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Mapping of Stochastic matrices into Polynomial form in the complex plane

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Mapping of Stochastic Matrices into Polynomial Form in the Complex Plane

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Introduction

This thesis originated from a specific problem from biology. Namely we need to study probabilistic models that represent molecular interactions that take place inside living cells, such as the number of molecular heat-shock proteins present in a cell. Because of the intrinsic discrete nature of the number of molecules present in cells, the fundamental mathematical models are based on Markov processes. For such processes a transition probability matrix describes the evolution of the state of the cell, whereas the state itself, i.e. the number of molecules present at a specific time, is described by a vector. The components of this vector represent the probabilities for finding specific molecule numbers. For example, consider a cell that contains between zero and ten heat-shock protein molecules. The number of heat-shock proteins that help repair a cell undergoing heat shock follows a random process. The components of our vector would represent the probabilities for having between zero and ten protein molecules. The transition probability matrix would include the probabilities of transitions between the number of molecules, in other words the probability that given the cell had five protein molecules that it would increase to six, remain at five, or decrease to four. Thus the next state of the cell is dependent upon the previous state of the cell.

We need also to consider, besides the probabilistic nature of the phenomenon, the appearance of thresholds in biology. Namely, many biological processes do not take place unless the population size of a molecular species reaches a threshold level. Once this level is reached, a cascade of events is opened and downstream events propagate through the cell. The threshold may not necessarily be an integer number, as it was with the molecular population size. So, on one side we need to work with discrete mathematical models built on integer numbers, but on another side we need to consider thresholds that are real numbers. This thesis focuses on this problem by investigating a mapping from discrete to continuous models. By mapping discrete Markov processes into a continuous model we hope to develop a tool to better analyze biological systems that have thresholds.

To be able to move from a discrete to a continuous model, we explore in this thesis a mapping from linear spaces of type R^{n+1} to the linear spaces of polynomial type Pol_n, the set of all polynomials of degree less than or equal to n with complex variable z and real coefficients. Instead of working with $n+1$ components of a vector v we work with a polynomial $\Psi(z)$ of degree n associated with the vector v by the following rule

$$\Psi(q^k) = v_k \tag{1}$$

for $k = 0, ..., n$ where q is a fixed number, and v_k represents the $(k+1)^{th}$ entry in the vector v. This

process is known as polynomial interpolation.

The coefficients Ψ_k of the polynomial $\Psi(z) = \Psi_0 + \Psi_1 z + \Psi_2 z^2 + ... \Psi_n z^n$ will depend on the components v_k of the vector v once the equation (1) is solved for Ψ_k. We now have a mapping from a discrete vector to a continuous complex function. Namely, the argument z of the polynomial $\Psi(z)$ will play the role of the continuous variable, whereas the discrete variable n , expressing the number of components of v, is captured by the degree of the polynomial $\Psi(z)$.

The variable z will be considered a complex variable, giving us the possibility to ask for the roots of $\Psi(z)$

$$\Psi(z) \;=\; a \prod_{m=1}^{n} (z - z_m) \tag{2}$$

The information about the components of the vector v is now transferred to the roots of $\Psi(z)$. We thus have new and more freedom in working in the complex plane, than we previously had when working with a set of n components of the vector v. Problems that are intractable in the component form become manageable in the polynomial space. For example, when attacking an eigenvalue problem that may be complicated we may work in the complex plane and find the roots z_m instead of finding the eigenvector components [1]. Moreover, the polynomial form lets us use the tools of calculus to integrate instead of doing summation for probabilities. In many instances integration is much easier to perform than a discrete sum. In order to perform this transition from discrete to continuous models, we need to map both our transition probability matrix and our vector into polynomial form in the complex plane. To do this we will define two mappings, Δ and J, which will be used to map matrices and vectors respectively. In Section 1, we will define our operator Δ, which maps polynomials to Laurent polynomials, $\Delta : Pol \mapsto Pol_L$. We are interested only in finding Δ operators that preserve the space Pol_n, so we focus, in Section 2, on restrictions of Δ to the space Pol_n. In this context Pol_n refers to polynomials up to degree n with no negative powers.We want the operator, when it acts on our function, to produce polynomials up to degree n with no negative powers, thus preserving the Pol_n space. Our operator Δ utilizes Laurent polynomials of order L, so we show in Section 3 that there exist Laurent polynomials for which Δ preserves Pol_n. In Sections 4 and 5 we consider an expansion of our operator Δ, which increases the size of our operator by increasing the number of Laurent polynomials utilized. We then switch gears in Section 6 to define our mapping $J : R^{n+1} \mapsto Pol_n$ with our function $\Psi(z)$. The function $\Psi(z)$ of degree $n+1$ is associated with the vector v by equating (1). We illustrate this with two specific examples. In Section 7 we associate a matrix M with our first operator, Δ_M, such that Δ_M can now map Ψ_v in Pol_n to $\Delta_M \Psi_v$ in Pol_n. We

then detail the mapping of two-dimensional and three-dimensional matrices. When considering four-dimensional matrices, finding Δ operators that will satisfy our mapping becomes difficult because we establish a system of equations with more equations than unknowns. To tackle this problem, we will use Kronecker products of two-dimensional matrices to determine Δ operators for a four-dimensional space.

1   Mapping R^{n+1} to Pol_n

Definition of the operator Δ acting on polynomials

Consider the linear space Pol of all polynomials of a complex variable z endowed with the usual algebraic operations. Also, consider the subspace Pol_n of Pol, consisting of all members of Pol of degree $\leqslant n$. For what follows we choose a complex number q to be fixed.

Definition

A function $f(z)$ of a complex variable z is called a **Laurent polynomial of order** L if it has the form

$$f(z) \;\;=\;\; \frac{F_{-L}}{z^L} + ... + \frac{F_{-3}}{z^3} + \frac{F_{-2}}{z^2} + \frac{F_{-1}}{z} + F_0^+ F_1 z + F_2 z^2 + F_3 z^3 + ... + F_L z^L$$

where the F_k are real numbers, with $F_{-L}^\alpha \neq 0$ and $F_L^\alpha \neq 0$

Definition

An operator $\Delta : Pol \to Pol_L$ is called a **Laurent operator of order L** if it can be expressed as

$$\Delta = f^+(z)T^+ + f^0(z)T^0 + f^-(z)T^- \tag{3}$$

where T^0 is the identity operator on Pol_n and T^+, T^- are the q-shift operators

$$T^+\Psi(z) \;\;=\;\; \Psi(qz)$$

$$T^-\Psi(z) \;\;=\;\; \Psi(q^{-1}z)$$

and each of the f's is a Laurent polynomial of order L.

For clarification on the index notation, the lower index specifies positive and negative powers of z while the upper index specifies positive and negative q shifts of $\Psi(z)$.

Decomposition of Δ

By rearranging our Laurent polynomials into constant, strictly positive, and strictly negative degrees of z, we can write our operator from (3) as

$$\Delta \;\; = \;\; (f_-^+(z) + f_0^+(z) + f_+^+(z))T^+ + (f_-^0(z) + f_0^0(z) + f_+^0(z))T^0 + \tag{4}$$

$$(f_-^-(z) + f_0^-(z) + f_+^-(z))T^- \tag{5}$$

where, for $\alpha = -, 0, +$,

$$f_-^\alpha(z) \;\; = \;\; \sum_{m=1}^{L} F_{-m}^\alpha \frac{1}{z^m} \tag{6}$$

$$f_+^\alpha(z) \;\; = \;\; \sum_{m=1}^{L} F_{+m}^\alpha z^m \tag{7}$$

and f_0^α is a constant.

Δ can also be decomposed into a lowering, raising, and diagonal operator

$$\Delta \;\; = \;\; \Delta_- + \Delta_0 + \Delta_+, \tag{8}$$

where

$$\Delta_+ \;\; = f_+^+(z)T^+ + f_+^0(z)T^0 + f_+^-(z)T^- \tag{9}$$

$$\Delta_- \;\; = f_-^+(z)T^+ + f_-^0(z)T^0 + f_-^-(z)T^- \tag{10}$$

$$\Delta_0 \;\; = f_0^+ T^+ + f_0^0 T^0 + f_0^- T^- \tag{11}$$

2 Restriction of Δ to Pol_n

We are interested in finding the Δ operators that preserve the Pol_n space, in other words we want

$\Delta\Psi \in Pol_n$ for every $\Psi \in Pol_n$, or equivalently

$$\Delta : Pol_n \to Pol_n. \tag{12}$$

We need to establish conditions on Δ for (12) to hold. We first consider Δ_+. We apply Δ_+ to the basis

$\{z^0, z^1, ..., z^n\}$ of Pol_n, and impose the condition that the transformed basis is a set in Pol_n. So we need to

focus on $\Delta_+(z^n), \Delta_+(z^{n-1}), ..., \Delta_+(z^0)$.

Lemma 1 *Necessary and sufficient conditions for Δ_+ to preserve Pol_n are:*

$$\Delta_+(z^n) = 0 \tag{13}$$

$$\Delta_+(z^k) = A_{k,k+1}z^{k+1} + A_{k,k+2}z^{k+2} + ... + A_{k,n}z^n, \, for \, 0 \leqslant k \leqslant n-1$$

where $A_{k,k+1}, A_{k,k+2}, ..., A_{k,n}$ are real numbers. In other words we can say $\Delta_+(z^k) \in Span\{z^0, z^1, ..., z^n\}$

Proof

Consider the Δ_+ operator on the basis function z^k for $k = 0, 1, ...n,$

$$\Delta_+(z^k) = A_{k,k+1}z^{k+1} + A_{k,k+2}z^{k+2} + ... + A_{k,n}z^{k+L} \tag{14}$$

If $k + L \leqslant n$, then $\Delta_+(z^k) \in Span\{z^0, z^1, ..., z^n\}$, and the result holds. We need restrictions if $k + L > n$, which would prevent $\Delta_+\Psi(z)$ being outside our Pol_n space. For $k = n$, $n + L > n$ because $L > 0$. So the constraint is

$$\Delta_+(z^n) = 0 \tag{15}$$

Which is our first condition of Lemma 1.

For $k = n - 1$, then $k + L = n - 1 + L$, therefore we must have constraints when $L > 1$ such that the polynomial does not exceed degree n. For example, for $L > 1$ and $k = n - 1$ we have the constraint

$$\Delta_+(z^{n-1}) = A_{n-1,n}z^n \tag{16}$$

For $k = n - 2$, then $k + L = n - 2 + L$, therefore we must have constraints when $L > 2$. Likewise, for $k = 0$, then $0 + L > n$, therefore we must have constraints when $L > n$.

Thus for $k > n - L$ we have

$$\Delta_+(z^k) = A_{k,k+1}z^{k+1} + A_{k,k+2}z^{k+2} + ... + A_{k,n}z^n \tag{17}$$

Which is the second condition from Lemma 1. $\quad \square$

Remark For $k \leqslant n - L$ we can still write

$$\Delta_+(z^k) = A_{k,k+1}z^{k+1} + A_{k,k+2}z^{k+2} + ... + A_{k,n}z^n \tag{18}$$

and look at it as a parameterization in terms of the constants $A_{k,m}$ and not as a constraint. There will be instances where we will need to produce more equations than just the necessary constraints in order to solve a system of equations. These conditions from Lemma 1 now become additional equations that will help us determine parameters for our unknown Laurent polynomials in the system, thus it is termed a parameterization. For example, in Section 7 our Δ operator for the two-dimensional case will not only depend on the entries from the associated matrix, but also on free parameters that come directly from the equations that appear in Lemma 1.

