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Analytic Continuation in Bergman Spaces and the Compression of Certain Toeplitz Operators

William T. Ross

Abstract. Let G be a Jordan domain and $K \subset G$ be relatively closed with Area$(K) = 0$. Let $A^2(G\backslash K)$ and $A^2(G)$ be the Bergman spaces on $G\backslash K$, respectively G and define $\mathcal{N} = A^2(G\backslash K) \ominus A^2(G)$. In this paper we show that with a mild restriction on K, every function in $\mathcal{N}$ has an analytic continuation across the analytic arcs of ∂G that do not intersect K. This result will be used to discuss the Fredholm theory of the operator $C_f = P_{\mathcal{N}} T_f|_{\mathcal{N}}$, where $f \in C(\bar{G})$ and T_f is the Toeplitz operator on $A^2(G\backslash K)$.

1. Introduction. Let U be a bounded, open, connected, non-empty subset of $\mathbf{C}$. Let $L^2(U)$ denote the Hilbert space of complex valued measurable functions (with respect to two-dimensional Lebesgue measure) on U which are square integrable. The inner product is given by $\langle f, g\rangle = \int_U f\bar{g}\, dA$ and the norm of a function $h \in L^2(U)$ will be given by $\|h\|_2 = \langle h, h\rangle^{1/2}$. The *Bergman space*, denoted $A^2(U)$, is the closed subspace of all functions in $L^2(U)$ which are analytic on U. We motivate this paper with the following example. Let D be the unit disk and $K \subset D$, K compact with Area$(K) = 0$. If $f \in A^2(D\backslash K)$, then in some annulus A contained in D, f has a Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n.$$

Define f_1 and f_2 in A by

$$f_1(z) = \sum_{n=0}^{\infty} c_n z^n \tag{1}$$

$$f_2(z) = \sum_{n=-\infty}^{-1} c_n z^n$$

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and note that f_1 and f_2 have unique analytic continuations to D, respectively $D\backslash K$. Thus $f_1 \in A^2(D)$ and $f_2 \in A^2(D\backslash K)$ with f_2 analytic across ∂D. The decomposition $f = f_1 + f_2$ is not, in general, orthogonal. However, if we define $\mathcal{N} = A^2(D\backslash K) \ominus A^2(D)$, then $f \in A^2(D\backslash K)$ has an orthogonal decomposition $f = f_D + f_{\mathcal{N}}$, where $f_D \in A^2(D)$ and $f_{\mathcal{N}} \in \mathcal{N}$. The function f_D is analytic on D and we will show that $f_{\mathcal{N}}$ is analytic across ∂D. Thus the orthogonal decomposition of a function in $A^2(D\backslash K)$ is also a decomposition into two functions, one of which behaves well inside the disk while the other behaves well across the boundary of the disk. Moreover, the two decompositions are always equal (i.e. $f_1 = f_D$, $f_2 = f_{\mathcal{N}}$ for all $f \in A(D\backslash K)$) only in the trivial case where $\mathcal{N} = 0$ (see Theorem 3.7).

This result can be generalized for Jordan regions G and $K \subset G$, K relatively closed with $Area(K) = 0$. After placing a mild technical condition on K, we show that if $\mathcal{N} = A^2(G\backslash K) \ominus A^2(G)$ then every function $f \in \mathcal{N}$ has an analytic continuation across the analytic arcs of ∂G which are "away" from K. In the Bergman space, point evaluations are bounded linear functionals whose norms are uniformly bounded on compact subsets. Using our analytic continuation, we will show that for functions in $\mathcal{N}$, not only are point evaluations continuous in $G\backslash K$ but they are also continuous for the extended functions outside the region $G\backslash K$. Hence for the space $\mathcal{N}$, not is only is analyticity preserved across the analytic arcs of ∂G but the continuity of the evaluation functional is also preserved.

Finally, our analytic continuation and bounded point evaluation results will be used in conjunction with techniques of [Ax] and [A-C-M] to discuss the Fredholm theory of the compression of certain Toeplitz operators on $A^2(G\backslash K)$ to the space $\mathcal{N}$.

Let $C(\bar{U})$ denote the set of complex valued continuous functions on $\bar{U}$ and let $A(U)$ denote the set of functions in $C(\bar{U})$ which are analytic on U. Let P_U denote the orthogonal projection of $L^2(U)$ onto $A^2(U)$ and for $f \in C(\bar{U})$, define the *Toeplitz operator* T_f^U on $A^2(U)$ by

$$T_f^U h = P_U(fh).$$

T_f^U is a bounded operator on $A^2(U)$ with $\|T_f^U\| \le \|f\|_\infty$, where $\|f\|_\infty = \sup\{|f(z)| : z \in \bar{U}\}$. When $f \in A(U)$, T_f^U is just a multiplication operator and when $f(z) = z$, T_f^U is called a *Bergman shift.* The authors in [Ax], and [A-C-M] have studied the Fredholm properties of T_f^U and have pointed out the relationship between the essential spectrum of T_f^U and certain boundary points of U where functions in $A^2(U)$ admit analytic continuation.

Let G be a Jordan domain and $K \subset G$ with Area$(K) = 0$ and $G\backslash K$ is open and connected. We define $\mathcal{N} = A^2(G\backslash K) \ominus A^2(G)$ and we let $P_{G\backslash K}$, P_G, and

$P_{\mathcal{N}}$ to be the orthogonal projections of $L^2(G\backslash K)$ onto $A^2(G\backslash K)$ (respectively $A^2(G)$ and $\mathcal{N}$). Let $f \in C(\bar{G})$. With respect to the decomposition $A^2(G\backslash K) = A^2(G) \oplus \mathcal{N}$ we can write $T_f^{G\backslash K}$ in matrix form as

$$T_f^{G\backslash K} = \begin{pmatrix} T_f^G & B_f \\ X_f & C_f \end{pmatrix},$$

where $B_f = P_G M_f|_{\mathcal{N}}$, $X_f = P_{\mathcal{N}} M_f|_{A^2(G)}$, and $C_f = P_{\mathcal{N}} M_f|_{\mathcal{N}}$. ($M_f$ is the multiplication operator $M_f(h) = fh$.)

The main object of study here is the operator C_f and the relationship between its Fredholm properties and the geometry of the set K. We also look at the Fredholm theory for T_f^G, X_f, and B_f, and how these contribute to the spectral and Fredholm picture of $T_f^{G\backslash K}$. If $f \in A(G)$, then $\mathcal{N}$ would be a *semi-invariant* subspace for $T_f^{G\backslash K}$ and C_f would be called the *compression* of $T_f^{G\backslash K}$ to $\mathcal{N}$. (Note that in this case $X_f = 0$, see [S].) In the special case of C_z, a rough analog has been studied by Conway in the Hardy space $H^2(G\backslash K)$, [Co3, Sections 4 and 5]. In Conway's case, the role of the decomposition $f = f_D + f_{\mathcal{N}}$ is replaced by the decomposition $f = f_1 + f_2$ defined above. We also mention that there are results on the lattice of invariant subspaces for the operator C_z, see [Ro].

Many of the techniques of [A-C-M] will be used here but our main tool will be the analytic continuation and bounded point evaluation properties of the space $\mathcal{N}$ which we will obtain using the reproducing kernel functions for $\mathcal{N}$ and various conformal mapping techniques.

2. Fredholm Theory and Analytic Continuation. As mentioned in the introduction, there is a close relationship between the essential spectrum of the Toeplitz operator T_f^U and analytic continuation of functions in $A^2(U)$ across certain boundary points of U. We now lay down some basic Fredholm theory facts and discuss the essential spectrum of the Toeplitz operator T_f^U.

Let H be a separable Hilbert space and let $\mathcal{B}(H)$ denote the set of all bounded operators on H. Let $\mathcal{K}(H)$ be the two sided ideal of compact operators on H. An operator $T \in \mathcal{B}(H)$ is said to be *Fredholm* if $\mathrm{Ran}(T)$ is closed and both $\ker(T)$ and $\ker(T^*)$ are finite dimensional. The *essential spectrum* of T, denoted by $\sigma_e(T)$, is the set of complex numbers λ such that $T - \lambda$ is not Fredholm. Let $\pi : \mathcal{B}(H) \to \mathcal{B}(H)/\mathcal{K}(H)$ be the natural map from $\mathcal{B}(H)$ onto the *Calkin algebra* $\mathcal{B}(H)/\mathcal{K}(H)$. A corollary to Atkinson's theorem says that T is Fredholm if and only if $\pi(T)$ is invertible in the Calkin algebra. The *essential norm* of T, $\|T\|_e$, is the norm of $\pi(T)$. That is

$$\|T\|_e = \inf\{\|T - K\| : K \in \mathcal{K}(H)\}.$$

The *index* of a Fredholm operator T, denoted by $\text{ind}(T)$, is defined by $\text{ind}(T) = \dim(\ker T) - \dim(\ker T^*)$. The index is a continuous map from the set of Fredholm operators (with the norm topology) to the integers (with the discrete topology). Finally, we mention that the essential spectrum, essential norm, and index are invariant under compact perturbations.

