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Restricted feedback control in discrete-time dynamical systems with memory

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When an equilibrium state of a physical or biological system suffers a loss of stability (e.g., via a bifurcation), it may be both possible and desirable to stabilize the equilibrium via closed-loop feedback control. Significant effort has been devoted towards using such control to prevent oscillatory or chaotic behavior in dynamical systems, both continuous-time and discrete-time. Regarding control in discrete-time systems, most prior attempts to stabilize unstable equilibria require that the system be perturbed once during each time step. However, there are examples of systems for which this is neither feasible nor possible. In this paper, we analyze a restricted feedback control method for discrete-time systems (restricted in the sense that the controller’s perturbations may be applied only in every other time step). We apply our theoretical analysis to a specific example from cardiac electrophysiology in which this sort of restricted feedback control is especially relevant. The example is a useful test case for the theory, and one for which an experimental setup is rather straightforward.

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1. INTRODUCTION

Closed-loop feedback can be an effective way to control instabilities in nonlinear dynamical systems. The underlying principle of feedback control is this: By applying small perturbations to an accessible system parameter, it may be possible to convert an unstable state of a system into a stable, attracting state, or to prevent undesirable bifurcations, oscillations, or chaos. Some readers may be familiar with the classical control problem of stabilizing an inverted pendulum [1] mounted to a moving cart, i.e., “balancing a broomstick on the palm of your hand.” There are two distinctions between that problem and the ones that we shall analyze in this article. First, and for emphasis, we consider feedback control applied via perturbations to a parameter, not directly to the state variable(s). For the inverted pendulum example, moving the cart would directly influence the angle $\theta$ of the pendulum relative to the upward vertical (a state variable), as well as its derivative. Second, our focus is on discrete-time dynamical systems. We shall use notation such as $x_n$ to denote the state of the system at time $n$, a nonnegative integer. Mappings of the form $x_{n+1} = f(x_n, \mu)$, where $f$ is a smooth function and $\mu$ is the “accessible system parameter,” will be of particular interest.

In the early 1990s, Ott, Grebogi, and Yorke (OGY) [2] proposed one of the first feedback control techniques for preventing periodic and aperiodic behavior in physical systems. Their method for stabilizing an unstable equilibrium state $x^*$ was to perturb some system parameter by an amount proportional to $x_0 - x^*$, the difference between the most recent state of the system and the equilibrium being targeted for stabilization. While this idea makes sense intuitively and can be an effective method for chaos control, it has one notable drawback: The OGY method requires advance knowledge of the unstable state $x^*$, a state which might be difficult or impossible to find in many applications. Pyragas [3] proposed an alternative, namely that the perturbations be chosen proportional to $x_n - x_{n-1}$, the difference between the two most recent states of the system. Applied to a mapping of the type mentioned above, such a method takes the form

$$x_{n+1} = f(x_n, \mu + \gamma(x_n - x_{n-1})), \quad (1)$$

where the proportionality constant $\gamma$ is commonly referred to as the feedback gain. Many subsequent studies (see, for example, [4–8]) have implemented Pyragas’ technique and minor variants thereof; following the authors of [7], we shall refer to Pyragas’ method as time-delay autosynchronization (TDAS). TDAS and some of its generalizations have been analyzed mathematically in the aforementioned studies with several goals in mind, such as (i) determining the method’s potential for successful stabilization of unstable states and (ii) optimizing its use in experimental setups.

In this paper, we adapt the TDAS method to the case in which control can be applied only in every other cycle, so that the controller operates in an ON-OFF pattern. There are systems for which this sort of constraint is natural, and throughout the paper we shall refer frequently to one such example from cardiac electrophysiology. In that context, the discrete-time model is appropriate due to the inherently discrete nature of the beating of the heart, and we shall use TDAS to prevent a period-doubling bifurcation associated with the onset of an arrhythmia known as T-wave alternans. Our mathematical analysis of the restricted TDAS method adapts and extends previous work [9,10] in two ways. First, we analyze control in a discrete-time system with short-term memory in the sense that the system depends upon more than one previous state. Second, we analyze a variant of restricted TDAS in which the perturbations applied by the controller must compete with impulsive forcing that is native to the system itself. In doing so, the example we have in mind is again from cardiac electrophysiology: competition between the heart’s native electrical stimuli and the stimuli applied via a pacer electrode. Our simulations suggest that, by automating a pacer electrode with an algorithm that does not require the electrode to fire a stimulus in every beat, it is still possible to terminate the abnormal alternans rhythm. Using less energy to achieve the same results may offer advantages in the form of prolonged battery life for an implantable pacemaker device.