So

$$\Delta_+(z^k) \quad = \quad A_{k,k+1}z^{k+1} + A_{k,k+2}z^{k+2} + ... + A_{k,n}z^n \tag{19}$$

for $0 \leqslant k \leqslant n-1$ and

$$\Delta_+(z^n) \quad = \quad 0 \tag{20}$$

Of course, for $k \leqslant n - L$

$$A_{k,n} \quad = \quad 0 \tag{21}$$

$$A_{,n-1} \quad = \quad 0 \tag{22}$$

$$\vdots \tag{23}$$

$$A_{k,k+L+1} \quad = \quad 0 \tag{24}$$

Let us now consider Δ_-. We apply Δ_- to the basis of Pol_n, $\{z^0, z^1, ..., z^n\}$, and impose the condition that the transformed basis is a set in Pol_n. So we need to focus on $\Delta_-(z^n), \Delta_-(z^{n-1}), ..., \Delta_-(z^0)$.

Lemma 2 *Necessary and sufficient conditions for Δ_- to preserve Pol_n are:*

$$\Delta_-(z^0) \quad = \quad 0 \tag{25}$$

$$\Delta_-(z^k) \quad = \quad B_{k,0}z^0 + B_{k,1}z^1 + ... + B_{k,k-1}z^{k-1}$$

for $k < L$ the conditions (25) impose constraints on Δ. For $k \geqslant L$ the conditions (25) are naturally fulfilled.

Proof

Consider the Δ_- operator on the basis z^k for $k = 0, 1, ...n$,

$$\Delta_-(z^k) \quad = \quad B_{k,k-L}z^{k-L} + ... + B_{k,k-2}z^{k-2} + B_{k,k-1}z^{k-1} \tag{26}$$

We need restrictions if $k - L < 0$, which would prevent a polynomial $\Delta_-\Psi(z)$ outside our Pol_n space. For $k = 0$, $-L < 0$ which is always true because L is positive. So the constraint is

$$\Delta_-(z^0) \quad = \quad 0 \tag{27}$$

which is our first condition of Lemma 2.

For $k = 1$, then $1 - L < 0$, therefore we must have constraints when $L > 1$ such that the polynomial does not contain negative degrees. For example, for $L > 1$ and $k = 1$ we have the constraint

$$\Delta_-(z^1) \quad = \quad B_{1,0}z^0 \tag{28}$$

For $k = 2$, then $2 - L < 0$, therefore we must have constraints when $L > 2$. Likewise, for $k = n$, then $n - L < 0$, therefore we must have constraints when $L > n$.

Thus for $k < L$ we have

$$\Delta_-(z^k) \quad = \quad B_{k,0}z^0 + B_{k,1}z^1 + ... + B_{k,k-1}z^{k-1} \tag{29}$$

which is the second condition from Lemma 2. $\square$

Remark Similarly to Δ_+, for $k > L$ we can still write

$$\Delta_-(z^k) \quad = \quad B_{k,0}z^0 + B_{k,1}z^1 + ... + B_{k,k-1}z^{k-1} \tag{30}$$

and look at it as a parametrization in terms of the constants $B_{k,m}$ and not as a constraint. So

$$\Delta_-(z^k) \quad = \quad B_{k,0}z^0 + B_{k,1}z^1 + ... + B_{k,k-1}z^{k-1} \tag{31}$$

for $0 < k \leqslant n$ and

$$\Delta_-(z^0) \quad = \quad 0 \tag{32}$$

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Of course, for $k > L$

$$B_{k,0} = 0 \tag{33}$$

$$B_{k,1} = 0 \tag{34}$$

$$\vdots \tag{35}$$

$$B_{k,k-L-1} = 0 \tag{36}$$

3 Δ Operator

Theorem 1

For $q \neq \pm 1$, there exist Laurent polynomials $f^\alpha(z), \alpha = -, 0, +$, for which Δ maps Pol_n into Pol_n for any

$n \in \mathbb{N}$

Proof

We have a total of three unknown Laurent polynomials $f^+(z), f^0(z), f^-(z)$. From Lemma 1 and Lemma 2,
$k > n-2$ from Δ_+ and $k < 2$ from Δ_-, which give us four constraints,

$$\Delta_+(z^n) = q^n f_+^+(z) + f_+^0(z) + q^{-n} f_+^-(z) = 0 \tag{37}$$

$$\Delta_+(z^{n-1}) = q^{n-1} f_+^+(z) + f_+^0(z) + q^{-n+1} f_+^-(z) = A_{n-1,n} z$$

$$\Delta_-(z^0) = f_-^+(z) + f_-^0(z) + f_-^-(z) = 0$$

$$\Delta_-(z^{-1}) = q f_-^+(z) + f_-^0(z) + q^{-1} f_-^-(z) = B_{1,0} z^{-1}$$

We can look at the four constraints as a system of equations for $f_+^+(z), f_+^0(z), f_+^-(z), f_-^+(z), f_-^0(z), f_-^-(z)$.
However, we have six unknowns and only four equations. We can add two more equations by noting the
following from Lemma 1 and Lemma 2, $\Delta_+(z^{n-2}) = A_{n-2,n} z^n + A_{n-2,n-1} z^{n-1}$ and
$\Delta_-(z^2) = B_{2,0} z^0 + B_{2,1} z^1$. This relation is not a constraint on Δ, but it provides a way of parameterizing
the six unknown functions in terms of $A_{n-2,n}, A_{n-2,n-1}, A_{n-2,n}, B_{1,0}, B_{2,0}, B_{2,1}$. So now we have two
additional equations

$$q^{n-2} f_+^+(z) + f_+^0(z) + q^{-n+2} f_+^-(z) = A_{n-2,n} z^2 + A_{n-2,n-1} z \tag{38}$$

$$q^2 f_-^+(z) + f_-^0(z) + q^{-2} f_-^-(z) = B_{2,0} z^{-2} + B_{2,1} z^{-1}$$

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This technique of determining these conditions was used by P.B. Weigmann and A.V. Zabrodin in [1]. In this thesis we generalize their method.

We are then able to determine the decompositions of the Laurent polynomials by solving this system of linear equations.The solutions for the six unknown polynomials are

$$
\begin{aligned}
f_+^+(z) &= \frac{-q^{2-n}z(A_{n-1,n} + qA_{n-1,n} - zA_{n-2,n} - A_{n-2,n-1})}{(q-1)^2(1+q)} \\
f_+^0(z) &= \frac{zA_{n-1,n} + q^2zA_{n-1,n} - qz^2A_{n-2,n} - qzA_{n-2,n-1}}{(q-1)^2} \\
f_+^-(z) &= \frac{q^n z(-A_{n-1,n} - qA_{n-1,n} + qzA_{n-2,n} + qA_{n-2,n-1})}{(q-1)^2(q+1)} \\
f_-^+(z) &= -\frac{zB_{1,0} + qzB_{1,0} - qB_{2,0} - qzB_{2,1}}{(q-1)^2(1+q)z^2} \\
f_-^0(z) &= -\frac{-zB_{1,0} - q^2zB_{1,0} + qB_{2,0} + qzB_{2,1}}{(q-1)^2z^2} \\
f_-^-(z) &= -\frac{q^2(zB_{1,0} + qzB_{1,0} - B_{2,0} - zB_{2,1})}{(q-1)^2(1+q)z^2}
\end{aligned}
$$

therefore now $f^+(z), f^0(z)$ and $f^-(z)$ are functions dependent on $A_{n-1,n}, A_{n-2,n}, A_{n-2,n-1}$, $B_{1,0}, B_{2,0}, B_{2,1}, f_0^+, f_0^0, f_0^-$ and q

$$
\begin{aligned}
f^+(z) &= \frac{-q^{2-n}z(A_{n-1,n} + qA_{n-1,n} - zA_{n-2,n} - A_{n-2,n-1})}{(q-1)^2(1+q)} \\
&\quad - \frac{zB_{1,0}qz + B_{1,0} - qB_{2,0} - qzB_{2,1}}{(q-1)^2(1+q)z^2} + f_0^+ \tag{39}
\end{aligned}
$$

$$
\begin{aligned}
f^0(z) &= -\frac{-zA_{n-1,n} - q^2zA_{n-1,n} + qz^2A_{n-2,n} + qzA_{n-2,n-1}}{(q-1)^2} \\
&\quad - \frac{-zB_{1,0} - q^2zB_{1,0} + qB_{2,0} + qzB_{2,1}}{(q-1)^2z^2} + f_0^0 \tag{40}
\end{aligned}
$$

$$
\begin{aligned}
f^-(z) &= \frac{q^n z(-A_{n-1,n} - qA_{n-1,n} + qzA_{n-2,n} + qA_{n-2,n-1})}{(q-1)^2(q+1)} \\
&\quad - \frac{q^2(zB_{1,0} + qzB_{1,0} - B_{2,0} - zB_{2,1})}{(q-1)^2(1+q)z^2} + f_0^- \tag{41}
\end{aligned}
$$

4 Expanding Δ for powers of q from λ to Λ

So far we have only considered q shifts up to the first degree. Let us expand our Δ operator to larger powers of q, in other words we can rewrite our Δ from (3) as

$$
\Delta = \sum_{\lambda=1}^{\Lambda} f^{\lambda+}(z)T^{\lambda+} + f^0(z)T^0 + \sum_{\lambda=1}^{\Lambda} f^{\lambda-}(z)T^{\lambda-} \tag{42}
$$

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where $T^{\lambda+}$ represents the positive q-shifts, $T^{\lambda+} = \Psi(q^\lambda z)$ for $\lambda = 1, 2, ...\Lambda$, and $T^{\lambda-}$ represents the negative q-shifts, $T^{\lambda-} = \Psi(q^{-\lambda} z)$ for $\lambda = 1, 2, ...\Lambda$. The $f^{\lambda+}(z)$ and $f^{\lambda-}(z)$ are the Laurent polynomials corresponding to their respective q-shift Ψ function that they act upon.

Definition of $\Delta_+, \Delta_0, \Delta_-$

Now we can rewrite the z-decomposition of Δ as the following:

$$\Delta_+ \quad = \quad \sum_{\lambda=1}^{\Lambda} f_+^{\lambda+}(z)T^{\lambda+} + f_+^0(z)T^0 + \sum_{\lambda=1}^{\Lambda} f_+^{\lambda-}(z)T^{\lambda-} \tag{43}$$

$$\Delta_0 \quad = \quad \sum_{\lambda=1}^{\Lambda} f_0^{\lambda+}(z)T^{\lambda+} + f_0^0(z)T^0 + \sum_{\lambda=1}^{\Lambda} f_0^{\lambda-}(z)T^{\lambda-} \tag{44}$$

$$\Delta_- \quad = \quad \sum_{\lambda=1}^{\Lambda} f_-^{\lambda+}(z)T^{\lambda+} + f_-^0(z)T^0 + \sum_{\lambda=1}^{\Lambda} f_-^{\lambda-}(z)T^{\lambda-} \tag{45}$$

Finding Δ

We now have $\Lambda + 1 + \Lambda = 2\Lambda + 1$ unknown Laurent polynomials, namely $f^{\lambda+}(z), f^0(z)$ and $f^{\lambda-}(z)$, $\lambda = 1, 2, ...\Lambda$. We will find the polynomials $f_+^{\lambda+}(z), f_+^0(z)$ and $f_+^{\lambda-}(z)$ from the constraints on Δ_+ and we will find the polynomials $f_-^{\lambda+}(z), f_-^0(z)$ and $f_-^{\lambda-}(z)$ from the constraints on Δ_-. The numbers $f_0^{\lambda+}, f_0^0$ and $f_0^{\lambda-}$ do not impose any constraints.

