument shows that an analogous result holds for the resolvent of $L(p)$ when $p \neq 0$.

Lemma 4.5. *Let $z \in \gamma_m$, let $0 \leq i, j \leq 1$, and let $M > 0$. Let $r \in C_0^\infty(0,1)$. There is a positive integer $N_1 > N$ depending only on M so that for all $p \in E$ with $\|p\| < M$ and every $m > N_1$, the estimate*

$$\|D^i(L(p)-z)^{-1}D^j r\| \leq C_{i+j} m^{i+j-\alpha}\|r\|_2$$

holds.

Proof. We will use equation (2.2). With R chosen sufficiently large, as in Theorem 2.4, so that (2.2) holds for $z \in \gamma_m$, we may estimate $\|A(z)\|$ and $\|B(z)\|$ for $z \in \gamma_m$ using Lemma A.5 and obtain

$$\|A(z)\| \leq Cm^{2-2\alpha}\|p_1\|_2$$

and

$$\|B(z)\| \leq Cm^{2-2\alpha}(1+\|p_1\|_2)\|p_2\|_2.$$

Since $2-2\alpha < -\alpha$ for $\alpha > 2$, we recover the estimate with $i = j = 0$. If $i = 1$ and $j = 0$, we compute from the second resolvent identity that

$$\begin{aligned} &D(L(p)-z)^{-1} \\ &\quad = D(L(0)-z)^{-1} - D(L(0)-z)^{-1}(Dp_1D + p_2)(L(p)-z)^{-1}. \end{aligned} \tag{4.4}$$

From Lemma A.5, we obtain

$$\|D(L(p)-z)^{-1}\| \leq c_1 m^{1-\alpha} + c_3\|p_1\|_2 m^{2-\alpha}\|D(L(p)-z)^{-1}\| + c_1\|p_2\|_2 m^{1-\alpha}.$$

By choosing m so large that $c_3\|p_1\|_2 m^{2-\alpha} < 1/2$, we can conclude that

$$\|D(L(p)-z)^{-1}\| \leq Cm^{1-\alpha}.$$

The proofs for $(i,j) = (0,1)$ and $(i,j) = (1,1)$ are similar. □

To finish the proof of Theorem 4.1, we use Lemmas 4.4 and 4.5 in conjunction with the identity (4.1) and the corresponding identity for the operator $D(L(p)-z)^{-1}D$ to estimate $\|\mathbb{A}(p,q;z)\|_{L^1\to L^\infty}$ and $\|\mathbb{B}(p,q;z)\|_{L^1\to L^\infty}$ and thereby show that (4.2) and (4.3) hold. For example, the norm of the second right-hand term in (4.1) is

$$\begin{aligned} &\|(L(0)-z)^{-1}V_{p-q}(L(p)-z)^{-1}V_p(L(0)-z)^{-1}\|_{L^1\to L^\infty} \\ &\quad = \|(L(0)-z)^{-1}(D(p_1-q_1)D + (p_2-q_2)) \\ &\qquad \times (L(p)-z)^{-1}(Dp_1D+p_2)(L(0)-z)^{-1}\|_{L^1\to L^\infty}. \end{aligned}$$

The term of highest order in m for $z \in \gamma_m$ comes from the term involving p_1, because it involves the highest number of differentiations. In what follows, let C denote a generic constant depending on only M, a bound for $\|p\|_E$ and $\|q\|_E$, and let $\|\ \cdot\ \|_{p,q}$ denote the $\mathcal{B}(L^p[0,1], L^q[0,1])$-operator norm. We can estimate

$$\begin{aligned}
&\|(L(0)-z)^{-1}D(p_1-q_q)D(L(p)-z)^{-1}Dp_1D(L(0)-z)^{-1}\|_{L^1\to L^\infty}\\
&\quad\le \|(L(0)-z)^{-1}D(p_1-q_1)\|_{2,\infty}\ \|D(L(p)-z)^{-1}p_1\|_{2,2}\ \|D(L(0)-z)^1\|_{1,2}\\
&\quad\le C(m^{1-\alpha}\|p_1-q_1\|_{L^2[0,1]})(m^{1-\alpha}\|p_1\|_{L^2[0,1]})(m^{1-\alpha})\\
&\quad\le Cm^{3-3\alpha}
\end{aligned}$$

using Lemma 4.4 for the first and third factors, and Lemma 4.5 for the second. Similar estimates on the remaining terms in (4.1) show that all terms can be bounded by $Cm^{2-2\alpha}\|p-q\|_E$ so that

$$\|\mathbb{A}(p,q;z)\|_{1,\infty} \le Cm^{2-2\alpha}\|p-q\|_E.$$

Analogous estimates show that

$$\|\mathbb{B}(p,q;z)\|_{1,\infty} \le Cm^{4-2\alpha}\|p-q\|_E.$$

Finally, we refine the crude eigenvalue asymptotics obtained in section 2. We let

$$p_{1,m} = \int_0^1 p_1(x)\cos(m\pi x)\,dx.$$

Note that the sequence $\{p_{1,m}\}$ belongs to $\ell^2(\mathbb{N})$.

THEOREM 4.6. *Let $p \in E$ with $\|p\|_E < M$, and $\alpha \in (2,3)$. There is a constant C depending only on α and M such that the estimate*

$$|\lambda_m(p) - \lambda_m(0) - m^2\pi^2\,(\overline{p_1} + p_{1,2m})\,| \le Cm^{4-\alpha}$$

holds for $n > N$. In particular, $m^{-2}(\lambda_m(p) - \lambda_m(0) - m^2\pi^2\overline{p_1})$ defines a sequence belonging to $\ell^2(\mathbb{N})$, and $\overline{p_1}$ may be recovered from the asymptotics of the $\lambda_m(p)$.

Proof. First suppose that $p \in C_0^\infty(0,1) \times C_0^\infty(0,1)$. It suffices to prove the estimate for such p since an arbitrary $q \in E$ can be approximated by such smooth p in norm and the eigenvalues are continuous functions of p. Let $\nu_m(t) = \lambda_m(tp)$ for $t \in [0,1]$. Then standard perturbative calculations show that

$$\begin{aligned}
&\nu_m(1) - \nu_m(0)\\
&\quad= \int_0^1\int_0^1 p_1(x)(z_m'(x;tp))^2\,dx\,dt\\
&\quad+ \int_0^1\int_0^1 p_2(x)(z_m(x,tp))^2\,dx\,dt.
\end{aligned}$$

Using Theorem 4.2 together with the explicit formula

$$z_m(x;0) = \sqrt{2}\sin(m\pi x),$$

we readily obtain the claimed asymptotics. $\square$

5. Analyticity. In this section, we prove the following theorem.

THEOREM 5.1. *The map μ is an analytic mapping from $\mathcal{E}$ into F.*

Proof. Theorem 4.6 already implies that the map μ has range in F. It remains to show that it has the required analyticity. It is not difficult to see that any particular μ_n is analytic in a small neighborhood of any $p \in \mathcal{E}$, where the size of the neighborhood may depend on n. To show analyticity of μ we must show, for any $p \in \mathcal{E}$, that the μ_n with $n > N$ are analytic in a fixed neighborhood of p independent of $n > N$.