Let W be a bounded connected region in $\mathbf{C}$. A point $\lambda \in \partial W$ is said to be *removable* with respect to $A^2(W)$ if there is an open neighborhood V of λ so that every function in $A^2(W)$ can be extented to be analytic in $W \cup V$. Define $\partial_r W = \{\lambda \in \partial W : \lambda \text{ is removable}\}$ and $\partial_e W = \partial W \backslash \partial_r W$. $\partial_e W$ is called the *Bergman essential boundary* of W. By definition, $\partial_r W$ is a relatively open subset of ∂W and hence $\partial_e W$ is compact. This next proposition lists some basic facts about the Bergman essential boundary. The proof can be found in [A-C-M].

Proposition 2.1.

(a) *If λ is an isolated point of ∂W, then $\lambda \in \partial_r W$.*
(b) *$\partial_r W$ has zero area.*
(c) *$\partial \bar{W} \subset \partial_e W$.*
(d) *Let $\lambda \in \partial W$. If the connected component of ∂W containing λ contains more than one point, then $\lambda \in \partial_e W$. (That is, $\partial_r W$ is totally disconnected.)*

We remark that there is a relationship between removable singularities in the Bergman space and logarithmic capacity, see [A-C-M, Lemma 15, Theorem 16], [Ca, p. 73], and [A-P, Theorem B]. We state these here for future reference.

Proposition 2.2. *Let K be a compact subset of a region U. Then $A^2(U \backslash K)$ is equal to $A^2(U)$ if and only if K has logarithmic capacity zero. Moreover, if K has positive logarithmic capacity, there is a measure μ on K whose Cauchy transform*

$$\hat{\mu}(z) = \int_K \frac{d\mu(\xi)}{\xi - z}$$

is a non-zero element of $A^2(U \backslash K)$.

Proposition 2.3. *A point $\lambda \in \partial U$ is removable if and only if there is a $\delta > 0$ such that $\overline{B(\lambda;\delta)} \backslash U$ has logarithmic capacity zero.*

We now state the connection between the Bergman essential boundary and the Fredholm theory for the Toeplitz operator T_f^W. The proof of this next proposition can be found in [A-C-M].

Proposition 2.4. *Let $f \in C(\overline{W})$*

(a) T_f^W *is compact if and only if f vanishes on $\partial_e W$.*

(b) $\sigma_e(T_f^W) = f(\partial_e W)$.

(c) $\|T_f^W\|_e = \|f|_{\partial_e W}\|_\infty$.

Let G be a Jordan region and $K \subset G$ be such that $\mathrm{Area}(K) = 0$ and $G \backslash K$ is open and connected. Define the *essential part of K* by

$$K_e = K \cap \partial_e(G \backslash K). \tag{2}$$

To facilitate our construction, we make the following mild restriction on K. We refer to this restriction as property $\mathcal{P}$.

Property $\mathcal{P}$. *$K_e \subset \tilde{K}$, where $\mathrm{Area}(\tilde{K}) = 0$ and $G \backslash \tilde{K}$ is simply connected.*

3. Reproducing Kernels and Analytic Continuation. A key element in the analysis of the space $\mathcal{N}$ and the operator C_f is the reproducing kernel. We now state the basic facts about the reproducing kernels for the Bergman space. For further information and proofs of these facts, see [Ar], [Be], [Kr], [N].

For any open set U and $w \in U$, the linear functional $\ell_w : A^2(U) \to \mathbf{C}$ defined by $\ell_w(f) = f(w)$ is continuous. Hence, there is a function $k^U(w,z)$ such that

$$f(w) = \int_U f(z)\overline{k^U(w,z)}dA(z). \tag{3}$$

The functions $k^U(w,z)$ are called the *reproducing kernels* for $A^2(U)$. As a function of two variables, $k^U(z,w)$ is analytic in the variable z and co-analytic in the variable w, where $z,w \in U$. We sometimes write $k_w^U(z)$ for $k^U(w,z)$. If P_U denotes the orthogonal projection of $L^2(U)$ onto $A^2(U)$, then P_U can be given in terms of kernel functions by the following formula

$$(P_U f)(w) = \int_U f(z)\overline{k_w^U(z)}dA(z) = \langle f, k_w^U \rangle. \tag{4}$$

We also mention that since $k_w^U(z)$ has the reproducing property, we have

$$\|k_w^U\|_2^2 = \langle k_w^U, k_w^U \rangle = k^U(w,w). \tag{5}$$

If φ is a conformal map from the region U_1 onto U_2, then φ will induce the unitary map from $A^2(U_2)$ to $A^2(U_1)$ by $f \to (f \circ \varphi)\varphi'$. Thus the kernel functions k^{U_1} and k^{U_2} are related by

$$k^{U_1}(w,z) = \overline{\varphi'(w)}\varphi'(z)k^{U_2}(\varphi(w),\varphi(z)). \tag{6}$$

The reproducing kernel for $A^2(D)$, where D is the unit disk, is given by $k^D(w,z) = \pi^{-1}(1-\bar{w}z)^{-2}$ and if U is any region that is mapped conformally onto D by φ, then applying (6), we get

$$k^U(w,z) = \frac{1}{\pi}\frac{\overline{\varphi'(w)}\varphi'(z)}{(1-\overline{\varphi(w)}\varphi(z))^2}. \tag{7}$$

(We remark that the kernel functions are unique so the above formula is independent of the choice of conformal map φ.) Let $\mathcal{N} = A^2(D\backslash K)\ominus A^2(D)$, where $K \subset D$ such that $D\backslash K$ is simply connected. Let $P_{D\backslash K}$, P_D, and $P_{\mathcal{N}}$ be the orthogonal projections from $L^2(D\backslash K)$ onto $A^2(D\backslash K)$ (respectively $A^2(D)$, $\mathcal{N}$). We know that $(P_D f)(w) = \langle f, k_w^D\rangle$ and $(P_{D\backslash K} f)(w) = \langle f, k_w^{D\backslash K}\rangle$, and since $P_{\mathcal{N}} = P_{D\backslash K} - P_D$ we see that

$$(P_{\mathcal{N}} f)(w) = \langle f, k_w^{D\backslash K} - k_w^D\rangle, \tag{8}$$

hence

$$k^{\mathcal{N}}(w,z) = k^{D\backslash K}(w,z) - k^D(w,z) \tag{9}$$

is the reproducing kernel for the space $\mathcal{N}$. Let φ be the conformal map from $D\backslash K$ on to D. By (7) we get

$$k^{D\backslash K}(w,z) = \frac{1}{\pi}\frac{\overline{\varphi'(w)}\varphi'(z)}{(1-\overline{\varphi(w)}\varphi(z))^2}. \tag{10}$$

Since Area$(K) = 0$, then $k^D(w,z) = \pi^{-1}(1-\bar{w}z)^{-2}$ and thus

$$k^{\mathcal{N}}(w,z) = \frac{1}{\pi}\frac{\overline{\varphi'(w)}\varphi'(z)}{(1-\overline{\varphi(w)}\varphi(z))^2} - \frac{1}{\pi}\frac{1}{(1-\bar{w}z)^2}. \tag{11}$$

Our first use of the kernel functions $k^{\mathcal{N}}(w,z)$ will be to show that functions in $\mathcal{N}$ have analytic continuation across $\partial D\backslash\overline{K_e}$ in the sense that there exists a neighborhood U of $D\backslash K_e$, containing $\partial D\backslash\overline{K_e}$, such that every $f \in \mathcal{N}$ has an analytic continuation to U, and for these analytic extensions, the linear functional $w \to f(w)$ is continuous for all $w \in U$. Our first step is to prove this for K such that $D\backslash K_e$ is simply connected. (Just so there is no confusion, we note that by Proposition 2.1, $A^2(D\backslash K) = A^2(D\backslash K_e)$.)

Theorem 3.1. *Let $K \subset D$ have the property $\mathcal{P}$ and such that $D\backslash K_e$ is simply connected. Define $\mathcal{N} = A^2(D\backslash K)\ominus A^2(D)$. Then, there exists a neighborhood U of $D\backslash K_e$, containing, $\partial D\backslash\overline{K_e}$ and a function $K^{\mathcal{N}}(w,z)$ of w and z on $U\times U$ with $K^{\mathcal{N}}(w,z)$ analytic in z for fixed w in U and co-analytic in w for fixed z in U and $K^{\mathcal{N}}(w,z) = k^{\mathcal{N}}(w,z)$ for all z and w in $D\backslash K_e$.*

Before proving this theorem, we must make a few remarks about φ, the conformal map between $D\backslash K_e$ and D.

For a non-zero complex number z, define $z^* = \bar{z}^{-1}$ and for any set of complex numbers B which does not contain the origin, define $B^* = \{b^* : b \in B\}$. We say that B is *symmetric* about an arc in the unit circle if $B = B^*$.

Let I be any relatively open arc compactly contained in $\partial D\backslash\overline{K_e}$ and let φ be the conformal map from $D\backslash K_e$ onto D. By a theorem of Carathéodory [M, p. 66–71], we get that φ is continuous on $(D\backslash K_e)\cup I$ and by the symmetry principle [M, p. 315], there is a neighborhood V of I such that V is symmetric with respect to I and φ has an analytic continuation to V. We identify the function φ on $D\backslash K_e$ with its analytic continuation to V.

Fix $e^{i\vartheta} \in I$ and $\zeta \in \partial D$ with $\zeta\varphi(e^{i\vartheta}) = e^{i\vartheta}$. A calculation shows that $\zeta\varphi'(e^{i\vartheta}) > 0$. Hence we can assume that our neighborhood V was chosen so that $\varphi'(w) \neq 0$ for all $w \in V$. This next lemma gives a formula for $\varphi(z)$ when z lies outside $D\backslash K_e$.