Lemma 3 *Necessary and sufficient conditions for Δ_+ to preserve Pol_n for an expanded Δ are:*

$$\sum_{\lambda=1}^{\Lambda} f_+^{\lambda+}(z)q^{\lambda k} + f_+^0(z) + \sum_{\lambda=1}^{\Lambda} f_+^{\lambda-}(z)q^{-\lambda k} \tag{46}$$

$$= \quad A_{k,k+1}z + A_{k,k+2}z^2 + ... + A_{k,n}z^{n-k} \tag{47}$$

and

$$\sum_{\lambda=1}^{\Lambda} f_+^{\lambda+}(z)q^{\lambda n} + f_+^0(z) + \sum_{\lambda=1}^{\Lambda} f_+^{\lambda-}(z)q^{-\lambda n} \quad = \quad 0 \tag{48}$$

For $k > n - L$ the relations (46) impose constraints on Δ_+. For $k \leqslant n - L$ the relations (46) are naturally fulfilled.

Proof We have the following Δ_+ constraints from Lemma 1

$$\Delta_+(z^n) \quad = \quad 0 \tag{49}$$

$$\Delta_+(z^k) \quad = \quad A_{k,k+1}z^{k+1} + A_{k,k+2}z^{k+2} + ... + A_{k,n}z^n \tag{50}$$

for $k > n - L$.

We now use (43) to find $\Delta_+(z^k)$ and $\Delta_+(z^n)$. The result will be equated with the above constraint.

$$\Delta_+(z^k) \;=\; \sum_{\lambda=1}^{\Lambda} f_+^{\lambda+}(z)T^{\lambda+}(z^k) + f_+^0(z)T^0(z^k) + \sum_{\lambda=1}^{\Lambda} f_+^{\lambda-}(z)T^{\lambda-}(z^k) \tag{51}$$

$$\Delta_+(z^n) \;=\; \sum_{\lambda=1}^{\Lambda} f_+^{\lambda+}(z)T^{\lambda+}(z^n) + f_+^0(z)T^0(z^n) + \sum_{\lambda=1}^{\Lambda} f_+^{\lambda-}(z)T^{\lambda-}(z^n) \tag{52}$$

which can be rewritten as

$$\Delta_+(z^k) \;=\; \sum_{\lambda=1}^{\Lambda} f_+^{\lambda+}(z)(zq^{\lambda})^k + f_+^0(z)z^k + \sum_{\lambda=1}^{\Lambda} f_+^{\lambda-}(z)(zq^{-\lambda})^k \tag{53}$$

for $k > n - L$ and

$$\Delta_+(z^n) \;=\; \sum_{\lambda=1}^{\Lambda} f_+^{\lambda+}(z)(zq^{\lambda})^n + f_+^0(z)z^n + \sum_{\lambda=1}^{\Lambda} f_+^{\lambda-}(z)(zq^{-\lambda})^n \tag{54}$$

Applying the constraints we get

$$\sum_{\lambda=1}^{\Lambda} f_+^{\lambda+}(z)(zq^{\lambda})^k + f_+^0(z)z^k + \sum_{\lambda=1}^{\Lambda} f_+^{\lambda-}(z)(zq^{-\lambda})^k \tag{55}$$

$$= \; A_{k,k+1}z^{k+1}A_{k,k+2}z^{k+2} + ... + A_{k,n}z^n \tag{56}$$

for $k > n - L$ and

$$\sum_{\lambda=1}^{\Lambda} f_+^{\lambda+}(z)(zq^{\lambda})^n + f_+^0(z)z^n + \sum_{\lambda=1}^{\Lambda} f_+^{\lambda-}(z)(zq^{-\lambda})^n \;=\; 0 \tag{57}$$

These can both be simplified to

$$\sum_{\lambda=1}^{\Lambda} f_+^{\lambda+}(z)q^{\lambda k} + f_+^0(z) + \sum_{\lambda=1}^{\Lambda} f_+^{\lambda-}(z)q^{-\lambda k} \tag{58}$$

$$= \; A_{k,k+1}z + A_{k,k+2}z^2 + ... + A_{k,n}z^{n-k} \tag{59}$$

for $k > n - L$ and

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$$\sum_{\lambda=1}^{\Lambda} f_+^{\lambda+}(z)q^{\lambda n} + f_+^0(z) + \sum_{\lambda=1}^{\Lambda} f_+^{\lambda-}(z)q^{-\lambda n} \quad = \quad 0 \tag{60}$$

These are the two relations that result in constraints that preserve Pol_n. $\quad\square$

Similar to the argument from earlier, if more equations are needed to establish a system of equations then we can parameterize using

$$\sum_{\lambda=1}^{\Lambda} f_+^{\lambda+}(z)q^{\lambda k} + f_+^0(z) + \sum_{\lambda=1}^{\Lambda} f_+^{\lambda-}(z)q^{-\lambda k} \tag{61}$$

$$= \quad A_{k,k+1}z + A_{k,k+2}z^2 + ... + A_{k,n}z^{n-k} \tag{62}$$

for $k \leqslant n - L$.

Lemma 4 *Necessary and sufficient conditions for Δ_- to preserve Pol_n for an expanded Δ are:*

$$\sum_{\lambda=1}^{\Lambda} f_+^{\lambda+}(z)q^{\lambda k} + f_+^0(z) + \sum_{\lambda=1}^{\Lambda} f_+^{\lambda-}(z)q^{-\lambda k} \tag{63}$$

$$= \quad B_{k,0}z^0 + B_{k,1}z^1 + ... + B_{k,k-1}z^{k-1} \tag{64}$$

and

$$\sum_{\lambda=1}^{\Lambda} f_+^{\lambda+}(z) + f_+^0(z) + \sum_{\lambda=1}^{\Lambda} f_+^{\lambda-}(z) \quad = \quad 0 \tag{65}$$

For $k < L$ the relations (63) impose constraints on Δ_-. For $k \geqslant L$ the relations (63) are naturally fulfilled.

Proof We have the following Δ_- constraints from Lemma 2

$$\Delta_-(z^0) \quad = \quad 0 \tag{66}$$

$$\Delta_-(z^k) \quad = \quad B_{k,0}z^0 + B_{k,1}z^1 + ... + B_{k,k-1}z^{k-1} \tag{67}$$

for $k < L$.

We now use (45) to find $\Delta_-(z^k)$ and $\Delta_-(z^0)$. The result will be equated with the above constraint.

$$\Delta_-(z^k) \quad = \quad \sum_{\lambda=1}^{\Lambda} f_-^{\lambda+}(z)T^{\lambda+}(z^k) + f_-^0(z)T^0(z^k) + \sum_{\lambda=1}^{\Lambda} f_-^{\lambda-}(z)T^{\lambda-}(z^k) \tag{68}$$

$$\Delta_-(z^0) \;=\; \sum_{\lambda=1}^{\Lambda} f_-^{\lambda+}(z)T^{\lambda+} + f_-^0(z)T^0 + \sum_{\lambda=1}^{\Lambda} f_-^{\lambda-}(z)T^{\lambda-} \tag{69}$$

which can be rewritten as

$$\Delta_-(z^k) \;=\; \sum_{\lambda=1}^{\Lambda} f_-^{\lambda+}(z)(zq^\lambda)^k + f_-^0(z)z^k + \sum_{\lambda=1}^{\Lambda} f_-^{\lambda-}(z)(zq^{-\lambda})^k \tag{70}$$

for $k < L$ and

$$\Delta_-(z^0) \;=\; \sum_{\lambda=1}^{\Lambda} f_-^{\lambda+}(z) + f_-^0(z) + \sum_{\lambda=1}^{\Lambda} f_-^{\lambda-}(z) \tag{71}$$

Applying the constraints we get

$$\sum_{\lambda=1}^{\Lambda} f_-^{\lambda+}(z)(zq^\lambda)^k + f_-^0(z)z^k + \sum_{\lambda=1}^{\Lambda} f_-^{\lambda-}(z)(zq^{-\lambda})^k \tag{72}$$
$$= \quad B_{k,0}z^0 + B_{k,1}z^1 + ... + B_{k,k-1}z^{k-1} \tag{73}$$

for $k < L$ and

$$\sum_{\lambda=1}^{\Lambda} f_-^{\lambda+}(z) + f_-^0(z) + \sum_{\lambda=1}^{\Lambda} f_-^{\lambda-}(z) \;=\; 0 \tag{74}$$

These can both be simplified to

$$\sum_{\lambda=1}^{\Lambda} f_-^{\lambda+}(z)q^{\lambda k} + f_-^0(z) + \sum_{\lambda=1}^{\Lambda} f_-^{\lambda-}(z)q^{-\lambda k} \tag{75}$$
$$= \quad B_{k,0}z^{-k} + B_{k,1}z^{1-k} + ... + B_{k,k-1}z^{-1} \tag{76}$$

for $k < L$ and

$$\sum_{\lambda=1}^{\Lambda} f_-^{\lambda+}(z)q^{\lambda n} + f_-^0(z) + \sum_{\lambda=1}^{\Lambda} f_-^{\lambda-}(z)q^{-\lambda n} \;=\; 0 \tag{77}$$

These are the two relations that result in constraints that preserve Pol_n. $\square$

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Similar to the argument from earlier, if more equations are needed to establish a system of equations then we can parameterize using

$$\sum_{\lambda=1}^{\Lambda} f_{-}^{\lambda+}(z)q^{\lambda k} + f_{-}^{0}(z) + \sum_{\lambda=1}^{\Lambda} f_{-}^{\lambda-}(z)q^{-\lambda k} \tag{78}$$

$$= B_{k,0}z^{-k} + B_{k,1}z^{1-k} + ... + B_{k,k-1}z^{-1} \tag{79}$$

for $k \geqslant L$

5 Δ Operator for $\Lambda \geqslant 1$

Theorem 2

Let $L, \Lambda \in \mathbb{N}$ and $q \neq \pm 1$. If $n \in \mathbb{N} \cup \{0\}, n \geqslant L - 1$, then there exist Laurent polynomials $f^{\lambda\alpha}(z), 1 \leqslant \lambda \leqslant \Lambda, k = -, 0, +$, each of order L, for which Δ_+ and Δ_- each map Pol_n into Pol_n.

Proof

Given $\Lambda \geqslant 1$, then we have $\Lambda + 1 + \Lambda = 2\Lambda + 1$ unknown polynomials, namely $f_{+}^{\lambda+}(z), f_{+}^{0}(z)$ and $f_{+}^{\lambda-}(z)$, $\lambda = 1, 2, ...\Lambda$.

We always have one constraint on Δ_+, namely $\Delta_+(z^n) = 0$. We have constraints on the basis z^k when $k > n - L$, in other words when $k = n - L + 1, n - L + 2, ..., n - 1$. This provides us with a total of $1 + L - 1 = L$ constraints. We do not want to have more constraints than unknown polynomials, so $L \leqslant 2\Lambda + 1$.

The largest L can be is $2\Lambda + 1$ and when $L < 2\Lambda + 1$ we need to add more equations to be able to solve for our unknown polynomials. By noting the relation on z^k when $k \leqslant n - L$ is naturally fulfilled, we can use this relation to add equations and parameterize our unknowns. So $0 < L \leqslant 2\Lambda + 1$.

