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The Norm of a Truncated Toeplitz Operator

Stephan Ramon Garcia and William T. Ross

ABSTRACT. We prove several lower bounds for the norm of a truncated Toeplitz operator and obtain a curious relationship between the $H^2$ and $H^\infty$ norms of functions in model spaces.

1. Introduction

In this paper, we continue the discussion initiated in [6] concerning the norm of a truncated Toeplitz operator. In the following, let $H^2$ denote the classical Hardy space of the open unit disk $\mathbb{D}$ and $K_\Theta := H^2 \cap (\Theta H^2)^\perp$, where $\Theta$ is an inner function, denote one of the so-called Jordan model spaces [2,4,7]. If $H^\infty$ is the set of all bounded analytic functions on $\mathbb{D}$, the space $K_\Theta^\infty := H^\infty \cap K_\Theta$ is norm dense in $K_\Theta$ (see [2, p. 83] or [9, Lemma 2.3]). If $P_\Theta$ is the orthogonal projection from $L^2 := L^2(\partial \mathbb{D}, |d\zeta|/2\pi)$ onto $K_\Theta$ and $\varphi \in L^2$, then the operator

$$A_\varphi f := P_\Theta(\varphi f), \quad f \in K_\Theta^\infty,$$

is densely defined on $K_\Theta$ and is called a truncated Toeplitz operator. Various aspects of these operators were studied in [3,5,6,9,10].

If $\|\cdot\|$ is the norm on $L^2$, we let

$$\|A_\varphi\| := \sup\{\|A_\varphi f\| : f \in K_\Theta^\infty, \|f\| = 1\}$$

and note that this quantity is finite if and only if $A_\varphi$ extends to a bounded operator on $K_\Theta$. When $\varphi \in L^\infty$, the set of bounded measurable functions on $\partial \mathbb{D}$, we have the basic estimates

$$0 \leq \|A_\varphi\| \leq \|\varphi\|_\infty.$$

However, it is known that equality can hold, in nontrivial ways, in either of the inequalities above and hence finding the norm of a truncated Toeplitz operator can be difficult. Furthermore, it turns out that there are many unbounded symbols $\varphi \in L^2$ which yield bounded operators $A_\varphi$. Unlike the situation for classical Toeplitz operators on $H^2$, for a given $\varphi \in L^2$, there many $\psi \in L^2$ for which $A_\varphi = A_\psi$ [9, Theorem 3.1].

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For a given symbol \( \varphi \in L^2 \) and inner function \( \Theta \), lower bounds on the quantity (1) are useful in answering the following nontrivial questions:

1. Is \( A_\varphi \) unbounded?
2. If \( \varphi \in L^\infty \), is \( A_\varphi \) norm-attaining (i.e., is \( \|A_\varphi\| = \|\varphi\|_\infty \))?

When \( \Theta \) is a finite Blaschke product and \( \varphi \in H^\infty \), the paper \([6]\) computes \( \|A_\varphi\| \) and gives necessary and sufficient conditions as to when \( \|A_\varphi\| = \|\varphi\|_\infty \). The purpose of this short note is to give a few lower bounds on \( \|A_\varphi\| \) for general inner functions \( \Theta \) and general \( \varphi \in L^2 \). Along the way, we obtain a curious relationship (Corollary 5) between the \( H^2 \) and \( H^\infty \) norms of functions in \( K_\Theta \).

## 2. Lower bounds derived from Poisson’s formula

For \( \varphi \in L^2 \), let

\[
(\mathcal{P}_\varphi)(z) := \int_{\partial \mathbb{D}} \frac{1 - |z|^2}{|\zeta - z|^2} \varphi(\zeta) \frac{|d\zeta|}{2\pi}, \quad z \in \mathbb{D},
\]

be the standard Poisson extension of \( \varphi \) to \( \mathbb{D} \). For \( \varphi \in C(\partial \mathbb{D}) \), the continuous functions on \( \partial \mathbb{D} \), recall that \( \mathcal{P}_\varphi \) solves the classical Dirichlet problem with boundary data \( \varphi \). Also note that

\[
k_\lambda(z) := \frac{1 - \Theta(\lambda)\Theta(z)}{1 - \lambda z}, \quad \lambda, z \in \mathbb{D},
\]

is the reproducing kernel for \( K_\Theta \) \([9]\).

