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# Indestructible Blaschke products

William T. Ross

*In memory of Alec L. Matheson*

## 1. Introduction

Consider the following set of linear fractional maps

$$\tau_a(z) := \frac{z-a}{1-\overline{a}z}, \quad |a| < 1.$$

Each $\tau_a$ is an automorphism of the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and $\tau_a(\partial\mathbb{D}) = \partial\mathbb{D}$. For an inner function $\phi$, the *Frostman shifts*

$$\phi_a(z) := \tau_a \circ \phi(z) = \frac{\phi(z)-a}{1-\overline{a}\phi(z)}$$

are certainly inner functions. A celebrated theorem of Frostman **[9]** says that $\phi_a$ is actually a Blaschke product for every $|a| < 1$ with the possible exception of a set of logarithmic capacity zero. In this survey paper, we explore the class of Blaschke products for which this exceptional set is empty. These Blaschke products are called *indestructible* and have some intriguing properties.

## 2. Frostman's theorems

If $(a_n)_{n\geq 1}$ is a sequence of points in $\mathbb{D}$, $p \in \mathbb{N} \cup \{0\}$, and $\gamma \in \mathbb{R}$, a necessary and sufficient condition that the infinite product

$$B(z) = e^{i\gamma} z^p \prod_{n=1}^{\infty} \frac{\overline{a_n}}{|a_n|} \frac{a_n - z}{1-\overline{a_n}z}$$

defines an analytic function on $\mathbb{D}$ is that the series

$$\sum_{n=1}^{\infty}(1-|a_n|)$$

converges. Such sequences $(a_n)_{n\geq 1}$ are called *Blaschke sequences* and the product $B$ is called a *Blaschke product.* The function $B$ is analytic on $\mathbb{D}$, has zeros precisely at the origin and the $a_n$'s (repeated according to their multiplicity), and satisfies

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$|B(z)| < 1$ for all $z \in \mathbb{D}$. Furthermore, by a well-known theorem of Fatou [3, Ch. 2] [8, Ch. 2], the radial limit function

$$B^*(\zeta) := \lim_{r \to 1^-} B(r\zeta)$$

exists and satisfies $|B^*(\zeta)| = 1$ for almost every $\zeta \in \partial\mathbb{D}$, with respect to (normalized) Lebesgue measure $m$ on $\partial\mathbb{D}$.

REMARK 2.1. (1) In what follows, we use the notation $B^*(\zeta)$ to denote the radial limit value of $B$ at $\zeta$ whenever it exists (whether or not it is unimodular).

(2) This paper will cover a selection of results about Blaschke products. All the basic properties of Blaschke products, and more, are covered in [3, 5, 6, 8, 12, 18, 26].

For a particular point $\zeta \in \partial\mathbb{D}$, there is the following refinement of Fatou's theorem [10] (see also [3, p. 33]).

THEOREM 2.2 (Frostman). *A necessary and sufficient condition that a Blaschke product $B$, with zeros $(a_n)_{n \geq 1}$, and all its subproducts have radial limits of modulus one at $\zeta \in \partial\mathbb{D}$ is that*

$$\sum_{n \geq 1} \frac{1 - |a_n|}{|\zeta - a_n|} < \infty. \tag{2.3}$$

The Frostman theorems (like Theorem 2.2 above and Theorem 2.13, Theorem 2.14, and Theorem 2.18 below) are not always standard material for many complex analysts and so, for the sake of completeness and to give the reader a sense of how all these ideas are related, we will outline parts of the proofs of his theorems.

In our discussion below, we will only use one direction of Theorem 2.2 so we prove this one direction and point the reader to [3, p. 34] for the proof of the other. Suppose, for fixed $\zeta \in \partial\mathbb{D}$, the condition in eq.(2.3) holds. We wish to show that

$$B^*(\zeta) := \lim_{r \to 1^-} B(r\zeta)$$

exists and $|B^*(\zeta)| = 1$. The proof of the same result for any sub-product will follow in a similar way. Without loss of generality, we can assume $\zeta = 1$. First check the following inequalities

$$|1 - ar| \geqslant 1 - r, \quad |1 - ar| \geqslant \frac{1}{2}|1 - a|, \quad 0 < r < 1, \quad |a| < 1. \tag{2.4}$$

Second, use induction to verify that for a sequence $(b_n)_{n \geq 1} \subset (0, 1)$, we have

$$\prod_{n=1}^{N} (1 - b_n) \geqslant 1 - \sum_{n=1}^{N} b_n, \quad \forall N \in \mathbb{N}. \tag{2.5}$$

Third, one can verify, via a routine computation, the identity

$$\left|\frac{r - a_n}{1 - \overline{a_n} r}\right|^2 = 1 - \frac{(1 - r^2)(1 - |a_n|^2)}{|1 - \overline{a_n} r|^2}$$

and so

$$
\begin{aligned}
|B(r)|^2 &= \prod_{n=1}^{\infty} \frac{|r-a_n|^2}{|1-\overline{a_n}r|^2} \\
&= \prod_{n=1}^{\infty} \left\{1 - \frac{(1-r^2)(1-|a_n|^2)}{|1-\overline{a_n}r|^2}\right\} \\
&\geqslant 1 - \sum_{n=1}^{\infty} \frac{(1-r^2)(1-|a_n|^2)}{|1-\overline{a_n}r|^2}, \qquad \text{(by eq.(2.5))} \\
&= 1 - \sum_{n=1}^{\infty} \frac{(1-r^2)}{|1-\overline{a_n}r|}\frac{(1-|a_n|^2)}{|1-\overline{a_n}r|}.
\end{aligned}
$$