The remainder of the paper is organized as follows. In Secs. II A and II B, linear stability analyses are used to predict parameter regimes in which the (unrestricted) TDAS and (restricted) ON-OFF TDAS algorithms could stabilize...
unstable equilibria. Section II C introduces a specific problem from cardiac electrophysiology: feedback control of alternans via perturbations to the timing of electrical stimuli. For reasons explained in that section, such perturbations need not be applied during every heartbeat, only in every other beat. A slight variant of the ON-OFF TDAS method is introduced to simulate competition between the heart’s native stimuli and the “artificial” stimuli supplied by a pacer electrode. In Sec. III, we analyze the (unrestricted) TDAS and ON-OFF TDAS as applied to a memoryless restitution mapping, and characterize the parameter regime in which those methods are predicted to succeed. Section IV contains similar analyses applied to a memoryless restitution mapping, and relates problems for further study.

II. TDAS CONTROL
A. Unrestricted TDAS
Here, we recall the control domain (defined below) for TDAS as applied to a simple, one-dimensional mapping; for a similar derivation, see [11]. Let \( x^* \) denote an unstable fixed point of the mapping

\[
x_{n+1} = f(x_n; \mu^*),
\]

that is, \( x^* = f(x^*; \mu^*) \). Suppose that we wish to stabilize \( x^* \) by applying perturbations to \( \mu^* \) during each iteration, replacing \( \mu^* \) by \( \mu^* + \epsilon_n \). TDAS chooses the perturbations \( \epsilon_n \) according to the rule

\[
\epsilon_n = \gamma (x_n - x_{n-1}),
\]

where \( \gamma \) is a proportionality constant called the feedback gain. In other words, TDAS perturbs the parameter \( \mu \) by an amount proportional to the difference between the two most recent states of the system.

Combining (2) and (3) results in a higher-dimensional mapping

\[
x_{n+1} = f(x_n; \mu^* + \gamma (x_n - x_{n-1})),
\]

for which \( x^* \) is still a fixed point. Introducing an auxiliary variable \( z_n = x_{n-1} \) allows us to write this higher-order recurrence as the first-order system

\[
x_{n+1} = f(x_n; \mu^* + \gamma (x_n - z_n)),
\]

\[
z_{n+1} = x_n.
\]

The vector \((x^*, x^*)\) is a fixed point of this system, and the primary goal of applying TDAS is to choose the feedback gain \( \gamma \) in such a way that the fixed point is stabilized. During the process of linearizing this system about the fixed point, one naturally encounters the Jacobian matrix

\[
J = \begin{bmatrix}
D_1 + \gamma D_2 & -\gamma D_2 \\
1 & 0
\end{bmatrix},
\]

where \( D_1 = D_1 f(x^*; \mu^*) \) and \( D_2 = D_2 f(x^*; \mu^*) \) are derivatives of \( f \) with respect to its first and second arguments, respectively, evaluated at the fixed point. Note that \(|D_1| \geq 1 \) since the fixed point was assumed to be unstable. Both eigenvalues of a \( 2 \times 2 \) matrix \( J \) have modulus less than 1 if and only if the trace and determinant obey the inequalities

(i) det \( J < 1 \),
(ii) tr \( J \) − det \( J < 1 \),
(iii) tr \( J \) + det \( J > -1 \).

Applied to (5) and assuming that \( D_2 > 0 \), these three conditions require

(i) \( \gamma < \frac{1}{D_2} \),
(ii) \( D_1 < 1 \),
(iii) \( \gamma > \frac{-1 - D_1}{2D_2} \).

The inequalities (7) define the control domain for unrestricted TDAS, the set of \( \gamma \) for which the unstable fixed point may be stabilized by feedback control.

To plot the control domain, suppose that \( D_1 < -1 \) so that the reason for instability is due to alternation of growing amplitude, i.e., the sort of instability that one associates with period-doubling bifurcations. In particular, the second inequality in (7) is automatically satisfied. Figure 1(a) illustrates the control domain (shaded region) under these assumptions, rendered as a plot of \( \gamma D_2 \) versus \( D_1 \). In the figure, we have opted not to extend the shading beyond \( D_1 = -1 \) since \( x^* \)
would be stable if $|D_1| < 1$. In fact, $x^*$ would remain stable even in the presence of “anticontrol” (negative $\gamma$), provided that $\gamma$ still obeys inequality (7)(iii).