Lemma 3.2. *$\varphi(z) = \varphi(z^*)^*$ for all $z \in V$.*

Proof. $V \supset I$ and $\varphi(z)\overline{\varphi(z^*)}$ is analytic on V and equal to 1 on I, hence equal to 1 on V. □

Lemma 3.3. *For all $z \in V$,*

$$\frac{\varphi'(z)z^2}{\varphi(z)^2\overline{\varphi'(z^*)}} = 1.$$

Proof. Since

$$h(z) = \frac{\varphi'(z)z^2}{\varphi(z)^2\overline{\varphi'(z^*)}}$$

is analytic on V, we will be done once we show that $h(z) = 1$ for all all $z \in I$. Fix $e^{i\vartheta} \in I$ and choose $\zeta \in \partial D$ with $\zeta\varphi(e^{i\vartheta}) = e^{i\vartheta}$. A calculation shows that $\zeta\varphi'(e^{i\vartheta}) > 0$, and so

$$\begin{aligned}\frac{\varphi'(e^{i\vartheta})e^{2i\vartheta}}{\varphi(e^{i\vartheta})^2\overline{\varphi'(e^{i\vartheta})}} &= \frac{\zeta^2\varphi'(e^{i\vartheta})e^{2i\vartheta}}{\zeta^2\varphi(e^{i\vartheta})^2\overline{\varphi'(e^{i\vartheta})}}\\ &= \frac{(\zeta\varphi'(e^{i\vartheta}))e^{2i\vartheta}}{\overline{\zeta\varphi'(e^{i\vartheta})}(\zeta\varphi(e^{i\vartheta}))^2} = 1.\end{aligned}$$ □

By Lemma 3.2, φ is univalent on $W = V\cup(D\backslash K_e)$ and if we define $g(w,z) = 1-\overline{\varphi(w)}\varphi(z)$ for z, w in W, we see that for fixed $z \in W$, $g(w,z)$ is co-analytic in the variable w on W with a zero of order one at $w = z^*$ (if $z^* \in W$) and no other zeros in W. Similarly, for fixed w in W, $g(w,z)$ is analytic in the variable z on W with a zero of order one at $z = w^*$ (if $w^* \in W$) and no other zeros in W.

Proof of Theorem 3.1. Fix $I \subset \partial D \backslash \overline{K_e}$, and let W be chosen as above. Define $K^{\mathcal{N}}(w,z)$ on $W \times W$ by

$$K^{\mathcal{N}}(w,z) = \frac{1}{\pi}\frac{\overline{\varphi'(w)}\varphi'(z)}{(1-\overline{\varphi(w)}\varphi(z))^2} - \frac{1}{\pi}\frac{1}{(1-\bar{w}z)^2}, \tag{12}$$

where φ is considered to be an analytic function on W. We first note that $K^{\mathcal{N}}(w,z) = k^{\mathcal{N}}(w,z)$ for all w and z in $D\backslash K_e$. We must now show that $K^{\mathcal{N}}(w,z)$ has the desired properties. Fix $z \in W$. If $z^* \notin W$ then $K^{\mathcal{N}}(w,z)$ is co-analytic on W and we are done. If, on the other hand, $z^* \in W$, then the functions

$$K_1(w,z) = \frac{1}{\pi}\frac{\overline{\varphi'(w)}\varphi'(z)}{(1-\overline{\varphi(w)}\varphi(z))^2} \tag{13}$$

and $K_2(w,z) = \pi^{-1}(1-\bar{w}z)^{-2}$ both have poles of order 2 at $w = z^*$ but no other poles in W. The rest of the proof will be dedicated to show that the principal parts of these poles are equal, hence the $K^{\mathcal{N}}(w,z) = K_1(w,z) - K_2(w,z)$ has a removable singularity at $w = z^*$. We can write the power series of φ in a neighborhood of z^* in the following form

$$\varphi(w) = \sum_{n=0}^{\infty} c_n(z)(1-w\bar{z})^n \tag{14}$$

where

$$c_n(z) = (-z^*)^n \frac{\varphi^{(n)}(z^*)}{n!}. \tag{15}$$

Since $K_1(w,z)$ has a pole of order 2 at $w = z^*$, it has a Laurent series which we can write in the following form

$$K_1(w,z) = \frac{1}{\pi}\left[\frac{d_{-2}(z)}{(1-\bar{w}z)^2} + \frac{d_{-1}(z)}{(1-\bar{w}z)} + \sum_{n=0}^{\infty} d_n(z)(1-\bar{w}z)^n\right]. \tag{16}$$

The function $K_1(w,z)$ can also be written in the usual form

$$\begin{aligned} K_1(w,z) &= \frac{1}{\pi}\frac{\overline{\varphi'(w)}\varphi'(z)}{(1-\overline{\varphi(w)}\varphi(z))^2} \\ &= \frac{1}{\pi}\frac{\varphi'(z)\sum_{n=1}^{\infty}(-n)z\overline{c_n(z)}(1-\bar{w}z)^{n-1}}{\left(\sum_{n=1}^{\infty}\varphi(z)\overline{c_n(z)}(1-\bar{w}z)^n\right)^2} \\ &= \frac{1}{\pi}\frac{a_0(z)+a_1(z)(1-\bar{w}z)+a_2(z)(1-\bar{w}z)^2+\cdots}{b_2(z)(1-\bar{w}z)^2+b_3(z)(1-\bar{w}z)^3+b_4(z)(1-\bar{w}z)^4+\cdots}. \end{aligned} \tag{17}$$

A computation reveals that

$$a_0(z) = -z\varphi'(z)\overline{c_1(z)}, \tag{18}$$

$$a_1(z) = -2z\varphi'(z)\overline{c_2(z)}, \tag{19}$$

$$b_2(z) = \varphi(z)^2\overline{c_1(z)}^2, \tag{20}$$

$$b_3(z) = 2\varphi(z)^2\overline{c_1(z)c_2(z)}. \tag{21}$$

Comparing the coefficients of $(1-\bar{w}z)^n$, we see that

$$a_0(z) = b_2(z)d_{-2}(z), \tag{22}$$

$$a_1(z) = b_2(z)d_{-1}(z) + d_{-2}(z)b_3(z). \tag{23}$$

By Lemma 3.3 and our calculations above,

$$d_{-2}(z) = \frac{a_0(z)}{b_2(z)} = \frac{z^2\varphi'(z)}{\varphi(z)^2\overline{\varphi(z^*)}} = 1, \tag{24}$$

and

$$\begin{aligned} d_{-1}(z) &= \frac{a_1(z) - b_3(z)d_{-2}(z)}{b_2(z)} \\ &= \frac{-2z\varphi'(z)\overline{c_2(z)} - 2\varphi(z)^2\overline{c_1(z)c_2(z)}}{\varphi(z)^2\overline{c_1(z)}^2} \end{aligned} \tag{25}$$

$$= \frac{\varphi'(z)z^2}{\varphi(z)^2\overline{\varphi'(z^*)}}\left(\frac{2\overline{c_2(z)}}{\overline{c_1(z)}}\right) - 2\left(\frac{\overline{c_2(z)}}{\overline{c_1(z)}}\right) = 0. \tag{26}$$

Thus

$$K_1(w,z) = \frac{1}{\pi}\left[\frac{1}{(1-\bar{w}z)^2} + \sum_{n=0}^{\infty} d_n(z)(1-\bar{w}z)^n\right], \tag{27}$$

yielding

$$K^{\mathcal{N}}(w,z) = K_1(w,z) - K_2(w,z) = \frac{1}{\pi}\sum_{n=0}^{\infty} d_n(z)(1-\bar{w}z)^n. \tag{28}$$

Hence the singularity at $w = z^*$ is removable, making $K^{\mathcal{N}}(w,z)$ co-analytic on W. In a similar fashion, we prove that for fixed $w \in W$, $K^{\mathcal{N}}(w,z)$ is analytic as a function of z on W.

Letting U be the union of all such W chosen above, gives us the desired result. □

Corollary 3.4. *Let U and $K^{\mathcal{N}}(w,z)$ be as in Theorem 3.1.*

(a) *If $w \in U$, then $K_w^{\mathcal{N}}(z) \in \mathcal{N}$ and*

$$sup\{\|K_w^{\mathcal{N}}\|_2 : w \in J\} < \infty$$

for all compact $J \subset U$.

(b) *If $f \in \mathcal{N}$, then $(Kf)(w) = \langle f, K_w^{\mathcal{N}} \rangle$ defines an analytic function on U which analytically continues f.*

(c) *If $w \in U$, then the evaluation functional $f \to (Kf)(w)$ defines a bounded linear functional on $\mathcal{N}$ with norm equal to $(K^{\mathcal{N}}(w,w))^{1/2}$.*

Proof. For $w \in U$, note that

$$\|K_w^{\mathcal{N}}\|_2^2 = \frac{1}{\pi^2} \int_{D \backslash K_e} \left| \frac{\overline{\varphi'(w)}\varphi'(z)}{(1 - \overline{\varphi(w)}\varphi(z))^2} - \frac{1}{(1 - \bar{w}z)^2} \right|^2 dA(z). \tag{29}$$

Hence by using Theorem 3.1, and the fact that $\varphi'(z) \in L^2(D \backslash K_e)$, we see that $\|K_w^{\mathcal{N}}\|_2 < \infty$ and

$$\sup\{\|K_w^{\mathcal{N}}\|_2 : w \in J\} < \infty$$

for all compact $J \subset U$. For $g \in L^2(D \backslash K_e)$, define a function $(Kg)(w)$ on U by

$$(Kg)(w) = \langle g, K_w^{\mathcal{N}} \rangle \tag{30}$$

and note that $(Kg)(w)$ is analytic on U. Since $K^{\mathcal{N}}(w,z) = k^{\mathcal{N}}(w,z)$ for all w, z in $D \backslash K_e$, then if $f \in \mathcal{N}$, one sees that $(Kf)(w)$ is an analytic continuation of f to U, proving (b).