Now use the inequalities in eq.(2.4) and the dominated convergence theorem to get

$$\lim_{r\to 1^-} |B(r)| = 1. \tag{2.6}$$

To finish, we need to show that

$$\lim_{r\to 1^-} \arg B(r)$$

exists. Use the identity

$$|a_n|\frac{\overline{a_n}}{|a_n|}\frac{a_n-r}{1-\overline{a_n}r} = 1 - \frac{1-|a_n|^2}{1-\overline{a_n}r}$$

to get

$$\arg B(r) = \sum_{n=1}^{\infty} \arg\left\{1 - \frac{1-|a_n|^2}{1-\overline{a_n}r}\right\}. \tag{2.7}$$

If $a_n = \alpha_n + i\beta_n$, where $\alpha_n, \beta_n \in \mathbb{R}$, some trigonometry will show that

$$\arg\left(1 - \frac{1-|a_n|^2}{1-\overline{a_n}r}\right) = \sin^{-1}\left(\frac{\beta_n r(1-|a_n|^2)}{|a_n||a_n-r||1-\overline{a_n}r|}\right).$$

From here, one can argue that the right-hand side of eq.(2.7) converges absolutely and uniformly in $r$ and so

$$\lim_{r\to 1^-} \arg B(r)$$

exists. Combine this with eq.(2.6) to complete one direction of the proof. See [3, p. 34] for the other direction.

REMARK 2.8. (1) If the zeros $(a_n)_{n\geqslant 1}$ do not accumulate at $\mathbb{T}$, the condition in eq.(2.3) is easily satisfied and in fact, $B$ extends analytically to an open neighborhood of $\mathbb{D}^-$ [18, p. 68].

(2) The zeros can accumulate at $\mathbb{T}$ and eq.(2.3) can still hold. For example, let $t_n \downarrow 0$ satisfy $\sum_n t_n < \infty$ and let

$$a_n = \frac{1}{2} + \frac{1}{2}e^{it_n}.$$

Notice how these zeros lie on the circle $|z - \frac{1}{2}| = \frac{1}{2}$, which is internally tangent to $\partial\mathbb{D}$ at $\zeta = 1$, and accumulate at $\zeta = 1$. A computation shows that

$$1 - |a_n|^2 = \frac{1}{2}(1-\cos t_n) \asymp t_n^2, \quad |1-a_n| = \frac{\sqrt{2}}{2}\sqrt{1-\cos t_n} \asymp t_n$$

and so

$$\sum_{n=1}^{\infty} \frac{1-|a_n|^2}{|1-a_n|} \leq \sum_{n=1}^{\infty} t_n < \infty.$$

Notice how this infinite Blaschke product $B$ with zeros $(a_n)_{n\geq 1}$ satisfies $|B^*(\zeta)| = 1$ for *every* $\zeta \in \partial\mathbb{D}$.

(3) With more work, one can even arrange the zeros of $B$ to satisfy the much stronger condition

$$\sup_{\zeta \in \partial\mathbb{D}} \sum_{n=1}^{\infty} \frac{1-|a_n|}{|\zeta - a_n|} < \infty.$$

We will get to this in the last section.

(4) So far, we have examined when $B^*(\zeta)$ exists and has modulus one. Frostman [9] showed that the Blaschke product with zeros $a_n = 1 - n^{-2}$ satisfies $B^*(1) = 0$.

(5) If $B^*(\zeta)$ exists for *every* $\zeta \in \partial\mathbb{D}$, then results in [1, 5] say that if $E$ is the set of accumulation points of $(a_n)_{n\geq 1}$, then (a) $E$ is a closed nowhere dense subset of $\partial\mathbb{D}$, (b) the function $\zeta \to B^*(\zeta)$ is discontinuous at $\zeta_0$ if and only if $\zeta_0 \in E$.

By Fatou's theorem, the radial limit function

$$\phi^*(\zeta) := \lim_{r \to 1^-} \phi(r\zeta),$$

for a bounded analytic function $\phi$ on $\mathbb{D}$, exists for m-almost every $\zeta \in \partial\mathbb{D}$ [8, p. 6]. If $|\phi^*(\zeta)| = 1$ for almost every $\zeta$, then $\phi$ is called an *inner function* and can be factored as

$$\phi(z) = \left\{ e^{i\gamma} z^p \prod_{n=1}^{\infty} \frac{\overline{a_n}}{|a_n|} \frac{a_n - z}{1 - \overline{a_n} z} \right\} \exp\left( -\int_{\partial\mathbb{D}} \frac{\zeta + z}{\zeta - z} \, d\mu(\zeta) \right). \tag{2.9}$$

Here $\mu$ is a positive finite measure on $\partial\mathbb{D}$ with $\mu \perp m$. The first factor in eq.(2.9) is the Blaschke factor and is an inner function. The second term in eq.(2.9) is called the *singular inner factor*. By a theorem of Fatou [8, p. 4],

$$\lim_{r \to 1^-} \int_{\partial\mathbb{D}} \frac{1 - r^2}{|\zeta - re^{i\theta}|^2} \, d\mu(\zeta) = (D\mu)(e^{i\theta}) \tag{2.10}$$

whenever $D\mu(e^{i\theta})$, the symmetric derivative *of* $\mu$ at $e^{i\theta}$, exists (and we include the possibility that $(D\mu)(e^{i\theta}) = \infty$). By the Lebesgue differentiation theorem, $D\mu$ exists at m-almost every $e^{i\theta}$. Moreover, since $\mu \perp m$, we know that

$$D\mu = 0 \text{ m-a.e.} \quad \text{and} \quad D\mu = \infty \ \mu\text{-a.e.} \tag{2.11}$$

See [30, p. 156 - 158] for the proofs of eq.(2.11). The first identity in eq.(2.11), along with the identity

$$\left| \exp\left( -\int_{\partial\mathbb{D}} \frac{\zeta + z}{\zeta - z} \, d\mu(\zeta) \right) \right| = \exp\left( -\int_{\partial\mathbb{D}} \frac{1 - |z|^2}{|\zeta - z|^2} \, d\mu(\zeta) \right), \tag{2.12}$$

shows that the radial limits of this second factor are unimodular m-almost everywhere and hence this factor is an inner function. Furthermore, if $\mu \not\equiv 0$ (i.e., the inner function $\phi$ has a non-trivial singular inner factor), we can use the second identity in eq.(2.11) along with eq.(2.12) once again to obtain the following theorem of Frostman [9].