To do this, we fix an $M > 0$ and choose N and R as in section 2. Let

$$P_n(p) = \frac{1}{2\pi i}\int_{\gamma_n} (L(p) - z)^{-1}\, dz\,,$$

where E_n is as defined in section 2. It follows from the analyticity of the resolvent that $P_n(p)$ is analytic in p. Moreover, for p and q with $\|p\| \le M$ and $\|q\| \le M$,

$$\begin{aligned}\|P_n(p) - P_n(q)\| &\le R n^{\alpha} \sup_{z\in\gamma_n} \|(L(p)-z)^{-1} - (L(q)-z)^{-1}\|\\ &\le C\|p-q\|_E\, n^{2-\alpha}\end{aligned}$$

by the estimate in the proof Theorem 4.2. For $n > N$ and $\alpha > 2$ we may choose $\|p - q\| < (2CN^{2-\alpha})^{-1}$ and guarantee that $\|P_n(p) - P_n(q)\| \le 1/2$ for all $n > N$. From the formula

$$\lambda_n(q) = \frac{\langle z_n(\cdot, p), L(q)P(q)u_n(p)\rangle}{\langle z_n(\cdot, p), P_n(q)u_n(p)\rangle},$$

we see that $\lambda_n(q)$ is analytic for $\|p - q\| < (2CN^{2-\alpha})^{-1}$, which defines a fixed neighborhood of p independent of $n > N$. Thus, given $p \in \mathcal{E}$, there is a fixed neighborhood U of p so that all of the $\mu_n(q)$ are analytic for $q \in U$. This shows the required analyticity. ◻

6. Linear independence: The Borg system. We now consider linear independence of the functions $g_n(p)$ constructed from eigenfunctions of $L(p)$ via (1.7) when $p \in \mathcal{D}$. We shall accomplish this by displaying a set of functions which is biorthogonal to the $g_n(p)$ in E. The development of this biorthogonal set will involve the spectral theory of a non-self-adjoint fifth-order system, the *Borg system*, satisfied by the functions

$$\widehat{g}_n(x;p) = \begin{pmatrix} (z_n(x;p))^2 \\ (z_n'(x;p))^2 \end{pmatrix}, \tag{6.1}$$

where $z_n(x;p)$ is the nth eigenfunction of $L(p)$.

First, we will define the Borg system and construct a basis for its solution space from solutions of the underlying fourth-order problems. Then, we will show how one may specify boundary conditions for this system so that the spectrum of the resulting boundary value problem for the Borg system coincides with the spectra of the fourth-order operators $L(p)$, $L_1(p)$, and $L_2(p)$. Finally, we will show that the resolvent of the Borg system has simple poles and rank-one residues; this will enable us to construct the desired biorthogonal set.

6.1. The Borg system. Now, we define the Borg system and establish some of its properties.

LEMMA 6.1. *For $p \in C_0^\infty(0,1) \times C_0^\infty(0,1)$, there exist differential operators $\mathcal{M}(p)$ and $\mathcal{B}(p)$, mapping $C_0^\infty(0,1) \times C_0^\infty(0,1)$ into itself, such that*

(i)

$$\mathcal{M} = \begin{pmatrix} D^5 + M_{11}(x,D) & M_{12}(x,D) \\ \\ M_{21}(x,D) & D^5 + M_{22}(x,D) \end{pmatrix},$$

where each M_{ij} *is a linear differential operator of order not exceeding four with smooth coefficients depending on p, and*

(ii)

$$\mathcal{B} = \begin{pmatrix} \frac{8}{3}D & 0 \\ \\ \mathcal{B}_{21}(x,D) & -24D \end{pmatrix},$$

where $\mathcal{B}_{21}$ *is a third-order linear differential operator with smooth coefficients depending on p.*

(iii) *If u and v are solutions of* $L(p)u = \lambda u$*, then*

$$\mathcal{M}(p)\phi = \lambda \mathcal{B}(p)\phi, \tag{6.2}$$

where $\phi = (uv, u'v')^T$*.*

The proof is a direct calculation and is omitted. The explicit forms of $\mathcal{M}$ and $\mathcal{B}$ are given in Appendix B.

Next we note a purely algebraic lemma. It will be used to furnish bases of solutions from which Green's function for the fifth-order system

$$(\mathcal{M} - \lambda\mathcal{B})u = f, \tag{6.3}$$

with boundary conditions dictated by the chosen basis, can be calculated.

LEMMA 6.2. *Fix* $\lambda \in \mathbb{C}$*. Let* $\{y_i\}_{i=1}^4$ *be a fundamental set of solutions for the differential equation* $L(p)u = \lambda u$*. The ten products* $\{y_i y_j : 1 \le i \le j \le 4\}$ *have nonvanishing Wronskian.*

Lemma 6.2 is a consequence of the following abstract result about symmetric tensor products. Recall that if V is a real n-dimensional vector space with basis $\{e_i\}_{i=1}^n$, the *symmetric tensor product* $V \otimes_s V$ is the $n(n+1)/2$-dimensional real vector space spanned by the tensors $e_i \otimes_s e_j = e_i \otimes e_j + e_j \otimes e_i$. If $A : V \to V$ is a linear transformation, $A \otimes_s A$ is the linear transformation on $V \otimes_s V$ acting on basis vectors by $(A \otimes_s A)(e_i \otimes_s e_j) = (Ae_i) \otimes_s (Ae_j)$ and extended to $V \otimes_s V$ by linearity.

LEMMA 6.3. *Let* $A : V \to V$ *be a linear operator. Then,*

$$\det(A \otimes_s A) = 2^{\frac{n(n+1)}{2}} \det(A)^{n+1}.$$

Proof. Suppose first that A is diagonal with eigenvalues $\{\lambda_i\}_{i=1}^n$. The eigenvalues of $A \otimes_s A$ are $2\lambda_i\lambda_j$ for $1 \le i \le j \le n$. The product over these $n(n+1)/2$ numbers gives $2^{\frac{n(n+1)}{2}} \left(\prod_{i=1}^n \lambda_i\right)^{n+1}$. This proves the formula for the dense set of $n \times n$ matrices which are similar to a diagonal matrix. The general result follows by continuity of the determinant function. ∎

We denote the 10×10 matrix of products $y_i y_j$, $1 \le i \le j \le 4$, and their derivatives of up to ninth order by Y_S.

Proof of Lemma 6.2. Let Ψ be the 10×10 matrix consisting of the symmetric derivatives $D^{(k)}y_iD^{(l)}y_j + D^{(l)}y_iD^{(k)}y_j$ for $1 \le i \le j \le 4$ and $0 \le k \le l \le 3$. By Lemma 6.3,

$$\det(\Psi) = 2^{10}\left(W(y_1, y_2, y_3, y_4)\right)^5 \neq 0.$$

A direct calculation shows that there is a nonsingular constant matrix G_1 for which

$$Y_S = \Psi G_1^T\,, \tag{6.4}$$

which establishes the result. ∎

LEMMA 6.4. *Let y_i be as in Lemma* 6.2*, and let Φ be the 10×10 matrix consisting of the y_iy_j and their derivatives of up to fourth order, and the $y_i'y_j'$ and their derivatives of up to fourth order. There is a nonsingular constant matrix C such that $\Phi = Y_SC^T$.*

The proof is a direct calculation and is omitted. The nonsingular constant matrix C maps any row vector consisting of y_iy_j and its derivatives up to ninth order, where y_i and y_j are solutions of the fourth-order problem, to a corresponding row vector whose entries are y_iy_j and its first four derivatives, followed by $y_i'y_j'$ and its first four derivatives.

We conclude from Lemma 6.4 and relation (6.4) that

$$\left\{\left(y_iy_j,\, y_i'y_j'\right)^T \,:\, 1 \le i \le j \le 4\right\}$$

forms a basis for the ten-dimensional solution space of the fifth-order system (6.2).

6.2. Boundary conditions for the Borg system. We are now ready to prescribe boundary conditions on the Borg system. We do so implicitly, by specifying a basis for the desired ten-dimensional solution space. To this end, choose a basis y_j of solutions to $L(p)u = \lambda u$ to satisfy the initial conditions $D^{i-1}y_j(0) = \delta_{ij}$, $1 \le i, j \le 4$, and similarly choose a basis z_j of solutions to $L(p)u = \lambda u$ to satisfy $D^{i-1}z_j(1) = \delta_{ij}$, $1 \le i, j \le 4$. Denote by B the 4×4 matrix with

$$y_j(x, \lambda) = \sum_{i=1}^{4} B_{ij}z_i(x, \lambda)\,, \quad 1 \le i, j \le 4;$$

the matrix B is a holomorphic function of λ with determinant 1. Bases of solutions for the fourth-order homogeneous problems $L(p)u = \lambda u$, $L_1(p)u = \lambda u$, and $L_2(p)u = \lambda u$ obeying the $x = 0$ and $x = 1$ boundary conditions are, respectively, $\{y_2, y_4, z_2, z_4\}$, $\{y_2, y_4, z_1, z_3\}$, and $\{y_2, y_4, z_3, z_4\}$. We denote the Wronskians of these sets, respectively, as $W_1(\lambda)$, $W_2(\lambda)$, and $W_3(\lambda)$; the respective zeros are exactly $\{\lambda_n(p)\}$, $\{\sigma_n(p)\}$, and $\{\tau_n(p)\}$. In terms of the matrix B,

$$\begin{aligned} W_1(\lambda) &= B_{1,4}B_{3,2} - B_{1,2}B_{3,4}\,, \\ W_2(\lambda) &= B_{2,4}B_{4,2} - B_{2,2}B_{4,4}\,, \\ W_3(\lambda) &= B_{1,2}B_{2,4} - B_{1,4}B_{2,2}\,. \end{aligned}$$

Let us denote, for each i and j,

$$\phi_{ij}^L = \begin{pmatrix} y_iy_j, \\ y_i'y_j', \end{pmatrix} \tag{6.5}$$

and

$$\phi_{ij}^R = \begin{pmatrix} z_i z_j, \\ \\ z_i' z_j', \end{pmatrix}. \tag{6.6}$$

From Lemmas 6.2 and 6.4 it follows that either of the sets ϕ_{ij}^L or ϕ_{ij}^R form a basis for the solution space of (6.2).