### B. Restricted TDAS

As before, suppose that $x^*$ is an unstable fixed point of the mapping $x_{n+1} = f(x_n; \mu^*)$, and that we wish to stabilize $x^*$ by appropriate perturbations of $\mu^*$. If the perturbations are restricted in that they are applied only at every other iteration, we obtain a system

$$
x_{n+1} = f(x_n; \mu^*), \quad \text{(control off)}
$$

$$
x_{n+2} = f(x_{n+1}; \mu^* + \gamma(x_{n+1} - x_n)). \quad \text{(control on)}
$$

Written as a single equation,

$$
x_{n+2} = f(f(x_n; \mu^*); \mu^* + \gamma(f(x_n; \mu^*) - x_n)).
$$

Linearizing about $x^*$, one obtains

$$
x_{n+2} - x^* \approx [D_2^1 + \gamma D_2(D_1 - 1)](x_n - x^*),
$$

where $D_1$ and $D_2$ denote derivatives of $f$ with respect to its first and second arguments, respectively, measured at the point $(x^*, \mu^*)$. It follows that the fixed point $x^*$ can be stabilized if $\gamma$ is chosen such that

$$-1 < D_1^2 + \gamma D_2(D_1 - 1) < 1. \quad (8)
$$

As in the previous section, suppose that $D_1 < -1$ and $D_2 > 0$. The control domain described by the inequalities (8) is depicted in Fig. 1(b) as a plot of $\gamma D_2$ versus $D_1$. (A similar figure appears in [10].) The lower boundary $\gamma = -(1 + D_1)$ is a line of slope -1; by comparison, the corresponding boundary for unrestricted TDAS derived in the previous section is a line of slope -1/2. Notice that the upper boundary $\gamma = -(1 + D_1^2)/(D_1 - 1)$ asymptotes to the lower boundary as $D_1 \to -\infty$ because the stabilization of $x^*$ should become increasingly difficult as the magnitude of $D_1$ increases. Interestingly, the control domain for restricted ON-OFF TDAS is unbounded, unlike the control domain for unrestricted TDAS shown in Fig. 1(a).

### C. Example: Two models of cardiac restitution

When a cardiac cell is repeatedly stimulated, or paced, by a sequence of impulsive electrical stimuli, each stimulus typically elicits a response known as an action potential. On a single-cell level, an action potential is a prolonged elevation of the potential $v$ across the cell’s membrane, following a stimulus. Action potential duration (APD) can be measured relative to some threshold voltage slightly above the resting potential; a precise definition is not important for our purposes. Suppose that pacing is periodic with period $B$, and let $A_n$ denote the APD following the $n$th stimulus. For large $B$ (slow pacing), cells normally phase lock into a response in which the sequence {\{A_n\}} is roughly constant, say $A^*$. If the pacing period $B$ is varied quasi-statically, this steady-state APD value $A^*$ also varies, a feature of cardiac tissue known as APD restitution. Usually, decreasing $B$ causes $A^*$ to decrease because rapid heart rates cause tissue fatigue, preventing cells from staying in an excited state (i.e., an action potential) for quite as long. If $B$ becomes critically low, $A^*$ may lose stability, resulting in an abnormal alternation of APD: alternans, illustrated in the top panel of Fig. 3. Alternans can set a substrate for the formation of potentially deadly arrhythmias; for details see, for example, [12–15]. In Secs. III and IV, we shall explore the feedback control of alternans using a variant of the restricted TDAS method described above.

In the language of nonlinear dynamics, the simplest model for APD restitution is presented as a one-dimensional mapping

$$
A_{n+1} = f(A_n; B). \quad (9)
$$

The onset of alternans can be interpreted as a period-doubling [16] or border-collision [17] bifurcation that occurs as $B$ is decreased. If alternans occurs for some pacing period $B$ and $A^*$ denotes the associated (unstable) fixed point of (9) satisfying $A^* = f(A^*; B)$, it is desirable to stabilize $A^*$ by feedback control. A specific example [5] of a restitution function $f$ that was fit to experimental data from bullfrogs is given by

$$
f(A_n; B) = \frac{\max - \beta e^{-(B-A_n)/\tau}}{1 + e^{-(B-A_n)/\tau}}, \quad (10)
$$

where $\max = 392$ ms, $\beta = 525$ ms, and $\tau = 40$ ms. For this mapping, the bifurcation to alternans occurs when $B \approx 455$ ms; for a bifurcation diagram of (9) and (10), see [18].