However, Lemma 3 does not hold for all n. Lemma 3 gives us the constraint

$$\sum_{\lambda=1}^{\Lambda} f_{+}^{\lambda+}(z)q^{\lambda k} + f_{+}^{0}(z) + \sum_{\lambda=1}^{\Lambda} f_{+}^{\lambda-}(z)q^{-\lambda n} \tag{80}$$

$$= A_{k,k+1}z + A_{k,k+2}z^2 + ... + A_{k,n}z^{n-k} \tag{81}$$

For $k > n - L$. We need to make sure k is always positive. Our necessary constraints on z^k are when $k = n - L + 1, n - L + 2, ..., n - 1$. So the smallest necessary constraint is when $k = n - L + 1$, therefore k will always be positive when $n - L + 1 \geqslant 0$, which gives us $n \geqslant L - 1$.

For example, if we let $\Lambda = 1$ and $L = 2\Lambda + 1$, so there are restrictions on z^k when $k = n - 2, n - 1$. If $n = 1$, so $n < L - 1$, then $k = -1, 0$ and we can not have k negative thus the restriction does not hold. Therefore given $\Lambda \geqslant 1$ then Δ_+ will preserve Pol_n if

$$
\begin{array}{ll}
L = 1 & n = 0, 1, 2, 3... \\
L = 2 & n = 1, 2, 3, ... \\
L = 3 & n = 2, 3, ... \\
\quad \vdots & \quad \vdots \\
L = 2\Lambda + 1 & n = 2\Lambda, 2\Lambda + 1, ...
\end{array}
$$

.

Similarly, given $\Lambda \geqslant 1$, then we have $\Lambda + 1 + \Lambda = 2\Lambda + 1$ unknown polynomials, namely $f_-^{\lambda+}(z), f_-^0(z)$ and $f_-^{\lambda-}(z)$, $\lambda = 1, 2, ...\Lambda$.

We always have one constraint on Δ_-, namely $\Delta_-(z^0) = 0$. We have constraints on the basis z^k when $k < L$, in other words when $k = L - 1, L - 2, ..., 1$. This provides us with a total of $1 + L - 1 = L$ constraints. We do not want to have more constraints then unknown polynomials, so $L \leqslant 2\Lambda + 1$.

The largest L can be is $2\Lambda + 1$ and when $L < 2\Lambda + 1$ we need to add more equations to be able to solve for our unknown polynomials. By noting the relation on z^k when $k \geqslant L$ is naturally fulfilled, we can use this relation to add equations and parameterize our unknowns. So $0 < L \leqslant 2\Lambda + 1$.

However, Lemma 4 does not hold for all n. Lemma 4 gives us the constraint

$$
\sum_{\lambda=1}^{\Lambda} f_-^{\lambda+}(z) q^{\lambda k} + f_-^0(z) + \sum_{\lambda=1}^{\Lambda} f_-^{\lambda-}(z) q^{-\lambda k} \tag{82}
$$
$$
= \quad B_{k,0} z^{-k} + B_{k,1} z^{1-k} + ... + B_{k,k-1} z^{-1} \tag{83}
$$

for $k < L$. We need to make sure k is always positive. Our necessary constraints on z^k are when $k = L - 1, L - 2, ..., 1$. We can not have our basis bigger than n, so $L - 1 \leqslant n$. This gives us the same conditions as Δ_+. Therefore there exist Laurent polynomials $f^{\lambda\alpha}(z), 1 \leqslant \lambda \leqslant \Lambda, k = -, 0, +$, for which Δ_+ and Δ_- each map Pol_n into Pol_n. $\quad \square$

6 Definition of the mapping J from R^{n+1} to Pol_n

Definition :

The mapping

$$
J_q : R^{n+1} \mapsto Pol_n \tag{84}
$$

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sends a vector $v = \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_n \end{pmatrix}$ into the polynomial $\Psi_v(z) = \Psi_0 + \Psi_1 z + \Psi_2 z^2 + ... + \Psi_n z^n$ whose coefficients

are determined by the following $n + 1$ rules

$$\Psi_v(q^k) = v_k \tag{85}$$

for $k = 0, 1, ..., n$. In principle $\Psi_0, \Psi_1, ... \Psi_n$ should carry the index v. However, for clarity of notation we will use the vector index v only for $\Psi_v(z)$.

Note that, to find the coefficients $\Psi_k, k = 0, .., n$, we can use the inverse of a Vandermonde matrix. The paper [2] defines a Vandermonde matrix as

$$v = \begin{pmatrix} 1 & 1 & ... & 1 \\ x_0 & x_1 & ... & x_n \\ x_0^2 & x_1^2 & ... & x_n^2 \\ \vdots & \vdots & \vdots & \vdots \\ x_0^n & x_1^n & ... & x_n^n \end{pmatrix} \tag{86}$$

Our rule $\Psi_v(q^k) = v_k$ implies

$$\Psi_0 + \Psi_1 + \Psi_2 + ... + \Psi_n \quad = \quad v_0 \tag{87}$$

$$\Psi_0 + \Psi_1 q + \Psi_2 q^2 + ... + \Psi_n q^n \quad = \quad v_1 \tag{88}$$

$$\vdots \quad = \quad \vdots \tag{89}$$

$$\Psi_0 + \Psi_1 q^n + \Psi_2 q^{2n} + ... + \Psi_n q^{n^2} \quad = \quad v_n \tag{90}$$

which can also be written as

$$\begin{pmatrix} 1 & 1 & ... & 1 \\ 1 & q & ... & q^n \\ 1 & q^2 & ... & q^{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & q^n & ... & q^{n^2} \end{pmatrix} \begin{pmatrix} \Psi_0 \\ \Psi_1 \\ \Psi_2 \\ \vdots \\ \Psi_n \end{pmatrix} = \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \tag{91}$$

Notice that the matrix above is a $(n + 1) \times (n + 1)$ Vandermonde matrix of the form $x_k = q^k$ for $k = 0, ..., n$. So, to solve for $\Psi_k, k = 0, 1, 2, ..., n$, we need the inverse Vandermonde matrix V^{-1}. Since V must be invertible, our q must satisfy the rule that $det(V) \neq 0$, thus V^{-1} does not exist for all q.

Lemma 5

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The mapping $J_q : R^{n+1} \mapsto Pol_n$ is linear.

Proof

We need to show that

(a)

$$J_q(\alpha v) \quad = \quad \alpha J_q(v) \tag{92}$$

Using the definition of J_q

$$\Psi_{\alpha v}(q^k) \quad = \quad (\alpha v)_k = \alpha(v_k) = \alpha \Psi_v(q^k) \tag{93}$$

where v_k is the k component of the vector v. We also need to show that

(b)

$$J_q(v_1 + v_2) \quad = \quad J_q(v_1) + J_q(v_2) \tag{94}$$

Using the definition of J_q, we have that

$$\Psi_{v_1+v_2}(q^k) \quad = \quad (v_1 + v_2)_k = v_{1_k} + v_{2_k} = \Psi_{v_1}(q^k) + \Psi_{v_2}(q^k) \tag{95}$$

where $(v_1 + v_2)_k$ is the k component of the vector $(v_1 + v_2)$. Therefore J_q is linear. $\quad \square$

In what follows we will work mainly with 2×2 and 3×3 matrices. For these cases we will work the

mapping J_q in detail below. For simplicity, we will omit the q subscript from J_q in what follows.

6.1 The mapping $J : R^2 \mapsto Pol_1$

The 2-dimensional case

$$v \quad = \quad \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \in R^2 \tag{96}$$

is mapped by J into $\Psi(z) = \Psi_0 + \Psi_1 z$. Using our rule that $\Psi(q^k) = v_k$,

$$v_0 \quad = \quad \Psi_0 + \Psi_1 \tag{97}$$

$$v_1 \quad = \quad \Psi_0 + \Psi_1 q \tag{98}$$

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or in matrix form

$$\begin{pmatrix} v_0 \\ v_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & q \end{pmatrix} \begin{pmatrix} \Psi_0 \\ \Psi_1 \end{pmatrix} \tag{99}$$

The coefficients matrix

$$V = \begin{pmatrix} 1 & 1 \\ 1 & q \end{pmatrix} \tag{100}$$

is invertible when $q \neq 1$, and

$$V^{-1} = \frac{1}{q-1} \begin{pmatrix} q & -1 \\ -1 & 1 \end{pmatrix} \tag{101}$$

We use the inverse Vandermonde to solve for Ψ_0 and Ψ_1,

$$\begin{pmatrix} \Psi_0 \\ \Psi_1 \end{pmatrix} = V^{-1} \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} = \begin{pmatrix} \frac{qv_0 - v_1}{q-1} \\ \frac{-v_0 + v_1}{q-1} \end{pmatrix} \tag{102}$$

Now our vector is mapped into the polynomial

$$\Psi(z) = \frac{qv_0 - v_1}{q-1} + \frac{-v_0 + v_1}{q-1} z \tag{103}$$

where the values for v_0, v_1 are taken from the original vector.

6.2 The mapping $J : R^3 \mapsto Pol_2$

Following a similar argument as the 2-dimensional case, we are now mapping a 3-dimensional vector

$$v = \begin{pmatrix} v_0 \\ v_1 \\ v_2 \end{pmatrix} \in R^3 \tag{104}$$

to $\Psi_v(z) = \Psi_0 + \Psi_1 z + \Psi_2 z^2$. Using our rule that $\Psi_v(q^k) = v_k$,

$$v_0 = \Psi_0 + \Psi_1 + \Psi_2 \tag{105}$$

$$v_1 = \Psi_0 + \Psi_1 q + \Psi_2 q^2 \tag{106}$$

$$v_2 = \Psi_0 + \Psi_1 q^2 + \Psi_2 q^4 \tag{107}$$

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or in matrix form

$$
\begin{pmatrix} v_0 \\ v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & q & q^2 \\ 1 & q^2 & q^4 \end{pmatrix} \begin{pmatrix} \Psi_0 \\ \Psi_1 \\ \Psi_2 \end{pmatrix} \tag{108}
$$

The coefficients matrix

$$
V = \begin{pmatrix} 1 & 1 & 1 \\ 1 & q & q^2 \\ 1 & q^2 & q^4 \end{pmatrix} \tag{109}
$$

$$
\tag{110}
$$

is invertible if $q \neq 0, \pm 1$, with inverse

$$
V^{-1} = \frac{1}{(q-1)^2} \begin{pmatrix} \frac{q^3}{q+1} & -q & \frac{1}{q+1} \\ -q & \frac{q^2+1}{q} & -\frac{1}{q} \\ \frac{1}{q+1} & -\frac{1}{q} & \frac{1}{q^2+q} \end{pmatrix} \tag{111}
$$

We use our V^{-1} to solve for Ψ_0, Ψ_1, Ψ_2,

$$
\Psi_0 = \frac{q^3 v_0 + q v_1 + q^2 v_1 - v_2}{(q-1)^2 (1+q)}
$$

$$
\Psi_1 = \frac{q^2 v_0 - v_1 - q^2 v_1 + v_2}{(q-1)^2 q}
$$

$$
\Psi_2 = \frac{-q v_0 + v_1 + q v_1 - v_2}{(q-1)^2 q (1+q)}
$$

where the values for v_0, v_1, v_2 are taken from the original vector. So,

$$
\Psi_v(z) = \frac{(q^2 - z)(q(q-z)v_0 + (1+q)(z-1)v_1) + (z-1)(z-q)v_2}{(q-1)^2 q (1+q)}
$$

which is defined for $q \neq 0, \pm 1$

7   Mapping a linear transformation $M : R^{n+1} \mapsto R^{n+1}$ to the operator $\Delta_M : Pol_n \mapsto Pol_n$

For each linear transformation defined by a matrix M from R^{n+1} to itself we are going to associate an

operator Δ_M, such that the following diagrams commute:

In other words, for each $v \in R^{n+1}$

$$
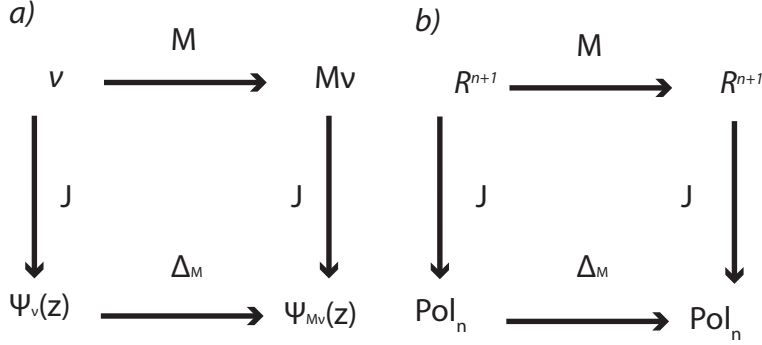\Delta_M J v = J M v \tag{112}
$$

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Figure 1: The mapping diagram, a)component b) space

The usefulness of the mapping of M into Δ will be clarified by the following theorem and its application.