In lieu of specifying explicit boundary conditions for the fifth-order system (6.2), we shall specify a ten-dimensional solution space for the Borg system by explicitly choosing a basis for the solution space. This basis, consisting of a subset $\mathcal{L}$ of the ϕ_{ij}^L and a subset $\mathcal{R}$ of the ϕ_{ij}^R, will be chosen so that the eigenfunctions of the three boundary value problems specified in section 2 will contribute to the eigenfunctions of (6.2) via (6.1). Explicitly, we choose

$$\mathcal{L} = (\phi_{22}^L, \phi_{44}^L, \phi_{24}^L)$$

and

$$\mathcal{R} = (\phi_{11}^R, \phi_{22}^R, \phi_{33}^R, \phi_{44}^R, \phi_{13}^R, \phi_{24}^R, \phi_{34}^R).$$

In what follows, we will also use the notation $\{\phi_i\}_{i=1}^{10}$ for these basis functions, where $1 \leq i \leq 3$ for the vectors in $\mathcal{L}$, and $4 \leq i \leq 10$ for the vectors in $\mathcal{R}$. We shall also designate the components of ϕ_i by

$$\phi_i(x;\lambda) = \begin{pmatrix} \zeta_i(x;\lambda), \\ \\ \eta_i(x;\lambda), \end{pmatrix}.$$

We need to verify that the ten functions in $\mathcal{L} \cup \mathcal{R}$ are linearly independent (and hence, a basis for the solution space). Computing an appropriate Wronskian determinant leads to an explicit eigenvalue condition. Recall that $\lambda_n(p)$, $\sigma_n(p)$, and $\tau_n(p)$ denote, respectively, the nth eigenvalues of $L(p)$, $L_1(p)$, and $L_2(p)$.

Lemma 6.5. *Let $W(\lambda)$ be the Wronksian of the solution set $\mathcal{L} \cup \mathcal{R}$, and let $p \in \mathcal{D}$. Then $W(\lambda)$ is a constant multiple of $W_1(\lambda)W_2(\lambda)W_3(\lambda)$, and $W(\lambda)$ has simple zeros at the eigenvalues $\{\lambda_n(p)\}$, $\{\sigma_n(p)\}$, $\{\tau_n(p)\}$.*

Proof. By Lemma 6.4 and the remarks following it, it suffices to show that the assertion of Lemma 6.5 is true for the Wronskian determinant of the ten functions z_1^2, z_2^2, z_3^2, z_4^2, $z_1 z_3$, $z_2 z_4$, $z_3 z_4$, and y_2^2, y_4^2, $y_2 y_4$. Evaluating the determinant at $x = 1$ leads to the determinant of a block upper triangular matrix

$$A = \begin{pmatrix} I & A_{12} \\ \\ 0 & A_{22} \end{pmatrix}, \tag{6.7}$$

where I is the 7×7 identity matrix, A_{12} is a 7×3 matrix whose entries are polynomials in the $B_{i,j}$, and A_{22} is the 3×3 matrix

$$A_{22}(\lambda) = \begin{pmatrix} 2\,B_{1,2}\,B_{2,2} & B_{1,2}\,B_{2,4} + B_{2,2}\,B_{1,4} & 2\,B_{1,4}\,B_{2,4} \\ \\ 2\,B_{1,2}\,B_{4,2} & B_{1,2}\,B_{4,4} + B_{1,4}\,B_{4,2} & 2\,B_{1,4}\,B_{4,4} \\ \\ 2\,B_{2,2}\,B_{3,2} & B_{2,2}\,B_{3,4} + B_{2,4}\,B_{3,2} & 2\,B_{2,4}\,B_{3,4} \end{pmatrix}.$$

Explicit calculation gives the formula

$$\det(A_{22}(\lambda)) = W_1(\lambda)W_2(\lambda)W_3(\lambda).$$

The fact that $W(\lambda)$ has simple zeros follows from a result of Everitt (see [12]) and the fact that $p \in \mathcal{D}$. ◻

Thus, the spectrum of the boundary value problem for the Borg system coincides exactly with the spectra of $L(p)$, $L_1(p)$, and $L_2(p)$. With some additional calculation, we can show the following lemma.

LEMMA 6.6. *Let $p \in \mathcal{D}$. At each zero of $W(\lambda)$, the kernel of $\mathcal{M} - \lambda\mathcal{B}$ is one-dimensional.*

Proof. It is enough to show that the matrix A defined in (6.7) has a one-dimensional kernel at such points λ. To see this, note that a nonzero solution of the homogeneous equation $(\mathcal{M} - \lambda\mathcal{B})u = 0$, which satisfies the left and right boundary conditions, exists if and only if the spans of the vectors in $\mathcal{L}$ and $\mathcal{R}$ have nonempty intersection. Recalling the notation $\{\phi_i\}_{i=1}^{10}$ we see that the dimension of the kernel of $\mathcal{M} - \lambda\mathcal{B}$ is the dimension of solutions $\{\alpha_i\}$ of the equation

$$\sum_{i=1}^{10} \alpha_i\phi_i = 0\,, \tag{6.8}$$

since the sets $\{\phi_i\}_{i=1}^{3}$ and $\{\phi_j\}_{j=4}^{10}$ are linearly independent. Let $\mathbf{\Phi}$ denote the 10×10 matrix containing the components of ϕ_i and their derivatives of up to fourth order. This matrix is related by a nonsingular constant matrix to the 10×10 Wronskian matrix containing the corresponding products y_iy_j, z_iz_j and their derivatives of up to ninth order. The solutions of (6.8) therefore correspond to the nullspace of the Wronskian matrix of the y_iy_j and z_iz_j so that the dimension of the space of solutions to (6.8) is exactly the dimension of the kernel of the matrix A in Lemma 6.5. Since A is upper triangular, $\dim\ker A = \dim\ker A_{22}$. The proof is completed by showing that $\dim\ker A_{22} = 1$ at each zero of $W(\lambda)$, which is the content of the next lemma. ◻

LEMMA 6.7. *If $W(\lambda) = 0$, then* $\dim\ker(A_{22}) = 1$.

Proof. By virtue of Lemma 6.5, $W(\lambda) = 0$ if and only if $W_j(\lambda) = 0$ for some j. Note that, since the spectra of the fourth-order problems do not overlap,

- $B_{1,2}$ and $B_{1,4}$ are never zero simultaneously (for otherwise, $W_1(\lambda) = W_3(\lambda) = 0$) and
- $B_{2,2}$ and $B_{2,4}$ are never zero simultaneously (for otherwise, $W_2(\lambda) = W_3(\lambda) = 0$).

Since the matrix A_{22} is symmetric (up to column-interchanges) with respect to the second index on the coefficients $B_{i,j}$, we may assume without loss of generality that $B_{2,2} \neq 0$ and express A_{22} equivalently as

$$A_{22} = \begin{pmatrix} 2B_{1,2} & B_{1,4} + B_{1,2}\alpha_2 & 2B_{1,4}\alpha_2 \\ 2B_{1,2}B_{4,2} & B_{1,2}B_{4,4} + B_{4,2}B_{1,4} & 2B_{1,4}B_{4,4} \\ 2B_{3,2} & B_{3,4} + B_{3,2}\alpha_2 & 2B_{3,4}\alpha_2 \end{pmatrix},$$

where

$$\alpha_2 \equiv \frac{B_{2,4}}{B_{2,2}}\,.$$

There are two possibilities: either $B_{1,2} \neq 0$ or $B_{1,4} \neq 0$. We consider the former only, the latter being essentially the same.

