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William T. Ross
University of Richmond, wross@richmond.edu

Stefan Richter

Carl Sundberg

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Hyperinvariant subspaces of the harmonic Dirichlet space

By *Stefan Richter**) at Knoxville, *William T. Ross* at Richmond and *Carl Sundberg* at Knoxville

1. Introduction

The *harmonic Dirichlet space* $\mathscr{D}$ is the space of functions f on the unit circle $\mathbb{T}$ for which

$$D(f) = \int_{\mathbb{T}}\int_{\mathbb{T}} \left|\frac{f(\zeta)-f(\xi)}{\zeta-\xi}\right|^2 \frac{|d\zeta|}{2\pi}\frac{|d\xi|}{2\pi} < \infty. \tag{1.1}$$

It is clear that $\mathscr{D} \subset L^2(\mathbb{T})$. Furthermore, as J. Douglas [8] pointed out, $\mathscr{D}$ consists of precisely those functions f in $L^2(\mathbb{T})$ whose harmonic extension to the open unit disk $\mathbb{D}$ has finite Dirichlet integral, in fact

$$D(f) = \frac{1}{\pi}\int_{\mathbb{D}} |\nabla f|^2 dA.$$

(Here dA represents two-dimensional Lebesgue measure.) If we define a norm on $\mathscr{D}$ by

$$\|f\|_{\mathscr{D}}^2 = \int_{\mathbb{T}} |f(\zeta)|^2 \frac{|d\zeta|}{2\pi} + D(f), \tag{1.2}$$

then a short computation shows that

$$\|f\|_{\mathscr{D}}^2 = \sum_{n\in\mathbb{Z}} (|n|+1)|\hat{f}(n)|^2, \tag{1.3}$$

where $\{\hat{f}(n)\}$ denotes the sequence of Fourier coefficients of f. Thus the operator

$$(Mf)(\zeta) = \zeta f(\zeta), \quad f \in \mathscr{D}$$

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is a bilateral weighted shift operator, the *bilateral Dirichlet shift.*

Our main results (Theorem 4.3 and Theorem 6.2) characterize the *hyperinvariant* subspaces of M, i.e. those (closed) subspaces $\mathcal{M}$ of $\mathcal{D}$ that are invariant for every operator that commutes with M. It is well known that the commutant of M can be identified with the multiplier algebra for $\mathcal{D}$, $M(\mathcal{D}) = \{\varphi \in L^2(\mathbb{T}) : \varphi\mathcal{D} \subset \mathcal{D}\}$, and it follows that $\mathcal{M}$ is hyperinvariant for M if and only if $\mathcal{M}$ is invariant for both M and M^{-1} [22], p. 91. We shall use $\operatorname{Lat}(M, M^{-1})$ to denote the lattice of hyperinvariant subspaces of M. If $f \in \mathcal{D}$, then

$$[f] = \operatorname{span}\{\zeta^n f(\zeta) : n \in \mathbb{Z}\}$$

will denote the hyperinvariant subspace generated by f.

For a statement of the main theorems, we quickly recall some facts about logarithmic capacity and quasi-topology. We shall supply more definitions and details in Sections 2 and 6. We say that a property holds *quasi-everywhere* (or q.e.) if it holds everywhere except on a set of (outer) logarithmic capacity zero. For a function $f \in \mathcal{D}$, the values $f(\zeta)$ are well defined quasi-everywhere via the sum of the Fourier series of f [3]. Furthermore, it turns out that for each set $E \subset \mathbb{T}$ the linear manifold

$$\mathcal{D}_E = \{f \in \mathcal{D} : f|_E = 0 \text{ q.e.}\}$$

is a closed subspace of $\mathcal{D}$, and of course hyperinvariant. A set $E \subset \mathbb{T}$ will be called *quasi-closed*, if there are open sets $V \subset \mathbb{T}$ of arbitrarily small logarithmic capacity such that $E \setminus V$ is closed. We shall write $E = F$ q.e. if the symmetric difference of E and F has logarithmic capacity zero. One checks that this defines an equivalence relation on the quasi-closed sets of $\mathbb{T}$.

With our convention that $f(\zeta)$ q.e. denotes the sum of the Fourier series of f at ζ, we will show that the set

$$Z(f) = \{\zeta : f(\zeta) = 0\}$$

is quasi-closed, thus for $f \in \mathcal{D}$,

$$[f] \subset D_{Z(f)}. \tag{1.4}$$

In Section 4, we shall prove that one always has equality in (1.4) and that this describes all of the hyperinvariant subspaces of M. More precisely, we have the following theorem:

Theorem 1.1. (a) *If E and F are quasi-closed subsets of $\mathbb{T}$, then*

$$\mathcal{D}_E = \mathcal{D}_F \Leftrightarrow E = F \quad q.e.$$

(b) *If $\mathcal{M} \in \operatorname{Lat}(M, M^{-1})$, then there exists a bounded non-negative function $f \in \mathcal{D}$ such that*

$$\mathcal{M} = [f] = \mathcal{D}_{Z(f)}.$$

Thus the hyperinvariant subspaces of M are in one-to-one correspondence with the equivalence classes of quasi-closed subsets of $\mathbb{T}$.

Part (a) of this theorem will follow easily from well known facts about the pointwise behavior of Sobolev space functions. For the proof of part (b) we shall use cut-off functions, the local Dirichlet integral, estimates from the theory of invariant subspaces in the analytic Dirichlet space [18], and Sobolev space techniques of Maz'ya and Shaposhnikova [16].

Our theorem may be considered as an analog of Wiener's theorem [14], p. 7, about the hyperinvariant subspaces of the bilateral unweighted shift B, i.e. multiplication by ζ on $L^2(\mathbb{T})$. Of course B is unitary, $B^{-1} = B^*$, and it follows that Wiener's theorem also characterizes the reducing subspaces of B. In contrast to this, we shall see that the bilateral Dirichlet shift is irreducible.

In Section 5, we will generalize our results to other types of Dirichlet spaces, namely the spaces $\mathscr{D}_\alpha$, $0 < \alpha < \infty$, of functions $f \in L^2(\mathbb{T})$ for which

$$\sum_{n \in Z} (1 + |n|)^\alpha |\hat{f}(n)|^2 < \infty.$$

For $\alpha > 1$, the hyperinvariant subspaces of $\mathscr{D}_\alpha$ are known, since in this case, $\mathscr{D}_\alpha$ is a Banach algebra of continuous functions and the hyperinvariant subspaces will be the closed ideals of $\mathscr{D}_\alpha$. Using Banach algebra techniques of Sarason, [21], p. 41, one can characterize these ideals in terms of their zero sets and show they are all of the form

$$\mathscr{D}_{\alpha, F} = \{f \in \mathscr{D}_\alpha : f|_F = 0\}$$

for some closed $F \subset \mathbb{T}$. Moreover, one can show there is a $g \in C^\infty(\mathbb{T})$ with $g^{-1}(0) = F$ and $\mathscr{D}_{\alpha,F} = [g]$. For $0 < \alpha < 1$, the situation becomes more complicated (since $\mathscr{D}_\alpha$ is not a Banach algebra of continuous functions) but we are still able to describe the hyperinvariant subspaces of $\mathscr{D}_\alpha$ by developing an analog of the machinery used above.

In Section 6 we will show, under the equivalent norm

$$\|f\|^2 = \frac{1}{2} |f(0)|^2 + D(f),$$

that every hyperinvariant subspace $\mathscr{M}$ of $\mathscr{D}$ can be generated by $P_{\mathscr{M}} 1$, the orthogonal projection of the function 1 onto $\mathscr{M}$, and moreover, this projection is not only a logarithmic potential but is also the solution to a certain capacitary extremal problem. This will imply the following theorem:

Theorem 1.2. *Let $E \subset \mathbb{T}$ be quasi-closed. For a non-negative finite Borel measure μ on $\mathbb{T}$ let*

$$u_\mu(\zeta) = 2 \int \log \frac{e}{|\zeta - \xi|} \, d\mu(\xi),$$

and set

$$M_E = \{\mu : \mu(\mathbb{T} \setminus E) = 0, \; u_\mu(\zeta) \leqq 1 \; q.e.\}.$$

Then $\mathscr{D}_E = [1 - u_{\mu_E}]$, *where* μ_E *is the equilibrium measure for E, i.e.* μ_E *satisfies* $\mu_E \in M_E$ *and*

$$\operatorname{cap}(E) = \mu_E(\mathbb{T}) = \sup\{\mu(\mathbb{T}) : \mu \in M_E\}.$$

(*Here* cap(E) *is defined in* (6.3).)

It will be clear from the general set up that if μ_E is the equilibrium measure for E, then the logarithmic potential u_{μ_E} equals one q.e. on E, thus $E \subset Z(1 - u_{\mu_E})$ q.e. The difficulty lies in proving the converse inclusion and our proof uses the harmonic extensions of logarithmic potentials u_μ to $\mathbb{C} \setminus \operatorname{supp}(\mu)$. The estimates in Section 6 were motivated by the connections between logarithmic capacity, harmonic measure, and escape probabilities of Brownian motion.

Before proceeding, we make the following conventions so as not to confuse the reader when speaking about functions on the unit circle or their harmonic extensions to the disk. We shall always use ζ and ξ for points on the unit circle $\mathbb{T}$ and z and w for points in the unit disk $\mathbb{D}$. When we write $f(\zeta)$ and $f(\xi)$ we always mean a function defined on the circle and $f(z)$ and $f(w)$ to mean its harmonic extension to the disk (or even $|z| > 1$). Also, H^2 will denote the usual Hardy space of the unit circle and $H^\infty = H^2 \cap L^\infty$.

2. Potentials and capacity

In order to consider the zero sets of functions in the Dirichlet space, we must first be specific about the points of definition and for this we introduce potentials. We follow [17] and [23] and refer the reader to these papers for proofs and further references.

For two integrable functions f and g on $\mathbb{T}$ we set

$$(f * g)(\zeta) = \int_{\mathbb{T}} f(\zeta\bar{\xi})\, g(\xi)\, \frac{|d\xi|}{2\pi}$$

and note that $(f * g)\hat{}\,(n) = \hat{f}(n)\,\hat{g}(n)$. Define the kernel

$$k(\zeta) = |1 - \zeta|^{-1/2}$$

and note there is a $\delta > 0$ such that

$$\delta(1 + |n|)^{-1/2} \leqq \hat{k}(n) \leqq \delta^{-1}(1 + |n|)^{-1/2} \quad \forall n \in \mathbb{Z}.$$

Thus

$$\text{(2.1)} \qquad \mathscr{D} = \{k * f : f \in L^2(\mathbb{T})\}$$

with $\|k * f\|_{\mathscr{D}}$ comparable to $\|f\|_{L^2}$.