Theorem from [1]

Let $n \in \mathbb{N}$. For any matrix M of the form

$$
\begin{pmatrix}
v_0 & a_0 & 0 & 0 & d_0 \\
d_1 & v_1 & a_1 & 0 & 0 \\
0 & d_2 & ... & ... & 0 \\
0 & 0 & ... & ... & a_{n-1} \\
a_n & 0 & 0 & d_n & v_n
\end{pmatrix},
\tag{113}
$$

there exists an operator $\Delta_M : Pol_n \mapsto Pol_n$, of the form (42) with $L = 2$ and $\lambda = 1$, for which

$\Delta_M J v = J M v.$

The matrices from the previous theorem are very special, they are tridiagonal. There are many situations for which the matrix is not tridiagonal. For example the transition matrix in a discrete Markov chain may not be tridiagonal for many applications encountered in epidemiology, demographics and biological system. We are thus led to explore the mapping of a general matrix M into the operator Δ.

7.1 2×2 **Matrix Mapping**

Consider a 2×2 matrix of the form

$$
M = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}.
\tag{114}
$$

We will now construct the corresponding Δ_M for $L = 2$ and $\lambda = 1$. For $\Psi_v \in Pol_1$, (3) yields

$$
\Delta_M(\Psi_v(z)) = f^+(z)\Psi_v(qz) + f^0(z)\Psi_v(z) + f^-(z)\Psi_v(q^{-1}z)
\tag{115}
$$

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From (39), (40), and (41), with $n = 1$ we can choose

$$
\begin{aligned}
f^+(z) &= \frac{-q^1 z(A_{0,1} + qA_{0,1} - zA_{-1,1} - A_{-1,0})}{(q-1)^2(1+q)} \\
&\quad - \frac{zB_{1,0}qz + B_{1,0} - qB_{2,0} - qzB_{2,1}}{(q-1)^2(1+q)z^2} + f_0^+ \\
f^0(z) &= -\frac{-zA_{0,1} - q^2 zA_{0,1} + qz^2 A_{-1,1} + qA_{-1,0}}{(q-1)^2} \\
&\quad - \frac{-zB_{1,0} - q^2 zB_{1,0} + qB_{2,0} + qzB_{2,1}}{(q-1)^2 z^2} + f_0^0 \\
f^-(z) &= \frac{q^1 z(-A_{0,1} - qA_{0,1} + qzA_{-1,1} + qA_{-1,0})}{(q-1)^2(q+1)} \\
&\quad - \frac{q^2(zB_{1,0} + qzB_{1,0} - B_{2,0} - zB_{2,1})}{(q-1)^2(1+q)z^2} + f_0^-
\end{aligned}
$$

where the coefficients $A_{j,k}$ and $B_{j,k}$ are defined in Lemmas 1 and 2.

Now we impose $\Psi_{Mv}(z) = \Delta_M(\Psi_v(z))$ for all $z \in C$ in order to solve for our parameters $A_{-1,1}, A_{-1,0}, A_{0,1}, B_{1,0}, B_{2,0}, B_{2,1}, f_0^-, f_0^+, f_0^0$. We know $\Psi_{Mv}(z)$ and $\Psi_v(z)$, so by equating $\Psi_{Mv}(z) = \Delta_M(\Psi_v(z))$ we can solve for the parameters in our unknown Laurent polynomials of Δ_M. The parameters depend on $p_{11}, p_{12}, p_{21}, p_{22}$, and are listed in the following system of equations:

$$
\begin{aligned}
A_{0,1} &= -\frac{-p_{11} + p_{12} - p_{21} - p_{22}}{q-1} \\
B_{1,0} &= -\frac{-qp_{11} - q^2 p_{12} + p_{21} + qp_{22}}{q-1} \\
f_0^0 &= -\frac{2q(p_{11} + p_{12} - p_{21} - p_{22})}{(q-1)^2} - \frac{-f_{10} + q^2 f_{10} + qp_{12} + p_{21} + p_{22} - qp_{22}}{q-1} \\
f_0^- &= -\frac{qf_{10} - q^2 f_{10} - qp_{12} - qp_{21}}{q-1} + \frac{(q+q^2)(p_{11} + p_{12} - p_{21} - p_{22})}{(q-1)^2}
\end{aligned}
$$

$A_{-1,1}, A_{-1,0}, B_{2,0}, B_{2,1}, f_0^+$ remain free parameters. Thus the Laurent polynomial mappings of a two dimensional matrix when $n = 1$ are

$$
\begin{aligned}
f^+(z) &= f_0^+ + \frac{(-1+q)qB_{2,0} + z((q-1)qB_{2,1} - (1+q)(-p_{21} + q(p_{11} + qp_{12} - p_{22})))}{(q-1)^3(1+q)z^2} \\
&\quad \frac{qz((q-1)zA_{-1,1} + (q-1)A_{-1,0} + (q+1)(p_{11} + p_{12} - p_{21} - p_{22}))}{(q-1)^3(1+q)} \\
f^0(z) &= -\frac{1}{(q-1)^2}(qz^2 A_{-1,1} + qzA_{-1,1} + \\
&\quad \frac{(q-1)qB_{2,0} + z((q-1)qB_{2,1} - (1+q^2)(-p_{21} + q(p_{11} + qp_{12} - p_{22})))}{(q-1)z^2}
\end{aligned}
$$

$$+2q(p_{11} + p_{12} - p_{21} - p_{22}) + \frac{z(p_{11} + p_{12} - p_{21} - p_{22})}{q - 1} + \frac{q^2 z(p_{11} + p_{12} - p_{21} - p_{22})}{q - 1}$$

$$+(q - 1)((q^2 - 1)f_0^+ + qp_{12} + p_{21} - (-(q - 1)p_{22})))$$

$$f^-(z) = \frac{1}{(q-1)^2}q((q-1)((q-1)f_0^+ + p_{12} + p_{21}) +$$

$$\frac{q((q-1)B_{2,0} + z((q-1)B_{2,1} - (1+q)(-p_{21} + q(p_{11} + qp_{12} - p_{22}))))}{(q^2 - 1)z^2}$$

$$+\frac{z((q-1)qzA_{-1,1} + (q-1)qA_{-1,0} + (1+q)(p_{11} + p_{12} - p_{21} - p_{22}))}{q^2 - 1} +$$

$$(1+q)(p_{11} + p_{12} - p_{21} - p_{22})$$

These Laurent polynomials still depend on a few free parameters, but when the operator Δ is applied to vector functions these free parameters disappear. When the operator Δ_M is applied on our function Ψ_v for the two-dimensional case, then these free parameters cancel out and do not appear in the resulting Ψ_{Mv}.

7.2 3×3 Matrix Mapping

Now consider a 3×3 matrix of the form

$$M = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix} \tag{116}$$

We will now construct the corresponding Δ_M for $L = 2$ and $\lambda = 1$. For $\Psi_v \in Pol_2$, (3) yields

$$\Delta_M(\Psi_v(z)) = f^+(z)\Psi_v(qz) + f^0(z)\Psi_v(z) + f^-(z)\Psi_v(q^{-1}z) \tag{117}$$

From (39), (40), and (41), with $n = 2$ we can choose

$$f^+(z) = \frac{-q^2 z(A_{1,2} + qA_{1,2} - zA_{0,2} - A_{0,1})}{(q - 1)^2(1 + q)}$$

$$-\frac{zB_{1,0}qz + B_{1,0} - qB_{2,0} - qzB_{2,1}}{(q - 1)^2(1 + q)z^2} + f_0^+$$

$$f^0(z) = -\frac{-zA_{1,2} - q^2 zA_{1,2} + qz^2 A_{0,2} + qA_{0,1}}{(q - 1)^2}$$

$$-\frac{-zB_{1,0} - q^2 zB_{1,0} + qB_{2,0} + qzB_{2,1}}{(q - 1)^2 z^2} + f_0^0$$

$$f^-(z) = \frac{q^2 z(-A_{1,2} - qA_{1,2} + qzA_{0,2} + qA_{0,1}}{(q - 1)^2(q + 1)}$$

$$-\frac{q^2(zB_{1,0} + qzB_{1,0} - B_{2,0} - zB_{2,1})}{(q - 1)^2(1 + q)z^2} + f_0^-$$

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where the coefficients $A_{j,k}$ and $B_{j,k}$ are defined in Lemmas 1 and 2.

Now we enforce $\Psi_{Mv}(z) = \Delta_M(\Psi_v(z))$ for all $z \in C$ in order to solve for our parameters $A_{-1,1}, A_{-1,0}, A_{0,1}, B_{1,0}, B_{2,0}, B_{2,1}, f_0^-, f_0^+, f_0^0$. Now all parameters can be solved for since there are nine equations, thus there will be no free parameters. The linear system of equations using the entries of the matrix M produces, as in the 2×2 case, the parameters

$$A_{1,2} = \frac{-p_{21} + p_{31} + q(p_{11} + qp_{12} + q^2 p_{13} - p_{21} - (q-1)(p_{22} + qp_{23}) + p_{32} + qp_{33})}{(q-1)^2 q(q+1)}$$

$$A_{0,2} = \frac{-p_{21} - p_{22} + q(p_{11} + p_{12} + p_{13} - p_{21} - p_{22} - p_{23}) - p_{23} + p_{31} + p_{32} + p_{33}}{(q-1)^2 q(1+q)}$$

$$A_{0,1} = \frac{p_{21} + p_{22} + p_{23} + q^2(-p_{11} - p_{12} - p_{13} + p_{21} + p_{22} + p_{23}) - p_{31} - p_{32} - p_{33}}{(q-1)^2 q}$$

$$B_{1,0} = \frac{q^3 p_{11} + q^4 p_{12} + p_{31} + q(q^4 p_{13} - (1+q)(p_{21} + q(p_{22} + qp_{23})) + p_{32} + qp_{33})}{(q-1)^2(1+q)}$$

$$B_{2,0} = \frac{q^3 p_{11} + q^5 p_{12} + p_{31} + q(q^6 p_{13} - (q+1)p_{21} + q(-q(1+q)(p_{22} + q^2 p_{23}) + p_{32} + q^2 p_{33}))}{(q-1)^2(1+q)}$$

$$A_{2,1} = \frac{p_{21} - p_{31} + q^2(-p_{11} + p_{21} + p_{22} - p_{32} + q^2(-p_{12} + p_{22} + p_{23} + q^2(-p_{13} + p_{23}) - p_{33}))}{(q-1)^2 q}$$