Our first result provides a general lower bound for \( \|A_\varphi\| \) which yields a number of useful corollaries:

**Theorem 1.** If \( \varphi \in L^2 \), then

\[
\sup_{\lambda \in \mathbb{D}} \frac{1 - |\lambda|^2}{1 - |\Theta(\lambda)|^2} \left| \int_{\partial \mathbb{D}} \varphi(z) \left| \frac{\Theta(z) - \Theta(\lambda)}{z - \lambda} \right|^2 \frac{|dz|}{2\pi} \right| \leq \|A_\varphi\|.
\]

In other words,

\[
\sup_{\lambda \in \mathbb{D}} \left| \int_{\partial \mathbb{D}} \varphi(z) d\nu_\lambda(z) \right| \leq \|A_\varphi\|
\]

where

\[
d\nu_\lambda(z) := \frac{1 - |\lambda|^2}{1 - |\Theta(\lambda)|^2} \left| \frac{\Theta(z) - \Theta(\lambda)}{z - \lambda} \right|^2 \frac{|dz|}{2\pi}
\]

is a family of probability measures on \( \partial \mathbb{D} \) indexed by \( \lambda \in \mathbb{D} \).

**Proof.** For \( \lambda \in \mathbb{D} \) we have

\[
\|k_\lambda\| = \sqrt{\frac{1 - |\Theta(\lambda)|^2}{1 - |\lambda|^2}},
\]

from which it follows that

\[
\|A_\varphi\| \geq \frac{1 - |\lambda|^2}{1 - |\Theta(\lambda)|^2} |\langle A_\varphi k_\lambda, k_\lambda \rangle| = \frac{1 - |\lambda|^2}{1 - |\Theta(\lambda)|^2} |\langle P_\Theta \varphi k_\lambda, k_\lambda \rangle|
\]

\[
= \frac{1 - |\lambda|^2}{1 - |\Theta(\lambda)|^2} |\langle \varphi k_\lambda, k_\lambda \rangle|
\]

\[
= \frac{1 - |\lambda|^2}{1 - |\Theta(\lambda)|^2} \left| \int_{\partial \mathbb{D}} \varphi(z) \left| \frac{\Theta(z) - \Theta(\lambda)}{z - \lambda} \right|^2 \frac{|dz|}{2\pi} \right|.
\]
That the measures $d\nu_\lambda$ are indeed probability measures follows from (4).

Now observe that if $\Theta(\lambda) = 0$, the argument in the supremum on the left hand side of (3) becomes the absolute value of the expression in (2). This immediately yields the following corollary:

**Corollary 1.** If $\varphi \in L^2$, then
\[
\sup_{\lambda \in \Theta^{-1}(\{0\})} |(\mathfrak{P}\varphi)(\lambda)| \leq \|A\varphi\|
\]
where the supremum is to be regarded as 0 if $\Theta^{-1}(\{0\}) = \emptyset$.

Under the right circumstances, the preceding corollary can be used to prove that certain truncated Toeplitz operators are norm-attaining:

**Corollary 2.** Let $\Theta$ be an inner function having zeros which accumulate at every point of $\partial \mathbb{D}$. If $\varphi \in C(\partial \mathbb{D})$ then $\|A\varphi\| = \|\varphi\|_\infty$.

**Proof.** Let $\zeta \in \partial \mathbb{D}$ be such that $|\varphi(\zeta)| = \|\varphi\|_\infty$. By hypothesis, there exists a sequence $\lambda_n$ of zeros of $\Theta$ which converge to $\zeta$. By continuity, we conclude that
\[
\|\varphi\|_\infty = \lim_{n \to \infty} |(\mathfrak{P}\varphi)(\lambda_n)| \leq \|A\varphi\| \leq \|\varphi\|_\infty
\]
whence $\|A\varphi\| = \|\varphi\|_\infty$.

The preceding corollary stands in contrast to the finite Blaschke product setting. Indeed, if $\Theta$ is a finite Blaschke product and $\varphi \in H^\infty$, then it is known that $\|A\varphi\| = \|\varphi\|_\infty$ if and only if $\varphi$ is the scalar multiple of the inner factor of some function from $K_\Theta$ [6] Theorem 2.

At the expense of wordiness, the hypothesis of Corollary 2 can be considerably weakened. A cursory examination of the proof indicates that we only need $\zeta$ to be a limit point of the zeros of $\Theta$, $\varphi \in L^\infty$ to be continuous on an open arc containing $\zeta$, and $|\varphi(\zeta)| = \|\varphi\|_\infty$.