While one-dimensional restitution models of the form (9) may be appropriate in some settings, $A_{n+1}$ is influenced not only by $A_n$, but also by a variety of other factors. For this reason, we shall also analyze TDAS as applied to a memory model [19]

$$
A_{n+1} = (1 - \alpha M_{n+1})G(B - A_n),
$$

$$
M_{n+1} = [1 - (1 - M_n)\beta e^{-(B-A_n)/\tau}]e^{-(\beta-A_n)/\tau}, \quad (11)
$$

where

$$G(x) = A_{\min} + \frac{E}{1 + e^{-(x-C)/D}}.
$$

The “memory” variable $M$ “charges up” (accumulates) during each action potential and decays during the resting period between consecutive action potentials, both with time constant $\tau$. During rapid pacing (small $B$), the increase in $M$ then modulates APD via the factor $\alpha M_{n+1}$, where $\alpha > 0$ is a parameter. The function $G$ is an alternative functional form to the restitution function $f$ in (10), sigmoidal as opposed to exponential. The parameters $A_{\min}$, $E$, $C$, and $D$ (all measured in milliseconds) merely translate and dilate $G$ in the horizontal and vertical directions. When numerical simulations are required, we use the parameter values

$$
\alpha = 0.2, \quad \tau = 1000 \text{ ms}, \quad A_{\min} = 88 \text{ ms},
$$

$$
C = 280 \text{ ms}, \quad D = 28 \text{ ms}, \quad E = 170 \text{ or } 250 \text{ ms.} \quad (12)
$$

We remark that these are not the same parameter choices that appeared originally in [19], and that we have listed two different choices for $E$. Our purpose in doing so is to amplify the alternans and enlarge the interval of $B$ over which it occurs, thereby allowing us to test the robustness of feedback control. Figure 2 shows the bifurcation diagrams for (11) with the parameters listed above. Note that by increasing the parameter $E$ from 170 to 250 ms, alternans can occur over a wider range of $B$ values, and with larger amplitude.

Variants of unrestricted TDAS have been previously analyzed in the context of controlling alternans; see, for
FIG. 2. Bifurcation diagrams for (11) with parameters chosen from (12). (a) Bifurcation diagram for \( E = 170 \) ms. (b) Bifurcation diagram for \( E = 250 \) ms. (Note the wider horizontal scale.)

example, [4,18,20–23]. Regarding the application of TDAS to a restitution mapping, there is an important physiological consideration to take into account. Perturbing \( B \) itself is complicated because \( B \) represents the period of the native electrical stimuli that are regulated by the autonomic nervous system. If an implanted electrode were to supply an artificial stimulus that stimulus would either (a) elicit an action potential if the cells were sufficiently well rested or (b) fail to elicit an action potential if the cells had been recently excited by one of the native stimuli. In principle, one only has the freedom to select the timing of the artificial stimuli, not the native ones, and there is no need to fire an artificial stimulus if a native stimulus has just been supplied. Referring to Fig. 3(a), suppose that alternans occurs if the native pacing period is \( B \) as is shown. In order for an artificial stimulus to elicit an action potential, it would need to preempt a native stimulus. Figure 3(b) illustrates how this could be accomplished. Following a short action potential duration \( A_n \), the cell would have plenty of rest prior to the next action potential. Suppose that an artificial stimulus is applied early, preempting the \((n + 1)\)th native stimulus (see figure), and effectively adjusting \( B \) by \( \epsilon_n = \gamma(A_n - A_{n-1}) < 0 \) in accordance with (3). After \( A_{n+1} \), an artificial stimulus is not applied: The controller is turned OFF because the TDAS computed perturbation \( \gamma(A_{n+1} - A_n) \) may well be positive, potentially requiring the artificial stimulus to occur right after a native stimulus. The interval between the \( n \)th and \((n + 1)\)st stimuli would be \( B + \epsilon_n \), while the interval between the \((n + 1)\)st and \((n + 2)\)nd stimuli would be \( B - \epsilon_n \).

III. MEMORYLESS RESTITUTION MAPPINGS: ANALYSIS

In this section, we assume that \( B \) is such that alternans would occur in the mapping (9) in the absence of feedback control, \( A^* \) denotes the (unstable) fixed point of the mapping, and \( D_1 = D_1(f(A^*; B) \) represents a slope of a restitution function at the unstable fixed point. According to the stability criterion for fixed points of one-dimensional mappings, \( D_1 \leq -1 \) in order for alternans to occur in (9).

A. Control domain for unrestricted TDAS

With TDAS control, the one-dimensional mapping (9) is replaced with a two-dimensional mapping

\[
A_{n+1} = f(A_n; B + \gamma(A_n - A_{n-1})),
\]

which may be written as a system of the form (4) with \( A_n = x_n, A_{n-1} = x_{n-1} \), and \( B = \mu^* \). Instead of rendering the control domain by plotting \( \gamma D_2 f(A^*; B) \) versus \( D_1 f(A^*; B) \) as in Fig. 1(a), in this context it is more natural to convert to a plot of \( \gamma \) versus \( B \). That way, given a heart rate for which alternans occurs, one is able to predict the range of feedback gains \( \gamma \) for which TDAS may suppress alternans. Figure 4(a) shows this alternate rendering of the control domain for the mapping (9) with the specific restitution function (10). Regarding the horizontal scale, the bifurcation to alternans occurs at \( B \approx 455 \) ms, and \( D_1 f(A^*; B) = -3 \) when \( B \approx 331 \) ms. As Fig. 1(a) predicted, TDAS can never stabilize \( A^* \) when the magnitude of \( D_1 f(A^*; B) \) exceeds 3.