Assume $B_{1,2} \neq 0$. The matrix A_{22} can then be written equivalently as

$$A_{22} = \begin{pmatrix} 2 & \alpha_1 + \alpha_2 & 2\alpha_1\alpha_2 \\ 2B_{4,2} & B_{4,4} + B_{4,2}\alpha_1 & 2B_{4,4}\alpha_1 \\ 2B_{3,2} & B_{3,4} + B_{3,2}\alpha_2 & 2B_{3,4}\alpha_2 \end{pmatrix},$$

where

$$\alpha_1 \equiv \frac{B_{1,4}}{B_{1,2}}.$$

It is straightforward to show that there exist nonsingular matrices $\mathbf{C}$ and $\mathbf{E}$ for which

$$A_{22} = \mathbf{CDE}^T,$$

where

$$\mathbf{D} = \begin{pmatrix} 2 & \alpha_1 + \alpha_2 & 2\alpha_1\alpha_2 \\ 0 & B_{4,4} - B_{4,2}\alpha_2 & 2\alpha_1(B_{4,4} - B_{4,2}\alpha_2) \\ 0 & B_{3,4} - B_{3,2}\alpha_1 & 2\alpha_2(B_{3,4} - B_{3,2}\alpha_1) \end{pmatrix} = \begin{pmatrix} 2 & \alpha_1 + \alpha_2 & 2\alpha_1\alpha_2 \\ 0 & -\frac{W_2}{B_{2,2}} & -2\frac{\alpha_1 W_2}{B_{2,2}} \\ 0 & -\frac{W_1}{B_{1,2}} & -2\frac{\alpha_2 W_1}{B_{1,2}} \end{pmatrix}.$$

Noting that row 1 of $\mathbf{D}$ never vanishes (and is independent of the other rows) and rows 2 and 3 cannot vanish simultaneously, we see that $\dim\ker A_{22} = \dim\ker \mathbf{D} \leq 1$ and is determined by the dimension of the kernel of the 2×2 submatrix

$$AA \equiv \begin{pmatrix} -\frac{W_2}{B_{2,2}} & -2\frac{\alpha_1 W_2}{B_{2,2}} \\ -\frac{W_1}{B_{1,2}} & -2\frac{\alpha_2 W_1}{B_{1,2}} \end{pmatrix}.$$

We have

$$\det(AA) = \frac{2}{B_{1,2}B_{2,2}} W_1 W_2(\alpha_2 - \alpha_1) = -\frac{2}{B_{1,2}^2 B_{2,2}^2} W_1 W_2 W_3,$$

and the result follows. □

As a consequence, we have not only that the eigenvalues $\{\nu_n\}$ of the boundary value problem (6.2) are precisely the eigenvalues of the three boundary value problems considered in section 2, but also that, for each such eigenvalue, the eigenfunction of (6.2) is $(z^2, (z')^2)^T$, where z is the eigenfunction of the corresponding fourth-order problem.

6.3. Biorthogonal set. We shall now construct the desired biorthogonal set from the residues of the resolvent $(\mathcal{M} - \lambda\mathcal{B})^{-1}$. It follows from explicit formulas for the Green's function in terms of the basis functions ϕ_{ij}^L, ϕ_{ij}^R and the Wronskian $W(\lambda)$ that the resolvent $(\mathcal{M} - \lambda\mathcal{B})^{-1}$ has simple poles with rank-one residues.

THEOREM 6.8. *Let $(\mathcal{M} - \lambda\mathcal{B})^{-1}$ be the resolvent of the non-self-adjoint boundary value problem* (6.2). *Then the poles of $(\mathcal{M} - \lambda\mathcal{B})^{-1}$ are simple and occur at the* *numbers ν_n. Their residue takes the form $(\chi_n, \cdot)\psi_n$, where $\psi_n \in \mathrm{Ker}(\mathcal{M} - \lambda\mathcal{B})$ and $\chi_n \in \mathrm{Ker}((\mathcal{M} - \lambda\mathcal{B})^*)$.*

Proof. The integral kernel of $(\mathcal{M} - \lambda\mathcal{B})^{-1}$ is the matrix-valued function

$$\begin{pmatrix} G_{11}(x,t;\lambda) & G_{12}(x,t;\lambda) \\ G_{21}(x,t;\lambda) & G_{22}(x,t;\lambda) \end{pmatrix},$$

where the G_{ij} obey

- $G_{11}(x,t;\lambda)$ and $G_{22}(x,t;\lambda)$ are continuous in $(x,t) \in [0,1] \times [0,1]$ together with their derivatives of up to order three and have a unit jump at $x = t$ in their fourth derivative,
- $G_{12}(x,t;\lambda)$ and $G_{21}(x,t;\lambda)$ are continuous in $(x,t) \in [0,1] \times [0,1]$ together with their derivatives up to order four

and solve the differential equations

$$(\mathcal{M} - \lambda\mathcal{B})_x G_{ij}(x,t;\lambda) = 0$$

for $x \neq t$. We can find explicit expressions for the G_{ij} by setting

$$\begin{pmatrix} G_{11}, \\ G_{21}, \end{pmatrix} = \begin{cases} \sum_{i\in I} \alpha_i(t)\phi_i(x) & x < t, \\ -\sum_{j\in J} \alpha_j(t)\phi_j(x) & x > t \end{cases}$$

and

$$\begin{pmatrix} G_{12}, \\ G_{22}, \end{pmatrix} = \begin{cases} \sum_{i\in I} \beta_i(t)\phi_i(x) & x < t, \\ -\sum_{j\in J} \beta_j(t)\phi_j(x) & x > t \end{cases}$$

and solving the linear equations

$$\begin{aligned} \sum_{i=1}^{10} \alpha_i(x)\zeta_i^{(k)}(x) &= 0, \qquad 0 \le k \le 3, \\ \sum_{i=1}^{10} \alpha_i(x)\zeta_i^{(4)}(x) &= 1, \\ \sum_{i=1}^{10} \alpha_i(x)\eta_i^{(k)}(x) &= 0, \qquad 0 \le k \le 4, \end{aligned}$$

and

$$\begin{aligned} \sum_{i=1}^{10} \beta_i(x)\zeta_i^{(k)}(x) &= 0, \qquad 0 \le k \le 4, \\ \sum_{i=1}^{10} \beta_i(x)\eta_i^{(k)}(x) &= 0, \qquad 0 \le k \le 3, \\ \sum_{i=1}^{10} \beta_i(x)\eta_i^{(4)}(x) &= 1. \end{aligned}$$

Using Cramer's rule to solve for the functions α_i and β_i yields an expression for Green's function in terms of the holomorphic functions $\phi_i(x;\lambda)$ and the Wronskian $W(\lambda)$. Using the known properties of $W(\lambda)$ we conclude that Green's function has simple poles.

A simple argument shows that the residue of $(\mathcal{M}-\lambda\mathcal{B})^{-1}$ has range in $\mathrm{Ker}(\mathcal{M}-\lambda\mathcal{B})$ and is therefore rank-one by Lemma 6.6. Writing the residue at $\lambda=\nu_n$ in the form $(\chi_n,\cdot)\psi_n$, it follows from the identity

$$((\mathcal{M}-\lambda\mathcal{B})^*)^{-1}=((\mathcal{M}-\lambda\mathcal{B})^{-1})^*$$

that $\chi_n\in\mathrm{Ker}((\mathcal{M}-\lambda\mathcal{B})^*)$. ◻

Suppose that ν_i and ν_j are distinct eigenvalues of the Borg operator. It is not difficult to see that the resolvent identity

$$(\mathcal{M}-\lambda\mathcal{B})^{-1}-(\mathcal{M}-\mu\mathcal{B})^{-1}=(\mu-\lambda)(\mathcal{M}-\lambda\mathcal{B})^{-1}\mathcal{B}(\mathcal{M}-\mu\mathcal{B})^{-1}$$

holds. Let P_i and P_j be the rank-one residues corresponding to these distinct eigenvalues. From the resolvent identity above, it is easy to see that the relations $P_i\mathcal{B}P_i=P_i$ and $P_i\mathcal{B}P_j=0$ hold for $i\neq j$. Writing $P_i=(\chi_i,\cdot)\psi_i$, we obtain the following theorem.