THEOREM 2.13 (Frostman). *If an inner function ¢ has a non-trivial singular inner factor, there is a point ( ∈ 81!)) such that ¢*(() = 0.*

From Remark 2.8 (4), the condition ¢*(() = 0 for some ( ∈ 81!)) does not completely determine the presence of a non-trivial inner factor. Another result of Frostman (see [9, p. 107] or [3, p. 32]) completes the picture.

THEOREM 2.14 (Frostman). *An inner function ¢ is a Blaschke product if and only if*

(2.15) lim_{r-+1-} 1_0^{211"} log*l<P(rei9)ld0* = 0.

Again, for the sake of giving the reader a feel for how all these ideas are related, and since this result will be used later, we outline a proof. We follow [3, p. 32]. Indeed, suppose ¢ = *B*, a Blaschke product. Let *Bn* be the product of the first *n* terms of *B* and, given ε > 0, choose a large *n* so that

$$\left|\frac{B}{B_n}(0)\right| > 1 - \varepsilon.$$

Thus,

0 ;;::: r rr-lall> log I B(r() ldm(()

= lim_{r-+1-} [_{la11>} log I BB_n (r() I *dm(()* − lim_{r-+1-} [_{la11>} log I Bn(r()ldm(()

= lim_{r-+1-} *f*_{Jall>} log I BB_n (r() I *dm(()*

;;::: log(l − c-).

The last inequality comes from the sub-mean value property applied to the subharmonic function log lB/Bnl [12, p. 36]. It follows that eq.(2.15) holds for ¢ = *B.*

Now suppose that ¢ is inner and eq.(2.15) holds. Factor ¢ = *Be9*, where

*g(z)* := − *f*_{la11>} ;+z / .,−z *dμ(()*

and notice, using the fact that *!Rg* is non-positive and harmonic along with the mean value property for harmonic functions, that if *!Rg* has a zero in II)), then *!Rg* = 0 on II)) and consequently μ = 0. Use the mean value property again to see that

[_{la11>} log 1¢(r()ldm(() = [_{la11>} log IB(r() ldm(() + *!Rg(O).*

As *r* ---+ 1-, the integral on the right-hand side approaches zero since *B* is a Blaschke product (see above) and the integral on the left-hand side approaches zero by assumption. This means that *!Rg(O)* = 0 and so, by what we said before, *μ* = 0 and so ¢ = B is a Blaschke product. This completes the proof.

The linear fractional maps

*Ta(z)* := (z − *a*) / (1 − *az*), lal < 1, are automorphisms of $\mathbb{D}$ (the complete set of automorphisms of $\mathbb{D}$) is $\{\zeta\tau_a : \zeta \in \partial\mathbb{D}, a \in \mathbb{D}\}$) and also satisfy $\tau_a(\partial\mathbb{D}) = \partial\mathbb{D}$. So certainly the Frostman shifts

$$\phi_a := \tau_a \circ \phi, \quad |a| < 1,$$

are all inner functions. However, some of them might not be Blaschke products - even if $\phi$ is a Blaschke product. For example (see eq.(4.6)) the function

$$B(z) := \tau_{1/2}\left(\exp\left(- \quad \right)\right)$$

turns out to be a Blaschke product. However,

$$B_{-1/2}(z) := \tau_{-1/2} \circ B(z) = \exp\left(- \quad \right)$$

is a singular inner function. Define the *exceptional set* $\mathcal{E}(\phi)$ for $\phi$ to be

(2.16) $$\mathcal{E}(\phi) := \{a \in \mathbb{D} : \tau_a \circ \phi \text{ is not a Blaschke product}\}.$$

This exceptional set $\mathcal{E}(\phi)$ has some very special properties. The first was observed in [16] (see also [22]).

PROPOSITION 2.17. *The exceptional set $\mathcal{E}(\phi)$ of an inner function $\phi$ is of type $F_\sigma$.*

PROOF. We follow the proof from [22, p. 53]. For each $a \in \mathbb{D}$, we the function

$$r \to \int_{\partial\mathbb{D}} \log|\phi_a(r\zeta)|\,dm(\zeta)$$

is increasing on $[0, 1)$ [8, p. 9] and so from Theorem 2.14, we see that $a \in \mathcal{E}(\phi)$ if and only if

$$\lim_{r\to 1^-} \int_{\partial\mathbb{D}} \log|\phi_a(r\zeta)|\,dm(\zeta) < 0.$$

For $r \in [0, 1)$ and $a \in \mathbb{D}$, let

$$I(r,a) := \int_{\partial\mathbb{D}} \log|\phi_a(r\zeta)|\,dm(\zeta)$$

and observe how, for fixed $r$, $I(r,a)$ is a continuous function of $a$.