B. Control domain for restricted TDAS

In the preceding section, we were able to recycle the calculations in Sec. II A to explore how unrestricted TDAS would be applied to a memoryless cardiac restitution mapping. A simple conversion also allowed us to rerender the general control domain from Fig. 1(a) in a more physiologically meaningful way [Fig. 4(a)]. To analyze the restricted ON-OFF TDAS applied to the restitution mapping, we cannot recycle the material in Sec. II B in the same way because those calculations were performed without accounting for the unique

FIG. 3. Schematic action potentials in a periodically stimulated cardiac cell. (a) Alternans in the absence of control. (b) Introduction of TDAS control in an ON-OFF pattern after the \( n \)th action potential. Native stimuli are indicated by bold dots, and the artificial stimulus is indicated by a hollow circle.
cardiac-specific restriction that was mentioned at the end of Sec. II C [see also Fig. 3(b)]. However, adapting the analysis is rather straightforward as we shall now demonstrate.

Referring to Fig. 3(b), the ON-OFF TDAS control algorithm would take the following form:

\[
\begin{bmatrix}
A_{n+1} \\
A_{n+2}
\end{bmatrix} = \Phi \begin{bmatrix}
A_{n-1} \\
A_n
\end{bmatrix} = \begin{bmatrix}
f(A_n; B + \epsilon_n ) \\
f(A_n; B - \epsilon_n )
\end{bmatrix},
\]

where \(\epsilon_n = \gamma(A_n - A_{n-1})\) is chosen according to the TDAS rule (3). Inserting this expression for \(\epsilon_n\) into the right-hand side of the preceding equation, the system can be written in the form

\[
\begin{bmatrix}
A_{n+1} \\
A_{n+2}
\end{bmatrix} = \Phi \begin{bmatrix}
A_{n-1} \\
A_n
\end{bmatrix} = \begin{bmatrix}
f(A_n; B + \epsilon_n ) \\
f(A_n; B - \epsilon_n )
\end{bmatrix}.
\]

That is, each subsequent pair of APD values can be calculated from the preceding pair. Linearizing (15) about its fixed point (i.e., the vector whose components are both \(A^*\)), one finds that the associated Jacobian matrix is

\[
J = D\Phi(A^*, A^*) = \begin{bmatrix}
-\gamma D_2 & D_1 + \gamma D_2 \\
-\gamma D_1 D_2 + \gamma D_2 & D_1 (D_1 + \gamma D_2) - \gamma D_2
\end{bmatrix}.
\]

where \(D_1 = D_1 f(A^*; B)\) and \(D_2 = D_2 f(A^*; B)\). Applied to the restitution mapping (9) and (10), the Jacobian matrix simplifies because \(D_2 = -D_1\):

\[
J = \begin{bmatrix}
\gamma D_1 & (1 - \gamma) D_1 \\
-\gamma D_1 (1 - D_1) & D_1 (\gamma - (\gamma - 1) D_1)
\end{bmatrix}.
\]

To stabilize the fixed point, the trace \(2\gamma D_1 - (\gamma - 1)D_1^2\) and determinant \(\gamma D_1^2\) of the matrix (17) must satisfy the inequalities (6) as follows:

(i) \(\gamma < \frac{1}{D_1^2}\), (ii) \(\gamma > \frac{D_1^2 - 1}{2(D_1^2 - D_1)}\), (iii) \(\gamma < \frac{D_1^2 + 1}{-2D_1}\).

The control domain described by these three inequalities is shown in Fig. 4(b); observe that inequality (iii) is redundant, as it is uniformly less restrictive than inequality (i). Not surprisingly, the control domain for this restricted version of TDAS in which control is ON in every other beat is a bit smaller than the control domain for unrestricted TDAS [Fig. 4(a)]. In particular, linear stability analysis predicts that ON-OFF TDAS could never completely suppress alternans if \(D_1 < -2\).