To complete the proof of (a), we must show that $K_w^{\mathcal{N}}(z) \in \mathcal{N}$ for all $w \in U$. Clearly $K_w^{\mathcal{N}} \in \mathcal{N}$ for $w \in D \backslash K_e$, thus if $n = 0, 1, 2, \ldots$, $(Kz^n)(w) = 0$ for all $w \in D \backslash K_e$. But since $(Kz^n)(w)$ is analytic on U, one sees that $(Kz^n)(w) = 0$ on U. However, $\{z^n : n \geq 0\}$ spans $A^2(D)$, so $K_w^{\mathcal{N}}(z) \in \mathcal{N}$ for all w in U. Thus we have proved (a).

For the proof of (c), define the linear functional $\ell_w : \mathcal{N} \to \mathbf{C}$ by $\ell_w(f) = (Kf)(w)$, where $w \in U$. By the Cauchy-Schwartz inequality,

$$|\ell_w(f)| = |\langle f, K_w^{\mathcal{N}} \rangle| \leq \|f\|_2 \; \|K_w^{\mathcal{N}}\|_2.$$

Hence $\|\ell_w\| \leq \|K_w^{\mathcal{N}}\|_2 = (K^{\mathcal{N}}(w,w))^{1/2}$. But $\ell_w(K_w^{\mathcal{N}}) = K^{\mathcal{N}}(w,w)$, hence $\|\ell_w\| = (K^{\mathcal{N}}(w,w))^{1/2}$, proving (c). □

We now expand our results to regions which are possibly not simply connected.

Theorem 3.5. *Let $K \subset D$ have the property $\mathcal{P}$ and define*

$$\mathcal{N} = A^2(D\backslash K) \ominus A^2(D).$$

Then there exists a open neighborhood U of $D\backslash K_e$, containing $\partial D\backslash \overline{K_e}$ such that the conclusions of Theorem 3.1 and Corollary 3.4 hold.

Proof. The conclusions of Corollary 3.4 will follow in the same way as before once we have shown that the kernel functions $k^{\mathcal{N}}(w,z)$ have 'extended' kernel functions $K^{\mathcal{N}}(w,z)$ in the sense of Theorem 3.1.

By property $\mathcal{P}$, there is a $\tilde{K}$ with $K_e \subset \tilde{K}$, Area$(\tilde{K}) = 0$, and $D\backslash\tilde{K}$ is simply connected. Note that

$$D\backslash\tilde{K} \subset D\backslash K_e \subset D \tag{31}$$

and so

$$A^2(D) \subset A^2(D\backslash K) \subset A^2(D\backslash\tilde{K}). \tag{32}$$

(Note that $A^2(D\backslash K) = A^2(D\backslash K_e)$ by Proposition 2.1.) If we let

$$\tilde{\mathcal{N}} = A^2(D\backslash\tilde{K}) \ominus A^2(D), \tag{33}$$

and

$$\mathcal{N} = A^2(D\backslash K) \ominus A^2(D), \tag{34}$$

then $\mathcal{N} \subset \tilde{\mathcal{N}}$ and one can show that $k_w^{\mathcal{N}} = P_{\mathcal{N}} k_w^{\tilde{\mathcal{N}}}$. Here $k^{\tilde{\mathcal{N}}}(w,z)$ is the reproducing kernel for the space $\tilde{\mathcal{N}}$.

By Theorem 3.1, there exists a neighborhood U' of $D\backslash\tilde{K}$, containing $\partial D\backslash\overline{\tilde{K}}$, such that the kernels $k^{\tilde{\mathcal{N}}}(w,z)$ have 'extensions' $K^{\tilde{\mathcal{N}}}(w,z)$ to U' in the sense of Theorem 3.1. Thus, as before, $K^{\tilde{\mathcal{N}}}(w,z) \in \tilde{\mathcal{N}}$ for all $w \in U'$ and if $g \in L^2(D)$, $(Kg)(w) = \langle g, K_w^{\tilde{\mathcal{N}}}\rangle$ is analytic on U'.

Since $\mathcal{N} \subset \tilde{\mathcal{N}}$, one sees that if z and w are in $D\backslash\tilde{K}$, then $k^{\mathcal{N}}(w,z) = \langle k_w^{\mathcal{N}}, K_z^{\tilde{\mathcal{N}}}\rangle$ which allows us to extend the kernels $k^{\mathcal{N}}(w,z)$ to U' in the sense of Theorem 3.1 by

$$K^{\mathcal{N}}(w,z) = \langle k_w^{\mathcal{N}}, K_z^{\tilde{\mathcal{N}}}\rangle. \tag{35}$$

If K_e were a compact subset of D, then $\overline{\tilde{K}}$ will be forced to intersect ∂D at some point z_0, which might introduce a possible singularity of $K^{\tilde{\mathcal{N}}}(w,z)$ at (z_0, z_0). This singularity can be avoided in the $K^{\mathcal{N}}(w,z)$ function by moving the point z_0 to another point z_0' (i.e. readjusting $\tilde{K}$).

Letting $U = U' \cup (D\backslash K_e)$, we see that $K^{\mathcal{N}}(w,z)$ has an obvious extension (in the sense of Theorem 3.1) to $U \times U$ and $K^{\mathcal{N}}(w,z) = k^{\mathcal{N}}(w,z)$ for all w and z in $D\backslash K_e$. □

We expand our analytic continuation results to Jordan domains with the aid of conformal mappings.

Theorem 3.6. *Let G be a Jordan domain and $K \subset G$ have the property $\mathcal{P}$. Let $\mathcal{N} = A^2(G\backslash K) \ominus A^2(G)$ and γ be the union of all relatively open analytic arcs contained in $\partial G\backslash\overline{K_e}$. Then, there exists a neighborhood U of $G\backslash K_e$, which contains γ such that the conclusions of Theorem 3.1 and Corollary 3.4 hold.*

Before we prove Theorem 3.6, we make the following observation which will be used several times in this paper.

Let G_1 and G_2 be regions in $\mathbf{C}$ and $K \subset G_1$ be relatively closed with $Area(K) = 0$. If φ is a conformal map between G_1 and G_2 then φ will induce a unitary operator $U : A^2(G_2\backslash\varphi(K)) \to A^2(G_1\backslash K)$ by $Uf = (f \circ \varphi)\varphi'$. One can conclude from this that $\varphi(K_e) = \varphi(K)_e$.

Proof of Theorem 3.6. Let φ be the conformal map from G to D and notice by our remarks above, $\varphi(K)$ will have the property $\mathcal{P}$. Let

$$\mathcal{M} = A^2(D\backslash\varphi(K)) \ominus A^2(D)$$

and notice that by Theorem 3.5, there exists a neighborhood V of $D\backslash\varphi(K_e)$, which contains $\partial D\backslash\overline{\varphi(K_e)}$, so that the kernels $k^{\mathcal{M}}(\lambda, \xi)$ for $\mathcal{M}$ have 'extensions' $K^{\mathcal{M}}(\lambda, \xi)$, in the sense of Theorem 3.1, to $V \times V$. By [M, p. 66-71 and p. 315], there exists a neighborhood U of $G\backslash K_e$ that contains γ such that φ analytically continues to U. We can assume, possibly by shrinking U, that $\varphi(U) \subset V$.

Extend $k^{\mathcal{N}}(w, z)$ to $U \times U$ in the sense of Theorem 3.1 by

$$K^{\mathcal{N}}(w, z) = \overline{\varphi'(w)}\varphi'(z)K^{\mathcal{M}}(\varphi(w), \varphi(z)). \tag{36}$$ □

For a compact $K \subset D$, K having the property $\mathcal{P}$, we have two decompositions of a function $f \in A^2(D\backslash K)$. We have the Laurent decomposition $f = f_1 + f_2$ mentioned in the introduction (see equation 1) and the orthogonal decomposition $f = f_D + f_{\mathcal{N}}$. One asks the question whether or not these two decompositions are always equal. More precisely, for what K do we have $f_1 = f_D$ and $f_2 = f_{\mathcal{N}}$ for all $f \in A^2(D\backslash K)$?

Theorem 3.7. *The following are equivalent*

(i) *$f_1 = f_D$ and $f_2 = f_{\mathcal{N}}$ for all $f \in A^2(D\backslash K)$.*
(ii) *K has logarithmic capacity zero.*
(iii) $\mathcal{N} = 0$.