For any set $E \subset \mathbb{T}$ we define the *capacity* $\operatorname{Cap}(E)$ by

(2.2) $$\operatorname{Cap}(E) = \inf\{\|f\|_{L^2}^2 : f \in L_+^2(\mathbb{T}),\ k * f \geqq 1 \text{ on } E\},$$

where we use $L_+^2(\mathbb{T})$ to denote the non-negative functions in $L^2(\mathbb{T})$. We note that Cap is a monotone, subadditive set function, and an outer capacity [17], Theorem 1, in the sense that for any set $E \subset \mathbb{T}$

$$\operatorname{Cap}(E) = \inf\{\operatorname{Cap}(G) : G \supset E,\ G \text{ open in } \mathbb{T}\}.$$

In fact, it is comparable to the square of the classical outer logarithmic capacity [17], Theorem 14 (i) (also see Section 6). Another important fact which will be used several times in this paper is that the inf in (2.2) is actually achieved by a function $f \in L_+^2(\mathbb{T})$, see [17], Theorem 9. That is to say, given any set $E \subset \mathbb{T}$ there is an $f \in L_+^2(\mathbb{T})$ such that $\operatorname{Cap}(E) = \|f\|_{L^2}^2$, $k * f = 1$ on E q.e. and $0 \leqq k * f \leqq 1$. We call such a $k * f$ a *capacitary potential* for E. From the definition of capacity we see that if $f \in L_+^2(\mathbb{T})$ and $a > 0$, then

(2.3) $$\operatorname{Cap}(\{\zeta : (k * f)(\zeta) \geqq a\}) \leqq \frac{1}{a^2}\|f\|_{L^2}^2.$$

Thus the potentials $k * f$ are finite valued q.e.

If for every $\varepsilon > 0$ there is a set E such that $\operatorname{Cap}(E) < \varepsilon$ and $f_n \to f$ uniformly on $\mathbb{T} \setminus E$, we say that $f_n \to f$ *quasi-uniformly*. An Egorov type result of Meyers [17], Theorem 4 (also see [2], Lemma 1), gives us the following string of implications: $f_n \to f$ in $L^2 \Rightarrow$ there is a subsequence $k * f_{n_k} \to k * f$ quasi-uniformly $\Rightarrow k * f_{n_k} \to k * f$ q.e.

The potentials $k * f$ are not, in general, continuous. To provide a substitution for continuity, we introduce the notion of quasi-continuity. We follow [2], [10], and [11]. A set $E \subset \mathbb{T}$ is said to be *quasi-closed* if for every $\varepsilon > 0$, there is an open set $W \subset \mathbb{T}$ with $\operatorname{Cap}(W) < \varepsilon$ such that $E \setminus W$ is closed. We say that a function f is *quasi-continuous* if for every $\varepsilon > 0$ there is an open set $W \subset \mathbb{T}$ with $\operatorname{Cap}(W) < \varepsilon$ such that f continuous on $\mathbb{T} \setminus W$. Facts which follow easily from the definitions are that all countable intersections and finite unions of quasi-closed sets are quasi-closed; if K is closed and f is quasi-continuous, then $f^{-1}(K)$ is quasi-closed; and if f is quasi-continuous and $f = g$ q.e. then g is quasi-continuous. For later reference, we record the following well known results as a proposition. The proofs of these facts are scattered throughout the literature and occasionally different definitions of capacity are used. So for the sake of completeness, we will give brief sketches of the proofs.

Proposition 2.1. (a) *If $f \in L^2(\mathbb{T})$, then $k * f$ is quasi-continuous.*

(b) *For any set $E \subset \mathbb{T}$, $\operatorname{Cap}(E)$ is comparable to* $\inf\|f\|_{\mathcal{D}}^2$, *where the* inf *is taken over all quasi-continuous $f \in \mathcal{D}$ with $0 \leqq f \leqq 1$ and $f = 1$ q.e. on E.*

(c) *If $f \in \mathcal{D}$ is quasi-continuous with $f = 0$ a.e. then $f = 0$ q.e.*

(d) *If $\{g_n : n \in \mathbb{N}\} \subset \mathcal{D}$ is a Cauchy sequence of quasi-continuous functions, then there exists a quasi-continuous function $g \in \mathcal{D}$ with $g_{n_k}(\zeta) \to g(\zeta)$ q.e. for some subsequence.*

Proof. (a) If $p \in C^\infty(\mathbb{T})$, then $k * p \in C^\infty(\mathbb{T})$. Hence, if we let $\{p_n : n \in \mathbb{N}\}$ be a sequence of trigonometric polynomials converging to f in L^2, then by the Egorov-type result mentioned above, a subsequence of $k * p_n$ will converge quasi-uniformly to $k * f$. This implies that $k * f$ is quasi-continuous.

(b) and (c). If $E \subset \mathbb{T}$ is compact, then one can show (see [2], proof of Theorem 2 (i), p. 263, and notice that the definition of capacity used there is given using C^∞ functions only) that

$$\begin{aligned} &\inf\{\|\phi\|_{\mathcal{D}}^2 : \phi \in C^\infty(\mathbb{T}),\ \phi \geqq 0,\ \phi \geqq 1 \text{ on } E\} \\ &= \inf\{\|f\|_{\mathcal{D}}^2, f \text{ quasi-continuous}, f \geqq 0, f \geqq 1 \text{ q.e. on } E\} \\ &= \inf\{\|f\|_{\mathcal{D}}^2, f \text{ quasi-continuous}, 0 \leqq f \leqq 1, f = 1 \text{ q.e. on } E\}. \end{aligned}$$

The second equality follows because if $f \geqq 0$ is quasi-continuous, then the function g defined by $g = \min\{f, 1\}$ is quasi-continuous with $\|g\|_{\mathcal{D}} \leqq \|f\|_{\mathcal{D}}$. From this (b) follows easily for compact sets E. This can be used to show that (c) is true (see [2], proof of Theorem 2 (iii), p. 263). Then using (c) and the remarks above, one easily concludes that (b) holds for all sets $E \subset \mathbb{T}$.

(d) It follows from (2.1) and (c) that $g_n = k * f_n$ q.e. for some $f_n \in L^2(\mathbb{T})$. Since $\|k * f_n\|_{\mathcal{D}}$ is comparable to $\|f_n\|_{L^2}$, we have $f_n \to f$ for some $f \in L^2(\mathbb{T})$. Hence, as before, there exists a subsequence $g_{n_k} = k * f_{n_k} \to k * f = g$ q.e. By (a), g is quasi-continuous. □

In the introduction, we made the convention that for $f \in \mathcal{D}$, $f(\zeta)$ denotes the sum of the Fourier series at ζ. We now show that f is quasi-continuous.

Proposition 2.2. *For $f \in \mathcal{D}$, $f(\zeta)$ defined as the sum of the Fourier series is quasi-continuous.*

Proof. As mentioned in the introduction, the Fourier series of f converges q.e. [3]. On the other hand, since the sequence of partial sums $\{p_n : n \in \mathbb{N}\}$ converges to f in norm, by Proposition 2.1 (d) a subsequence will converge q.e. to a quasi-continuous function. Hence f, the sum of the Fourier series, must be quasi-continuous. □

Remark. We note that it now follows from Abel's theorem, a result of Landau [15], p. 65–66, and the above that for $f \in \mathcal{D}$ the radial limit function and the sum of the Fourier series agree everywhere, and they both equal the quasi-continuous representative of f, which is uniquely defined except for sets of capacity zero.

If $f = k * g$, $g \in L^2(\mathbb{T})$, it may happen that the radial limit $\lim_{r \to 1} f(re^{i\theta})$ exists but $k * g(e^{i\theta})$ does not exist in the Lebesgue sense. (For example, take

$$g(e^{i\theta}) = \begin{cases} (\sqrt{\theta}\,\log\theta^{-1})^{-1}, & 0 < \theta < 1/2, \\ (-\sqrt{|\theta|}\,\log|\theta|^{-1})^{-1}, & -1/2 < \theta < 0, \\ 0, & 1/2 < |\theta| < \pi. \end{cases}$$

Then $g \in L^2(\mathbb{T})$, $\int k(e^{i\theta})|g(e^{-i\theta})|\,d\theta = \infty$, and $f(r1) = 0$ for all $0 < r < 1$.) However, the following fact (which is not necessary for our arguments but we include it for the sake of completeness) is true for all θ:

Proposition 2.3. *If either*

$$\lim_{r \to 1} f(re^{i\theta})$$

or

$$\text{P.V. } k * g(e^{i\theta})$$

exists, then so does the other, and they are equal.

Proof. Define $k_\varepsilon(e^{i\theta})$ to be $|1 - e^{i\theta}|^{-1/2}$ if $\varepsilon < |\theta| < \pi$ and zero if $|\theta| < \varepsilon$. Let $l_\varepsilon(e^{i\theta})$ be equal to $1/2\varepsilon$ if $|\theta| < \varepsilon$ and zero if $\varepsilon < |\theta| < \pi$. Also, let $P_r(\theta)$ be the Poisson kernel. We will prove this proposition by showing that

$$\lim_{\varepsilon \to 0} [P_{1-\varepsilon} * f(e^{i\theta}) - k_\varepsilon * g(e^{i\theta})] = 0 \quad \forall \theta. \tag{2.4}$$

Since $f \in \mathcal{D} \subset \text{VMO}$ [24], we have

$$\lim_{\varepsilon \to 0} \|P_{1-\varepsilon} * f - l_\varepsilon * f\|_\infty = 0.$$

So to prove (2.4) we need to show

$$\lim_{\varepsilon \to 0} [l_\varepsilon * f(e^{i\theta}) - k_\varepsilon * g(e^{i\theta})] = 0 \quad \forall \theta. \tag{2.5}$$

To show (2.5) we may assume without loss of generality that $e^{i\theta} = 1$. Writing $f = k * g$ we see that $l_\varepsilon * f = (l_\varepsilon * k) * g$, thus we need to show that

$$\lim_{\varepsilon \to 0} \int [(l_\varepsilon * k)g - k_\varepsilon g]\,d\theta = 0 \quad \forall g \in L^2(\mathbb{T}), \tag{2.6}$$

i.e. that $l_\varepsilon * k - k_\varepsilon \to 0$ weakly in L^2. Since (2.6) is true for continuous g, we need to show that

$$\|l_\varepsilon * k - k_\varepsilon\|_{L^2}$$

remains bounded as $\varepsilon \to 0$. By symmetry we just need to show

$$\int_0^\pi (l_\varepsilon * k - k_\varepsilon)^2\,d\theta$$

is uniformly bounded in ε.