To test the above theory, we applied both unrestricted and ON-OFF TDAS to the one-dimensional restitution mapping (9) and (10). Sample results are illustrated in Fig. 5: after 20 beats of (uncontrolled) alternans, we apply unrestricted TDAS [Fig. 5(a)] and ON-OFF TDAS [Fig. 5(b)] using a
feedback gain of $\gamma = 0.2$. In both figures, $B = 440$ ms, which corresponds to $D_1 \approx -1.2$. Because $(B, \gamma) = (440, 0.2)$ lies in the intersection of the control domains shown in the two panels of Fig. 4, the above theory predicts that the fixed point is stabilized regardless of whether control is applied in every beat or every other beat. This is precisely what is shown in Fig. 5: the sequence $\{A_n\}$ converges to $A^*$ in each case, with a slightly longer transient if ON-OFF TDAS is used. We remark that for $\gamma = 0.15$, similar results are observed (not pictured here), but that the transient is longer for the unrestricted TDAS. Of course, there are $(B, \gamma)$ pairs for which only the unrestricted TDAS can stabilize $A^*$, and $(B, \gamma)$ pairs for which neither version of TDAS can succeed.

IV. RESTITUTION MAPPING WITH MEMORY: ANALYSIS

Applying TDAS to a memory model such as (11), either the unrestricted or ON-OFF version, is equally straightforward but more tedious. Again, $B$ is the accessible parameter that we perturb according to (3). Suppose that alternans occurs if the native pacing period is $B$, but more tedious. Again, the unrestricted or ON-OFF version, is equally straightforward with the entire right-hand side of (19b), upon which the vector

$$
\begin{bmatrix}
A_{n+1} \\
M_{n+1} \\
A_{n+2} \\
M_{n+2}
\end{bmatrix} = \Psi
\begin{bmatrix}
A_{n} \\
M_{n} \\
A_{n-1} \\
M_{n-1}
\end{bmatrix}
$$

is the accessible parameter that $B$ corresponds to.

A. Control domain for unrestricted TDAS

With (unrestricted) TDAS, the model equations (11) take the form

(a) $A_{n+1} = (1 - \alpha M_{n+1})G[B + \gamma(A_n - z_n) - A_n],$

(b) $M_{n+1} = [1 - (1 - M_n)e^{-A_n/\tau}]e^{-(B + \gamma(A_n - z_n) - A_n)/\tau},$

(c) $z_{n+1} = A_n,$

where, as in the analysis of (13), we have introduced an auxiliary variable $z_n = A_{n-1}$ to write the equation as a system of one-dimensional mappings. Note that the $M_{n+1}$ appearing on the right-hand side of (19a) can be replaced with the entire right-hand side of (19b), upon which the vector $(A_{n+1}, M_{n+1}, z_{n+1})$ is expressed as a function of the vector $(A_n, M_n, z_n)$. The fixed point of (19) that we wish to stabilize is $(A^*, M^*, A^*)$, and the range of $\gamma$ for which stabilization is possible is obtained by characterizing the magnitudes of the eigenvalues of the $3 \times 3$ Jacobian matrix $J$ at the fixed point. Using the Jury stability test, the eigenvalues of $J$ are the roots of the polynomial

$$p(\lambda) = \lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3 = 0,$$

where $a_1 = -\text{trace} J$, $a_2 = \frac{1}{2}[(\text{trace} J)^2 - \text{trace}(J^2)]$, and $a_3 = -\det J$. All eigenvalues have modulus strictly less than 1 (implying stability) if and only if

(i) $p(1) > 0$,

(ii) $p(-1) < 0$,

(iii) $3 + a_1 - a_2 - 3a_3 > 0$,

(iv) $1 + a_1a_3 - a_2 - a_3^2 > 0$.

These requirements produce four inequalities that $\gamma$ must satisfy in order for (unrestricted) TDAS to succeed. Rather than plotting the control domain defined by these inequalities (similar plots are shown in [22]), we provide the analogous plot for the novel case of restricted ON-OFF TDAS in the next section.

B. Control domain for restricted TDAS

Referring to Fig. 3(b), applying ON-OFF TDAS control to the model (11) yields

(a) $A_{n+1} = (1 - \alpha M_{n+1})G[B + \gamma(A_n - A_{n-1}) - A_n],$

(b) $M_{n+1} = [1 - (1 - M_n)e^{-A_n/\tau}]e^{-(B + \gamma(A_n - A_{n-1}) - A_n)/\tau},$

(c) $z_{n+1} = A_n,$

(d) $M_{n+2} = [1 - (1 - M_{n+1})e^{-A_n/\tau}]e^{-(B - \gamma(A_n - A_{n-1}) - A_n)/\tau}.$

Note that the right-hand sides of all four of these equations can be written in terms of the variables $A_{n-1}, A_n, A_{n+1}$ and $M_n$. For instance, the right-hand side of (22b) can be substituted for $M_{n+1}$ appearing in (22a) and (22d). Hence, the vector $(A_{n+1}, M_{n+1}, A_{n+2}, M_{n+2})$ may be expressed as a function $\Psi$ of the vector consisting of the two previous pairs of iterates:

$$
\begin{bmatrix}
A_{n+1} \\
M_{n+1} \\
A_{n+2} \\
M_{n+2}
\end{bmatrix} = \Psi
\begin{bmatrix}
A_{n} \\
M_{n} \\
A_{n-1} \\
M_{n-1}
\end{bmatrix}
$$