THEOREM 6.9. *The biorthogonality relations*

$$(\chi_i,\mathcal{B}\psi_j)=\delta_{ij} \tag{6.9}$$

hold. Consequently, the eigenfunctions $\{\psi_j\}$ form a linearly independent set in E.

Proof. The conclusion that the set $\{\psi_j\}$ is linearly independent in E follows immediately from (6.9), once it is established that

$$\chi_i\in D(\mathcal{B}^*) \tag{6.10}$$

for each i. Appendix B gives explicitly the boundary conditions which determine $D((\mathcal{M}-\lambda\mathcal{B})^*)$ and $D(\mathcal{B}^*)$. Direct comparison shows that $D((\mathcal{M}-\lambda\mathcal{B})^*)\subset D(\mathcal{B}^*)$, so (6.10) holds. ◻

We now order the poles, ν_n, of the Borg operator so that $\nu_{3n}=\lambda_n$, $\nu_{3n+1}=\sigma_n$, $\nu_{3n+2}=\tau_n$. The vectors ψ_{3n} are, up to normalization, exactly the vectors $\widehat{g}_n$. Thus these vectors form a linearly independent set, and their orthogonal complement is an infinite-dimensional space spanned by the vectors

$$\{\mathcal{B}^*\chi_{3n+1},\mathcal{B}^*\chi_{3n+2}\}_{n=1}^{\infty}.$$

To conclude that the same is true of the gradients g_n for the direct spectral map, we need the following lemma.

LEMMA 6.10. *The kernel of $\mathcal{B}$ is the one-dimensional subspace of $L^2[0,1]\times L^2[0,1]$ spanned by the vector $\widehat{g}_0=(0,1)^T$.*

This is a direct computation using the formulas for $\mathcal{B}$ and its boundary conditions recorded in Appendix B.

Now consider the family of vectors $\{\tilde{g}_n\}_{n=0}^{\infty}$, where $\tilde{g}_0=(0,1)$ and $\tilde{g}_n=(n^2\pi^2)^{-1}\widehat{g}_n-\tilde{g}_0$. Since $\tilde{g}_0\in\ker(\mathcal{B})$, we have the biorthogonality relations

$$\langle\mathcal{B}^*\chi_n,\tilde{g}_m\rangle=c_n\delta_{nm}$$

for $n\in\mathbb{N}$, $c_n>0$, and $m\in 0\cup\mathbb{N}$. Thus the family $\{\tilde{g}_n\}_{n=0}^{\infty}$ is linearly independent. Since the gradients g_n are obtained from $\tilde{g}_n$ by permuting the first and second entries, we have proved the following theorem.

THEOREM 6.11. *For any $p\in\mathcal{D}$, the gradients $\{g_n(x;p)\}_{n=0}^{\infty}$ are linearly independent. Moreover, the complement of their span has infinite dimension in E.*

7. Proofs of the main theorems. We now prove Theorem 1.3, first proving that $d\mu(q)$ is a linear isomorphism from $\mathcal{E}_v(q)$ onto F, and then proving that the space $\mathcal{E}_h(q)$ is complementary. In light of the discussion following the statement of Theorem 1.3, and in view of Theorem 5.1 and the estimates of section 4, the first assertion will be proved once the following result is established.

LEMMA 7.1. *For each $p \in \mathcal{D}$, $d\mu(p)$ is a linear isomorphism from $\mathcal{E}_v(p)$ onto F.*

In proving this result, we will make use of some results on Riesz bases. Recall that if $\mathcal{H}$ is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, a basis $\{e_n\}$ is called a *Riesz basis* for $\mathcal{H}$ if there exist $a, b \in \mathbb{R}^+$ for which

$$a\|h\|^2 \le \sum_n |\langle h, e_n\rangle|^2 \le b\|h\|^2 \quad \forall\, h \in \mathcal{H}\,. \tag{7.1}$$

LEMMA 7.2. *Let $\mathcal{H}$ be a Hilbert space, and let $\{e_n\}$ be a Riesz basis for $\mathcal{H}$. Then, there is a unique set $\{\epsilon_m\} \subseteq \mathcal{H}$ for which*

(1) *$\langle e_n, \epsilon_m\rangle = \delta_{m,n}$ for all $m, n \in \mathbb{N}$,*

(2) *There exist $\alpha, \beta \in \mathbb{R}^+$ so that*

$$\alpha\|h\|^2 \le \sum_m |\langle h, \epsilon_m\rangle|^2 \le \beta\|h\|^2 \quad \forall h \in \mathcal{H}\,.$$

Proof. Let $Th = \sum_n \langle h, e_n\rangle e_n$. Then, T is self-adjoint, and

$$\langle h, Th\rangle = \left\langle h, \sum_n \langle h, e_n\rangle e_n \right\rangle = \sum_n |\langle h, e_n\rangle|^2.$$

It then follows from (7.1) that the spectrum of T is contained in the interval $[a, b]$ so that T^{-1} exists. By the spectral theorem, we have

$$b^{-2}\|h\|^2 \le \|T^{-1}h\|^2 \le a^{-2}\|h\|^2. \tag{7.2}$$

Let $\epsilon_n \equiv T^{-1}e_n$ for each $n \in \mathbb{N}$. Then

$$e_n = T\epsilon_n = \sum_m \langle \epsilon_n, e_m\rangle e_m,$$

so $\langle \epsilon_m, e_n\rangle = \delta_{nm}$ by the linear independence of the e_n; this establishes (1). Finally, it follows from relation (7.2) and the definition of ϵ_n that $\sum_n |\langle h, \epsilon_n\rangle|^2 = \sum_n |\langle T^{-1}h, e_n\rangle|^2$ obeys the inequality

$$ab^{-2}\|h\|^2 \le \sum_n |\langle h, \epsilon_n\rangle|^2 \le ba^{-2}\|h\|^2\,,$$

which establishes (2). □

LEMMA 7.3. *Let $\{d_n\}$ be a Riesz basis for $\mathcal{H}$. Then the linear map $\mathcal{A} : \mathcal{H} \mapsto \ell^2(\mathbb{N})$ defined by*

$$\mathcal{A}x = \{\langle x, d_n\rangle\}_{n\ge 1}\,,\ x \in \mathcal{H},$$

is an isomorphism.

Proof. We will show that $\mathcal{A}$ is a bounded bijection from $\mathcal{H}$ to ℓ^2. The conclusion of the lemma will then follow from the open mapping theorem. First, it is clear that

$\mathcal{A}x = 0$ only when $\langle x, d_n \rangle = 0$ for each n. Since $\{d_n\}$ is a basis for $\mathcal{H}$, we must have $x \equiv 0$ so that $\mathcal{A}$ is one-to-one. Further, for $x \in \mathcal{H}$ we have, from (7.1),

$$\|\mathcal{A}x\|^2 = \|\{\langle x, d_n \rangle\}\|_{\ell^2}^2 = \sum_n |\langle x, d_n \rangle|^2 \leq b\|x\|^2 ,$$

so $\mathcal{A}$ is bounded.