For fixed $r \in (0, 1)$ and $k \in \mathbb{N}$, let

$$F(r,k) := \left\{a \in \mathbb{D} : I(r,a) \le - \right\}.$$

Notice that $F(r,k)$ is relatively closed in $\mathbb{D}$. Finally, we observe that

$$\mathcal{E}(\phi) = \bigcup F\left(1 - \ ,k\right)$$

which proves the result. □

The exceptional set $\mathcal{E}(\phi)$ satisfies one more special property. In order to explain this, we need the definition of logarithmic capacity. We follow [12, p. 78]. For a compact set $K \subset \mathbb{D}$ and positive finite measure supported on $K$, consider the Green potential

$$G\sigma(z) := \int \log\left|1 - \bar{\zeta}z \right|\,d\sigma(\zeta)$$

and note that

$$0 \leq G_\sigma(z) \leq \infty, \quad z \in \mathbb{D}.$$

Since $K$ is a compact subset of $\mathbb{D}$, $G_\sigma$ is continuous near $\partial\mathbb{D}$ and in fact

$$G_\sigma(\zeta) = 0, \quad \zeta \in \partial\mathbb{D}.$$

We will say $K$ has *positive logarithmic capacity* if there is a positive (non-zero) measure $\sigma$ supported on $K$ such that $G_\sigma$ is bounded on $\mathbb{D}$. Otherwise, we say that $K$ has *zero logarithmic capacity*. We say a Borel set $E \subset \mathbb{D}$ has positive logarithmic capacity if it contains a compact subset of positive logarithmic capacity. For example, if $A$ denotes two-dimensional Lebesgue area measure in the plane and $A(E) > 0$, a computation shows that $G_A$ is bounded on $\mathbb{D}$. Thus any set of positive area has positive logarithmic capacity. However sets of zero logarithmic capacity are much 'thinner'. For example, Borel subsets of logarithmic capacity zero must have zero area and compact subsets of zero logarithmic capacity must be totally disconnected. There are various other ways to define logarithmic capacity, depending on the particular application. However, they all have the same sets of logarithmic capacity zero. Two excellent sources which sort all this out are **[11, 29]**.

This next result of Frostman **[9]** says that $\mathcal{E}(\phi)$ is a small set.

THEOREM 2.18 (Frostman). *For an inner function* $\phi$, $\mathcal{E}(\phi)$ *has logarithmic capacity zero.*

PROOF. Suppose $\mathcal{E}(\phi)$ has positive logarithmic capacity. By Theorem 2.14, there is a compact subset $K$ of positive logarithmic capacity such that

$$\lim_{r \to 1^-} \int_{\partial\mathbb{D}} \log\left|\frac{1-\overline{w}\phi(r\zeta)}{w-\phi(r\zeta)}\right| dm(\zeta) > 0, \quad \forall w \in K. \tag{2.19}$$

Moreover, by the definition of logarithmic capacity, there is a positive non-zero measure $\sigma$ supported on $K$ such that $G_\sigma$ is bounded on $\mathbb{D}$. We than have

$$\begin{aligned}
0 &= \lim_{r\to 1^-} \int_{\partial\mathbb{D}} G_\sigma(\phi(r\zeta))\,dm(\zeta) && \text{(dominated convergence theorem)}\\
&= \lim_{r\to 1^-} \int_K \left(\int_{\partial\mathbb{D}} \log\left|\frac{1-\overline{w}\phi(r\zeta)}{w-\phi(r\zeta)}\right| dm(\zeta)\right) d\sigma(w) && \text{(Fubini's theorem)}\\
&\geq \int_K \liminf_{r\to 1^-} \int_{\partial\mathbb{D}} \log\left|\frac{1-\overline{w}\phi(r\zeta)}{w-\phi(r\zeta)}\right| dm(\zeta)\, d\sigma(w) && \text{(Fatou's lemma)}\\
&> 0 && \text{(by eq.(2.19))}
\end{aligned}$$

which is a contradiction. □

Let us make a few remarks about the limits of Theorem 2.18.

REMARK 2.20. (1) Frostman **[9,** p. 113] showed that if $E$ is relatively closed in $\mathbb{D}$ and has logarithmic capacity zero, then there is an inner function $\phi$ with $\mathcal{E}(\phi) = E$ (see also [3, p. 37] and the next two comments).

(2) Recall from Proposition 2.17 and Theorem 2.18 that $\mathcal{E}(\phi)$ is an $F_\sigma$ set of logarithmic capacity zero. The authors in **[22]** showed that if $E \subset \mathbb{D}$ is of type $F_\sigma$ and has logarithmic capacity zero, then there is an inner function $\phi$ such that $\mathcal{E}(\phi) = E$.

(3) Suppose that $E$ is a closed subset of $\mathbb{D}$, $0 \notin E$, and $E$ has logarithmic capacity zero. We claim that there is a Blaschke product $B$ such that $B_a := \tau_a \circ B$ is a Blaschke product whenever $a \in \mathbb{D} \setminus E$ and $B_a$ is a singular inner function whenever $a \in E$. To see this, let $B$ be the universal covering map from $\mathbb{D}$ onto $\mathbb{D} \setminus E$ **[7,** p. 125]. Notice that $B^*(\zeta) \in \partial\mathbb{D} \cup E$. First note that $B$ is inner. Indeed, suppose that $|B^*(\zeta)| < 1$ for $\zeta \in A$ and $m(A) > 0$. Then $B^*(A) \subset E$ and, since $E$ has logarithmic capacity zero, we see that $B \equiv 0$ **[3,** p. 37] which is a contradiction. Second, note that $B_a$ is a Blaschke product for all $a \in \mathbb{D} \setminus E$. Indeed, $B_a$ maps $\mathbb{D}$ onto $\mathbb{D} \setminus \tau_a(E)$ and $0 \notin \tau_a(E)$. Moreover, $B_a^*(\zeta) \in \partial\mathbb{D} \cup \tau_a(E)$ and so $B_a^*(\zeta)$ can never be zero. An application of Theorem 2.13 completes the proof. Third, $B_a$ is a singular inner function whenever $a \in E$. To see this, note that $B$ maps $\mathbb{D}$ onto $\mathbb{D} \setminus E$ and so $a \notin B(\mathbb{D})$ which means the inner function $B_a$ has no zeros. Thus $B_a$ must be a singular inner function.