Because $(A^*, M^*)$ was a fixed point of the uncontrolled mapping (11), it follows that $(A^*, M^*, A^*, M^*)$ is a fixed point of (23). It would appear that one must characterize the dependence of the eigenvalues of a $4 \times 4$ Jacobian matrix on the parameter $\gamma$ to determine the control domain. However, because the right-hand side of (22) actually does not involve $M_{n-1}$, each entry in the second column of the Jacobian matrix $J = D \Psi(A^*, M^*, A^*, M^*)$ is zero. Disregarding the zero eigenvalue, the other three eigenvalues determine the stability of the fixed point, and because they are roots of a cubic polynomial, we may apply the stability test (21) to that polynomial. Again, we obtain four inequalities that $\gamma$ must satisfy to stabilize the fixed point.

We shall not write out these inequalities, as they are quite long and not especially illuminating. Instead, we shall plot the solution of the inequalities (i.e., the control domain) using the two specific parameter sets in (12) above. We remark that the control domains for these two parameter sets are representative of what one sees generally for parameter sets capable of producing alternans in (11). Figure 6 shows the control domain for the two different choices of the parameter $E$. The shaded region in Fig. 6(a) indicates the set of $(B, \gamma)$ pairs for which restricted TDAS control can successfully suppress alternans by stabilizing the fixed point of (23). Importantly, there are choices of $\gamma$ which bridge across all values of $B$. The implication is that, according to our linear stability analysis, each such $\gamma$ could potentially terminate alternans regardless of the underlying pacing period $B$. By contrast, note that the analogous region in Fig. 6(b) indicates that there are $B$ for which no choice of $\gamma$ could possibly stabilize the fixed point. The differences between these two control domains can be understood by referring to their corresponding bifurcations.
diagrams in Fig. 2. For $E = 250$ ms, there are pacing periods $B$ for which the amplitude of alternans is substantially higher than in the case $E = 170$ ms. Certainly, it makes sense that control should be harder to achieve in the more severe case in which $E = 250$ ms.

Figure 7 shows a bifurcation diagram for (22), the memory model with ON-OFF control, using a feedback gain of $\gamma = 0.25$ and the usual parameter sets (12). Figure 7(a) shows that for $E = 170$ ms, the fixed point $A^*$ is stabilized for all $B$ and alternans is completely eliminated. Figure 7(b) shows that for $E = 250$ ms, there is still a window of $B$ for which alternans cannot be eliminated by ON-OFF TDAS. Still, in comparison to the corresponding bifurcation diagram in the absence of control [Fig. 2(b)], both the size of the alternans window and the amplitude of alternans are substantially reduced. Incidentally, we selected $\gamma = 0.25$ under the guidance of the theoretical control domain in Fig. 6, and it is encouraging that the results in Fig. 7 appear consistent with what the theory predicts.

C. Basins of attraction

The preceding local stability analyses are useful in predicting how an ON-OFF controller device should be automated to achieve stabilization. Because those results are merely local, it is useful to test the robustness of ON-OFF TDAS, we calculated numerically the basins of attraction of the stabilized fixed points both for the memoryless restitution mapping (10) and for the memory model (11). The numerical experiments with (9) and (10) were performed as follows.

1. For each choice of $B$, compute the fixed point that satisfies $A^* = f(A^*; B)$. Then select a fixed choice for $\gamma$.
2. Choose some separation $\delta > 0$ between the initial APD and the fixed point APD. Set $A_{-1} = A^* + \delta$, and then generate $A_0 = f(A_{-1}; B)$ by performing one iteration of the mapping without control.
3. Starting from the initial pair $(A_{-1}, A_0)$, use ON-OFF TDAS control (14) to generate the pairs $(A_1, A_2), (A_3, A_4), \ldots$ up to $(A_{499}, A_{500})$.
4. If $|A_{500} - A^*| < 10^{-1}$, regard $A_{-1} = A^* + \delta$ as belonging to the basin of attraction for $A^*$. Record the point $(B, \delta)$. Repeat this process for various choices of $B$ and $\delta$. We considered $\delta$ as large as 200 ms, a very large discrepancy between the initial APD and the fixed point APD for these mapping models. Simulations for the memory model (11) were performed in a similar manner, choosing initial APD values at various distances $\delta$ from $A^*$. Regarding the initial conditions on the memory variable $M$, in every simulation we set $M_{-1} = 1.1M^*$, ten percent larger than the fixed point value of $M$, and generated $M_0$ by performing one iteration of the uncontrolled mapping. Subsequent pairs of iterates were obtained recursively from (22).