Proof. (ii) $\Leftrightarrow$ (iii) by [A-C-M, Lemma 15] and $(ii) \Rightarrow (i)$ is trivial. So suppose that K has positive logarithmic capacity. Then by Proposition 2.2, there is a non-zero measure μ on K whose Cauchy transform

$$\hat{\mu}(z) = \int_K \frac{d\mu(\xi)}{\xi - z} \tag{37}$$

belongs to $A^2(D \backslash K)$. Let $\hat{\mu} = f_1 + f_2$ be the Laurent decomposition and notice that

$$\hat{\mu}(z) = \int_K \frac{d\mu(\xi)}{\xi - z} = -\sum_{n=0}^{\infty} \left(\int_K \xi^n d\mu(\xi) \right) z^{-(n+1)} \tag{38}$$

So $f_1(z) = 0$.

If the decomposition $\hat{\mu} = f_1 + f_2$ were orthogonal, then $f_2 \perp A^2(D)$. So for all $m \geq 0$,

$$0 = \langle \hat{\mu}, z^m \rangle = \int_K \int_D \frac{\bar{z}^m}{\xi - z} \, dA(z) \, d\mu(\xi). \tag{39}$$

Now,

$$\int_D \frac{\bar{z}^m}{\xi - z} \, dA(z) = \int_{|z|<|\xi|} + \int_{|\xi|<|z|<1} \tag{40}$$

and

$$\int_{|z|<|\xi|} \frac{\bar{z}^m}{\xi - z} \, dA(z) = \int_{|z|<|\xi|} \left(\sum_{n=0}^{\infty} \frac{\bar{z}^m z^n}{\xi^{n+1}} \right) dA(z). \tag{41}$$

Converting to polar coordinates, one sees that

$$\int_{|z|<|\xi|} \frac{\bar{z}^m}{\xi - z} \, dA(z) = \left(\frac{\pi}{m+1} \right) \bar{\xi}^{\,m+1}. \tag{42}$$

Also,

$$\int_{|\xi|<|z|<1} \frac{\bar{z}^m}{\xi - z} \, dA(z) = -\int_{|\xi|<|z|<1} \bar{z}^m \left(\sum_{n=0}^{\infty} \frac{\xi^n}{z^{n+1}} \right) dA(z). \tag{43}$$

Converting to polar coordinates again yields

$$\int_{|\xi|<|z|<1} \frac{\bar{z}^m}{\xi - z} \, dA(z) = 0. \tag{44}$$

$$0 = \langle \hat{\mu}, z^m \rangle = \left(\frac{\pi}{m+1} \right) \int_K \overline{\xi}^{\,m+1} \, d\mu(\xi) \tag{45}$$

for all $m \geq 0$.

Let $d\nu(\xi) = \xi d\mu(\xi)$ and note that

$$\int_K p(\xi) \, d\nu(\xi) = 0 \tag{46}$$

for all polynomials $p(\xi)$. Since $D \backslash K$ is connected and $\text{Area}(K) = 0$, an application of Lavrentiev's theorem [Co2, p. 343] gives us that polynomials are uniformly dense in the continuous functions on K. Thus $d\nu = 0$, which means that $d\mu$ must be a point mass at the origin. But then $\hat{\mu}(z) = Cz^{-1}$ which does not belong to $L^2(D)$, a contradiction. □

4. Some Matrix Calculations. Let $f \in C(\overline{G})$ and form the Toeplitz operator $T_f^{G\backslash K}$. With respect to the decomposition $A^2(G\backslash K) = A^2(G) \oplus \mathcal{N}$ we write $T_f^{G\backslash K}$ in matrix form as

$$T_f^{G\backslash K} = \begin{pmatrix} T_f^G & B_f \\ X_f & C_f \end{pmatrix}. \tag{47}$$

From [A-C-M, Proposition 8] we get that if $W = G\backslash K$ or $W = G$, and if $f, g \in C(\overline{G})$, then $T_{fg}^W - T_f^W T_g^W$ and $T_f^W T_g^W - T_g^W T_f^W$ are compact operators. We now prove a similar result for the operators C_f and C_g.

Proposition 4.1. *For f, $g \in C(\overline{G})$,*

(a) *B_f and X_f are compact operators,*

(b) *$C_{fg} - C_f C_g$ and $C_f C_g - C_g C_f$ are compact operators.*

Proof.

$$T_{fg}^{G\backslash K} = \begin{pmatrix} T_{fg}^G & B_{fg} \\ X_{fg} & C_{fg} \end{pmatrix} \tag{48}$$

and

$$T_f^{G\backslash K} T_g^{G\backslash K} = \begin{pmatrix} T_f^G T_g^G + B_f X_g & * \\ * & X_f B_g + C_f C_g \end{pmatrix}. \tag{49}$$

Applying [A-C-M, Proposition 8] we get that $T_{fg}^{G\backslash K} - T_f^{G\backslash K} T_g^{G\backslash K}$ is a compact operator and by (48) and (49), is equal to

$$\begin{pmatrix} (T_{fg}^G - T_f^G T_g^G) - B_f X_g & * \\ * & (C_{fg} - C_f C_g) - X_f B_g \end{pmatrix}. \tag{50}$$

Since $T_{fg}^G - T_f^G T_g^G$ is compact as is every entry in the above matrix, then $B_f X_g$ is compact for all $f, g \in C(\overline{G})$. Note that since $B_f = X_{\overline{f}}^*$ and $X_f = B_{\overline{f}}^*$, then $B_f X_g = B_f B_{\overline{g}}^*$ is compact for all $f, g \in C(\overline{G})$. Letting $g = \overline{f}$ we see that $B_f B_f^*$ is compact, and applying the polar decomposition and the spectral theorem, we get B_f is compact. A similar argument shows X_f is compact, proving (a).

Again looking at the matrix representation for $T_{fg}^{G\backslash K} - T_f^{G\backslash K} T_g^{G\backslash K}$ (50), we see $(C_{fg} - C_f C_g) - X_f B_g$ is compact, and now using (a) we obtain $C_{fg} - C_f C_g$ is compact. Finally, $C_f C_g - C_g C_f = (C_f C_g - C_{fg}) + (C_{gf} - C_g C_f)$ is compact, proving (b). □

Since

$$T_f^{G\backslash K} = \begin{pmatrix} T_f^G & 0 \\ 0 & C_f \end{pmatrix} + \begin{pmatrix} 0 & B_f \\ X_f & 0 \end{pmatrix}, \tag{51}$$

we can use Proposition 4.1 to conclude that the second matrix in the above sum is a compact operator. Since the essential spectrum of an operator remains unchanged under compact perturbations we get

$$\sigma_e(T_f^{G\backslash K}) = \sigma_e\left(\begin{pmatrix} T_f^G & 0 \\ 0 & C_f \end{pmatrix} \right) = \sigma_e(T_f^G) \cup \sigma_e(C_f). \tag{52}$$

By Proposition 2.4, $\sigma_e(T_f^{G\backslash K}) = f(\partial_e(G\backslash K))$ and $\sigma_e(T_f^G) = f(\partial_e G) = f(\partial G)$ (Proposition 2.1). Since $K_e = K \cap \partial_e(G\backslash K)$, we get $\sigma_e(C_f) \supset f(\overline{K_e})$. By using techniques of [A-C-M], we will ultimately show that $\sigma_e(C_f) = f(\overline{K_e})$ but first we will need some information on the index of the operator C_z.

Proposition 4.2. *Let $\lambda \in \overline{G}$.*

(a) *For $\lambda \notin \sigma_e(C_z)$, $\operatorname{ind}(C_z - \lambda) = 0$.*

(b) *For $\lambda \in \overline{K_e}$, $C_z - \lambda$ does not have closed range.*

Proof.

(a) Since $\sigma_e(T_z^{G\backslash K}) = \partial_e(G\backslash K)$ and $\sigma_e(T_z^G) = \partial_e G = \partial G$ (Proposition 2.1), then if $\lambda \notin \partial_e(G\backslash K)$, one sees that $\lambda \notin \sigma_e(T_z^{G\backslash K}) \cup \sigma_e(T_z^G)$. Thus $\operatorname{ind}(T_{z-\lambda}^{G\backslash K})$ and $\operatorname{ind}(T_{z-\lambda}^G)$ are well defined and a calculation [A-C-M, Theorem 5] shows that $\operatorname{ind}(T_{z-\lambda}^{G\backslash K}) = \operatorname{ind}(T_{z-\lambda}^G) = -1$. Since

$$T_{z-\lambda}^{G\backslash K} = \begin{pmatrix} T_{z-\lambda}^G & B_z \\ 0 & C_{z-\lambda} \end{pmatrix} \tag{53}$$

and B_z is compact, then

$$\begin{aligned} -1 &= \operatorname{ind}(T_{z-\lambda}^{G\backslash K}) \\ &= \operatorname{ind}\begin{pmatrix} T_{z-\lambda}^G & 0 \\ 0 & C_{z-\lambda} \end{pmatrix} \\ &= \operatorname{ind}(T_{z-\lambda}^G \oplus C_{z-\lambda}) \qquad (54) \\ &= \operatorname{ind}(T_{z-\lambda}^G) + \operatorname{ind}(C_{z-\lambda}) \\ &= -1 + \operatorname{ind}(C_{z-\lambda}). \end{aligned}$$

Hence $\operatorname{ind}(C_{z-\lambda}) = 0$ for all $\lambda \notin \partial_e(G\backslash K)$. By the remarks before Proposition 4.2, $\sigma_e(C_z) \subset \partial_e(G\backslash K)$, so if $\lambda \notin \sigma_e(C_z)$ but contained in $\partial_e(G\backslash K)$ we can choose a sequence $\{\lambda_n\}$ contained in $\overline{G}\backslash\partial_e(G\backslash K)$ so that $\lambda_n \to \lambda$. Since $\lambda \notin \sigma_e(C_z)$ then $C_{z-\lambda}$ is Fredholm, thus $\operatorname{ind}(C_{z-\lambda})$ is well defined. But, since the index is continuous, $\operatorname{ind}(C_{z-\lambda_n}) = 0$ for all n, and $C_{z-\lambda_n} \to C_{z-\lambda}$, then $\operatorname{ind}(C_{z-\lambda}) = 0$, proving (a).