(4) If one is willing to work even harder in the previous example, one can find an *interpolating* Blaschke product $B$ such that $B_a$ is an interpolating Blaschke product for all $a \in \mathbb{D} \setminus E$ while $B_a$ is a singular inner function whenever $a \in E$ **[14,** Theoerm 1.1]. In fact, the above proof is part of this one.

## 3. Indestructible Blaschke products

From Frostman's theorem (Theorem 2.18), we know that the exceptional set $\mathcal{E}(\phi)$ of an inner function $\phi$ is small. A Blaschke product $B$ is *indestructible* if $\mathcal{E}(B) = \emptyset$. This next technical result from **[21]** helps show that indestructible Blaschke products actually exist.

PROPOSITION 3.1. *If $B$ is a Blaschke product such that $B^*(\zeta)$ is never equal to $a \in \mathbb{D} \setminus \{0\}$, then $B$ is indestructible.*

PROOF. Suppose that for some $a \in \mathbb{D} \setminus \{0\}$, $B_a = \tau_a \circ B$ has a non-trivial singular inner factor. By Theorem 2.13, there is a $\zeta \in \partial\mathbb{D}$ such that $B_a^*(\zeta) = 0$. However, for $0 < r < 1$,

$$|B_a(r\zeta)| \geq \frac{1}{2}|B(r\zeta) - a|$$

and so, taking limits as $r \to 1^-$, we see that $B^*(\zeta) = a$, which contradicts our assumption.

□

COROLLARY 3.2. *If $B$ is a Blaschke product whose zeros $(a_n)_{n \geq 1}$ satisfy*

$$\sum_{n=1}^{\infty} \frac{1-|a_n|}{|\zeta - a_n|} < \infty \tag{3.3}$$

*for every $\zeta \in \partial\mathbb{D}$, then $B$ is indestructible.*

PROOF. By Theorem 2.2, $|B^*(\zeta)| = 1$ for every $\zeta \in \mathbb{T}$. Now apply Proposition 3.1. □

Certainly any finite Blaschke product satisfies eq.(3.3). The infinite Blaschke product in Remark 2.8 (2) also satisfies eq.(3.3) and thus is indestructible.

Let us say a few words about the origins of the concept of indestructibility. The following idea was explored by Heins **[15, 16]** for analytic functions on Riemann surfaces but, for the sake of simplicity, we outline this idea when the Riemann surface is the unit disk. Our discussion has not only historical value, but will be useful when we discuss a fascinating example of Morse later on.

If $f : \mathbb{D} \to \mathbb{D}$ is analytic and $a \in \mathbb{D}$, the function $z \mapsto -\log|f_a(z)|$, where $f_a = \tau_a \circ f$, is superharmonic on $\mathbb{D}$ (i.e., $\log|f_a|$ is subharmonic on $\mathbb{D}$). Using the classical inner-outer factorization theorem [8, Ch. 2], one can show that

$$-\log|f_a(z)| = \sum_{f(w)=a} -n(w)\log|\tau_w(z)| + u_a(z), \tag{3.4}$$

where $n(w)$ is the multiplicity of the zero of $f(z) - a$ at $z = w$, and $u_a$ is a non-negative harmonic function on $\mathbb{D}$. The focus of Heins' work is the residual term $u_a$. His first observation is that $u_a$ is the greatest harmonic minorant of $-\log|f_a|$. Moreover, since $u_a$ is a non-negative harmonic function on $\mathbb{D}$, Herglotz's theorem [8, p. 2] yields a positive measure $\mu_a$ on $\partial\mathbb{D}$ such that

$$u_a(z) = (P\mu_a)(z) = \int_{\partial\mathbb{D}} \frac{1-|z|^2}{|\zeta - z|^2}\, d\mu_a(\zeta),$$

the Poisson integral of $\mu_a$. Heins proves that if $\mu_a = \nu_a + \sigma_a$ is the Lebesgue decomposition of $\mu_a$, where $\nu_a \ll m$ and $\sigma_a \perp m$, then the $m$-almost everywhere defined function

$$q_a(\zeta) := \log\left|\frac{1-\bar{a}f^*(\zeta)}{a - f^*(\zeta)}\right|$$

is integrable on $\partial\mathbb{D}$ and

$$d\nu_a = q_a\, dm. \tag{3.5}$$

In the general setting, and the actual focus of his work, Heins examines the residual term $u_a$ in Lindelöf's theorem

$$G_{S_1}(f(z),a) = \sum_{f(w)=a} n(w) G_{S_2}(z,w) + u_a(z),$$

where $S_1$ and $S_2$ are Riemann surfaces with positive ideal boundary, $f$ is a conformal map from $S_1$ to $S_2$, and $G_{S_j}$ is the Green's function for $S_j$. To study the residual term $u_a$ in this general setting, Herglotz's theorem and the Lebesgue decomposition theorem are replaced by an old decomposition theorem of Parreau [27, Théorème 12] (see also [17, p. 7]). When $S_1 = S_2 = \mathbb{D}$, observe that

$$G_{S_j}(z,a) = -\log\left|\frac{a-z}{1-\bar{a}z}\right|.$$

We state this next theorem in the special case of the disk but refer the reader to Heins' paper where an analog of this theorem holds for Riemann surfaces.