To show $\mathcal{A}$ is onto $\ell^2(\mathbb{N})$, we introduce $\{\delta_m\}$ as the Riesz basis biorthogonal to $\{d_n\}$, the existence of which is guaranteed by Lemma 7.2. Then, given $\{y_m\} \in \ell^2(\mathbb{N})$, set $y = \sum_m y_m \delta_m$. From (7.1),

$$\|y\|^2 \leq \frac{1}{a} \sum_n |\langle y, d_n \rangle|^2 = \frac{1}{a} \sum_n \left| \sum_m y_m \langle \delta_m, d_n \rangle \right|^2 = \frac{1}{a} \sum_n |y_n|^2 < \infty$$

so that $y \in \mathcal{H}$. Also,

$$\mathcal{A}y = \{\langle y, d_n \rangle\}_{n \geq 1} = \left\{ \sum_m y_m \langle \delta_m, d_n \rangle \right\}_{n \geq 1} = \{y_n\}_{n \geq 1} ,$$

which shows that $\mathcal{A}$ maps onto $\ell^2(\mathbb{N})$. By the open mapping theorem, $\mathcal{A}$ is an isomorphism. $\square$

For each $n \in \mathbb{N}$, let z_n and u_n denote, respectively, the nth eigenfunction of $L(p)$ and $L(0)$, and as before let

$$g_n(x, p) = \begin{pmatrix} \dfrac{z_n'(x, p)^2}{n^2\pi^2} - 1 \\ \\ \dfrac{z_n(x; p)^2}{n^2\pi^2} \end{pmatrix} .$$

By explicit computation

$$g_n(x, 0) = \begin{pmatrix} \cos(2n\pi x) \\ \\ \dfrac{1 - \cos(2n\pi x)}{n^2\pi^2} \end{pmatrix} . \tag{7.3}$$

Recalling the form (1.6) of $d\mu(p)$, it suffices, by virtue of Lemma 7.3, to show that $\{g_n(\ \cdot\ ; p)\}$ is a Riesz basis for $\mathcal{E}_v(p) = \operatorname{span}\{g_n(\ \cdot\ ; p)\}$. Using Fourier theory, one can show directly that $\{g_n(\ \cdot\ ; 0)\}$ is a Riesz basis for its span. To show that the same is true for $\{g_n(\ \cdot\ ; p)\}$, we note that $\{g_n(\ \cdot\ ; p)\}$ is linearly independent, by virtue of Theorem 6.11, and use the following stability result for Riesz bases.

LEMMA 7.4. *Let $\mathcal{H}$ be a Hilbert space, and let $\{e_n\} \subseteq \mathcal{H}$ be a Riesz basis for $\mathcal{H}_e \equiv \operatorname{span}\{e_n\}$. Let $\{d_n\} \subseteq \mathcal{H}$ be a linearly independent set for which*

$$\sum_n \|d_n - e_n\|^2 \equiv M < \infty.$$

Then, $\{d_n\}$ is a Riesz basis for $\mathcal{H}_d \equiv \operatorname{span}\{d_n\}$; i.e., there exist α, $\beta \in \mathbb{R}^+$ so that, for each $h \in \mathcal{H}_d$,

$$\alpha\|h\|^2 \leq \sum_n |\langle h, d_n \rangle|^2 \leq \beta\|h\|^2. \tag{7.4}$$

Proof. Define a map $A : \mathcal{H}_e \longrightarrow \mathcal{H}_d$ by $Ae_n = d_n$ for each $n \in \mathbb{N}$, extended by linearity. Then, for $h \in \mathcal{H}_d$ and each n,

$$\langle h, d_n\rangle = \langle h, Ae_n\rangle = \langle A^* h, e_n\rangle. \tag{7.5}$$

Let a and b be the constants for which (7.1) holds for $\{e_n\}$ on $\mathcal{H}_e$. Then, from (7.4),

$$a\|A^*h\|^2 \leq \sum_n |\langle A^*h, e_n\rangle|^2 = \sum_n |\langle h, d_n\rangle|^2 \leq b\|A^*h\|^2. \tag{7.6}$$

We claim that A (and hence A^*) is boundedly invertible. If this is true, then (7.6) leads to

$$\frac{a}{\|(A^*)^{-1}\|^2}\|h\|^2 \leq \sum_n |\langle h, d_n\rangle|^2 \leq b\|A^*\|^2\|h\|^2,$$

which establishes (7.4).

To show A is invertible, we note that the linear independence of $\{d_n\}$ implies the injectivity of A. To see that the range of A is all of $\mathcal{H}_d$, note that any $h \in \mathcal{H}_d$ can be written as $h = \sum_n h_n d_n$, where $\{h_n\} \in \ell^2(\mathbb{N})$. Then, setting $x \equiv \sum_n h_n e_n$, one easily sees that $x \in \mathcal{H}_e$ and $Ax = h$. Hence, A maps $\mathcal{H}_e$ onto $\mathcal{H}_d$.

Finally, we show that A is bounded. Choose $x \in \mathcal{H}_e$, and write as

$$x = \sum_n x_n e_n = \sum_n \langle x, \epsilon_n\rangle e_n,$$

where $\{\epsilon_n\}$ is the Riesz basis for $\mathcal{H}_e$ which is biorthogonal to $\{e_n\}$. Then,

$$Ax = \sum_n x_n d_n = \sum_n x_n e_n + \sum_n x_n(d_n - e_n) = x + \sum_n x_n(d_n - e_n),$$

which yields

$$\begin{aligned}
\|Ax\|^2 &\leq \|x\|^2 + \left(\textstyle\sum_n |x_n|^2\right)\left(\textstyle\sum_n \|d_n - e_n\|^2\right)\\
&= \|x\|^2 + M\textstyle\sum_n |\langle x, \epsilon_n\rangle|^2\\
&\leq \|x\|^2 + M\beta\|x\|^2,
\end{aligned}$$

where (7.4) was used in the last inequality. Thus,

$$\|A\|^2 \leq 1 + M\beta < \infty,$$

and A is bounded. By the open mapping theorem, A has a bounded inverse, as asserted. ∎

Proof of Lemma 7.1. From the estimates of Theorem 4.1, one can show that

$$\sum_n \|g_n(\,\cdot\,;p) - g_n(\,\cdot\,;0)\|^2 < \infty, \tag{7.7}$$

so, by Lemma 7.4, $\{g_n\}$ is a Riesz basis for $\mathcal{E}_v(p)$. The result now follows from Lemma 7.3. ∎

Finally, we prove that for any $q \in \mathcal{E}$, the complementary space $\mathcal{E}_h(q)$ is infinite-dimensional. We first observe that, by the explicit formula (7.3) and Fourier analysis, the gradients $g_n(\cdot, 0)$ are orthogonal vectors, and the complementary space $\mathcal{E}_h(0)$ has infinite dimension. Since the gradients satisfy (7.7), the second part of Theorem 1.3 will follow from the next lemma.

LEMMA 7.5. *Let $\{v_n\}$ be an orthogonal set of vectors in a Hilbert space $\mathcal{H}$. Let $\{w_n\} \subset \mathcal{H}$ be a linearly independent set of vectors which satisfy $\sum_n \|v_n - w_n\|^2 < \infty$. Set $V \equiv \operatorname{span}\{v_n\}$ and $W \equiv \operatorname{span}\{w_n\}$. If $V^\perp$ has infinite dimension, then $W^\perp$ also has infinite dimension.*

Proof. Suppose not, and choose an infinite sequence of orthogonal unit vectors $\{e_n\}$ from $V^\perp$ so that $e_n \to 0$ weakly. Let $\epsilon > 0$ be given. Writing $e_n = P_W e_n + P_{W^\perp} e_n$ we see that $P_{W^\perp} e_n \to 0$ if $W^\perp$ has finite dimension. We will obtain a contradiction by showing that $\|P_W e_n\|$ is also small for large n. First observe that by hypothesis, for M sufficiently large and all n,

$$\sum_{m=M+1}^{\infty} |\langle e_n, w_m\rangle|^2 < \epsilon.$$

On the other hand, using the weak convergence again,

$$\sum_{m=1}^{M} |\langle e_n, w_m\rangle|^2 \to 0$$

as $n \to \infty$ for a fixed M. It follows that for any $\epsilon > 0$,

$$\limsup_{n\to\infty} \left(\sum_m |\langle e_n, w_m\rangle|^2 \right) \leq \epsilon.$$

Since $\{w_n\}$ is a Riesz basis, by virtue of Lemma 7.4, this means that $\limsup_{n\to\infty} \|P_W e_n\| \leq \epsilon$, a contradiction. ∎

Appendix A. The free Green's function and free resolvent operator. In this appendix, we prove some useful technical estimates on the Green's kernel for the differential operator $L_\#(0)$, where $L_\#(0)$ is one of the operators $L(0)$, $L_1(0)$, or $L_2(0)$ defined in (1.3)–(1.5). Let $G_0^\#(x, y; z)$ be the integral kernel of $(L_\#(0) - z)^{-1}$, and let $C^\#_{R,\alpha}$ be defined as in section 2.