Theorem 1 yields yet another lower bound for $\|A\varphi\|$. Recall that an inner function $\Theta$ has a finite angular derivative at $\zeta \in \partial \mathbb{D}$ if $\Theta$ has a nontangential limit $\Theta(\zeta)$ of modulus one at $\zeta$ and $\Theta'$ has a finite nontangential limit $\Theta'(\zeta)$ at $\zeta$. This is equivalent to asserting that
\[
\frac{\Theta(z) - \Theta(\zeta)}{z - \zeta}
\]
has the nontangential limit $\Theta'(\zeta)$ at $\zeta$. If $\Theta$ has a finite angular derivative at $\zeta$, then the function in (6) belongs to $H^2$ and
\[
\lim_{r \to 1^-} \int_{\partial \mathbb{D}} \left| \frac{\Theta(z) - \Theta(r\zeta)}{z - r\zeta} \right|^2 |dz| = \frac{1}{2\pi} \int_{\partial \mathbb{D}} \left| \frac{\Theta(z) - \Theta(\zeta)}{z - \zeta} \right|^2 |dz|.
\]
Furthermore, the above is equal to
\[
\lim_{r \to 1^-} \frac{1 - |\Theta(r\zeta)|^2}{1 - r^2} = |\Theta'(\zeta)| > 0.
\]

See [1,8] for further details on angular derivatives. Theorem 1 along with the preceding discussion and Fatou's lemma yield the following lower estimate for $\|A\varphi\|$.
Corollary 3. For an inner function $\Theta$, let $D_{\Theta}$ be the set of $\zeta \in \partial \mathbb{D}$ for which $\Theta$ has a finite angular derivative $\Theta'(\zeta)$ at $\zeta$. If $\varphi \in L^\infty$ or if $\varphi \in L^2$ with $\varphi \geq 0$, then

$$
\sup_{\zeta \in D_{\Theta}} \frac{1}{|\Theta'(\zeta)|} \left| \int_{\partial \mathbb{D}} \varphi(z) \left| \frac{\Theta(z) - \Theta(\zeta)}{z - \zeta} \right|^2 \frac{|dz|}{2\pi} \right| \leq \|A_{\varphi}\|.
$$

In other words,

$$
\sup_{\zeta \in D_{\Theta}} \left| \frac{1}{|\Theta'(\zeta)|} \int_{\partial \mathbb{D}} \varphi(z) d\nu_{\lambda}(z) \right| \leq \|A_{\varphi}\|,
$$

where

$$
d\nu_{\lambda}(z) := \frac{1}{|\Theta'(\zeta)|} \left| \frac{\Theta(z) - \Theta(\zeta)}{z - \zeta} \right|^2 \frac{|dz|}{2\pi}
$$

is a family of probability measures on $\partial \mathbb{D}$ indexed by $\zeta \in D_{\Theta}$.

3. Lower bounds and projections

Our next several results concern lower bounds on $\|A_{\varphi}\|$ involving the orthogonal projection $P_{\Theta}: L^2 \to K_{\Theta}$.

Theorem 2. If $\Theta$ is an inner function and $\varphi \in L^2$, then

$$
\|P_{\Theta}(\varphi) - \overline{\Theta(0)}P_{\Theta}(\Theta \varphi)\| \leq \|A_{\varphi}\|.
$$

Proof. First observe that $\|k_0\| = (1 - |\Theta(0)|^2)^{1/2}$. Next we see that if $\varphi \in L^2$ and $g \in K_{\Theta}$ is any unit vector, then

$$(1 - |\Theta(0)|^2)^{1/2}\|A_{\varphi}\| \geq |\langle A_{\varphi}k_0, g \rangle| = |\langle P_{\Theta}(\varphi k_0), g \rangle| = |\langle P_{\Theta}(\varphi) - \overline{\Theta(0)}P_{\Theta}(\Theta \varphi), g \rangle|. $$

Setting

$$
g = \frac{P_{\Theta}(\varphi) - \overline{\Theta(0)}P_{\Theta}(\Theta \varphi)}{\|P_{\Theta}(\varphi) - \overline{\Theta(0)}P_{\Theta}(\Theta \varphi)\|}
$$

yields the desired inequality. $\square$

In light of the fact that $P_{\Theta}(\Theta \varphi) = 0$ whenever $\varphi \in H^2$, Theorem 2 leads us immediately to the following corollary:

Corollary 4. If $\Theta$ is inner and $\varphi \in H^2$, then

$$
\|P_{\Theta}(\varphi)\| \leq \|A_{\varphi}\|.
$$

It turns out that (7) has a rather interesting function-theoretic implication. Let us first note that for $\varphi \in H^\infty$, we can expect no better inequality than

$$
\|\varphi\| \leq \|\varphi\|_\infty
$$

(with equality holding if and only if $\varphi$ is a scalar multiple of an inner function). However, if $\varphi$ belongs to $K_{\Theta}^\infty$, then a stronger inequality holds.