as well as three f parameters,

$$f_0^- = \frac{1}{(q-1)^4 q(1+q)^2}(2q^2(1+q+q^2)p_{11} + p_{13} - p_{21} + q(q^2(1+q(4+q))p_{12} \qquad (118)$$
$$+(1+q(2+q+q^2))p_{13} - 3p_{21} - p_{22} + p_{31} + 2p_{32} + p_{33} + q(- * 4 + q(3+q))p_{21}$$
$$-(3+q(4+q(3+q)))p_{22} - 2p_{23} + p_{33} + q(-(3+2q)(1+q^2)p_{23} + (1+q(2+q+q^3))p_{31}$$
$$+2p_{32} + (2+q+q^2)p_{33})))$$

$$f_0^0 = \frac{1}{(q-1)^4 q}(q^2(-2+q-2q^2)p_{11} - p_{13} + p_{21} + q(p_{21} + p_{22} - p_{32} +$$
$$q(1+q+q^2)p_{12} - (1+q^2)p_{13} + 2p_{21} + p_{23} - p_{31} - p_{32} - 2p_{33} + q((1+q)p_{21} +$$
$$(4+q^2)p_{22} + p_{23} - p_{32} + q((2+q+q^2)p_{23} - (1+q^2)p_{31} - 2p_{33}))))$$

$$f_0^+ = \frac{1}{(q-1)^4(1+q)^2}(p_{13} + q(q(1+q^2)(1+q+q^2)p_{11} + 2p^3(1+q+q^2)p_{12} -$$
$$2p_{21} + q((1+q(2+q+q^3))p_{13} - (3+2q)(1+q^2)p_{21} - p_{22} + p_{32} + q((-3 - q(4+$$
$$q(3+q)))p_{22} - (1+q)^2(1+q+q^2)p_{23} + p_{31} + 4p_{32} + 2p_{33} + q((1+q(2+$$
$$q+q^2))p_{31} + p_{32} + 2(1+q)p_{33}))))))$$

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8 Kronecker product as a tool to find mappings for matrices of higher dimensionality than 2 or 3

Once we reach matrices of four-dimensions, we are no longer able to follow this same technique. The reason for this is that we have 16 entries in our probability matrix, but only 9 parameters, $A_{-1,1}, A_{-1,0}, A_{0,1}, B_{1,0}, B_{2,0}, B_{2,1}, f_0^-, f_0^+, f_0^0$, thus we are no longer able to solve our system of equations. To tackle this problem we will build four-dimensional matrices and vectors from two-dimensional matrices and vectors using Kronecker Product.

Definition of Kronecker Product of Vectors

The Kronecker Product of two vectors $v, u \in R^2$ is

$$v \otimes u = \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \otimes \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} v_0 u_0 \\ v_0 u_1 \\ v_1 u_0 \\ v_1 u_1 \end{pmatrix}. \tag{119}$$

We will denote by $R^2 \otimes R^2$ the space of all Kronecker products of pairs of vectors in R^2. Next, we define the mapping $J_\otimes : R^2 \otimes R^2 \mapsto Pol_3$ as follows:

$$J_\otimes(v \otimes u)(z) = \Psi_{v \otimes u}(z) \tag{120}$$

where $\Psi_{v \otimes u}(z)$ satifies

$$\Psi_{v \otimes u}(q^0) = v_0 u_0 = \Psi_v(q^0)\Psi_u(q^0) \tag{121}$$

$$\Psi_{v \otimes u}(q^1) = v_0 u_1 = \Psi_v(q^0)\Psi_u(q^1) \tag{122}$$

$$\Psi_{v \otimes u}(q^2) = v_1 u_0 = \Psi_v(q^1)\Psi_u(q^0) \tag{123}$$

$$\Psi_{v \otimes u}(q^3) = v_1 u_1 = \Psi_v(q^1)\Psi_u(q^1), \tag{124}$$

where $\Psi_u(z) = Ju(z)$ and $\Psi_v(z) = Jv(z)$.

Lemma 6

The operator $J_\otimes$ is slotwise linear, i.e. for each $v_1, v_2, u_1, u_2 \in R^2$ and $\alpha_1, \alpha_2, \beta_1, \beta_2 \in R$,

$$J_\otimes((\alpha_1 v_1 + \alpha_2 v_2) \otimes u_1) = \alpha_1 J_\otimes(v_1 \otimes u_1) + \alpha_2 J_\otimes(v_2 \otimes u_1) \tag{125}$$

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and

$$J_\otimes(v_1 \otimes (\beta_1 u_1 + \beta_2 u_2)) \quad = \quad \beta_1 J_\otimes(v_1 \otimes u_1) + \beta_2 J_\otimes(v_1 \otimes u_2). \tag{126}$$

Proof

We need to show that

(a)

$$J_\otimes((v_1 + v_2) \otimes u) \quad = \quad J_\otimes(v_1 \otimes u) + J_\otimes(v_2 \otimes u) \tag{127}$$

Using our definition of $J_\otimes(v \otimes u)(z) = \Psi_{v\otimes u}(z)$ we have

$$
\begin{aligned}
J_\otimes((v_1 + v_2) \otimes u)(q^k) \quad &= \quad \Psi_{v_1+v_2}(q^k)\Psi_u(q^k) = [\Psi_{v_1}(q^k) + \Psi_{v_2}(q^k)]\Psi_u(q^k) \\
&\qquad \Psi_{v_1}(q^k)\Psi_u(q^k) + \Psi_{v_2}(q^k)\Psi_u(q^k) = J_\otimes(v_1 \otimes u)(q^k) + J_\otimes(v_2 \otimes u)(q^k)
\end{aligned}
$$

for $k = 0, 1, 2, 3$

since

$$\Psi_{v_1+v_2}(q^k) \quad = \quad (v_1 + v_2)_k = v_{1_k} + v_{2_k} = \Psi_{v_1}(q^k) + \Psi_{v_2}(q^k) \tag{128}$$

where $(v_1 + v_2)_k$ is the k component of the vector $(v_1 + v_2)$.

(b) We also need to show that

$$J_\otimes((\alpha v) \otimes u) \quad = \quad \alpha J_\otimes(v \otimes u) \tag{129}$$

Using the definition of $J(v \otimes u)(z) = \Psi_{v\otimes u}(z)$ we have that

$$
\begin{aligned}
J_\otimes((\alpha v) \otimes u)(q^k) \quad &= \quad \Psi_{(\alpha v)\otimes u}(q^k) = \Psi_{\alpha v}(q^k)\Psi_u(q^k) \\
&= \quad \alpha \Psi_v(q^k)\Psi_u(q^k) = \alpha \Psi_{v\otimes u}(q^k) = \alpha J_\otimes(v \otimes u)(q^k)
\end{aligned} \tag{130}
$$

since

$$\Psi_{\alpha v}(q^k) \quad = \quad (\alpha v)_k = \alpha(v_k) = \alpha \Psi_v(q^k) \tag{131}$$

where v_k is the k component of the vector v.

Therefore $J_\otimes : R^2 \otimes R^2 \mapsto Pol_3$ is slotwise linear. $\quad\square$

Theorem 3

There exist four functions of q and z $S_{00}(z), S_{01}(z), S_{10}(z), S_{11}(z)$ such that

$$\Psi_{v\otimes u}(z) = S_{00}\Psi_v(z)\Psi_u(z) + S_{01}\Psi_v(z)\Psi_u(qz) + S_{10}\Psi_v(qz)\Psi_u(z) + S_{11}\Psi_v(qz)\Psi_u(qz)$$

Proof

Let's consider the corresponding basis

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{132}$$

for R^2. The basis for $R^2 \otimes R^2$ is

$$e_1 \otimes e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, e_1 \otimes e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, e_2 \otimes e_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, e_2 \otimes e_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

We want to show that there exist four functions of z and q such that:

$$\Psi_{v\otimes u}(z) = S_{00}\Psi_v(z)\Psi_u(z) + S_{01}\Psi_v(z)\Psi_u(qz) + S_{10}\Psi_v(qz)\Psi_u(z) + \tag{133}$$

$$S_{11}\Psi_v(qz)\Psi_u(qz)$$

For the left side of (133) we can decompose $\Psi_{v\otimes u}(z)$ into its basis components so we have

$$\Psi_{v\otimes u}(z) = \Psi_{(\beta_1 e_1 + \beta_2 e_2)\otimes(\gamma_1 e_1 + \gamma_2 e_2)} \tag{134}$$

Applying the slotwise linearity of the $J_\otimes$ mapping, we now have

$$\Psi_{(\beta_1 e_1 + \beta_2 e_2)\otimes(\gamma_1 e_1 + \gamma_2 e_2)} = \beta_1\gamma_1\Psi_{e_1\otimes e_1} + \beta_1\gamma_2\Psi_{e_1\otimes e_2} \tag{135}$$

$$+\beta_2\gamma_1\Psi_{e_2\otimes e_1} + \beta_2\gamma_2\Psi_{e_2\otimes e_2}$$

where

$$\Psi_{e_1\otimes e_1} = \Psi_{(1,0,0,0)} = \frac{(z-q)(z-q^2)(z-q^3)}{(1-q)(1-q^2)(1-q^3)} \tag{136}$$

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$$\Psi_{e_1 \otimes e_2} = \Psi_{(0,1,0,0)} = \frac{(z - q^0)(z - q^2)(z - q^3)}{(q - q^0)(q - q^2)(q - q^3)} \tag{137}$$

$$\Psi_{e_2 \otimes e_1} = \Psi_{(0,0,1,0)} = \frac{(z - q^0)(z - q)(z - q^3)}{(q^2 - q^0)(q^2 - q)(q^2 - q^3)} \tag{138}$$

$$\Psi_{e_2 \otimes e_2} = \Psi_{(0,0,0,1)} = \frac{(z - q^0)(z - q)(z - q^2)}{(q^3 - q^0)(q^3 - q)(q^3 - q^2)} \tag{139}$$

which we obtain by solving for the $\Psi(z)$ functions of the four-dimensional basis vectors. When we substitute the $\Psi(z)$ basis kronecker products with their third degree polynomials we have

$$\Psi_{v \otimes u}(z) = \beta_1 \gamma_1 \Psi_{e_1 \otimes e_1} + \beta_1 \gamma_2 \Psi_{e_1 \otimes e_2} + \beta_2 \gamma_1 \Psi_{e_2 \otimes e_1} + \beta_2 \gamma_2 \Psi_{e_2 \otimes e_2} = \tag{140}$$
$$\beta_1 \gamma_1 \frac{(z - q)(z - q^2)(z - q^3)}{(1 - q)(1 - q^2)(1 - q^3)} + \beta_1 \gamma_2 \frac{(z - q^0)(z - q^2)(z - q^3)}{(q - q^0)(q - q^2)(q - q^3)}$$
$$\beta_2 \gamma_1 \frac{(z - q^0)(z - q)(z - q^3)}{(q^2 - q^0)(q^2 - q)(q^2 - q^3)} + \beta_2 \gamma_2 \frac{(z - q^0)(z - q)(z - q^2)}{(q^3 - q^0)(q^3 - q)(q^3 - q^2)}$$