In what follows, $\beta = (\beta_x, \beta_y)$, where β_x and β_y are nonnegative integers, and $\partial^\beta = \partial_x^{\beta_x} \partial_y^{\beta_y}$. We wish to derive estimates on $\partial^\beta G_0^\#(x, y; z)$. These estimates will involve series of the form

$$\sum_{n=1}^{\infty} \frac{D_x^{\beta_x} u_n^\#(x) D_y^{\beta_y} u_n^\#(y)}{\lambda_n^\#(0) - z}\,, \tag{A.1}$$

where $\lambda_n^\#(0)$ and $u_n^\#$ are the eigenvalues and eigenfunctions, respectively, of $L_\#(0)$. It can be verified directly that for each set of boundary conditions, the eigenfunctions obey

$$|D_x^j u_n^\#(x)| \leq C n^j\,, \quad 0 \leq j \leq 2\,,$$

for some constant C. Thus, to estimate series of the form (A.1), it suffices to majorize the numerical series

$$\sum_{n=1}^{\infty} \frac{n^{|\beta|}}{|\lambda_n^{\#}(0) - z|}.$$

In so doing, we will require the following technical result, which may be easily verified.

LEMMA A.1. *Let $R > 1$ and $\alpha \in (2, 3)$. Then, for $m > (8R)^{\frac{1}{3-\alpha}}$,*

$$\int_1^{m-1} \frac{t^2}{m^4\pi^4 - t^4\pi^4}\, dt \le \frac{1}{m\pi^4} \ln m, \tag{A.2}$$

$$\int_{m+2}^{\infty} \frac{t^2}{t^4\pi^4 - [(m+1)^4\pi^4 + R(m+1)^\alpha]}\, dt \le \frac{C}{m} \ln m\,, \tag{A.3}$$

where the constant C depends only on α.

First of all, we have the following lemma.

LEMMA A.2. *Let $R > 0$, let $\alpha \in (2, 3)$, and let N be an integer satisfying the bounds* (2.1). *Then for any $z \notin C^{\#}_{R,\alpha}$:*

(a) *for $|\beta| \le 1$,*

$$\sup_{x,y\in[0,1]} |\partial^\beta G_0^{\#}(x, y; z)| \le c_{|\beta|} R^{-1};$$

(b) *for $|\beta| = 2$,*

$$\sup_{x,y\in[0,1]} |\partial^\beta G_0^{\#}(x, y; z)| \le c_2 R^{-1} \ln(R),$$

where c_2 is a numerical constant depending only on α.

Proof. We consider the case $\lambda_n(0) = n^4\pi^4$, the computations for other boundary conditions being similar.

For $|\beta| \le 1$ we have the simple majorization

$$\sum_{n=1}^{\infty} \frac{n}{|n^4\pi^4 - z|} \le \sum_{n=1}^{\infty} \frac{n}{Rn^\alpha}\,,$$

which gives estimates of the desired form.

For $|\beta| = 2$, we seek to estimate the series

$$\sum_{n=1}^{\infty} \frac{n^2}{|n^4\pi^4 - z|}$$

for $z \notin C_{R,\alpha}$. We split the sum into

$$T_1 = \sum_{n=1}^{N} \frac{n^2}{|n^4\pi^4 - z|}$$

and

$$T_2 = \sum_{n=N+1}^{\infty} \frac{n^2}{|n^4\pi^4 - z|}\,,$$

where $N > (8R)^{\frac{1}{3-\alpha}}$.

To estimate T_1, we use the fact that for $1 \le n \le N$, $|z - n^4\pi^4| \ge |z_N - n^4\pi^4|$ where $z_N = N^4\pi^4 + RN^\alpha$. The integral test then yields the bound

$$\begin{aligned} T_1 &\le \sum_{n=1}^{N-2} \frac{n^2}{|n^4\pi^4 - N^4\pi^4|} + \frac{(N-1)^2}{z_N - (N-1)^4\pi^4} \\ &\quad + \frac{N^2}{|N^4\pi^4 - z_N|} \\ &\le \int_1^{N-1} \frac{t^2}{N^4\pi^4 - t^4\pi^4} + CR^{-1} . \end{aligned}$$

Using (A.2) and the fact that $N = O(R^{\frac{1}{3-\alpha}})$, we conclude that $T_1 \le C(\alpha)R^{-1}\ln R$.

To estimate T_2, we consider the two cases $\Re(z) \le z_N$ and $\Re(z) \ge z_N$ separately. If $\Re(z) \le z_N$, then $|m^4\pi^4 - z| \ge |m^4\pi^4 - z_N|$, from which we obtain

$$T_2 \le \sum_{n=N+1}^{\infty} \frac{n^2}{\pi^4 n^4 - [\pi^4 N^4 + RN^\alpha]}.$$

We may estimate this sum by

$$\frac{(N+1)^2}{\pi^4(N+1)^2 - z_N} + \int_{N+1}^{\infty} \frac{t^2}{t^4\pi^4 - [\pi^4 N^4 + CRN^\alpha]}\, dt ,$$

which, in conjunction with (A.3), yields an estimate of the desired form.

If $\Re(z) > z_N$, we divide the half-plane $\Re(z) > z_N$ into strips

$$S_m = \{z \in \mathbb{C} : \Re(z) \in [m^4\pi^4 + Rm^\alpha, (m+1)^4\pi^4 + R(m+1)^\alpha)\}$$

and fix m so that $z \in S_m$. We can then estimate T_2 by letting $x = \Re(z)$ and splitting

$$\begin{aligned} \sum_{n=N+1}^{\infty} \frac{n^2}{|n^4\pi^4 - z|} &= \sum_{n=N+1}^{m-2} \frac{n^2}{|n^4\pi^4 - z|} \\ &\quad + \left(\frac{(m-1)^2}{z - (m-1)^4\pi^4} + \frac{m^2}{|z - m^4\pi^4|} \right. \\ &\quad \left. + \frac{(m+1)^2}{|z - (m+1)^4\pi^4|} + \frac{(m+2)^2}{|z - (m+2)^4\pi^4|} \right) \\ &\quad + \sum_{n=m+3}^{\infty} \frac{n^2}{n^4\pi^4 - x} \\ &= T_{21} + T_{22} + T_{23} . \end{aligned}$$

From the definition of $C_{R,\alpha}$, it is clear that

$$T_{22} \le CR^{-1} ,$$

which bounds T_{22}. Finally, using the fact that, for x in the interval $[m^4\pi^4+Rm^\alpha, (m+1)^4\pi^4 + R(m+1)^\alpha)$,

$$T_{21} \leq \int_0^{m-1} \frac{t^2}{m^4\pi^4 - t^4\pi^4}\, dt,$$
$$T_{23} \leq \int_{m+2}^{\infty} \frac{t^2}{t^4\pi^4 - [(m+1)^4\pi^4 + R(m+1)^\alpha]}\, dt\,,$$

we can use (A.2) and (A.3) to bound T_{21} and T_{23}, respectively. ∎

We will also need estimates on the free resolvent kernel on contours surrounding the sets $E_m^\#$.

LEMMA A.3. *Let $R > 0$, let $\alpha \in (2, 3)$, let N be an integer satisfying the bounds* (2.1)*, and let $E_m^\#$ be defined as in section* 2*. Let γ_m be the contour bounding $E_m^\#$. If $m > N$ and $|\beta| \leq 2$, then the estimate*

$$\sup_{\substack{x, y \in [0,1] \\ z \in \gamma_m}} |\partial^\beta G_0^\#(x, y; z)| \leq C_{|\beta|} m^{|\beta| - \alpha}$$

holds. Here $C_{|\beta|}$ is a numerical constant which diverges as $\alpha \uparrow 3$.