If $\varepsilon < \theta < \pi - \varepsilon$, then $l_\varepsilon * k(e^{i\theta}) \sim \varepsilon^{-1}(\sqrt{\theta + \varepsilon} - \sqrt{\theta - \varepsilon})$. If $0 < \theta < \varepsilon$ then $l_\varepsilon * k(e^{i\theta}) \sim \varepsilon^{-1}(\sqrt{\varepsilon + \theta} + \sqrt{\varepsilon - \theta})$. If $\pi - \varepsilon < \theta < \pi$ then $l_\varepsilon * k(e^{i\theta})$ and $k(e^{i\theta})$ are both bounded by one. Thus

$$\int_{\varepsilon<\theta<\pi-\varepsilon} (l_\varepsilon * k - k_\varepsilon)^2 d\theta \sim \int_\varepsilon^{\pi-\varepsilon} \left(\frac{1}{\varepsilon}[\sqrt{\theta+\varepsilon}-\sqrt{\theta-\varepsilon}] - \frac{1}{\sqrt{\theta}}\right)^2 d\theta$$

$$= \int_\varepsilon^{\pi-\varepsilon} \left(\frac{2\varepsilon}{\varepsilon[\sqrt{\theta+\varepsilon}+\sqrt{\theta-\varepsilon}]} - \frac{1}{\sqrt{\theta}}\right)^2 d\theta = \int_\varepsilon^{\pi-\varepsilon} \left(\frac{2\sqrt{\theta}-\sqrt{\theta+\varepsilon}-\sqrt{\theta-\varepsilon}}{(\sqrt{\theta+\varepsilon}+\sqrt{\theta-\varepsilon})\sqrt{\theta}}\right)^2 d\theta$$

$$< \int_\varepsilon^{\pi} \left(\frac{2\sqrt{\theta}-\sqrt{\theta+\varepsilon}-\sqrt{\theta-\varepsilon}}{\sqrt{\theta}}\right)^2 \frac{d\theta}{\theta} = \int_\varepsilon^{\pi} \left(2-\sqrt{1+\frac{\varepsilon}{\theta}}-\sqrt{1-\frac{\varepsilon}{\theta}}\right)^2 \frac{d\theta}{\theta}$$

$$= \int_{\varepsilon/\pi}^{1} (2-\sqrt{1+s}-\sqrt{1-s})^2 \frac{ds}{s} < \int_0^1 (2-\sqrt{1+s}-\sqrt{1-s})^2 \frac{ds}{s} < \infty .$$

Also

$$\int_{0<\theta<\varepsilon} (l_\varepsilon * k - k_\varepsilon)^2 d\theta = \int_{0<\theta<\varepsilon} (l_\varepsilon * k)^2 d\theta \sim \int_0^\varepsilon \left[\frac{1}{\varepsilon}(\sqrt{\varepsilon+\theta}+\sqrt{\varepsilon-\theta})\right]^2 d\theta$$

$$< (\sqrt{2}+1)^2 \int_0^\varepsilon \left(\frac{1}{\varepsilon}\sqrt{\varepsilon}\right)^2 d\theta = (\sqrt{2}+1)^2 .$$

Finally, it is clear that

$$\int_{\pi-\varepsilon<\theta<\pi} (l_\varepsilon * k - k_\varepsilon)^2 d\theta$$

remains bounded as $\varepsilon \to 0$. □

The following lemma is a slight generalization of a well known fact for compact sets (see [17], Theorem 7) and is found in [11], Theorem 2.10 (see Footnote 2 on p. 133).

Lemma 2.4. *If* $\{E_i : i \in \mathbb{N}\}$ *is a sequence of quasi-closed sets with* $E_i \downarrow E$, *then* $\operatorname{Cap}(E_i) \to \operatorname{Cap}(E)$.

3. Weak convergence and cut-off functions

For an arbitrary function $f \in L^2(\mathbb{T})$, and a.e. $\zeta \in \mathbb{T}$, $f(\zeta)$ exists as the radial limit of $f(z)$ at ζ. We follow [18] and define the *local Dirichlet integral* $D_\zeta(f)$ by the formula

$$D_\zeta(f) = \int_{\mathbb{T}} \left|\frac{f(\zeta)-f(\xi)}{\zeta-\xi}\right|^2 \frac{|d\xi|}{2\pi},$$

and $D_\zeta(f) = \infty$ if $f(\zeta)$ does not exist. Thus we can norm $\mathcal{D}$ in terms of the local Dirichlet integral

$$\|f\|_{\mathcal{D}}^2 = \|f\|_{L^2}^2 + \|D_\zeta(f)\|_{L^1} . \tag{3.1}$$

Since $\mathcal{D} \subset L^2(\mathbb{T})$, every $h \in \mathcal{D}$ can be written as $h = h_+ + h_-$, where $h_+ \in \mathcal{D} \cap H^2$ and $h_- \in \mathcal{D} \cap (H^2)^\perp$.

Proposition 3.1. *Let $h = h_+ + h_-$ where $h_+ \in \mathscr{D} \cap H^2$ and $h_- \in \mathscr{D} \cap (H^2)^\perp$. Then*

$$D_\zeta(h) = D_\zeta(h_+) + D_\zeta(h_-) \quad a.e.$$

Proof. Let $\zeta \in \mathbb{T}$ such that $D_\zeta(h_+)$ and $D_\zeta(h_-)$ are finite. We note that

$$D_\zeta(h) = \int_{\mathbb{T}} \left| \frac{h_+(\zeta) - h_+(\xi)}{\zeta - \xi} - \overline{(\zeta\xi)}\, \frac{h_-(\zeta) - h_-(\xi)}{\bar{\zeta} - \bar{\xi}} \right|^2 \frac{|d\xi|}{2\pi}.$$

If we let $f_1(\xi)$ be the first function in the above integrand and $f_2(\xi)$ the second, we see that $f_1 \in H^2$ and $f_2 \in (H^2)^\perp$. Thus f_1 and f_2 are orthogonal in L^2 norm, hence $D_\zeta(h) = D_\zeta(h_+) + D_\zeta(h_-)$. □

In [18], it was shown that if g belongs to the analytic Dirichlet space, i.e. $g \in \mathscr{D} \cap H^2$, and $g_r(\zeta)$, $0 < r < 1$, is the analytic extension of g evaluated at $r\zeta$, then $D_\zeta(g_r) \leqq 4D_\zeta(g)$. From this it follows that if $\varphi \in H^\infty$ and $f \in \mathscr{D} \cap H^2$ with $\varphi f \in \mathscr{D}$ then $\varphi_r f \to \varphi f$ weakly in $\mathscr{D} \cap H^2$. This can be used to prove that if $f, g \in \mathscr{D} \cap H^2$ with $|g(z)| \leqq |f(z)|$ for all $z \in \mathbb{D}$, then

$$g \in \operatorname{span}\{z^n f(z) : n \in \mathbb{N} \cup \{0\}\}.$$

In this section, we plan to prove analogous results for the harmonic Dirichlet space which will enable us to work in a dense algebra via cut-off functions.

The harmonic Dirichlet space $\mathscr{D}$ is not an algebra. However, if $f, g \in \mathscr{D} \cap L^\infty$, then a routine calculation yields $\|fg\|_{\mathscr{D}} \leqq C(\|f\|_\infty \|g\|_{\mathscr{D}} + \|g\|_\infty \|f\|_{\mathscr{D}})$. Hence $\mathscr{D} \cap L^\infty$ is an algebra. In the characterization of $\operatorname{Lat}(M, M^{-1})$, we wish to describe the hyperinvariant subspace generated by a single vector $f \in \mathscr{D}$, that is

$$[f] = \operatorname{span}\{\zeta^n f(\zeta) : n \in \mathbb{Z}\}.$$

To do this, we wish to work off a dense algebra in $[f]$ and a natural candidate for this algebra is $[f] \cap L^\infty$. Thus we introduce the cut-off function

$$|f|_M = \min\{|f|, M\},$$

where $M \in \mathbb{N}$. Note that $D(|f|_M) \leqq D(|f|)$ so $|f|_M \in \mathscr{D}$. (To avoid any confusion we note that $|f(\zeta)| = |f|(\zeta)$ and $|f(\zeta)|_M = |f|_M(\zeta)$ almost everywhere. Hence by the quasi-continuity of $|f(\zeta)|$, $|f|(\zeta)$, $|f(\zeta)|_M$, $|f|_M(\zeta)$, and Proposition 2.1, the equalities hold quasi-everywhere.) One hopes that $|f|_M \in [f]$ and moreover that $[|f|_M] = [f]$. This is indeed the case and we state one of the main results of this section which is reminiscent of [18], Corollary 5.5.

Theorem 3.2. *Let $g, h \in \mathscr{D}$ with $|g(\zeta)| \leqq |h(\zeta)|$ almost everywhere. Then $g \in [h]$.*

This theorem and many of our other results will depend on the following important lemma, which has analogs in other function spaces. It allows us to use weak convergence instead of norm convergence, normally difficult to work with in the harmonic Dirichlet space.

Lemma 3.3. *Let $\{f_n : n \in \mathbb{N}\}$ be a sequence of functions in $\mathcal{D}$ which are uniformly bounded in Dirichlet norm and which converge to zero pointwise almost everywhere. Then $f_n \to 0$ weakly in $\mathcal{D}$.*

Proof. By using Egorov's theorem and the fact that the sequence $\{f_n : n \in \mathbb{N}\}$ is uniformly bounded in L^2 norm, one shows that $\langle f_n, h\rangle_{L^2} \to 0$ for all $h \in L^\infty$, hence $f_n \to 0$ weakly in L^2.

If $z \in \mathbb{D}$ and P_z is the Poisson kernel at z, then for $f \in \mathcal{D}$

$$|f(z)| = |\langle f, P_z\rangle_{L^2}| \leqq \|f\|_{L^2} \|P_z\|_{L^2} \leqq \|f\|_{\mathcal{D}} \|P_z\|_{L^2}.$$

Thus evaluation at $z \in \mathcal{D}$ defines a continuous linear functional on $\mathcal{D}$, i.e. there is a $k_z \in \mathcal{D}$ such that $f(z) = \langle f, k_z\rangle_{\mathcal{D}}$. By the weak convergence in L^2 and the identity

$$\langle f_n, k_z\rangle_{\mathcal{D}} = \langle f_n, P_z\rangle_{L^2},$$

we have $\langle f_n, h\rangle_{\mathcal{D}} \to 0$ whenever h is a finite linear combination of functions k_z, $z \in \mathbb{D}$. But these finite linear combinations are dense in $\mathcal{D}$, hence $\{f_n : n \in \mathbb{N}\}$ is a norm bounded sequence that converges to zero weakly on a dense set, hence $f_n \to 0$ weakly in $\mathcal{D}$. □

For a function $h \in L^2(\mathbb{T})$ and $0 < r < 1$, we define h_r on $\mathbb{T}$ by setting $h_r(\zeta)$ to be the harmonic extension of h evaluated at $r\zeta$.