(b) Let $\lambda \in \overline{K_e}$, then, by the remarks before Proposition 4.2, $\lambda \in \sigma_e(C_z)$. Note that $\ker(C_{z-\lambda}) = \{0\}$ for all $\lambda \in \mathbf{C}$. (If $f \in \mathcal{N}$ with $C_{z-\lambda}f = 0$, then $(z-\lambda)f \in A^2(G)$ which implies that $f \in A^2(G)$. But $f \in \mathcal{N}$, hence $f = 0$.) Thus, if $C_{z-\lambda}$ has closed range, then $C_{z-\lambda}$ would be left invertible and hence left Fredholm. This would imply that $\operatorname{ind}(C_{z-\lambda})$ would be well defined. But since the index is a continuous map and $\operatorname{ind}(C_{z-\mu})=0$ for all $\mu \notin \sigma_e(C_z)$ then $\operatorname{ind}(C_{z-\lambda}) = 0$ which implies that $C_{z-\lambda}$ is Fredholm, contradicting the fact that $\lambda \in \sigma_e(C_z)$. □

5. Compactness of the Compression. In this section we prove a compactness criterion for C_f similar to Proposition 2.4 in almost the same way as [A-C-M, Theorem 7] did for the Toeplitz operator $T_f^{G\backslash K}$ except that our analytic continuation results of Section 3 will provide a suitable substitution for the analytic continuation used to prove the original result. We first prove our theorem under the simplifying assumption that $G = D$ and then move to the general case by means of a conformal mapping trick. Let $K \subset D$ have the property $\mathcal{P}$ and letting $f \in C(\overline{D})$, define $\mathcal{N}$, $T_f^{D\backslash K}$, and C_f in the usual way.

Theorem 5.1. *Let $f \in C(\overline{D})$. Then C_f is a compact operator if and only if f vanishes on $\overline{K_e}$.*

Proof. Suppose that C_f is compact and that $\lambda \in \overline{K_e}$ we wish to show that $f(\lambda) = 0$. Since $\lambda \in \overline{K_e}$, Proposition 4.2 says that $C_{z-\lambda}$ does not have closed range. Hence there is a sequence $\{h_n\}$ in $\mathcal{N}$ with $\|h_n\|_2 = 1$, for all n and $\|C_{z-\lambda}h_n\|_2 \to 0$. By Banach-Alaoglu and passing to a subsequence if necessary, we can assume that $h_n \to h$ weakly in $\mathcal{N}$. Using the facts that $C_{z-\lambda}h_n \to C_{z-\lambda}h$ weakly in $\mathcal{N}$, $\|C_{z-\lambda}h_n\|_2 \to 0$, and that $\ker(C_{z-\lambda}) = 0$, we can conclude that $h_n \to 0$ weakly in $\mathcal{N}$.

Let $\varepsilon > 0$ be given and let U be a neighborhood of λ so that

$$\|(f - f(\lambda))|_{D\cap U}\|_\infty < \varepsilon. \tag{55}$$

Then

$$\begin{aligned}\left\|(f - f(\lambda)h_n)\right\|_2^2 &= \int_{(D\setminus K)\cap U} |f - f(\lambda)|^2 \; |h_n|^2 \, dA \\ &\quad + \int_{(D\setminus K)\setminus U} |f - f(\lambda)|^2 |h_n|^2 \, dA \quad \le \\ &\le \varepsilon^2 + \int_{(D\setminus K)\setminus U} \left|\frac{f - f(\lambda)}{z - \lambda}\right|^2 |z - \lambda|^2 \; |h_n|^2 \, dA \\ &\le \varepsilon^2 + \left\| \left.\frac{f - f(\lambda)}{z - \lambda}\right|_{(D\setminus K)\setminus U} \right\|_\infty^2 \; \|T_{z-\lambda}^{D\setminus K} h_n\|_2^2. \end{aligned} \tag{56}$$

But

$$T_{z-\lambda}^{D\setminus K} h_n = \begin{pmatrix} T_{z-\lambda}^{D} & B_z \\ 0 & C_{z-\lambda} \end{pmatrix} \begin{pmatrix} 0 \\ h_n \end{pmatrix} = \begin{pmatrix} B_z h_n \\ C_{z-\lambda} h_n \end{pmatrix} \tag{57}$$

so

$$\|T_{z-\lambda}^{D\setminus K} h_n\|_2^2 = \|B_z h_n\|_2^2 + \|C_{z-\lambda} h_n\|_2^2. \tag{58}$$

But since $h_n \to 0$ weakly and B_z is compact (Proposition 4.1), then $\|B_z h_n\|_2 \to 0$. Thus $\|T_{z-\lambda}^{D\setminus K} h_n\|_2 \to 0$, and hence $\|(f - f(\lambda))h_n\|_2 \to 0$. Thus

$$|f(\lambda)| = \|f(\lambda)h_n\|_2 \tag{59}$$

$$\le \|C_f h_n - f(\lambda)h_n\|_2 + \|C_f h_n\|_2 \tag{60}$$

$$= \|P_{\mathcal{N}}(fh_n - f(\lambda))h_n)\|_2 + \|C_f h_n\|_2 \tag{61}$$

$$\le \|(f - f(\lambda))h_n\|^2 + \|C_f h_n\|_2 \tag{62}$$

By our earlier calculation, $\|(f - f(\lambda))h_n\|_2 \to 0$. Since C_f is compact and $h_n \to 0$ weakly, then $\|C_f h_n\|_2 \to 0$, hence, by above, $|f(\lambda)| = 0$.

Conversely, suppose that $f \in C(\overline{D})$ and vanishes on $\overline{K_e}$. Let $\varepsilon > 0$ be given and choose a $g \in C(\overline{D})$ so that $g = 0$ on a neighborhood U of $\overline{K_e}$ and $\|f - g\|_\infty < \varepsilon$. Let $\{h_n\}$ be a sequence in $\mathcal{N}$ such that $h_n \to 0$ weakly. Using Theorem 3.5, we get that each h_n has an analytic continuation across $\partial D \backslash \overline{K_e}$. Thus if we let $L = \overline{\{z \in D : g(z) \neq 0\}}$ and use Theorem 3.5, we get $h_n \to 0$ uniformly on L. Hence

$$\|C_g h_n\|_2 = \|P_{\mathcal{N}}(gh_n)\|_2 \leq \|g\|_2 \|h_n|_L\|_\infty \to 0 \tag{63}$$

so C_g is compact. Since $\|C_f - C_g\| \leq \|f - g\|_\infty$, then C_f is the norm limit of compact operators hence compact. □

To get Theorem 5.1 for the case where G is a Jordan domain, we make use of the unitary operator induced by the conformal map between D and G.

Let G be a Jordan domain and $K \subset G$ have the property $\mathcal{P}$. Since G is simply connected, there is a conformal map φ form D onto G and by a theorem of Carathéodory [M, p. 70], φ is continuous on the closure of D. Define the unitary operator $U : L^2(G) \to L^2(D)$ by

$$(Uh) = (h \circ \varphi)\varphi'.$$

Note that $UA^2(G\backslash K) = A^2(D\backslash K')$, where $K' = \varphi^{-1}(K)$, and $UA^2(G) = A^2(D)$. Thus letting $V = U|_{\mathcal{N}}$ then V becomes a unitary map from $\mathcal{N} = A^2(G\backslash K) \ominus A^2(G)$ onto $\mathcal{N}' = A^2(D\backslash K') \ominus A^2(D)$. For $h \in C(\overline{G})$ and $k \in C(\overline{D})$ define the operators $C_h^{\mathcal{N}} = P_{\mathcal{N}} M_h|_{\mathcal{N}}$ and $C_k^{\mathcal{N}'} = P_{\mathcal{N}'} M_k|_{\mathcal{N}'}$.