Corollary 5. If $\Theta$ is an inner function, then

$$
\|\varphi\| \leq (1 - |\Theta(0)|^2)^{1/2}\|\varphi\|_\infty
$$

holds for all $\varphi \in K_{\Theta}^\infty$. If $\Theta$ is a finite Blaschke product, then equality holds if and only if $\varphi$ is a scalar multiple of an inner function from $K_{\Theta}$.
PROOF. First observe that the inequality
\[ \|\varphi\| \leq (1 - |\Theta(0)|^2)^{\frac{1}{2}} \|\varphi\|_\infty \]
follows from Corollary 4 and the fact that \( P_\Theta \varphi = \varphi \) whenever \( \varphi \in K_\Theta \). Now suppose that \( \Theta \) is a finite Blaschke product and assume that equality holds in the preceding for some \( \varphi \in K_\Theta^\infty \). In light of (7), it follows that \( \|A_\varphi\| = \|\varphi\|_\infty \). From [6, Theorem 2] we see that \( \varphi \) must be a scalar multiple of the inner part of a function from \( K_\Theta \). But since \( \varphi \in K_\Theta^\infty \), then \( \varphi \) must be a scalar multiple of an inner function from \( K_\Theta \). \( \square \)

When \( \Theta \) is a finite Blaschke product, then \( K_\Theta \) is a finite dimensional subspace of \( H^2 \) consisting of bounded functions [3, 5, 9]. By elementary functional analysis, there are \( c_1, c_2 > 0 \) so that
\[ c_1 \|\varphi\| \leq \|\varphi\|_\infty \leq c_2 \|\varphi\| \]
for all \( \varphi \in K_\Theta \). This prompts the following question:

**Question.** What are the optimal constants \( c_1, c_2 \) in the above inequality?

4. Lower bounds from the decomposition of \( K_\Theta \)

A result of Sarason [9, Theorem 3.1] says, for \( \varphi \in L^2 \), that
\[ A_\varphi \equiv 0 \iff \varphi \in \Theta H^2 + \overline{\Theta} H^2. \]

It follows that the most general truncated Toeplitz operator on \( K_\Theta \) is of the form \( A_{\psi + \chi} \) where \( \psi, \chi \in K_\Theta \). We can refine this observation a bit further and provide another canonical decomposition for the symbol of a truncated Toeplitz operator.

**Lemma 1.** Each bounded truncated Toeplitz operator on \( K_\Theta \) is generated by a symbol of the form
\[ \varphi = \begin{cases} \psi & \text{in } H^2 \\ \chi/\overline{\Theta} & \text{in } \Theta H^2 \end{cases} \]
where \( \psi, \chi \in K_\Theta \).

Before getting to the proof, we should remind the reader of a technical detail. It follows from the identity \( K_\Theta = H^2 \cap \Theta z H^2 \) (see [2, p. 82]) that
\[ C : K_\Theta \to K_\Theta, \quad Cf := \overline{z} \Theta, \]
is an isometric, conjugate-linear, involution. In fact, when \( A_\varphi \) is a bounded operator we have the identity \( CA_\varphi C = A_{\varphi^*} [9, \text{Lemma 2.1}] \).

**Proof of Lemma 1.** If \( T \) is a bounded truncated Toeplitz operator on \( K_\Theta \), then there exists some \( \varphi \in L^2 \) such that \( T = A_\varphi \). We claim that this \( \varphi \) can be chosen to have the special form (10). First let us write \( \varphi = f + zg \) where \( f, g \in H^2 \). Using the orthogonal decomposition \( H^2 = K_\Theta \oplus \Theta H^2 \), it follows that \( \varphi \) may be further decomposed as
\[ \varphi = (f_1 + \Theta f_2) + z(g_1 + \Theta g_2) \]
where \( f_1, g_1 \in K_\Theta \) and \( f_2, g_2 \in H^2 \). By (9), the symbols \( \Theta f_2 \) and \( \overline{\Theta (z g_2)} \) yield the zero truncated Toeplitz operator on \( K_\Theta \). Therefore we may assume that
\[ \varphi = f + zg \]
for some \( f, g \in K_\Theta \). Since \( Cg = \overline{gz} \Theta \), we have \( \overline{zg} = (Cg)\overline{\Theta} \) and hence (10) holds with \( \psi = f \) and \( \chi = Cg \). \( \square \)
Corollary 6. Let $\Theta$ be an inner function. If $\psi_1, \psi_2 \in K_\Theta$ and $\varphi = \psi_1 + \psi_2 \overline{\Theta}$, then
\[
\frac{\|\psi_1 - \overline{\Theta(0)}\psi_2\|}{(1 - |\Theta(0)|^2)^{1/2}} \leq \|A_{\varphi}\|.
\]

Proof. If $\varphi = \psi_1 + \psi_2 \overline{\Theta}$, then, since $\psi_1, \psi_2 \in K_\Theta$ and $\psi_2 \overline{\Theta} \in \mathbb{z}H^2$, we have
\[
P_\Theta(\varphi) - \overline{\Theta(0)}P_\Theta(\Theta \varphi) = \psi_1 - \overline{\Theta(0)}\psi_2.
\]
The result now follows from Theorem 2.

References