Proposition 3.4. *Let $\varphi \in L^\infty$ and $f \in \mathcal{D}$ with $\varphi f \in \mathcal{D}$. Then $\varphi_r f \to \varphi f$ weakly as $r \to 1$.*

Proof. Note that $\varphi_r f - \varphi f \to 0$ almost everywhere, so to complete the proof, we just need to show (by Lemma 3.3) that $\|\varphi_r f\|_{\mathcal{D}}$ remains bounded as $r \to 1$. By (3.1), we just need to get estimates on $D_\xi(\varphi_r f)$ for which we have

$$D_\xi(\varphi_r f) \leqq 2\|\varphi\|_\infty^2 D_\xi(f) + 2|f(\xi)|^2 D_\xi(\varphi_r). \tag{3.2}$$

Note that $\varphi_r = \varphi_r^+ + \varphi_r^- = (\varphi^+)_r + (\varphi^-)_r$ so we can use an estimate of [18], Theorem 5.2, and Proposition 3.1 to get

$$D_\xi(\varphi_r) = D_\xi(\varphi_r^+) + D_\xi(\varphi_r^-) \leqq 4\big(D_\xi(\varphi^+) + D_\xi(\varphi^-)\big) = 4D_\xi(\varphi).$$

Thus

$$|f(\xi)|^2 D_\xi(\varphi_r) \leqq 4|f(\xi)|^2 D_\xi(\varphi) \leqq 8D_\xi(\varphi f) + 8\|\varphi\|_\infty^2 D_\xi(f),$$

where the second inequality follows from a computation similar to the one needed to verify (3.2) (see [18], p. 376). Combining this with (3.2), we see that $\|\varphi_r f\|_{\mathcal{D}}$ remains bounded as $r \to 1$. □

Corollary 3.5. *With φ and f as above, $\varphi f \in [f]$.*

Proof. Since $\varphi_r \in C^\infty(\mathbb{T})$, for $0 < r < 1$, we can find a sequence of trigonometric polynomials that converge to φ_r uniformly in all derivatives, thus $\varphi_r f \in [f]$ for all $0 < r < 1$. Now apply Proposition 3.4 to get $\varphi f \in [f]$. □

Proof of Theorem 3.2. Let $f = h$ and $\varphi(\zeta) = g(\zeta)/h(\zeta)$, if $h(\zeta) \neq 0$ and $\varphi(\zeta) = 0$ otherwise, and note that $\varphi \in L^\infty$. Now apply Corollary 3.5 to see that $\varphi f = g \in [h]$. □

Corollary 3.6. *If $\mathcal{M} \in \mathrm{Lat}(M, M^{-1})$, then $\mathcal{M} \cap L^\infty$ is dense in $\mathcal{M}$.*

Proof. Let $g \in \mathcal{M}$. It follows from Theorem 3.2 that $[|g|] = [g]$, thus we may assume $g \geqq 0$. For $M \in \mathbb{N}$, let $g_M = \min\{g, M\}$. By Theorem 3.2, $g_M \in [g] \cap L^\infty$ and it follows from Lemma 3.3 that $g_M \to g$ weakly in $\mathcal{D}$. □

4. Invariant subspaces

For a quasi-closed set $E \subset \mathbb{T}$, define

$$\mathcal{D}_E = \{f \in \mathcal{D} : f|_E = 0 \text{ q.e.}\},$$

and notice, by Proposition 2.1 (d), that $\mathcal{D}_E$ is a closed subspace of $\mathcal{D}$.

Proposition 4.1. *Let E_1 and E_2 be quasi-closed sets in $\mathbb{T}$. Then $\mathcal{D}_{E_1} \subset \mathcal{D}_{E_2}$ if and only if* $\mathrm{Cap}(E_2 \backslash E_1) = 0$. *Consequently, $\mathcal{D}_{E_1} = \mathcal{D}_{E_2}$ if and only if* $\mathrm{Cap}(E_1 \Delta E_2) = 0$.

Proof. Clearly $\mathrm{Cap}(E_2 \backslash E_1) = 0$ implies $\mathcal{D}_{E_1} \subset \mathcal{D}_{E_2}$. For the other direction, assume $\mathrm{Cap}(E_2 \backslash E_1) > 0$ and choose V open with $\mathrm{Cap}(V) < \mathrm{Cap}(E_2 \backslash E_1)/2$ and $E_1 \backslash V$ closed. Let g be the capacitary potential function for V (i.e. $g = 1$ q.e. on V, $0 \leqq g \leqq 1$, and $\|g\|_{\mathcal{D}}^2 = \mathrm{Cap}(V)$) and pick $\varphi \in C^\infty(\mathbb{T})$ such that $\varphi^{-1}(0) = E_1 \backslash V$. ($E_1 \backslash V$ is a compact subset of $\mathbb{T}$, so this is always possible.) Then the function $f = \varphi(1-g) \in \mathcal{D}_{E_1}$. If $f \in \mathcal{D}_{E_2}$, then $1 - g = 0$ q.e. on $E_2 \backslash E_1$. Thus g is a test function for $E_2 \backslash E_1$, i.e.

$$\mathrm{Cap}(E_2 \backslash E_1) \leqq \|g\|_{\mathcal{D}}^2 < \mathrm{Cap}(E_2 \backslash E_1)/2,$$

a contraction. Hence $f \in \mathcal{D}_{E_1} \backslash \mathcal{D}_{E_2}$, so $\mathcal{D}_{E_1} \neq \mathcal{D}_{E_2}$. □

Remark. We point out that for the analytic Dirichlet space $\mathcal{D} \cap H^2$, one defines the simply invariant subspaces $\mathcal{D}_E \cap H^2$, see [4], p. 295. However for this case, the structure of these subspaces is quite different and equality relations, as in Proposition 4.1, become more complicated.

Clearly $\mathcal{D}_E \in \mathrm{Lat}(M, M^{-1})$ for each quasi-closed set $E \subset \mathbb{T}$. Our main result is that these subspaces exhaust all of $\mathrm{Lat}(M, M^{-1})$. We first prove this for the hyperinvariant subspaces generated by a single vector. Recall that $Z(f) = f^{-1}(0)$.

Lemma 4.2. *If $f \in \mathscr{D}$, then $[f] = \mathscr{D}_{Z(f)}$.*

Proof. Clearly $[f] \subset \mathscr{D}_{Z(f)}$. If $|f|_1$ denotes the cut-off function

$$|f|_1(\zeta) = \min\{|f(\zeta)|, 1\},$$

then $Z(f) = Z(|f|_1)$ and since $|f|_1(\zeta) \leqq |f(\zeta)|$, it follows from Theorem 3.2 that

$$[|f|_1] \subset [f] \subset \mathscr{D}_{Z(f)} = \mathscr{D}_{Z(|f|_1)}.$$

Thus we may assume $0 \leqq f \leqq 1$. Furthermore, it follows from Corollary 3.6 that it suffices to show $\mathscr{D}_{Z(f)} \cap L^\infty \subset [f]$.

To this end, let $g \in \mathscr{D}_{Z(f)} \cap L^\infty$ and notice, by Theorem 3.2, we may assume $g \geqq 0$. For $n \in \mathbb{N}$, let

$$g_n(\zeta) = \max\left\{g(\zeta) - \frac{1}{n}, 0\right\}.$$

It follows from Lemma 3.3 that $g_n \to g$ weakly, hence it is enough to show $g_n \in [f]$ for each $n \in \mathbb{N}$.

For the rest of the proof, we fix $n \in \mathbb{N}$. For $t \geqq 0$ define

$$N_t = \{\zeta \in \mathbb{T} : g_n(\zeta) \neq 0, f(\zeta) \leqq t\}.$$

The functions f and g are quasi-continuous, hence the sets

$$M_t = \left\{\zeta \in \mathbb{T} : g(\zeta) \geqq \frac{1}{n}, f(\zeta) \leqq t\right\}$$

are quasi-closed and for each $t \geqq 0$ they satisfy $N_t \subset M_t$. Now $M_0 \subset Z(f) \setminus Z(g)$, hence by the assumption on g and Lemma 2.4, we have

$$\operatorname{Cap}(N_t) \leqq \operatorname{Cap}(M_t) \to \operatorname{Cap}(M_0) = 0$$

as $t \to 0$. By Proposition 2.1 (b), we can find a family $0 \leqq w_t \leqq 1$ of functions in $\mathscr{D}$ with $w_t = 1$ quasi-everywhere on N_t and $\|w_t\|_{\mathscr{D}} \to 0$ as $t \to 0$. For $t, \delta > 0$ we consider the function

$$\varphi_{t,\delta} = \frac{1 + w_t}{f + \delta}\, g_n.$$

It is easy to verify $(f + \delta)^{-1} \in \mathscr{D} \cap L^\infty$ and thus $\varphi_{t,\delta} \in \mathscr{D} \cap L^\infty$, since it is the product of bounded Dirichlet functions. Furthermore, for the same reason $\varphi_{t,\delta} f \in \mathscr{D} \cap L^\infty$, in fact, by Theorem 3.2, $\varphi_{t,\delta} f \in [f]$. We shall conclude the proof by showing that we can choose $t, \delta \to 0$ so that

$$\varphi_{t,\delta} f \to g_n$$

weakly in $\mathscr{D}$.

First we show that one can choose $\delta = \delta(t)$ so that the Dirichlet norms of $\varphi_{t,\delta} f$ stay uniformly bounded as $t \to 0$. Note that

$$\varphi_{t,\delta} f = (1 - w_t) g_n - \delta \frac{(1 - w_t) g_n}{f + \delta}. \tag{4.1}$$

It is clear that

$$\|(1 - w_t) g_n\|_{\mathscr{D}} \leqq \|1 - w_t\|_\infty \|g_n\|_{\mathscr{D}} + \|g_n\|_\infty \|1 - w_t\|_{\mathscr{D}},$$

thus the first summand on the right hand side of (4.1) remains bounded, because $0 \leqq w_t \leqq 1$ and $\|w_t\|_{\mathscr{D}} \to 0$ as $t \to 0$. Hence it suffices to show that

$$\left\| \frac{(1 - w_t) g_n}{f + \delta} \right\|_{\mathscr{D}} \leqq C(t) \tag{4.2}$$

independently of $\delta > 0$.

We write

$$h(\zeta) = (1 - w_t(\zeta))\, g_n(\zeta)$$

and let

$$A = N_t \cup Z(g_n) \subset Z(h) \quad \text{q.e.}$$

We shall estimate the size of the Douglas integral (1.1) of $h(f + \delta)^{-1}$ by distinguishing several cases:

First: If $\zeta, \xi \in A$, then

$$\left| \frac{h(\zeta)}{f(\zeta) + \delta} - \frac{h(\xi)}{f(\xi) + \delta} \right|^2 = 0.$$

Second: If $\zeta \in A$, $\xi \notin A$, then $f(\xi) > t$, so

$$\left| \frac{h(\zeta)}{f(\zeta) + \delta} - \frac{h(\xi)}{f(\xi) + \delta} \right|^2 \leqq \frac{1}{t^2} |h(\xi)|^2 = \frac{1}{t^2} |h(\zeta) - h(\xi)|^2.$$

Third: The case $\zeta \notin A$, $\xi \in A$ is similar to the second case.