Lemma 5.2. *If $f \in C(\overline{G})$, then $VC_f^{\mathcal{N}} = C_{f\circ\varphi}^{\mathcal{N}'} V$.*

Proof. Let $f \in C(\overline{G})$. Let M_f and $M_{f\circ\varphi}$ be the multiplication operators on $L^2(G)$, respectively $L^2(D)$ and notice that $UM_f = M_{f\circ\varphi}U$. Also notice that $UP_{G\backslash K} = P_{D\backslash K'}U$ and $UP_G = P_D U$, hence $UP_{\mathcal{N}} = P_{\mathcal{N}'}U$. Thus, letting $V = U|_{\mathcal{N}}$ we get that $VC_f^{\mathcal{N}} = C_{f\circ\varphi}^{\mathcal{N}'} V$. □

Corollary 5.3. *Let $f \in C(\overline{G})$. C_f is compact if and only if f vanishes on $\overline{K_e}$.*

Proof. Let φ be the conformal map between D and G and let V, $\mathcal{N}'$, and $C_{f\circ\varphi}^{\mathcal{N}'}$ be as before. Since $C_f = V^* C_{f\circ\varphi}^{\mathcal{N}'} V$, then C_f is compact if and only if $C_{f\circ\varphi}^{\mathcal{N}'}$ is compact. Making use of the fact that $f \circ \varphi \in C(\overline{D})$ and applying Theorem 5.1, we see that $C_{f\circ\varphi}^{\mathcal{N}'}$ is compact if and only if $f \circ \varphi$ vanishes on $\overline{K_e'}$. But since $\varphi^{-1}(K) = K'$, then $K_e' = \varphi^{-1}(K_e)$. Hence C_f is compact if and only if f vanishes on $\overline{K_e}$. □

6. The Essential Spectrum of the Compression. Using the same techniques as in [A-C-M] and our previous results, we will now compute the essential spectrum and essential norm of C_f. The proofs in this section are almost exactly the same as the proofs used in [A-C-M] to compute the essential spectrum and essential norm of the Toeplitz operator $T_f^{G\setminus K}$ except that in our case the functions in $\mathcal{N}$ have analytic continuation across the analytic parts of ∂G that do not intersect $\overline{K_e}$. Hence, as in Corollary 5.3, our attention is restricted to the behavior of the symbol f on $\overline{K_e}$.

Let G be a Jordan region and $K \subset G$ have the property $\mathcal{P}$. Let $\mathcal{N}$ be as usual and define $\mathcal{B}(\mathcal{N})$ to be the set of bounded operators on $\mathcal{N}$, $\mathcal{K}(\mathcal{N})$ to be the set of compact operators on $\mathcal{N}$, $\mathcal{T}(\mathcal{N})$ be the algebra generated by $\{C_f : f \in C(\overline{G})\}$, and $\mathcal{J}(\mathcal{N})$ be the commutator ideal of $\mathcal{T}(\mathcal{N})$. That is $\mathcal{J}(\mathcal{N})$ is the smallest norm closed two sided ideal of $\mathcal{T}(\mathcal{N})$ which contains $\{AB - BA : A, B \in \mathcal{T}(\mathcal{N})\}$. Our aim now is similar to that of Theorem 9 of [A-C-M], which is to give a complete description of the C^*-algebra $\mathcal{T}(\mathcal{N})$. The proofs of these next two propositions use similar techniques used in the proof of Theorem 9 of [A-C-M].

Proposition 6.1. *The commutator ideal $\mathcal{J}(\mathcal{N})$ of $\mathcal{T}(\mathcal{N})$ is $\mathcal{K}(\mathcal{N})$.*

Proof. We first show that $\mathcal{T}(\mathcal{N})$ is irreducible. Suppose that $\mathcal{T}(\mathcal{N})$ was reducible. Then, there are non-zero subspaces $\mathcal{M}_1$ and $\mathcal{M}_2$ of $\mathcal{N}$ with $\mathcal{N} = \mathcal{M}_1 \oplus \mathcal{M}_2$ and $C_f\mathcal{M}_1 \subset \mathcal{M}_1$ and $C_f\mathcal{M}_2 \subset \mathcal{M}_2$ for all $f \in C(\overline{G})$. Letting $g \in \mathcal{M}_1$ and $h \in \mathcal{M}_2$, we see that

$$0 = \langle C_f g, h\rangle = \langle P_{\mathcal{N}}(fg), h\rangle = \langle fg, h\rangle \tag{64}$$

for all $f \in C(\overline{G})$. This implies that $g\overline{h} = 0$, hence $\mathcal{M}_1 = \mathcal{M}_2 = 0$, a contradiction. Next, we show that $\mathcal{T}(\mathcal{N})$ contains a non-zero compact operator. We know from Proposition 4.1 that $C_zC_{\bar{z}} - C_{|z|^2}$ is a compact operator belonging to $\mathcal{T}(\mathcal{N})$. Suppose that $C_zC_{\bar{z}} - C_{|z|^2} = 0$. Then for all $f \in \mathcal{N}$,

$$\begin{aligned} 0 &= \langle (C_zC_{\bar{z}} - C_{|z|^2})f, f\rangle \\ &= \langle z(P_{\mathcal{N}}(\bar{z}f) - \bar{z}f), f\rangle \\ &= \langle (P_{\mathcal{N}} - I)\bar{z}f, \bar{z}f\rangle \\ &= -\|(I - P_{\mathcal{N}})(\bar{z}f)\|^2. \end{aligned} \tag{65}$$

Thus $(I - P_{\mathcal{N}})(\bar{z}f) = 0$ for all $f \in \mathcal{N}$, contradicting the fact that $\bar{z}f$ cannot be analytic for all $f \in \mathcal{N}$. So $\mathcal{T}(\mathcal{N})$ is an irreducible C^*-algebra which contains a nonzero element. Thus by [Arv p. 18, Corollary 1], $\mathcal{T}(\mathcal{N}) \supset \mathcal{K}(\mathcal{N})$. By Proposition 4.1, $\mathcal{J}(\mathcal{N}) \subset \mathcal{K}(\mathcal{N})$. Since $\mathcal{J}(\mathcal{N})$ is a two-sided ideal of $\mathcal{T}(\mathcal{N})$ and $\mathcal{K}(\mathcal{N}) \subset \mathcal{T}(\mathcal{N})$, it is clear that $\mathcal{J}(\mathcal{N})$ is a two-sided ideal of $\mathcal{K}(\mathcal{N})$. Also note that $\mathcal{J}(\mathcal{N})$ is non-zero since $\mathcal{K}(\mathcal{N}) \subset \mathcal{T}(\mathcal{N})$ and so $\mathcal{T}(\mathcal{N})$ is not a commutative algebra. By using [Arv, p. 18, Corollary 1], we can conclude that $\mathcal{J}(\mathcal{N}) = \mathcal{K}(\mathcal{N})$. □

Proposition 6.2. *$\mathcal{T}(\mathcal{N})/\mathcal{K}(\mathcal{N})$ and $C(\overline{K_e})$ are isometrically *-isomorphic C^* algebras with an isomorphism that maps $C_f + \mathcal{K}(\mathcal{N})$ to $f|_{\overline{K_e}}$ for all $f \in C(\bar{G})$.*

Proof. Define a map $\alpha : C(\bar{G}) \to \mathcal{T}(\mathcal{N})/\mathcal{K}(\mathcal{N})$ by

$$\alpha(f) = C_f + \mathcal{K}(\mathcal{N}). \tag{66}$$

By Proposition 4.1, α is a homomorphism, hence $\alpha(C(\bar{G}))$ is a dense subalgebra of $\mathcal{T}(\mathcal{N})/\mathcal{K}(\mathcal{N})$. Define

$$Z(G) = \left\{ f \in C(\bar{G}) : f\Big|_{\overline{K_e}} = 0 \right\}. \tag{67}$$

By Corollary 5.3, $Z(G)$ is exactly the kernel of α. Thus there is a homomorphism

$$\tilde{\alpha} : C(\bar{G})/Z(G) \to \mathcal{T}(\mathcal{N})/\mathcal{K}(\mathcal{N}) \tag{68}$$

which is injective and

$$\tilde{\alpha}(f + Z(G)) = C_f + \mathcal{K}(\mathcal{N}). \tag{69}$$

Since $\tilde{\alpha}$ is an injective C^* homomorphism it is an isometry, thus $\tilde{\alpha}$ has dense range. But α and $\tilde{\alpha}$ have the same range, hence $\tilde{\alpha}$ is a C^* isomorphism. Define $F : C(\bar{G})/Z(G) \to C(\overline{K_e})$ by

$$F(f + Z(G)) = f\Big|_{\overline{K_e}}. \tag{70}$$

F is an isometric isomorphism and $F \circ \tilde{\alpha}^{-1}$ is our desired isometric isomorphism with

$$F \circ \tilde{\alpha}^{-1}(C_f + \mathcal{K}(\mathcal{N})) = f\Big|_{\overline{K_e}}. \tag{71}$$

□

Theorem 6.3. *If $f \in C(\bar{G})$, then $\sigma_e(C_f) = f(\overline{K_e})$.*

Proof. The spectrum of $f|\overline{K_e}$ in $C(\overline{K_e})$ is $f(\overline{K_e})$. Thus by Proposition 6.2, the spectrum of the coset $C_f + \mathcal{K}(\mathcal{N})$ in the C^* algebra $\mathcal{T}(\mathcal{N})/\mathcal{K}(\mathcal{N})$ is $f(\overline{K_e})$. Using the fact that the spectrum of an element of a C^* algebra remains unchanged when the algebra is enlarged, we see that the spectrum of the coset $C_f + \mathcal{K}(\mathcal{N})$ in the algebra $\mathcal{B}(\mathcal{N})/\mathcal{K}(\mathcal{N})$ is $f(\overline{K_e})$. Thus $\sigma_e(C_f) = f(\overline{K_e})$. □

Corollary 6.4. *If $f \in C(\bar{G})$, then $\|C_f\|_e = \left\| f|_{\overline{K_e}} \right\|_\infty$.*

Proof. By Proposition 4.1 and the fact that $C_f^* = C_{\overline{f}}$ we see that the coset $C_f + \mathcal{K}(\mathcal{N})$ is a normal element of $\mathcal{B}(\mathcal{N})/\mathcal{K}(\mathcal{N})$. Hence the norm of $C_f + \mathcal{K}(\mathcal{N})$ (ie. $\|C_f\|_e$) is equal to its spectral radius. Thus

$$\|C_f\|_e = \sup\{|\lambda| : \lambda \in \sigma_e(C_f)\} \tag{72}$$

$$= \sup\{|f(z)| : z \in \overline{K_e}\}. \tag{73}$$

□

We can apply Theorem 6.3 to the special case where $f(z) = z$ to compute the spectrum of the compression C_z.