THEOREM 3.6 (Heins). *The functions $u_a$ and $P\nu_a$ satisfy the following properties.*

(1) *Either $P\nu_a(z) = 0$ for all $(a,z) \in \mathbb{D}\times\mathbb{D}$ or $P\nu_a(z) \neq 0$ for all $(a,z) \in \mathbb{D}\times\mathbb{D}$.*
(2) *The set $\{a \in \mathbb{D} : u_a - P\nu_a > 0\}$ is an $F_\sigma$ set of logarithmic capacity zero.*

PROOF. Observe that if $a \in \mathbb{D}$ is fixed and $P\nu_a(z) = 0$ for some $z \in \mathbb{D}$, we can use the fact that $P\nu_a$ is a non-negative harmonic function along with the mean value property of harmonic functions to argue that $P\nu_a = 0$. Thus, from eq.(3.5), we have, this particular $a$,

$$0 = \lim_{r \to 1^-} P\nu_a(r\zeta) = \log \left|\frac{1-\bar{a}f^*(\zeta)}{a - f^*(\zeta)}\right|, \quad \text{a.e. } \zeta \in \partial\mathbb{D}.$$

Whence it follows that $|f^*(\zeta)| = 1$ almost everywhere, i.e., $f$ is inner. The fact that $f$ is inner along with the fact that $\tau_b$ maps $\partial\mathbb{D}$ to $\partial\mathbb{D}$ for each $b \in \mathbb{D}$ shows that

$$\lim_{r \to 1^-} P\nu_b(r\zeta) = 0 \quad \text{a.e. } \zeta \in \partial\mathbb{D}.$$

Thus, from eq.(2.10), we see that for each $b \in \mathbb{D}$,

$$0 = \lim_{r \to 1^-} P\nu_b(r\zeta) = D\nu_b(\zeta) \quad \text{a.e. } \zeta \in \partial\mathbb{D}$$

and so $\nu_b \perp m$. But since $\nu_b \ll m$ it must be the case that $\nu_b = 0$. Thus we have shown part (1) of the theorem.

To avoid some technicalities, and to keep our focus on Blaschke products, let us prove part (2) of the theorem in the special case when $P\nu_a = 0$ for some (equivalently all) $a$. Note that $f$ is inner. If $U_a$ has a zero in $\mathbb{D}$, then, as argued before using the mean value property of harmonic functions, $U_a = 0$. Recall from our earlier discussion that

$$U_a = \mathbf{m}(-\log|f_a|),$$

where **m** denotes the greatest harmonic minorant. If we factor $f_a = bg$ as the product of a Blaschke product b and a singular inner function $g$, one can argue that

$$\mathbf{m}(-\log|f_a|) = \mathbf{m}(-\log|b|) + \mathbf{m}(-\log|g|).$$

It follows from Theorem 2.14 and a technical fact about greatest harmonic minorants **[12,** p. 38], that $\mathbf{m}(-\log|b|) = 0$. But since we are assuming $U_a = 0$, we have $\mathbf{m}(-\log|g|) = 0$. However, $g$ has no zeros in $\mathbb{D}$ and so $-\log|g|$ is a non-negative harmonic function on $\mathbb{D}$ and thus

$$0 = \mathbf{m}(-\log|g|) = -\log|g|.$$

Hence $g = e^{ic}$, $c \in \mathbb{R}$, equivalently, $f_a$ is a Blaschke product. Thus we have shown $U_a = 0 \Rightarrow f_a$ is a Blaschke product. If $f_a$ is a Blaschke product, then, as pointed out before, $U_a = \mathbf{m}(-\log|f_a|) = 0$. It follows that

$$\{a \in \mathbb{D} : U_a > 0\} = \mathcal{E}(f). \tag{3.7}$$

Now use Proposition 2.17 and Theorem 2.18. □

In summary, $U_a = 0$ if and only if $f_a$ is a Blaschke product. Moreover, $U_a = 0$ for every $a \in \mathbb{D}$ if and only if $f$ is an indestructible Blaschke product. Heins did not coin the term 'indestructible' in his work. McLaughlin **[21]** was the first to use this term and to explore the properties of these products.

## 4. Zeros of indestructible Blaschke products

McLaughlin **[21]** determined a characterization of the indestructible Blaschke products in terms of their level sets. Suppose $\phi$ is inner and $a \in \mathbb{D} \setminus \{\phi(0)\}$. Let $(w_j)_{j\geq 1}$ be the solutions to $\phi(z) - a = 0$ and factor

$$\varphi_a = \frac{\phi - a}{1 - \bar{a}\phi} = b \cdot s,$$

where $b$ is a Blaschke product whose zeros are $(w_j)_{j=1}^{\infty}$ and $s$ is a singular inner function. Taking absolute values of both sides of the above equation and evaluating at $z = 0$, we get

$$\left|\frac{\phi(0)-a}{1-\bar{a}\phi(0)}\right| = \left(\prod |w_j|\right) |s(0)|.$$

As discussed in the proof of Theorem 2.14, notice that $|s(0)| = 1$ if and only if $s$ is a unimodular constant, i.e., $\phi_a$ is a Blaschke product. In other words, for $a \neq \phi(0)$, $\phi_a$ is a Blaschke product if and only if

$$\left|\frac{\phi(0)-a}{1-\bar{a}\phi(0)}\right| = \prod_{j=1}^{\infty} |w_j|.$$