Fourth: If $\zeta, \xi \notin A$, then $f(\zeta), f(\xi) > t$, hence

$$\begin{aligned} \left| \frac{h(\zeta)}{f(\zeta) + \delta} - \frac{h(\xi)}{f(\xi) + \delta} \right|^2 &= \left| \frac{h(\zeta) - h(\xi)}{f(\zeta) + \delta} - h(\xi) \frac{f(\zeta) - f(\xi)}{(f(\zeta) + \delta)(f(\xi) + \delta)} \right|^2 \\ &\leqq 2 \left\{ \frac{1}{t^2} |h(\zeta) - h(\xi)|^2 + \frac{1}{t^4} \|h\|_\infty^2 |f(\zeta) - f(\xi)|^2 \right\}. \end{aligned}$$

Thus it follows that

$$D\left(\frac{(1-w_t)g_n}{f+\delta}\right) = \int_T\int_T \frac{1}{|\zeta-\xi|^2}\left|\frac{h(\zeta)}{f(\zeta)+\delta} - \frac{h(\xi)}{f(\xi)+\delta}\right|^2 \frac{|d\zeta|}{2\pi}\frac{|d\xi|}{2\pi}$$
$$\leqq \frac{2}{t^4}\{t^2 D(h) + \|h\|_\infty^2 D(f)\}.$$

Note that the above is independent of $\delta > 0$, hence (4.2) has been proved.

Finally, we show that $\varphi_{t_j,\delta(t_j)} f \to g_n$ almost everywhere for some sequence $t_j \to 0$. Since $\|w_t\|_{\mathscr{D}} \to 0$ as $t \to 0$, there is a sequence $t_j \to 0$ such that $w_{t_j} \to 0$ a.e. as $j \to \infty$. Thus

$$\varphi_{t_j,\delta(t_j)} f = (1-w_{t_j}) g_n \frac{f}{f+\delta(t_j)} \to g_n \quad \text{a.e.} \quad \text{on } \mathbb{T}\setminus Z(f).$$

This is all we need, since $\varphi_{t_j,\delta(t_j)} f = g_n = 0$ a.e. on $Z(g_n)$ and $Z(f)\setminus Z(g_n)$ has measure zero. □

Theorem 4.3. *Let $\mathscr{M} \in \mathrm{Lat}(M, M^{-1})$. Then there is a bounded non-negative $f \in \mathscr{D}$ with $\mathscr{M} = [f] = \mathscr{D}_{Z(f)}$.*

Proof. Since $\mathscr{M}$ is separable,

$$\mathscr{M} = \operatorname{span}\{[f_n] : f_n \in \mathscr{D}, f_n \not\equiv 0, n \in \mathbb{N}\}.$$

By Theorem 3.2 we may assume $f_n \geqq 0$ for all n. Let $\varepsilon_n = \|f_n\|_{\mathscr{D}}^{-1} 2^{-n}$, then

$$g = \sum_{n\in\mathbb{N}} \varepsilon_n f_n \in \mathscr{D} \tag{4.3}$$

and $Z(g) = \bigcap_n Z(f_n)$ quasi-everywhere. (Note by Proposition 2.1 (d), the pointwise limit of the sum on the right hand side of (4.3) equals the sum of the Fourier series of g q.e.) We now have

$$g \in \operatorname{span}\{[f_n] : n \in \mathbb{N}\} = \mathscr{M} \subset \mathscr{D}_{\bigcap_n Z(f_n)} = \mathscr{D}_{Z(g)},$$

hence by Lemma 4.2,

$$[g] \subset \mathscr{M} \subset \mathscr{D}_{Z(g)} = [g].$$

Now let $f = g_1 = \min\{g, 1\}$ and note that $Z(f) = Z(g)$ q.e. so

$$\mathscr{M} \subset [g] = [f] = \mathscr{D}_{Z(f)}. \quad \square$$

Remark. For any set $E \subset \mathbb{T}$ (not necessarily quasi-closed) we can define the set

$$\mathscr{D}_E = \{f \in \mathscr{D} : f|_E = 0 \quad \text{q.e.}\}$$

and see that by Proposition 2.1, $\mathscr{D}_E$ is a closed hyperinvariant subspace of $\mathscr{D}$.Thus, by Theorem 4.3, $\mathscr{D}_E = \mathscr{D}_F$ for some quasi-closed $F \subset \mathbb{T}$. What is F in terms of E? For notational purposes, we say that a set E_1 is *quasi-contained* in E_2 if $\operatorname{Cap}(E_1 \setminus E_2)$ is zero and that E_1 is *quasi-equivalent* to E_2 if $\operatorname{Cap}(E_1 \Delta E_2)$ is zero. For a set $A \subset \mathbb{T}$ we let $\mathscr{H}_A$ be the family of quasi-closed sets that quasi-contain A. A theorem of Fuglede [11], Theorem 2.7, states that $\mathscr{H}_A$ has a "quasi-minimal" element, i.e. a quasi-closed $A^* \in \mathscr{H}_A$ such that A^* is quasi-contained in every $E \in \mathscr{H}_A$. We call this quasi-minimal element A^* the *quasi-closure* of A. (A^* is actually an equivalence class with respect to q.e.) Returning to our question, we now see that $\mathscr{D}_E = \mathscr{D}_{E^*}$ as follows: Clearly $\mathscr{D}_{E^*} \subset \mathscr{D}_E$. If $f \in \mathscr{D}_E$ then $f = 0$ q.e. on E and using the quasi-continuity of f and the fact that $Z(f)$ is quasi-closed, we get $f = 0$ q.e. on E^* so $f \in \mathscr{D}_{E^*}$. The notation of quasi-closed is used to distinguish $\mathscr{D}_{E_1}$ from $\mathscr{D}_{E_2}$ (E_1 and E_2 are quasi-closed sets) as in Proposition 4.1. The analog of Proposition 4.1 is not true for $\mathscr{D}_E$ if E is an arbitrary subset of $\mathbb{T}$. An example of this can be derived from the remark following this next corollary.

Also worth mentioning here is that the notion of quasi-closed is necessary here since $\mathscr{D}_E$ cannot always be written as $\mathscr{D}_F$ for some closed $F \subset \mathbb{T}$. To see this, notice that for any open arc I in the unit circle (sufficiently small), we have (see [23], p. 122, and Lemma 2.5)

$$\operatorname{Cap}(I) \sim \left(\log\left(\frac{1}{|I|}\right)\right)^{-1}.$$

Using this estimate and a Cantor type construction, one can construct a closed set K of positive capacity with $E = \mathbb{T} \setminus K$ quasi-closed and dense in $\mathbb{T}$. (Note that E will be a countable, disjoint union of arcs.) A straightforward argument yields $\operatorname{Cap}(E \Delta F) > 0$ for every closed set F, which means, by Proposition 4.1, that $\mathscr{D}_E$ is not equal to $\mathscr{D}_F$ for any closed set $F \subset \mathbb{T}$.

A hyperinvariant subspace $\mathscr{M}$ of M is called *lattice complemented* if there is another hyperinvariant subspace $\mathscr{N}$ of M such that $\mathscr{M} \cap \mathscr{N} = (0)$ and $\mathscr{M} \vee \mathscr{N} = \mathscr{D}$.

Corollary 4.4. *Let $E \subset \mathbb{T}$ be quasi-closed. Then $\mathscr{D}_E$ is lattice complemented in* $\operatorname{Lat}(M, M^{-1})$ *if and only if $\mathbb{T} \setminus E$ is quasi-closed. If $\mathscr{D}_E$ is lattice complemented, then $\mathscr{D}_{T \setminus E}$ is the unique lattice complementary hyperinvariant subspace.*

Proof. If $\mathbb{T} \setminus E$ is quasi-closed, then

$$\mathscr{D}_E \cap \mathscr{D}_{T \setminus E} = (0)$$

and

$$\mathscr{D}_E \vee \mathscr{D}_{T \setminus E} = \mathscr{D}.$$

In fact, the first identity is clear. To see the second one, let f and g be non-negative functions in $\mathscr{D}$ such that $[f] = \mathscr{D}_E$ and $[g] = \mathscr{D}_{T \setminus E}$. Then $f + g \in \mathscr{D}_E \vee \mathscr{D}_{T \setminus E}$ and $\operatorname{Cap}(Z(f+g)) = 0$, hence $[f+g] = \mathscr{D}$. Thus $\mathscr{D}_E$ is lattice complemented.

Conversely, if $\mathscr{D}_E$ is lattice complemented, then there is a quasi-closed set $F \subset \mathbb{T}$ such that

$$(0) = \mathcal{D}_E \cap \mathcal{D}_F = \mathcal{D}_{E \cup F}$$

and

$$\mathcal{D} = \mathcal{D}_E \vee \mathcal{D}_F.$$

By Proposition 4.1 we must have $\operatorname{Cap}(\mathbb{T} \setminus (E \cup F)) = 0$ and $\operatorname{Cap}(E \cap F) = 0$, so $\mathbb{T} \setminus E = F$ q.e. and the corollary follows. □

Remark. Intervals are examples of quasi-closed sets whose complement is also quasi-closed. On the other hand, if $E \subset \mathbb{T}$ has Lebesgue measure zero, then any function in $\mathcal{D}_{T \setminus E}$ equals 0 a.e., so $\mathcal{D}_{T \setminus E} = (0)$. Thus any quasi-closed set of Lebesgue measure zero and of positive capacity is an example of a set whose complement is not quasi-closed.

Corollary 4.5. *M is irreducible.*

Proof. If $\mathcal{M}$ and $\mathcal{M}^\perp$ are invariant for M, then $\mathcal{M}$ and $\mathcal{M}^\perp$ are invariant for M^{-1}. By Corollary 4.4, $\mathcal{M} = \mathcal{D}_E$ and $\mathcal{M}^\perp = \mathcal{D}_{T \setminus E}$, where E and $\mathbb{T} \setminus E$ are quasi-closed.