For the right side of (133) we combine the linearity of J_q from Lemma 5:

$$\Psi_v = \Psi_{\beta_1 e_1 + \beta_2 e_2} = \beta_1 \Psi_{e_1} + \beta_2 \Psi_{e_2} \tag{141}$$

$$\Psi_u = \Psi_{\gamma_1 e_1 + \gamma_2 e_2} = \gamma_1 \Psi_{e_1} + \gamma_2 \Psi_{e_2} \tag{142}$$

and equation (103), which yields

$$\Psi_{e_1}(z) = \frac{q}{q - 1} - \frac{1}{q - 1} z \tag{143}$$

$$\Psi_{e_2}(z) = \frac{-1}{q - 1} + \frac{1}{q - 1} z \tag{144}$$

$$\Psi_{e_1}(qz) = \frac{q}{q - 1} - \frac{q}{q - 1} z = -q \Psi_{e_2}(z) \tag{145}$$

$$\Psi_{e_2}(qz) = \frac{-1}{q - 1} + \frac{q}{q - 1} z = \Psi_{e_1} + (1 + q) \Psi_{e_2} \tag{146}$$

to obtain

$$S_{00} \Psi_v(z) \Psi_u(z) + S_{01} \Psi_v(z) \Psi_u(qz) + S_{10} \Psi_v(qz) \Psi_u(z) + S_{11} \Psi_v(qz) \Psi_u(qz) = \tag{147}$$
$$S_{00}(q, z)[\beta_1 \Psi_{e_1(z)} + \beta_2 \Psi_{e_2}(z)][\gamma_1 \Psi_{e_1}(z) + \gamma_2 \Psi_{e_2}(z)] +$$
$$S_{01}[\beta_1 \Psi_{e_1}(z) + \beta_2 \Psi_{e_2}(z)][\gamma_1 \Psi_{e_1}(qz) + \gamma_2 \Psi_{e_2}(qz) +$$

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$$S_{10}[\beta_1 \Psi_{e_1}(qz) + \beta_2 \Psi_{e_2}(qz)][\gamma_1 \Psi_{e_1}(z) + \gamma_2 \Psi_{e_2}(z)] +$$

$$S_{11}[\beta_1 \Psi_{e_1}(qz) + \beta_2 \Psi_{e_2}(qz)][\gamma_1 \Psi_{e_1}(qz) + \gamma_2 \Psi_{e_2}(qz)]$$

$$= \quad S_{00}(q,z)[\beta_1(\frac{q}{q-1} - \frac{1}{q-1}z) + \beta_2(\frac{-1}{q-1} + \frac{1}{q-1}z)][\gamma_1(\frac{q}{q-1} - \frac{1}{q-1}z) + \gamma_2(\frac{-1}{q-1} + \frac{1}{q-1}z)] +$$

$$S_{01}[\beta_1(\frac{q}{q-1} - \frac{1}{q-1}z) + \beta_2(\frac{-1}{q-1} + \frac{1}{q-1}z)][\gamma_1(\frac{q}{q-1} - \frac{q}{q-1}z) + \gamma_2(\frac{-1}{q-1} + \frac{q}{q-1}z)] +$$

$$S_{10}[\beta_1(\frac{q}{q-1} - \frac{q}{q-1}z) + \beta_2(\frac{-1}{q-1} + \frac{q}{q-1}z)][\gamma_1(\frac{q}{q-1} - \frac{1}{q-1}z) + \gamma_2(\frac{-1}{q-1} + \frac{1}{q-1}z)] +$$

$$S_{11}[\beta_1(\frac{q}{q-1} - \frac{q}{q-1}z) + \beta_2(\frac{-1}{q-1} + \frac{q}{q-1}z)][\gamma_1(\frac{q}{q-1} - \frac{q}{q-1}z) + \gamma_2(\frac{-1}{q-1} + \frac{q}{q-1}z)]$$

When we compare and equate (140) with (147) and match powers of z, we obtain four equations that we can use to solve for our four unknowns, $S_{00}, S_{01}, S_{10}, S_{11}$:

$$S_{00} = \frac{1}{(q-1)^4 q^2 (1+q)(1+q+q^2)z^2} * -q^5(1+q) + q(1+q+q^2 - q^3 + 2q^4 + 3q^5 + 2q^6 - q^7)z +$$

$$(1+q)(1+q+q^2)(-1 + q^3(-1 + (q-2)(q-1)q)z^2 + (2+q+2q^2+q^4+q^5+q^6)z^3 - (1+q^3)z^4$$

$$S_{01} = \frac{(q-z)(z-1)(z + q^2(-q^2 + q(-1+q+q^2)z - z^2))}{(q-1)^4 q^2 (1+q+q^2)z^2}$$

$$S_{10} = \frac{(z-1)(-q^6(1+q) - q^2(1+q+q^2)(-1+(q-2)q^4)z - q(2+q+2q^2+q^4)z^2 + (q+1)z^3}{(q-1)^4 q^3 (1+q)(1+q+q^2)z^2}$$

$$S_{11} = -\frac{(q-z)(z-1)(-q^5 + q(1+q^4)z + (-1+q-q^2)z^2)}{(q-1)^4 q^3 (1+q)(1+q+q^2)z^2} \quad \square$$

Definition of Kronecker Product of Matrices

The Kronecker Product of matrices A, B is

$$A \otimes B = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \otimes \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} = \tag{148}$$

$$\begin{pmatrix} p_{11}\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} & p_{12}\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \\ p_{21}\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} & p_{22}\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} p_{11}r_{11} & p_{11}r_{12} & p_{12}r_{11} & p_{12}r_{12} \\ p_{11}r_{21} & p_{11}r_{22} & p_{12}r_{21} & p_{12}r_{22} \\ p_{21}r_{11} & p_{21}r_{12} & p_{22}r_{11} & p_{22}r_{12} \\ p_{21}r_{21} & p_{21}r_{22} & p_{22}r_{21} & p_{22}r_{22} \end{pmatrix} \tag{149}$$

where A and B are 2×2 matrices. Next, we define the operator $\Delta_{A \otimes B}$ as follows:

Definition of $\Delta_{A \otimes B} : Pol_3 \mapsto Pol_3$

Define $\Delta_{A \otimes B}$ such that

$$\Delta_{A \otimes B} J(v \otimes u) = J(Av \otimes Bu) \tag{150}$$

where

$$J(Av) = \Delta_A J(v) \tag{151}$$

Theorem 4 *We can find three Laurent polynomials, $f_{A\otimes B}^+, f_{A\otimes B}^0, f_{A\otimes B}^-$ such that*

$$\Delta_{A\otimes B} = f_{A\otimes B}^+ T^+ + f_{A\otimes B}^0 T^0 + f_{A\otimes B}^- T^- \tag{152}$$

for four specific matrix pairs A and B:

$$
\begin{aligned}
A_1 &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & B_1 &= \begin{pmatrix} \frac{1+q+q^2}{q} & \frac{1+q+q^2}{q^2(1+q)} \\ \frac{q^2}{1+q} & 1 \end{pmatrix} \\[2mm]
A_2 &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & B_2 &= \begin{pmatrix} \frac{1+q+q^2+q^3}{q^3} & \frac{-1}{q^3} \\ 1 & 0 \end{pmatrix} \\[2mm]
A_3 &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & B_3 &= \begin{pmatrix} 0 & \frac{1}{1+q+q^2+q^3} \\ \frac{-q^3}{1+q+q^2+q^3} & 1 \end{pmatrix} \\[2mm]
A_4 &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & B_4 &= \begin{pmatrix} \frac{q}{1+q+q^2} & \frac{1}{1+2q+2q^2+q^3} \\ \frac{q}{1+q} & 1 \end{pmatrix}
\end{aligned}
\tag{153}
$$

Proof

Following the operator application from (112) we have that

$$
\begin{aligned}
\Delta_{A\otimes B}\Psi_{v\otimes u} &= (f_{A\otimes B}^+ T^+ + f_{A\otimes B}^0 T^0 + f_{A\otimes B}^- T^-)\Psi_{v\otimes u}(z) = \\
&\quad f_{A\otimes B}^+ \Psi_{v\otimes u}(qz) + f_{A\otimes B}^0 \Psi_{v\otimes u}(z) + f_{A\otimes B}^- \Psi_{v\otimes u}(q^{-1}z)
\end{aligned}
\tag{154}
$$

In Theorem 3 we proved that there exist $S_{00}, S_{01}, S_{10}, S_{11}$ for which

$$\Psi_{v\otimes u}(z) = S_{00}\Psi_v(z)\Psi_u(z) + S_{01}\Psi_v(z)\Psi_u(qz) + S_{10}\Psi_v(qz)\Psi_u(z) + S_{11}\Psi_v(qz)\Psi_u(qz)$$

So now

$$\Delta_{A\otimes B}J(v\otimes u) = \Delta_{A\otimes B}\Psi_{v\otimes u} =$$

$$f_{A\otimes B}^+ S_{00}(qz)\Psi_v(qz)\Psi_u(qz) + S_{01}(qz)\Psi_v(qz)\Psi_u(q^2z) + S_{10}(qz)\Psi_v(q^2z)\Psi_u(qz)$$

$$+S_{11}(qz)\Psi_v(q^2z)\Psi_u(q^2z) +$$

$$f_{A\otimes B}^0 S_{00}(z)\Psi_v(z)\Psi_u(z) + S_{01}(z)\Psi_v(z)\Psi_u(qz) + S_{10}(z)\Psi_v(qz)\Psi_u(z) + S_{11}(z)\Psi_v(qz)\Psi_u(qz) +$$

$$f_{A\otimes B}^- S_{00}(q^{-1}z)\Psi_v(q^{-1}z)\Psi_u(q^{-1}z) + S_{01}(q^{-1}z)\Psi_v(q^{-1}z)\Psi_u(z) + S_{10}(q^{-1}z)\Psi_v(z)\Psi_u(q^{-1}z)$$

$$+S_{11}(q^{-1}z)\Psi_v(z)\Psi_u(z) \tag{155}$$

Once again using Theorem 3, if we now consider the right side of (150) we have

$$
\Psi_{Av\otimes Bu} = S_{00}(z)\Psi_{Av}(z)\Psi_{Bu}(z) + S_{01}(z)\Psi_{Av}(z)\Psi_{Bu}(qz) + \tag{156}
$$
$$
S_{10}(z)\Psi_{Av}(qz)\Psi_{Bu}(z) + S_{11}(z)\Psi_{Av}(qz)\Psi_{Bu}(qz)
$$

$$
=
$$

$$
S_{00}(z)(\Delta_A\Psi_v)(z)(\Delta_B\Psi_u)(z) +
$$
$$
S_{01}(z)(\Delta_A\Psi_v)(z)(\Delta_B\Psi_u)(qz) +
$$
$$
S_{10}(z)(\Delta_A\Psi_v)(qz)(\Delta_B\Psi_u)(z) +
$$
$$
S_{11}(z)(\Delta_A\Psi_v)(qz)(\Delta_B\Psi_u)(qz)
$$

$$
=
$$

$$
S_{00}(z)[f_A^+(z)\Psi_v(qz) + f_A^0(z)\Psi_v(z) + f_A^-(z)\Psi_v(q^{-1}z)]
$$
$$
[f_B^+(z)\Psi_u(qz) + f_B^0(z)\Psi_u(z) + f_B^-(z)\Psi_u(q^{-1}z)] +
$$
$$
S_{01}(z)[f_A^+(z)\Psi_v(qz) + f_A^0(z)\Psi_v(z) + f_A^-(z)\Psi_v(q^{-1}z)]
$$
$$
[f_B^+(qz)\Psi_u(q^2z) + f_B^0(qz)\Psi_u(qz) + f_B^-(qz)\Psi_u(z)] +
$$
$$
S_{10}(z)[f_A^+(qz)\Psi_v(q^2z) + f_A^0(qz)\Psi_v(qz) + f_A^-(qz)\Psi_v(z)]
$$
$$
[f_B^+(z)\Psi_u(qz) + f_B^0(z)\Psi_u(z) + f_B^-(z)\Psi_u(q^{-1}z)] +
$$
$$
S_{11}(z)[f_A^+(qz)\Psi_v(q^2z) + f_A^0(qz)\Psi_v(qz) + f_A^-(qz)\Psi_v(z)]
$$
$$
[f_B^+(qz)\Psi_u(q^2z) + f_B^0(qz)\Psi_u(qz) + f_B^-(qz)\Psi_u(z)]
$$