What happens when $a = \phi(0)$? Let

$$\phi(z) - \phi(0) = b_n z^n + b_{n+1} z^{n+1} + \cdots$$

be the Taylor series of $\phi - \phi(0)$ about $z = 0$ and let $(z_j)_{j=1}^{\infty}$ be the non-zero zeros of $\phi(z) - \phi(0)$. As before, write

$$\frac{1}{z^n}\frac{\phi - \phi(0)}{1-\overline{\phi(0)}\phi} = b \cdot s,$$

where $s$ is a singular inner function and $b$ is the Blaschke product whose zeros are $(z_j)_{j=1}^{\infty}$. Again, take absolute values of both sides of the above expression and evaluate at $z = 0$ to get

$$\frac{|b_n|}{|\phi(0)|^2} = \left(\prod_{j=1}^{\infty} |z_j|\right) |s(0)|.$$

Moreover, $\phi = \phi_0$ is a Blaschke product if and only if

$$\frac{|b_n|}{1-|\phi(0)|^2} = \prod_{j=1}^{\infty} |z_j|.$$

Combining these observations, we have shown the following theorem.

THEOREM 4.1 (McLaughlin). *Using the notation above, a Blaschke product $B$ is indestructible if and only if*

$$\left|\frac{B(0)-a}{1-\bar{a}B(0)}\right| = \prod_{j=1}^{\infty} |w_j|, \quad \forall a \neq B(0),$$

*and*

$$\frac{|b_n|}{|\phi(0)|^2} = \prod_{j=1}^{\infty} |z_j|.$$

Though the above theorem gives necessary and sufficient conditions (in terms of the level sets of $B$) to be indestructible, characterizing indestructibility just in terms of the *zeros* of $B$ seems almost impossible. Consider the following theorem of Morse [23].

THEOREM 4.2 (Morse). *There is a Blaschke product $B$ for which $\mathcal{E}(B) \neq 0$ but such that if $c$ is any zero of $B$, then $\mathcal{E}(B/\tau_c) = 0$.*

In other words, there are 'destructible' Blaschke products which become indestructible when one of their zeros are removed. We will not give all of the technical details here since they are done thoroughly in Morse's paper. However, since they do relate directly to the earlier work of Heins, from the previous section, we will give an outline of Morse's theorem.

Suppose $B$ is a Blaschke product such that the set

$$\{\zeta \in \partial\mathbb{D} : |B^*(\zeta)| < 1\}$$

is at most countable. For $a \in \mathbb{D}$, let

$$U_a := m(-\log|B_a|),$$

be the greatest harmonic minorant of the non-negative superharmonic function $-\log|B_a|$. This function is the residual function covered in the previous section (see eq.(3.4)). Since $U_a$ is a non-negative harmonic function on $\mathbb{D}$, Herglotz's theorem says that

$$U_a = P\mu_a,$$

the Poisson integral of a measure $\mu_a$ on $\partial\mathbb{D}$. Moreover, since $\log|B(\zeta)| = 0$ for $m$-almost every $\zeta$, it follows that $u(\zeta) = 0$ $m$-almost everywhere. By Fatou's theorem (see eq.(2.10)),

$$u(\zeta) = (D\mu_a)(\zeta)$$

at every point where $(D\mu_a)(\zeta)$ exists (and we count the possibility that $(D\mu_a)(\zeta)$ might be equal to $+\infty$). We see two things from this. First, $(D\mu_a)(\zeta) = 0$ for $m$-almost every $\zeta$ and so, by the Lebesgue decomposition theorem, $\mu_a \perp m$. Second, since we are assuming that $\{\zeta \in \partial\mathbb{D} : |B^*(\zeta)| < 1\}$ is at most countable, we can use the facts that

$$\{\zeta \in \partial\mathbb{D} : (D\mu_a)(\zeta) = +\infty\} = \{\zeta : u(\zeta) = +\infty\} \subset \{\zeta : |B^*(\zeta)| < 1\}$$

and $\{\zeta : (D\mu_a)(\zeta) = +\infty\}$ is a carrier for $\mu_a$ (since $\mu_a \perp m$) [30, p. 158] to see that $\mu_a$ is a discrete measure. It might be the case that $\mu_a = 0$, i.e., $B_a$ is a Blaschke product (see eq.(3.7)).

If we make the further assumption that not only is $\{\zeta : |B^*(\zeta)| < 1\}$ at most countable but $B$ is also destructible, i.e., $B_a$ is not a Blaschke product for some $a \in \mathbb{D}$, we see (see eq.(3.7)) that $U_a > 0$ and so, for this particular $a$, the discrete measure $\mu_a$ above is not identically zero. Define $Q(B)$ to be the union of the carriers of the measures $\{\mu_a : U_a = P\mu_a > 0\}$. Notice that

$$Q(B) \subset \{\zeta \in \partial\mathbb{D} : |B^*(\zeta)| < 1\}, \tag{4.3}$$

and hence is at most countable, and that $Q(B)$ is contained in the accumulation points of the zeros of $B$. We also see in this case that $B$ is destructible if and only if $Q(B) \neq \emptyset$.

A technical theorem of Morse [23, Proposition 3.2] says that if $\zeta \in Q(B)$, then there is an inner function $g$, a point $a \in \mathbb{D}$, and a $\beta > 0$ such that

$$B_a(z) = g(z) \exp\left(-\beta \frac{\zeta + z}{\zeta - z}\right).$$

Morse says in this case that $B$ is *exponentially destructible* at $\zeta$. It follows from here that for some $\alpha > 0$

$$|B(r\zeta) - a| = O(e^{\frac{-\alpha}{1-r}}), \quad r \to 1^-. \tag{4.4}$$

An argument using this growth estimate (see [23, Proposition 3.4]) shows that if *c* is any zero of *B,* then

(4.5) $$Q(B) \cap Q(B/ rc) = 0.$$

Morse gives a treatment of exponentially destructible Blaschke products beyond what we cover here.