Suppose $\mathcal{M} = \mathcal{D}_E \neq (0)$ and choose $f \in \mathcal{M}$, $f \not\equiv 0$. If $g \in \mathcal{D}_{T \setminus E}$, $g \geqq 0$, then

$$0 = \langle g, f \rangle_{\mathcal{D}} = \int_T g(\zeta) f(\zeta) \frac{|d\zeta|}{2\pi} + \int_T \int_T \frac{(g(\zeta) - g(\xi))(f(\zeta) - f(\xi))}{|\zeta - \xi|^2} \frac{|d\zeta|\,|d\xi|}{4\pi^2}$$

$$= -2 \int_E g(\xi) \left(\int_{T \setminus E} \frac{f(\zeta)}{|\zeta - \xi|^2} \frac{|d\zeta|}{2\pi} \right) \frac{|d\xi|}{2\pi}.$$

But since $f \not\equiv 0$ and $f \geqq 0$, the function

$$h(\xi) = \int_{T \setminus E} \frac{f(\zeta)}{|\zeta - \xi|^2} \frac{|d\zeta|}{2\pi} > 0$$

for each ξ. Thus $g(\xi) = 0$ a.e., hence $\mathcal{D}_{T \setminus E} = (0)$. □

5. Other types of Dirichlet spaces

In this section, we generalize our results to other types of Dirichlet spaces that lie between $L^2(\mathbb{T})$ and $\mathcal{D}$. For $0 < \alpha \leqq 1$, define the *harmonic Dirichlet space of order* α, denoted by $\mathcal{D}_\alpha$, to be the space of $L^2(\mathbb{T})$ functions which have finite Douglas type integral

$$S_\alpha(f) = \int_T \int_T \frac{|f(\zeta) - f(\xi)|^2}{|\zeta - \xi|^{1+\alpha}} \frac{|d\zeta|}{2\pi} \frac{|d\xi|}{2\pi}.$$

We norm $\mathcal{D}_\alpha$ by

$$\|f\|_\alpha^2 = \int_T |f(\zeta)|^2 \frac{|d\zeta|}{2\pi} + S_\alpha(f)$$

which by [7], Theorem 3 (c), is comparable to

$$\sum_{n\in Z}(1+|n|)^{\alpha}|\hat{f}(n)|^2.$$

If $f(z)$ represents the harmonic extension of $f(\zeta)$ to $\mathbb{D}$, then S_α is also comparable to

$$\int_D |\nabla f(z)|^2(1-|z|)^{1-\alpha}\,dA(z).$$

For $\lambda\in\mathbb{D}$ and $h\in L^2(\mathbb{T})$, we can define the local Dirichlet integral $D_\lambda(h)$ by

$$D_\lambda(h)=\int_T\left|\frac{h(\zeta)-h(\lambda)}{\zeta-\lambda}\right|^2\frac{|d\zeta|}{2\pi}.$$

(Here $h(\lambda)$ is the harmonic extension of h evaluated at λ.) As in Proposition 3.1, one verifies that if $h=h_++h_-$, $h_+\in H^2$, $h_-\in(H^2)^\perp$, then

$$D_\lambda(h)=D_\lambda(h_+)+D_\lambda(h_-), \tag{5.1}$$

and a computation analogous to the one in Lemma 5.1 and Theorem 5.2 of [18] shows that if $h_r(\zeta)=h(r\zeta)$ then

$$D_\lambda(h_r)\leqq 4D_\lambda(h) \tag{5.2}$$

for each $0\leqq r<1$ and $h\in H^2$. Thus (5.1) implies that (5.2) holds for each $h\in L^2(\mathbb{T})$.

In [1] is was shown that for an analytic function $f(z)$ on $\mathbb{D}$, $w(r)\in C^2[0,1]$, $w(r)$ decreasing, concave, and $w(r)\to 0$ as $r\to 1$, (e.g. $w(r)=(1-r)^{1-\alpha}$, $0<\alpha\leqq 1$)

$$\int_D |f'(z)|^2 w(|z|)\,dA(z)=-\frac{1}{4}\int_D \Delta\tilde{w}(z)(1-|z|^2)D_z(f)\,dA(z),$$

where $\tilde{w}(z)=w(|z|)$. As in Section 3 (with the obvious modifications) we show that if $g,h\in\mathcal{D}_\alpha$ with $|g(\zeta)|\leqq|h(\zeta)|$ a.e. then $g\in[h]$.

To discuss the pointwise behavior of functions in $\mathcal{D}_\alpha$, we develop an appropriate capacity for the $\mathcal{D}_\alpha$ spaces. For $0<\alpha\leqq 1$ define

$$k_\alpha(\zeta)=|1-\zeta|^{\frac{\alpha}{2}-1}$$

and note there is a $\delta>0$ with

$$\delta(1+|n|)^{-\alpha/2}\leqq\hat{k}_\alpha(n)\leqq\delta^{-1}(1+|n|)^{-\alpha/2}.$$

Hence $\mathcal{D}_\alpha=\{k_\alpha * f: f\in L^2(\mathbb{T})\}$ with $\|k_\alpha * f\|_\alpha$ comparable to $\|f\|_{L^2}$. For any set $E\subset\mathbb{T}$ we define the capacity $C_\alpha(E)$ by

$$C_\alpha(E)=\inf\{\|f\|^2_{L^2}: f\in L^2_+(\mathbb{T}),\ k_\alpha * f\geqq 1 \text{ on } E\}$$

and note that if we define α-quasi-everywhere (α-q.e.), α-quasi-closed, and α-quasi-continuous analogously as in Section 2, we have similar types of properties as in Proposition 2.1. A result of Salem and Zygmund [20] shows that the Fourier series of $f \in \mathcal{D}_\alpha$ converges α-q.e. and a similar argument as in Section 2 shows that if we define $f(\zeta)$ to be the α-q.e. sum of its Fourier series, then f is α-quasi-continuous.

For an α-quasi-closed set $E \subset \mathbb{T}$, we let

$$\mathcal{D}_{\alpha,E} = \{f \in \mathcal{D}_\alpha : f|_E = 0 \quad \alpha\text{-q.e.}\}$$

and note that by Proposition 2.1, $\mathcal{D}_{\alpha,E}$ is a closed subspace of $\mathcal{D}_\alpha$. One also proves, as in Lemma 4.2, that if $f \in \mathcal{D}_\alpha$ then $[f] = \mathcal{D}_{\alpha,Z(f)}$ and the following theorem holds:

Theorem 5.1. (a) *If E and F are α-quasi-closed subsets of $\mathbb{T}$, then*

$$\mathcal{D}_{\alpha,E} = \mathcal{D}_{\alpha,F} \Leftrightarrow E = F \quad \alpha\text{-q.e.}$$

(b) *If $\mathcal{M} \in$ Lat(M, M^{-1}), then there is a bounded non-negative function $f \in \mathcal{D}_\alpha$ such that*

$$\mathcal{M} = [f] = \mathcal{D}_{\alpha,Z(f)}.$$

(c) *Let $E \subset \mathbb{T}$ be α-quasi-closed. Then $\mathcal{D}_{\alpha,E}$ is lattice complemented in* Lat(M, M^{-1}) *if and only if $\mathbb{T} \setminus E$ is α-quasi-closed. If $\mathcal{D}_{\alpha,E}$ is lattice complemented, then $\mathcal{D}_{\alpha,\mathbb{T}\setminus E}$ is the unique lattice complementary hyperinvariant subspace.*

(d) *M on $\mathcal{D}_\alpha$ is irreducible.*

Remark. Using a similar type of construction as in the remark following Theorem 4.3, and the fact that for an open arc $I \subset \mathbb{T}$ (sufficiently small), [23], p. 122 and Lemma 2.5,

$$C_\alpha(I) \sim |I|^{1-\alpha}, \quad 0 < \alpha < 1,$$

we can construct α-quasi-closed sets E for which $\mathcal{D}_{\alpha,E}$ cannot always be written as $\mathcal{D}_{\alpha,F}$ for some closed $F \subset \mathbb{T}$. This is a stark contrast to the $\alpha > 1$ case where, as mentioned in the introduction, every hyperinvariant subspace can be written as $\mathcal{D}_{\alpha,F}$ for some closed F.

6. Extremal functions and generators of invariant subspaces

In Section 4 we saw that every hyperinvariant subspace of M is generated by a single function. In the case of the Dirichlet space, we shall now describe such a function as the solution to a certain extremal problem.

As all Hilbert spaces do, the Dirichlet space carries many equivalent norms. While the invariant subspaces do not depend on the particular norm chosen, the orthogonal projections will. For this section, we fix a new norm on the Dirichlet space by setting

$$\|f\|^2 = \frac{1}{2}|f(0)|^2 + D(f), \quad f \in \mathcal{D}. \tag{6.1}$$

This norm is obviously equivalent to the one defined in (1.2) and (1.3). We shall see that with this norm each hyperinvariant subspace $\mathscr{M}$ of M is generated by the orthogonal projection of the constant function 1 onto $\mathscr{M}$. We point out that this result is analogous to the situation in $L^2(\mathbb{T})$, where the hyperinvariant subspaces of the bilateral shift are generated by characteristic functions, and these are just the orthogonal projections of 1 (with respect to the usual norm on $L^2(\mathbb{T})$) onto the hyperinvariant subspaces.

Below, we will make use of the representation of capacitary potential functions as logarithmic potentials with respect to measures of finite energy. We will recall a few known facts and follow [12], Chapter 3. However, our choice of norm (6.1) is different from the one in [12], so we explicitly state properties of the norm that are used in [12] as a lemma.

Lemma 6.1. *Let $f, g \in \mathscr{D}$ be real-valued and set $u = \min\{f, g\}$ and $v = \max\{f, g\}$. Then*

(a) $\| |f| \| \leqq \|f\|$,

(b) $\|u\|^2 + \|v\|^2 \leqq \|f\|^2 + \|g\|^2$.

Proof. (a) Let $A = \{\zeta : f(\zeta) \geqq 0\}$ and $B = \mathbb{T} \setminus A$. A straightforward computation shows that

$$\|f\|^2 - \| |f| \|^2 = \int_A \int_B f(\zeta) f(\xi) \left(2 - \frac{8}{|\zeta - \xi|^2}\right) \frac{|d\zeta|}{2\pi} \frac{|d\xi|}{2\pi} \geqq 0,$$

and thus (a) follows.