Proof. Here again we consider the case $\lambda_n(0) = n^4\pi^4$ and $|\beta| = 2$. We must majorize the numerical series

$$\sum_{n=1}^{\infty} \frac{n^2}{|z - n^4\pi^4|},$$

where $|z - m^4\pi^4| = Rm^\alpha$. We split the sum into

$$\sum\nolimits_{n=1}^{m-2} \frac{n^2}{|z - n^4\pi^4|}$$
$$+ \left[\frac{(m-1)^2}{|z - (m-1)^4\pi^4|} + \frac{m^2}{|z - m^4\pi^4|} + \frac{(m+1)^2}{|z - (m+1)^4\pi^4|} \right]$$
$$+ \sum\nolimits_{n=m+2}^{\infty} \frac{n^2}{|z - n^4\pi^4|}.$$

The three bracketed terms are easily estimated by $6Rm^{2-\alpha}$. We can estimate the first and last terms using (A.2) and (A.3). ∎

Now we derive estimates on operators involving compositions of the resolvent $(L_\#(0) - z)^{-1}$ with the operator of differentiation, D, and the operator of multiplication by a function $r \in C_0^\infty(0, 1)$. These compositions are initially defined on $C_0^\infty(0, 1)$ and extended by density to bounded operators. It follows from this definition and an integration by parts that the operator $D^{\beta_x}(L(0) - z)^{-1}D^{\beta_y}$ has integral kernel $(-1)^{\beta_y}(\partial^\beta G_0)(x, y; z)$. From the kernel estimates in Lemma A.2 and integration by parts, the following estimates are easily demonstrated.

LEMMA A.4. *Suppose that $r \in C_0^\infty(0, 1)$. For $z \notin C_{R,\alpha}$, there exist $c_0, c_1, c_2 \in \mathbb{R}$ so that the following estimates hold:*

(a) $\|(L_\#(0) - z)^{-1}\| \leq c_0 R^{-1}$.
(b) $\|(L_\#(0) - z)^{-1} r\| \leq c_0 R^{-1} \|r\|_2$.
(c) $\|D(L_\#(0) - z)^{-1}\| \leq c_1 R^{-1}$.

(d) $\|(L_\#(0)-z)^{-1}Dr\| \le c_1 R^{-1}\,\|r\|_2$.
(e) $\|D(L_\#(0)-z)^{-1}Dr\| \le c_2 R^{-1}\ln(R)\,\|r\|_2$.

The following bounds are used to estimate the resolvent $(L_\#(p)-z)^{-1}$ for $z\in\gamma_m$, the contour determined by the boundary of the set E_m defined in section 2. We denote by $\|A\|_{p,q}$ the norm of the linear operator A from $L^p[0,1]$ to $L^q[0,1]$, where $1\le p,q\le\infty$. Using the strong pointwise estimates on the free resolvent and its partial derivatives from Lemma A.3, we easily obtain the following lemma.

LEMMA A.5. *Let $z\in\gamma_m$ and $r\in C_0^\infty(0,1)$, and let $0\le i,j\le 1$. Then for any p,q with $1\le p,q\le\infty$, the estimate*

$$\|D^i(L_\#(0)-z)^{-1}D^j\|_{p,q}\le C_{i+j}m^{i+j-\alpha}$$

holds, and for any p,q with $1\le p\le 2$ and $1\le q\le\infty$, the estimate

$$\|D^i(L_\#(0)-z)^{-1}D^j r\|_{p,q}\le C_{i+j}m^{i+j-\alpha}\|r\|_2$$

holds. Here C_{i+j} is the numerical constant defined in Lemma A.3.

Appendix B. Domains related to the Borg system. In this appendix, we collect some useful calculations regarding the Borg system $\mathcal{M}-\lambda\mathcal{B}$ of section 6. First of all, for $p=(p_1,p_2)\in\mathcal{D}$, the matrix-valued differential operators $\mathcal{M}$ and $\mathcal{B}$ which define the Borg system can be computed to be

$$\mathcal{M}=\begin{pmatrix} M_1 & M_2\\ M_3 & M_4\end{pmatrix}\quad \textit{and}\quad \mathcal{B}=\begin{pmatrix} B_1 & B_2\\ B_3 & B_4\end{pmatrix},$$

where

$$\begin{aligned}
M_1 &= D^5 - p_1D^3 - 2p_1'D^2 + \left(\frac{8}{3}p_2 - p_1''\right)D + 2p_1',\\
M_2 &= -\frac{10}{3}D^3 + \frac{4}{3}p_1D + \frac{2}{3}p_1',\\
M_3 &= -3p_1'D^4 + 10p_2D^3 + (3p_1p_1' + 5p_2')D^2 + \left(p_2'' - 4p_1p_2 + 3(p_1')^2\right)D - 6p_1'p_2,\\
M_4 &= D^5 - 5p_1D^3 - p_1'D^2 + (4p_1^2 - 8p_1'' - 24p_2)D + (2p_1p_1' - 2p_1''' - 6p_2'),\\
B_1 &= \frac{8}{3}D,\\
B_2 &= 0,\\
B_3 &= 10D^3 + 4p_1D - 6p_1',\\
B_4 &= -24D.
\end{aligned}$$

Boundary conditions are to be specified so that if w is an eigenfunction (with eigenvalue λ) of one of the three underlying fourth-order problems (1.3), (1.4), (1.5), then $z=[w^2,\,(w')^2]^T$ will be in the kernel of $\mathcal{M}-\lambda\mathcal{B}$. The three fourth-order operators $L(p)$, $L_1(p)$, and $L_2(p)$ underlying the Borg system carry the following boundary conditions (see section 1):

$$L(p): w(0)=w''(0)=w(1)=w''(1)=0,$$

$$L_1(p): w(0)=w''(0)=w'(1)=w'''(1)=0,$$

$$L_2(p): w(0)=w''(0)=w(1)=w'(1)=0.$$

Setting $u \equiv w^2$ and $v \equiv (w')^2$, one can compute directly that the boundary conditions which determine the domain of $\mathcal{M}$ are:

- At $x = 0$: $u = u' = u''' = v' = v''' = u'' - 2v = u^{(4)} - 4v'' = 0$,
- At $x = 1$: $u' = u''' = v' = 0$.

The operator $\mathcal{B}$ is to be viewed as a perturbation of $\mathcal{M}$, so we take $D(\mathcal{B}) = D(\mathcal{M})$, from which it follows that $D(\mathcal{M} - \lambda\mathcal{B}) = D(\mathcal{M})$.

It will also be useful to determine the boundary conditions which define the domains of certain adjoint operators related to the Borg system. First, consider $D(\mathcal{M}^*)$, where $\mathcal{M}^*$ denotes the Hilbert space adjoint of the unbounded operator $\mathcal{M}$.

Let $z = [u, v]^T \in D(\mathcal{M})$. An element $\sigma = [\phi, \psi]^T \in D(\mathcal{M}^*)$ must satisfy

$$\langle \mathcal{M}z, \sigma\rangle - \langle z, \mathcal{M}^\dagger \sigma\rangle = 0\,,$$

where $\mathcal{M}^\dagger$ denotes the formal adjoint of $\mathcal{M}$. One can compute directly the following boundary conditions which define $D(\mathcal{M}^*)$:

- At $x = 0$: $\psi = \psi'' + \frac{2}{3}\phi = \psi^{(4)} - \frac{4}{3}\phi'' + \frac{8}{3}p_1\phi = 0$,
- At $x = 1$: $\phi = \phi'' = \phi^{(4)} = \psi = \psi' = \psi'' = \psi^{(4)} = 0$.

Similarly, one can compute the following boundary conditions which define $D(\mathcal{B}^*)$:

- At $x = 0$: $\psi = 0$,
- At $x = 1$: $\psi = 5\psi'' + \frac{4}{3}\phi = 0$.

One immediately observes the following containment:

$$D(\mathcal{M}^*) \subset D(\mathcal{B}^*)\,.$$

Finally, proceeding as above, one may compute the following set of boundary conditions which defines $D((\mathcal{M} - \lambda\mathcal{B})^*)$:

- At $x = 0$: $\psi = \psi'' + \frac{2}{3}\phi = \psi^{(4)} - \frac{4}{3}\phi'' + \frac{8}{3}p_1\phi = 0$,
- At $x = 1$: $\phi = \phi'' = \phi^{(4)} = \psi = \psi' = \psi'' = \psi^{(4)} = 0\,$,

which shows that $D((\mathcal{M} - \lambda\mathcal{B})^*) = D(\mathcal{M}^*)$.

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