We are now ready to discuss Morse's example. Choose *a* E ID> \ {O} and define

(4.6) $$B(z) := \tau_a \left(\exp\left(-\frac{1+z}{1-z}\right)\right).$$

One can see that B is an inner function, B*(() exists for every ( E 81Dl, and

IB*(()i = { 1, f ( E 81Dl \ {1};
JaJ, if ( = 1.

By Theorem 2.13, B is a Blaschke product. It is also the case, by direct computation, that the zeros of *B* can only accumulate at ( = 1. Finally, notice from eq.(4.3) and the identity

*B_a( z)* = exp (– 1 +z / 1 – z) ,

that

$$Q(B) = \{1\}$$

and so *B* is destructible, in fact exponentially destructible at 1.

We claim that if *c* is a zero of *B*, then *B/ rc* (*B* with the zero at *c* divided out) is indestructible. Indeed, since

{( : J (B/rc)*(() J < 1} = {( : J B*(() J < 1} = {1}

we can apply eq.(4.3) to get

*Q(B/ rc)* C {1}.

However, from eq.(4.5) we see that *Q(B/ rc)* = 0 which means, from our discussion above, that *B/Tc* is indestructible.

## 5. Classes of indestructible Blaschke products

So far, we have discussed conditions on a Blaschke product that make it indestructible. We now examine a refinement of this question. Suppose that '.B is a particular class of Blaschke products and *B* E '.B. What extra assumptions are required of *B* so that *Ba* E '.B for all *a* E ID>?

We focus on the class (and certain sub-classes) of e, the *Carleson-Newman* Blaschke products. These are Blaschke products *B* whose zeros *(an)n l* satisfy the so-called 'conformal invariant' version of the Blaschke condition

oo
)1 – *Jani)* < oo,
n=1

i.e.,

sup { (1 – Jcp(an) I) : <p E Aut(ID>)} < oo.

There are several equivalent definitions of e. For example, *B* E e <==? *B* is the finite product of interpolating Blaschke products <==? the measure *En(l – l anl 2 )8an* is a Carleson measure. The standard reference for this is [12] but another nice exposition with further references is [24, Theorem 2.2]. Two important examples of Blaschke products which belong to e are 'J, the *thin* Blaschke products *B* which satisfy the condition

$$\lim_{n\to\infty}(1-|a_n|^2)|B'(a_n)| = 1,$$

and *9",* the *F'rostman* Blaschke products which satisfy the condition

sup *L.,;* lr 1-lan l l < 00.
{E8llli n=l ., – *an*

An example of a thin Blaschke product is one whose zeros *(an)n:;,,1* satisfy

lim 1 – lan+il = 0
*n--+oo* 1 – lanl

**[13,** Prop. 1.1], while an example of a Frostman Blaschke product is one with zeros *an = rnei9n*, where (rn)n:;,,1 C (0, 1), (On)n:;,,1 C (0, 1),

sup { *(} +1* : *n* 1 } < 1 and *L.,;* 1-*rn* < 00
*(}n* n=l *(}n*

**[2,** p. 130].

The thin Blaschke products relate to Douglas algebras and the structure of the bounded analytic functions **[31]** as well as composition operators on the Bloch space **[4].** The Frostman Blaschke products turn out to be the only inner multipliers of the space of Cauchy transforms of measures on ()IIJ) **[19]** (see also **[2]).**

Following the definition of *f.(B),* the exceptional set of a Blaschke product in eq.(2.16), define

f.e(B) := {a E IIJ) : *Ba* </. e};
*£.,(B)* := {*a* E IIJ) : *Ba* </. 'J};
*f.:7(B)* := {a E IIJ) : *Ba* </. 9"}.

Gorkin and Martini **[14,** Lemma 3.2] use a result of Tolokonnikov **[31,** p. 884] to show the following.

THEOREM 5.1. *If B* E 'J *then f..,(B)* = 0.

The current author and Matheson **[20]** use the theory of inner multipliers for the space of Cauchy transforms and the ideas of Tolokonnikov **[31]** and Pekarskil **[28]** to prove the following.

THEOREM 5.2. *If* B E *9", then f.:J'(B)* = 0.

Nicolau **[25]** states necessary and sufficient conditions, in terms of the zeros of *B,* so that f.e(B) = 0 - which are a bit technical to get into here. We do point out the following.

THEOREM 5.3. (1) *If B is a Blaschke product, then* f.e(B) *is closed in* IIJ).

(2) *Given any* 0 < *s* < 1, *there is a* B E e *such that* f.e(B) = {s lzl < 1}.

REMARK 5.4. (1) Compare this to *f.(B)* which is an *Fu* set of logarithmic capacity zero (Proposition 2.17 and Theorem 2.18).

(2) The first result of the above theorem is contained in **[25,** Lemma 1]. A version of the second result is found in **[25,** §3]. See **[14,** Theorem 4.2] for the version we state here.

There is a sizable literature of deep results which relate the class of Blaschke products

$$'.P := \{B : ee(B) = 0\}$$

to many ideas in function algebras. We refer the reader to **[24,** p. 287] for a discussion of this and for the exact references.

So far we have discussed when a Blaschke product has the property that all its Frostman shifts belong to a certain class of Blaschke products. We point out two papers **[14, 24]** which discuss when an inner function $\phi$ (not necessarily a Blaschke product) has the property that $\phi_a$ belongs to a certain class of Blaschke products (the class $e$ for example) for all $a \in \mathbb{D} \setminus \{0\}$.

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