(b) This follows from (a) and the parallelogram law, because $u = \frac{1}{2}(f + g - |f - g|)$ and $v = \frac{1}{2}(f + g + |f - g|)$. □

If $\mathscr{M} \in \operatorname{Lat}(M, M^{-1})$, then by Theorem 4.3 there is a quasi-closed set $E \subset \mathbb{T}$ such that $\mathscr{M} = \mathscr{D}_E$. The projection of 1 onto $\mathscr{M}$, $P_{\mathscr{M}} 1$, is the unique function that solves the extremal problem

$$\inf\{ \|1 - g\|^2 : g \in \mathscr{D}_E\}.$$

If $g \in \mathscr{D}_E$, then $\operatorname{Re}(g) \in \mathscr{D}_E$ and $\|1 - \operatorname{Re}(g)\|^2 \leqq \|1 - g\|^2$, hence it is clear that $P_{\mathscr{M}} 1$ is real valued. Thus, as in the proof of Proposition 2.1 (b), one sees (using absolute values and cut-off functions with Lemma 6.1) that the unique solution f_E to

$$\inf\{ \|f\|^2 : 0 \leqq f \leqq 1,\ f = 1 \quad \text{q.e. on } E\} \tag{6.2}$$

satisfies $f_E = 1 - P_{\mathscr{M}} 1$. Comparing (6.2) with Proposition 2.1 (b), we see that f_E is the capacitary potential function for the capacity

$$\operatorname{cap}(E) = \inf\{ \|f\|^2 : 0 \leqq f \leqq 1,\ f = 1 \quad \text{q.e. on } E\}. \tag{6.3}$$

Remark. This capacity is comparable to the one defined in (2.2) (see Proposition 2.1 (b) and notice the small c versus the capital C in our original definition of capacity). The reason for redefining capacity will become clear in our proof of Theorem 6.2. However, it is not clear to us whether this is just a technicality necessitated by our proof or not (see also the remarks following the proof).

Theorem 6.2. *Let $E \subset \mathbb{T}$ be quasi-closed, and let f_E be the capacitary potential function for E, i.e. the unique solution to* (6.3) *using the norm* (6.1). *Then*

$$\mathscr{D}_E = [1 - f_E]. \tag{6.4}$$

Proof. Let $F = Z(1 - f_E)$. Then F is quasi-closed, and it is clear that $E \subset F$ q.e. Thus, by Lemma 4.2 and Proposition 4.1, it suffices to show that $\operatorname{cap}(F \setminus E) = 0$. Since f_E is the capacitary potential for E we have

$$\|f_E\|^2 = \operatorname{cap}(E) \leqq \operatorname{cap}(F) \leqq \|f_E\|^2.$$

This implies $\operatorname{cap}(E) = \operatorname{cap}(F)$, hence we shall be done, once the following lemma has been established. □

Lemma 6.3. *Let $E \subset F$ be subsets of $\mathbb{T}$ with E quasi-closed. If* $\operatorname{cap}(F) = \operatorname{cap}(E)$, *then* $\operatorname{cap}(F \setminus E) = 0$.

Remark. If the hypothesis that E be quasi-closed is dropped in Lemma 6.3, then its conclusion becomes false. To see this, let K be the Cantor set, $E = \mathbb{T} \setminus K$, and $F = \mathbb{T}$. Indeed, as remarked after the proof of Corollary 4.4, up to q.e. the only quasi-closed superset of E is $\mathbb{T}$. Thus $\mathbb{T} = F$ is the quasi-closure of E and hence $\operatorname{cap}(E) = \operatorname{cap}(F)$, while $\operatorname{cap}(F \setminus E) > 0$.

Before we prove Lemma 6.3 we need to recall a few known facts about potentials (see [12], Chapter 3). We say that a non-negative finite Borel measure μ on $\mathbb{T}$ has *finite energy integral* if

$$\int_{\mathbb{T}} |g| \, d\mu \leqq C \|g\|$$

for all $g \in \mathscr{D} \cap C(\mathbb{T})$ and some positive constant C independent of g. We denote the set of non-negative finite Borel measures on $\mathbb{T}$ with finite energy by $E_+(\mathbb{T})$. Thus $\mu \in E_+(\mathbb{T})$ if and only if there is a function $u_\mu \in \mathscr{D}$ such that

$$(g, u_\mu) = \int_{\mathbb{T}} g \, d\mu$$

for each $g \in \mathscr{D} \cap C(\mathbb{T})$. (Here $(\cdot, \cdot)$ is the inner product induced by the norm (6.1).) One checks that u_μ is of the form

$$u_\mu(\zeta) = 2 \int \log \frac{e}{|\zeta - \xi|} \, d\mu(\xi). \tag{6.5}$$

Note that $u_\mu(\zeta)$ is defined for each $\zeta \in \mathbb{T}$ with values in $[0, \infty]$ and that u_μ naturally extends to be harmonic on $\mathbb{C} \setminus \operatorname{supp}(\mu)$. This will be used below and it is the reason for our choice of norm (6.1) and the definition of cap in (6.3). Also note that a routine estimate yields

(6.6) $$u_\mu(\zeta) = \lim_{r \to 1^-} u_\mu(r\zeta)$$

for all $\zeta \in \mathbb{T}$. If μ has a finite energy integral, then μ puts no mass on any set of capacity zero and

(6.7) $$(g, u_\mu) = \int_{\mathbb{T}} g \, d\mu$$

for each $g \in \mathcal{D}$. Furthermore, if $A \subset \mathbb{T}$ is compact, then there is a measure μ with finite energy integral and $\operatorname{supp}(\mu) \subset A$ such that $f_A = u_\mu$ (see [12], Section 3.3). By use of Fatou's lemma, the facts that

$$u_\mu\left(\frac{1}{\bar{z}}\right) = u_\mu(z) + 2\log|z|\,\mu(\mathbb{T}),$$

and $u_\mu(z) \leqq 1$ for all $z \in \mathbb{C}$, and (6.6), one shows that

$$\lim_{z \to \zeta} f_A(z) = \lim_{z \to \zeta} u_\mu(z) = 1$$

for quasi-every $\zeta \in A$ (the unrestricted limit). Thus u_μ is continuous on $\mathbb{C}$, except possibly on a set of capacity zero.

Lemma 6.4. *Let $E \subset \mathbb{T}$ be quasi-closed. Then the set*

(6.8) $$\mathcal{L}_E = \{u_\mu : \mu \in E_+(\mathbb{T}),\ \mu(\mathbb{T} \setminus E) = 0,\ u_\mu(\zeta) \leqq 1 \ q.e.\}$$

is closed in $\mathcal{D}$. Furthermore $f_E \in \mathcal{L}_E$. Thus all capacitary extremal functions are logarithmic potentials. Also note that by (6.6) *these extremal functions have radial limits at each point $\zeta \in \mathbb{T}$.*

Proof. Let $\{u_{\mu_n} : n \in \mathbb{N}\}$ be a sequence in $\mathcal{L}_E$ with $u_{\mu_n} \to f$ in $\mathcal{D}$. We must first show that $f = u_\mu$ for some measure $\mu \in E_+(\mathbb{T})$ with $\mu(\mathbb{T} \setminus E) = 0$. Each μ_n is a non-negative measure and

$$\mu_n(\mathbb{T}) = (1, u_{\mu_n}) \to (1, f),$$

hence a subsequence of the measures, also denoted by μ_n will converge to a non-negative measure μ in the weak-star topology of $M(\mathbb{T})$. If $g \in \mathcal{D} \cap C(\mathbb{T})$, then

$$\int |g| \, d\mu = \lim_n \int |g| \, d\mu_n \leqq \lim_n \|g\| \, \|u_{\mu_n}\| = \|g\| \, \|f\|,$$

so $\mu \in E_+(\mathbb{T})$ and $u_\mu \in \mathcal{D}$. Furthermore, for fixed $z \in \mathbb{D}$ the function

$$2\log \frac{e}{|z - \zeta|}$$

is continuous on $\mathbb{T}$. Thus, by (6.5), the harmonic extensions of u_{μ_n} converge pointwise on $\mathbb{D}$ to $u_\mu(z)$. This implies that $u_\mu = f$ and $0 \leqq u_\mu \leqq 1$.

Next we show that $\mu(\mathbb{T}\setminus E) = 0$. It is clear that $\mathscr{L}_E \perp \mathscr{D}_E$, thus $\int f\,d\mu = 0$ for all $f \in \mathscr{D}_E$. Now let $f \in \mathscr{D}_E$ be such that $Z(f) = E$ q.e. (Theorem 4.3). Then $\int \zeta^n f\,d\mu = 0$ for all $n \in \mathbb{Z}$, so $f\,d\mu \equiv 0$, thus $\mu(\mathbb{T}\setminus Z(f)) = 0$. This implies $\mu(\mathbb{T}\setminus E) = 0$.

Finally, to see that $f_E \in \mathscr{L}_E$, we choose an increasing sequence of compact subsets E_n of E with $\operatorname{cap}(E_n) \to \operatorname{cap}(E)$. Then $f_{E_n} = u_{\mu_n} \in \mathscr{L}_E$ and using (6.7), one easily checks that $u_{\mu_n} \to f_E$, i.e. $f_E \in \mathscr{L}_E$. □

Next we recall that if $A \subset B$, then $f_A \leqq f_B$ q.e. on $\mathbb{T}$ (Lemma 6.1 allows us to apply the argument of [9], p. 157). We shall need a stronger statement, namely, that $f_A(\zeta) \leqq f_B(\zeta)$ for each $\zeta \in \mathbb{T}$. This follows from the weaker statement, because the harmonic extensions to $\mathbb{D}$ satisfy $f_A(z) \leqq f_B(z)$ for each $z \in \mathbb{D}$, hence by Lemma 6.4 we obtain $f_A(\zeta) \leqq f_B(\zeta)$ for each $\zeta \in \mathbb{T}$.

Proof of Lemma 6.3. We will show that the hypothesis implies

$$\operatorname{cap}(F\setminus L) = \operatorname{cap}(E\setminus L) \tag{6.9}$$

for any closed subset L of E. This will finish the proof, because since E is quasi-closed, we can find a sequence $\{L_n : n \in \mathbb{N}\}$ of closed subsets of E with $\operatorname{cap}(E\setminus L_n) \to 0$ and thus

$$\operatorname{cap}(F\setminus E) \leqq \operatorname{cap}(F\setminus L_n) = \operatorname{cap}(E\setminus L_n) \to 0\,.$$

In order to show (6.9) we fix a closed subset L of E. It suffices to show that

$$f_{E\setminus L}(\zeta) = 1 \quad \text{q.e.} \quad \text{on } F\setminus E \tag{6.10}$$

for this will imply that $f_{E\setminus L}$ is a test function for the capacitary problem (6.3) for the set $F\setminus L$, thus

$$\operatorname{cap}(E\setminus L) \leqq \operatorname{cap}(F\setminus L) \leqq \|f_{E\setminus L}\|^2 = \operatorname{cap}(E\setminus L)\,.$$

Let $E_n \subset E$ be an increasing sequence of closed sets such that $\operatorname{cap}(E_n) \to \operatorname{cap}(E)$ as $n \to \infty$. The hypothesis implies that $f_F = f_E$, so by the remarks before the proof, we have that $f_{E_n}(\zeta) \to f_E(\zeta) = 1$ q.e. on $F\setminus E$. Let $\zeta_0 \in F\setminus E$ be such that $f_{E_n}(\zeta_0) \to 1$ and set

$$\delta = \frac{1}{2}\operatorname{dist}(L, \zeta_0) > 0\,.$$

Furthermore, let

$$F_n = E_n \cap \{\zeta \in \mathbb{T} : |\zeta - \zeta_0| \leqq \delta\}\,,$$

then $F_n \subset E_n\setminus L \subset E\setminus L$. Hence

$$f_{F_n}(\zeta_0) \leqq f_{E\setminus L}(\zeta_0)$$

and (6.10) will follow (and the proof will be finished), once we show that $f_{F_n}(\zeta_0) \to 1$.