Corollary 6.5. $\sigma(C_z) = \overline{K_e}$.

Proof. Since $\sigma_e(C_z) = \overline{K_e}$, then $\overline{K_e} \subset \sigma(C_z)$. For the reverse inclusion, let $\lambda \notin \overline{K_e}$. Then $\lambda \notin \sigma_e(C_z)$ so by Proposition 4.2, $\text{ind}(C_{z-\lambda}) = 0$. Since $\ker(C_{z-\lambda}) = \{0\}$, then $C_{z-\lambda}$ is invertible, hence $\lambda \notin \sigma(C_z)$. □

7. Similarity.

In this last part we use Corollary 6.5 in conjunction with results in [A-F-V], [A-C-M], and [Ro] to prove a similarity theorem for the compression C_z. It is remarkable that similarity class of C_z depends only on K_e and not the domain G.

Let G_1 and G_2 be two Jordan regions and $K_1 \subset G_1$ and $K_2 \subset G_2$ each compact and having the property $\mathcal{P}$. For $i = 1, 2$, let C_i be C_z acting on $\mathcal{N}_i$. That is $C_i = P_{\mathcal{N}_i} T_z^{G_i \backslash K_i}|_{\mathcal{N}_i}$, where $\mathcal{N}_i = A^2(G_i \backslash K_i) \ominus A^2(G_i)$.

Theorem 7.1. *The following are equivalent:*

(i) *C_1 and C_2 are similar.*
(ii) $(K_1)_e = (K_2)_e$.
(iii) *$K_1 \backslash K_2$ and $K_2 \backslash K_1$ have zero logarithmic capacity.*

Before we proceed to the proof of Theorem 7.1, we must first set up some preliminaries. For more information and details, see [A-F-V] and [Ro].

For a bounded region U in $\mathbf{C}$, let $C_0^\infty(U)$ be the set of infinitely differentiable functions with compact support on U. Define the Sobolev space $W_1^{2,0}(U)$ to be the completion of $C_0^\infty(U)$ in the following norm

$$\|v\|_{W_1^{2,0}(U)} = \left(\int_U |\nabla v|^2 \, dA\right)^{1/2}.$$

Let $B^2(U) = L^2(U) \ominus A^2(U)$ and note that by a result of [A-F-V] (also see [Ro]) that $B^2(U)$ and $W_1^{2,0}(U)$ are isomorphic via the unitary differential operator

$D_U = 2\frac{\partial}{\partial z}$, where $\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$. Let S_U denote the multiplication operator on $A^2(U)$ defined by

$$(S_U f)(z) = zf(z).$$

By the definition of $B^2(U)$, we can define R_U on $B^2(U)$ by

$$(R_U g)(z) = \bar{z}g(z).$$

(Note that $B^2(U)$ is R_U-invariant since $A^2(U)$ is S_U-invariant.) We can also define M_U on $W_1^{2,0}(U)$ by

$$(M_U h)(z) = \bar{z}h(z).$$

By [A-F-V] (also se [Ro]), R_U and M_U are continuous and are unitarily equivalent via D_U with $D_U M_U = R_U M_U$.

Letting G_1 and G_2 be as in Theorem 7.1, one sees that for $i = 1, 2$, R_{G_i} will induce a continuous operator $\tilde{R_{G_i}}$ on the quotient space $B^2(G_i)/B^2(G_i \backslash K_i)$ by $\tilde{R}_{G_i}[g] = [R_{G_i} g]$, where $[g]$ denotes the coset of g. Also note that M_{G_i} $(i = 1, 2)$ will induce a continuous operator $\tilde{M}_{G_i}$ on the quotient space

$$W_1^{2,0}(G_i)/W_1^{2,0}(G_i \backslash K_i)$$

by $\tilde{M}_{G_i}[h] = [M_{G_i} h]$. The differential operator D_{G_i} will induce the obvious isomorphism $\tilde{D_{G_i}}$ from $W_1^{2,0}(G_i)/W_1^{2,0}(G_i \backslash K_i)$ onto $B^2(G_i)/B^2(G_i \backslash K_i)$ with

$$\tilde{D}_{G_i} \tilde{M}_{G_i} = \tilde{R}_{G_i} \tilde{D}_{G_i}. \tag{74}$$

One sees that $\mathcal{N}_i = B^2(G_i) \ominus B^2(G_i \backslash K_i)$, and hence $\mathcal{N}_i$ is isomorphic to the quotient space $B^2(G_i)/B^2(G_i \backslash K_i)$ via the natural map $q_i(f) = [f]$. Finally, notice that $q_i C_i^* = \tilde{R}_{G_i} q_i$. Hence C_i^* is equivalent to $\tilde{R}_{G_i}$, which, by above, makes C_i^* unitarily equivalent to $\tilde{M}_{G_i}$ $(i = 1, 2)$.

For a set $E \subset \mathbf{R}^2$, let Cap(E) denote the logarithmic capacity of E, (see [Ca]). We say a property holds *quasi-everywhere* if it holds except on a set of logarithmic capacity zero. A function $f : \mathbf{R}^2 \to \mathbf{C}$ is said to be *quasi-continuous* if for every $\varepsilon > 0$, there is an open set W with $Cap(W) < \varepsilon$ and the restriction of f to $\mathbf{R}^2 \backslash W$ is continuous. For further explanations, see [Ba]. For a bounded region U, every function in $W_1^{2,0}(U)$ has a quasi-continuous representative [M-H], and a result of Bagby [Ba] will imply that if $K \subset U$ is compact, then a quasi-continuous $f \in W_1^{2,0}(U)$ belongs to $W_1^{2,0}(U \backslash K)$ if and only if f vanishes quasi-everywhere on K.

Proof of Theorem 7.1.

(i) $\Rightarrow$ (ii): Use Corollary 6.5.

(ii) $\Rightarrow$ (i): On the other hand, suppose that $(K_1)_e = (K_2)_e = K$. By Proposition 2.1, $A^2(G_i\backslash K_i) = A^2(G_i\backslash K)$, hence by our above discussion, C_i^* is unitarily equivalent to $\tilde{M}_{G_i}$ on the quotient space $W_1^{2,0}(G_i)/W_1^{2,0}(G_i\backslash K)$ $(i = 1,2)$. We will complete the proof by showing that $\tilde{M}_{G_1}$ and $\tilde{M}_{G_2}$ are similar.

Let $G = G_1 \cap G_2$. Then, the injection

$$J_i : W_1^{2,0}(G) \to W_1^{2,0}(G_i) \tag{75}$$

is an isometry, and using the above mentioned result of Bagby [Ba], J_i will induce the injective operator $\tilde{J}_i : W_1^{2,0}(G)/W_1^{2,0}(G\backslash K) \to W_1^{2,0}(G_i)/W_1^{2,0}(G_i\backslash K)$ by $\tilde{J}_i[h] = [h]$. $\tilde{J}_i$ is also surjective since if $[h] \in W_1^{2,0}(G_i)/W_1^{2,0}(G_i\backslash K)$, then letting $\varphi \in C_0^\infty(G)$ with $\varphi = 1$ on K, will give us that $[\varphi h] \in W_1^{2,0}(G)/W_1^{2,0}(G\backslash K)$ with $\tilde{J}_i[\varphi h] = [h]$. Thus $\tilde{J}_i$ is invertible for $i = 1,2$ and one easily checks that $\tilde{J}_i\tilde{M}_G = \tilde{M}_{G_i}\tilde{J}_i$ $(i = 1,2)$. Thus $\tilde{M}_{G_1}$ and $\tilde{M}_{G_2}$ are similar, which gives us the similarity of C_1 and C_2.

(ii) $\Rightarrow$ (iii): By Proposition 2.2, $\mathrm{Cap}(K_i\backslash(K_i)_e) = 0$ $(i = 1,2)$. Thus if $(K_1)_e = (K_2)_e$, then $K_1\backslash K_2$ and $K_2\backslash K_1$ have zero logarithmic capacity.

(iii) $\Rightarrow$ (ii): Conversely, suppose that $K_1\backslash K_2$ and $K_2\backslash K_1$ have zero logarithmic capacity. Then $\mathrm{Cap}((K_1)_e\backslash(K_2)_e) = 0$ and $\mathrm{Cap}((K_2)_e\backslash(K_1)_e) = 0$. If, for example, $(K_1)_e\backslash(K_2)_e \neq \emptyset$, then, letting $\lambda \in (K_1)_e\backslash(K_2)_e$ and using the fact that $(K_1)_e\backslash(K_2)_e$ is relatively open, one sees that there is a positive δ so that

$$\overline{B(\lambda;\delta)} \cap (K_1)_e \subset (K_1)_e\backslash(K_2)_e.$$

Since $\mathrm{Cap}(\overline{B(\lambda;\delta)} \cap (K_1)_e) = 0$, then by above, $\mathrm{Cap}(\overline{B(\lambda;\delta)} \cap K_1) = 0$. By Proposition 2.3, λ is removable, a contradiction. □

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