Some of our results are shown in Fig. 8, rendered as plots of $\delta$ versus $B$ as opposed to $A^* + \delta$ versus $B$. In each panel, the basins are shaded. The panels in the first row correspond to a relatively weak feedback gain $\gamma = 0.1$, and the panels in the
First, by comparing the small-$\gamma$ panels in Figs. 8(a) to 8(c) with their moderate-$\gamma$ counterparts in Figs. 8(d) to 8(f), the basins of attraction are considerably larger in the latter case. In these simulations, the choice $\gamma = 0.25$ was nearly optimal for the purpose of creating large basins of attraction. Increasing $\gamma$ much beyond 0.25 causes the basins to retreat, and choosing very large $\gamma$ has a destabilizing effect. Next, the basins of attraction are surprisingly large for the systems in these three examples. Figure 8(e) shows that ON-OFF TDAS always stabilizes $A^*$ regardless of $B$, even if the initial APD iterate is a whopping 200 ms away from $A^*$. Apart from Fig. 8(d), ON-OFF TDAS either stabilizes $A^*$ for every initial separation $\delta$ up to 200 ms, or for none at all. The observation that varying $B$ can cause an abrupt transition from an empty basin of attraction (apart from $A^*$ itself) to a huge basin of attraction came as a surprise to us, and we were skeptical at first. However, further numerical experiments showing time series of $A_n$ for various $\gamma$ and $\delta$ confirmed our findings.

Of all of the panels in Fig. 8, only Fig. 8(d) was aligned with our expectations. As the amplitude of alternans increases and the magnitude of the Floquet multiplier $D_1 = D_1 f(A^*; B)$ increases further above 1, the repulsiveness of the fixed point becomes stronger. When the stabilization of $A^*$ is possible, the basin of attraction ought to shrink as the repulsiveness of the fixed point increases. Importantly, general mapping models might have multiple fixed points (unlike the restitution model examples studied here), which could lead to smaller basins of attraction.

V. DISCUSSION AND CONCLUDING REMARKS

We have provided careful stability analyses of a feedback control method for stabilizing unstable fixed points of discrete-time systems. The method is a restricted version of a previously reported technique known as TDAS [3,11]; it is restricted in the sense that the system cannot be perturbed during each time step. When applied to two examples of cardiac restitution mapping models, our local stability analyses predict how the feedback gain should be chosen so as to prevent unwanted alternation of action potential duration. For the restitution mappings that we analyzed, it is encouraging that when restricted TDAS successfully stabilizes a fixed point, the basin of attraction of the fixed point is substantially larger than one might rightfully expect.

There are potential advantages to applying feedback control in an ON-OFF pattern. First, and perhaps most importantly, is the possibility of stabilizing a fixed point with less energy usage than would be required if perturbations were applied during every iteration. In the cardiac example, this could imply longer battery life for an implanted medical device capable of delivering electrical stimuli. Second, when one compares the two panels of Fig. 1, it is potentially useful that the control domain for restricted ON-OFF TDAS is actually unbounded, unlike the control domain for TDAS. However, this observation could be misleading for two reasons: (i) the quantities $D_1$ and $D_2$ are usually related to one another (e.g., $D_1 = -D_2$ for the memoryless cardiac restitution mapping) and (ii) in the restricted case, the upper and lower boundaries of the control domain merge as $D_1$ decreases, imposing a tight restriction on $\gamma$ that could render such control impractical.

One might wonder about generalizations of the restricted feedback control method described in this paper. For instance, what would happen if perturbations could only be applied in every third cycle, so that the controller fires in an ON-OFF-OFF pattern? Because the upper boundaries of the control domains shown in Fig. 4 are $\gamma = -1/D_1$ [Fig. 4(a)] and
γ = 1/D² [Fig. 4(b)], it seems natural to conjecture that the upper boundary of the corresponding domain for ON-OFF-OFF control would be γ = −1/D³. Certainly, the less frequently that feedback control is turned ON, the smaller the control domain should become (see also the figure in [10], which supports this). We did not perform the analysis of ON-OFF-OFF TDAS control because, for the purpose of controlling a period-doubling bifurcation, it seems unnatural to implement control every third beat to stop a period-2 response.

Finally, we should mention that (unrestricted) TDAS has been used to successfully control alternans experimentally [5]. To our knowledge, the restricted version of TDAS described in this paper has not been used to control TDAS or to prevent the onset of similar instabilities in other systems. We hope that our mathematical analysis will serve as a guide to those seeking to stabilize a variety of physical or biological systems for which control need not be applied all the time.

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