The following part of the proof was motivated by the connections between logarithmic capacity, harmonic measure, and escape probabilities of Brownian motion (though no probability is used in the proof). More precisely, when in our statement we see the quantity $f_A(\zeta_0)$ for a set $A \subset \mathbb{T}$, we think of the probability that Brownian motion starting at ζ_0 leaves a large disc (say of radius 4) minus A for the first time through A. Thus, our hypothesis that $f_{E_n}(\zeta_0) \to 1$ can roughly be interpreted as saying that the Brownian traveler with high probability hits E_n before he hits $\{z \in \mathbb{C} : |z| = 4\}$. Our goal is to show that he must hit E_n near ζ_0, i.e. in F_n. We shall accomplish this below with a comparison argument for harmonic functions by putting up little barriers at the ends of F_n.

Let $G \subset \mathbb{C}$ be the open set that is symmetric with respect to $\mathbb{T}$ and satisfies

$$\bar{\mathbb{D}} \cap G = \bar{\mathbb{D}} \cap \{z \in \mathbb{C} : |z - \zeta_0| < \delta\}$$

and fix κ, $0 < \kappa < \delta$.

For each $n \in \mathbb{N}$, let v_n be the harmonic function in $G \backslash F_n$ which has boundary values zero q.e. on

$$\partial G \cap \{z \in \mathbb{C} : |z| > 1 + \kappa\}$$

and boundary values one q.e. on

$$F_n \cup (\partial G \cap \{z \in \mathbb{C} : |z| \leqq 1 + \kappa\}).$$

(Here we use the general solution of the Dirichlet problem due to Wiener, see [13], p. 243.) For $z \in G \backslash F_n$ let

$$w_n(z) = v_n(z) + v_n(1/\bar{z}) - 1.$$

Then w_n and f_{F_n} are harmonic in $G \backslash F_n$ and by considering their values on $\partial(G \backslash F_n)$, we see that

$$w_n(\zeta_0) \leqq f_{F_n}(\zeta_0) + \omega(\delta, \kappa) \tag{6.11}$$

for each $n \in \mathbb{N}$, where $\omega(\delta, \kappa)$ denotes the harmonic measure for G at ζ_0 of the set

$$\left\{z \in \partial G : \frac{1}{1+\kappa} \leqq |z| \leqq 1 + \kappa\right\}.$$

For small values of δ, G looks like a disc with radius δ, thus $\omega(\delta, \kappa) \simeq 4\kappa/\delta$ as $\kappa \to 0$.

Furthermore, since f_{E_n} is the logarithmic potential of the form (6.5), it takes negative values on the circle $\{z \in \mathbb{C} : |z| = 4\}$. Of course, $0 \leqq f_{E_n} \leqq 1$ on $\mathbb{T}$, hence in the annulus $\{z \in \mathbb{C} : 1 < |z| < 4\}$, we have $1 - f_{E_n}(z) \geqq \log|z|/\log 4$. This implies

$$1 - f_{E_n}(z) \geqq \frac{\log(1+\kappa)}{\log 4} \quad \text{for } z \in \partial G \cap \{z \in \mathbb{C} : |z| > 1 + \kappa\}.$$

On the set

$$F_n \cup (\partial G \cap \{z \in \mathbb{C} : |z| \leqq 1 + \kappa\})$$

we have $1 - f_{E_n}(z) \geqq 1 - v_n(z)$, because the left hand side is non-negative and the right hand side is zero. It follows that

$$1 - f_{E_n}(z) \geqq (1 - v_n(z)) \frac{\log(1+\kappa)}{\log 4}$$

for all $z \in \partial(G \setminus F_n)$ and so

$$1 - f_{E_n}(\zeta_0) \geqq (1 - v_n(\zeta_0)) \frac{\log(1+\kappa)}{\log 4} \tag{6.12}$$

for each $n \in \mathbb{N}$. Combining (6.11) with (6.12) and using $w_n(\zeta_0) = 2v_n(\zeta_0) - 1$ we obtain

$$1 - f_{F_n}(\zeta_0) \leqq 2 \frac{\log 4}{\log(1+\kappa)} (1 - f_{E_n}(\zeta_0)) + \omega(\delta, \kappa)$$

for all n. Now let $n \to \infty$ and then $\kappa \to 0$, thus $\lim\limits_{n \to \infty} f_{F_n}(\zeta_0) \geqq 1$ and so, in fact $f_{F_n}(\zeta_0) \to 1$ as $n \to \infty$. □

The question which we have not been able to answer is for which capacities does Lemma 6.3 remain true. As pointed out earlier, Lemma 6.3 and Theorem 6.2 imply that $P_{\mathscr{D}_E}1$ generates $\mathscr{D}_E$ when equipped with the norm (6.1). The following example shows that this fact becomes false for some other equivalent norms on $\mathscr{D}$.

Example. Let $E \subset \mathbb{T}$ be quasi-closed with $(0) \neq \mathscr{D}_E \neq \mathscr{D}$ and take $g \in \mathscr{D}$, $0 \leqq g \leqq 1$, $1 - g \in \mathscr{D}_E$, but $[1 - g] \neq \mathscr{D}_E$ (e.g. $g = 1$). Then $g \notin \mathscr{D}_E$, and

$$\mathscr{M} = \{f + \alpha g : f \in \mathscr{D}_E, \alpha \in \mathbb{C}\}$$

is closed, hence every $h \in \mathscr{D}$ has a unique representation $h = f + \alpha g + h_1$, where $f + \alpha g \in \mathscr{M}$ and $h_1 \perp \mathscr{M}$ with respect to the usual inner product on $\mathscr{D}$ defined in (1.2). We define a new norm by

$$\|h\|_*^2 = \|f\|_{\mathscr{D}}^2 + |\alpha|^2 \|g\|_{\mathscr{D}}^2 + \|h_1\|_{\mathscr{D}}^2.$$

One checks that $\|h\|^2 \leqq 2\|h\|_*^2$, thus by the open mapping theorem the two norms are equivalent. In this new norm $g \perp \mathscr{D}_E$ and, of course, $1 - g \in \mathscr{D}_E$, hence $P_{\mathscr{D}_E}1 = 1 - g$. However, by our choice of g we have $[P_{\mathscr{D}_E}1] \neq \mathscr{D}_E$.

The capacitary potential function f_E for a quasi-closed set E is a logarithmic potential function u_μ. We shall now seee that the *quasi-support* of μ (defined below) is equal to E. For $\mu \in E_+(\mathbb{T})$ (the non-negative finite Borel measures on $\mathbb{T}$ with finite energy integral) we let

$$\mathscr{K}_\mu = \{A \subset \mathbb{T} : A \text{ is quasi-closed}, \mu(\mathbb{T} \setminus A) = 0\}.$$

We note that since μ puts no mass on any set of capacity zero, it follows that if two sets satisfy $A = B$ q.e., then either A and B are both in $\mathscr{K}_\mu$ or neither one is. Also, it is easy to verify that $\mathscr{H}_\mu$ is closed under countable intersections. Thus by a theorem of Fuglede [11], Theorem 2.7, $\mathscr{H}_\mu$ has a *quasi-minimal* element $A_* \in \mathscr{H}_\mu$. That is to say $\operatorname{cap}(A_* \setminus A) = 0$ for all $A \in \mathscr{H}_\mu$. A_* is called the *quasi-support* of μ. The equivalence class of A_* (with respect to q.e.) is unique and every set in this equivalence class is also a quasi-support of μ. Thus, when we speak of the quasi-support of μ we should keep in mind that we are thinking about an equivalence class of sets rather than about a set itself. We also note that considerations about the existence of quasi-supports by Choquet and Getoor [6] were the motivation for Fuglede's theorem, which we used here to prove the existence of the quasi-support. Fuglede's theorem is true for capacities other than the ones considered here, but we point out that for our special case, Fuglede's theorem can easily be deduced from the results of Section 4 and the separability of $\mathscr{D}$.

Theorem 6.5. *Let $E \subset \mathbb{T}$ be quasi-closed and let $f_E = u_{\mu_E}$ be the capacity potential for E (in the norm* (6.1)*). Then the quasi-support of μ equals E. Also, μ_E is the unique measure in $E_+(\mathbb{T})$ with $\mu(\mathbb{T} \setminus E) = 0$, $u_{\mu_E}(\zeta) \leqq 1$ q.e., and*

$$\text{(6.13)} \quad \operatorname{cap}(E) = \mu_E(\mathbb{T}) = \sup\{\mu(\mathbb{T}) : u_\mu(\zeta) \leqq 1 \text{ q.e.}, \ \mu(\mathbb{T} \setminus E) = 0, \ \mu \in E_+(\mathbb{T})\}.$$

Proof. It is well known (and at this point elementary to show) that μ_E solves the extremal problem (6.13) and that $\mu_E(\mathbb{T}) = \operatorname{cap}(E)$ (see e.g. [17], Theorems 13 and 14).

To finish the proof we must show that the quasi-support of μ_E equals E q.e. Let A denote the quasi-support of μ_E. Since $\mu_E(\mathbb{T} \setminus E) = 0$, we have $A \subset E$ q.e. The measure μ_E is a test measure for the extremal problem (6.13) for the set A, hence

$$\mu_E(\mathbb{T}) \leqq \operatorname{cap}(A) \leqq \operatorname{cap}(E) = \mu_E(\mathbb{T}) \ .$$

This implies $\operatorname{cap}(A) = \operatorname{cap}(E)$. Thus, by Lemma 6.3, $\operatorname{cap}(A \setminus E) = 0$, i.e. $A = E$ q.e.

For the uniqueness part, suppose that for two measures μ and ν we have $u_\mu = u_\nu$ q.e. Then the harmonic extensions agree everywhere in $\mathbb{D}$. But for all $z \in \mathbb{D}$,

$$u_\mu(z) = \mu(\mathbb{T}) + \frac{1}{2} \sum_{n \in \mathbb{N}} \frac{\hat{\mu}(n)}{n} z^n + \frac{1}{2} \sum_{n \in \mathbb{N}} \frac{\hat{\mu}(-n)}{n} \bar{z}^n$$

and the result follows. □

Together with Theorem 6.2, this theorem implies Theorem 1.2.

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Department of Mathematics, University of Tennessee, Knoxville, Tennessee 37996-1300, USA

Department of Mathematics, University of Richmond, Richmond, Virginia 23173, USA

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