By equating (155) with (156) we get a system of sixteen equations,

$$
S_{11}(qz)f_{A\otimes B}^+(z) = S_{11}(z)f_A^+(qz)f_B^+(qz) \tag{157}
$$
$$
S_{10}(qz)f_{A\otimes B}^+(z) = S_{10}(z)f_A^+(qz)f_B^+(z) + S_{11}(z)f_A^+(qz)f_B^0(qz)
$$
$$
0 = S_{10}(z)f_A^+(qz)f_B^0(z) + S_{11}(z)f_A^+(qz)f_B^-(qz)
$$
$$
0 = S_{10}(z)f_A^+(qz)f_B^-(z)
$$
$$
S_{01}(qz)f_{A\otimes B}^+(z) = S_{01}(z)f_A^+(z)f_B^+(qz) + S_{11}(z)f_A^0(qz)f_B^+(qz)
$$
$$
S_{00}(qz)f_{A\otimes B}^+(z) + S_{11}(z)f_{A\otimes B}^0(z) = S_{00}(z)f_A^+(z)f_B^+(z) + S_{01}(z)f_A^+(z)f_B^0(qz) + S_{10}(z)f_A^0(qz)f_B^+(z) +
$$
$$
S_{11}(z)f_A^0(qz)f_B^0(qz)
$$

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$$S_{10}(z)f_{A\otimes B}^0(z) = S_{00}(z)f_A^+(z)f_B^0(z) + S_{01}(z)f_A^+(z)f_B^-(qz) + S_{10}(z)f_A^0(qz)f_B^0(z) +$$
$$S_{11}(z)f_A^0(qz)f_B^-(qz)$$

$$0 = S_{00}(z)f_A^+(z)f_B^-(z) + S_{10}(z)f_A^0(qz)f_B^-(z)$$

$$0 = S_{01}(z)f_A^0(z)f_B^+(qz) + S_{11}(z)f_A^-(qz)f_B^+(qz)$$

$$S_{01}(z)f_{A\otimes B}^0(z) = S_{00}(z)f_A^0(z)f_B^+(z) + S_{01}(z)f_A^0(z)f_B^0(qz) + S_{10}(z)f_A^-(qz)f_B^+(z) +$$
$$S_{11}(z)f_A^-(qz)f_B^0(qz)$$

$$S_{00}(z)f_{A\otimes B}^0(z) + S_{11}(q^{-1}z)f_{A\otimes B}^-(z) = S_{00}(z)f_A^0(z)f_B^0(z) + S_{01}(z)f_A^0(z)f_B^-(qz) +$$
$$S_{10}(z)f_A^-(qz)f_B^0(z) + S_{11}(z)f_A^-(qz)f_B^-(qz)$$

$$S_{10}(q^{-1}z)f_{A\otimes B}^-(z) = S_{00}(z)f_A^0(z)f_B^-(z) + S_{10}(z)f_A^-(qz)f_B^-(z) \tag{158}$$

$$0 = S_{01}(z)f_A^-(z)f_B^+(qz)$$

$$0 = S_{00}(z)f_A^-(z)f_B^+(z) + S_{01}(z)f_A^-(z)f_B^0(qz)$$

$$S_{01}(q^{-1}z)f_{A\otimes B}^-(z) = S_{00}(z)f_A^-(z)f_B^0(z) + S_{01}(z)f_A^-(z)f_B^-(qz)$$

$$S_{00}(q^{-1}z)f_{A\otimes B}^-(z) = S_{00}(z)f_A^-(z)f_B^-(z)$$

However, we can not solve this system of equations for all matrices A and B because the mapping of Kronecker product vector multiplication, $J(Av \otimes Bu)$ is nonlinear. However, we were able to successfully solve for our unknown polynomials $f_{A\otimes B}^+, f_{A\otimes B}^0, f_{A\otimes B}^-$ for specific matrices. when we decompose our matrix A into basis form

$$A = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = p_{11}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + p_{12}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + p_{21}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + p_{22}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

We can then use the unit basis vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{159}$$

to help solve our system of equations. We are now able to solve the system, which also provides us with specific conditions for the B matrix. Thus there are at least four pairs of A and B that can now be mapped as a Kronecker product,

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} B_1 = \begin{pmatrix} \frac{1+q+q^2}{q} & \frac{1+q+q^2}{q^2(1+q)} \\ \frac{q^2}{1+q} & 1 \end{pmatrix} \tag{160}$$

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$$A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad B_2 = \begin{pmatrix} \frac{1+q+q^2+q^3}{q^3} & \frac{-1}{q^3} \\ 1 & 0 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad B_3 = \begin{pmatrix} 0 & \frac{1}{1+q+q^2+q^3} \\ \frac{-q^3}{1+q+q^2+q^3} & 1 \end{pmatrix}$$

$$A_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad B_4 = \begin{pmatrix} \frac{q}{1+q+q^2} & \frac{1}{1+2q+2q^2+q^3} \\ \frac{q}{1+q} & 1 \end{pmatrix}$$

The three polynomial functions, $f_{A\otimes B}^+, f_{A\otimes B}^0, f_{A\otimes B}^-$, are

$$
\begin{aligned}
f_{A\otimes B}^+ \; = \; & \frac{1}{(q-1)^4(q+1)^2(1+q+q^2)(q^4-z)z^2}(-q(q^2-z)(q^3-z)(z-1)p_{11} \hspace{2cm} (161)\\
& (-q^2(1+q+q^2)(q-z)(-1+qz)r_{11}+q^4(1+q)(q-z)^2r_{12}+(1+q+q^2)\\
& (z-1)((-1-q-q^2)q^2-q)r_{21}+q^2(q+1)(q-z)r_{22}))+(q-z)(q^4\\
& (q^2-z)^2(q^3-z)p_{12}(q^2(q-z)r_{11}-(1+q+q^2)(z-1)r_{21}+\\
& (z-1)(-q^3(q-z)(q^2-z)p_{22}((1+q+q^2)(q^3-z)r_{11}+(z-q^2)r_{22}\\
& (z-1)p_{21}(-(1+q+q^2)^2(q^3-z)(qz-1)r_{11}+q^2(1+q)(1+q+q^2)\\
& (q-z)(q^3-z)r_{12}+(q^2-z)((1+q+q^2)(qz-1)r_{21}-q^2(1+q)(q-z)r_{22})))))) \\[6pt]
f_{A\otimes B}^0 \; = \; & \frac{1}{(q-1)^4q^2(1+q)(1+q+q^2)(q^3-z)q^4-z)z^2}\\
& (q^3-zq^{-1})(q(q^2-z)(q^3-z)p_{11}(q^2(1+q+q^2)(q-z)^2(qz-1)r_{11}+\\
& (z-1)((q-z)(q^4(1+q)(q^2-z)r_{12}+(1+q+q^2)^2(-1+qz)r_{21})+q^2(1+q)\\
& (1+q+q^2)(q^2-z)(z-1)r_{22}))-(q-z)(q^4(q^2-z)(q^3-z)^2p_{12}(q^2\\
& (q-z)r_{11}+(1+q+q^2)(-1+z)r_{21})+(z-1)(q^3(q-z)(q^3-z)p_{22}\\
& ((-1-q-q^2)(q^3-z)r_{11}+(q^2-z)r_{21})+p_{21}((1+q+q^2)^2(q-z)(q^3-z)\\
& qz-1)r_{11}+(q^2-z)(q^2(1+q)(1+q+q^2)(q^3-z)(z-1)r_{12}-(1+q+q^2)(q-z)\\
& (-1+qz)r_{21}-q^2(1+q)(q^2-z)(z-1)r_{22})))) \\[6pt]
f_{A\otimes B}^- \; = \; & \frac{1}{(q-1)^4q^3(1+q)^2(1+q+q^2)(q^2-z)(q^4-z)z^2}\\
& (q^3-zq^{-1})(q^3(q^2-z)^2(q^3-z)^2p_{12}(q^2(q-z)r_{11}+(1+q+q^2)(z-1)r_{21})\\
& +q(q-z)(q^2-z)(q^3-z)p_{11}(-q^2(1+q+q^2)(q-z)(z-1)((-1-q-q^2)(z-1)r_{21}\\
& +q(1+q)(q^2-z)r_{22}))-(q-z)(z-1)q^2(q^2(q^2-z)(q^3-z)p_{22}((1+q+q^2)\\
& (q^3-z)r_{11}+(z-q^2)r_{21})+(q-z)p_{21}(-(1+q+q^2)^2(q^3-z)(z-1)r_{11}
\end{aligned}
$$

$$(q^2 - z)((1 + q + q^2)(q(1 + q)(q^3 - z)r_{12} + (z - 1)r_{21} - q(q + 1)(q^2 - z)r_{22})))))$$

where $p_{11}, p_{12}, p_{21}, p_{22}$ are the fixed entrees in the A basis matrix and $r_{11}, r_{12}, r_{21}, r_{22}$ are the entrees from the corresponding B matrix.

Now we have that

$$
\begin{aligned}
\Delta_{A \otimes B} \quad = \quad & \Delta_{(p_{11}\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right))+p_{12}\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)+p_{21}\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)+p_{22}\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right))}^{\otimes B} \\
& \Delta_{p_{11}\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\otimes B_1 + p_{12}\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\otimes B_2 + p_{21}\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)\otimes B_3 + p_{22}\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)\otimes B_4}
\end{aligned}
$$

We now have a set of $4x4$ matrices that can be mapped using our technique. Namely, when A is fixed, $A \otimes B$ can be mapped when we have our B matrix such that (160) is satisfied. □

Conclusion

The goal of this thesis project was to study the mappings of discrete stochastic models into the complex plane in polynomial form. We were successful in establishing two mappings, namely Δ and J, that were based off our operator Δ and our function Ψ that enabled us to complete this transition from discrete to continuous. By associating a matrix M with our operator Δ and a vector v with our function $\Psi(z)$ we can compute $\Delta_M \Psi_v(z)$ instead of the matrix product Mv. We detailed both the 2-dimensional case and the 3-dimensional case, and illustrated how operators associated with Kronecker products of 2-dimensional matrices can be used to map a subset of 4-dimensional matrices. Namely, when a 2-dimensional matrix A is broken down into basis decomposition, each basis has a matrix B paired with it and these pairs of matrices can successfully be mapped using $\Delta_{A \otimes B} \Psi_{v \otimes u}$. So while not all of 4-dimensional space can be mapped using our technique, a subset of matrices can be. We now have more freedom in working in the complex plane than we previously had when working with a discrete set.

We also considered an expansion of our Δ operator, which opens up possibilities for further work on this topic. We narrowed our focus on $\Lambda = 2$ and $L = 2$, but further work could focus on the expansion of our operator and considering polynomial spaces of higher degree using larger degree Laurent polynomials for Δ. Another area of exploration is in using our mappings for representing matrices in larger polynomial spaces, so rather than $R^{n+1} \mapsto Pol_n$ we could have $R^{n+1} \mapsto Pol_k$ where $k = n + 2, n + 3, n + 4, ...$ and study the representation of these mappings in larger spaces. By understanding these representations further we could possibly look at combinations of networks with differing degrees, which would be impossible in linear algebra where matrix-vector multiplication is restricted to matrices and vectors of the same dimension.

References
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